

Shapley-Scarf Housing Markets: Respecting Improvement, Integer Programming, and Kidney Exchange

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Abstract

In a housing market of Shapley and Scarf [46], each agent is endowed with one indivisible object and has preferences over all objects. An allocation of the objects is in the (strong) core if there exists no (weakly) blocking coalition. We show that for strict preferences the unique strong core allocation “respects improvement”: if an agent’s object becomes more desirable for some other agents, then the agent’s allotment in the unique strong core allocation weakly improves. We obtain similar results for the domain of weak preferences.

Respecting improvements is an important property for applications of the housing markets model such as kidney exchange: it incentivises each patient to bring the best possible set of donors to the market. We conduct computer simulations using markets that resemble the pools of kidney exchange programmes. We compare the game-theoretical solutions with current techniques (maximum size and maximum weight allocations) in terms of price of fairness, number of blocking cycles, and number of violations of the respecting improvement property. We find that game-theoretical solutions fare much better at respecting improvements and they do so at a low efficiency cost. As a stepping-stone for our simulations, we provide novel integer programming formulations for computing core, strong core, and competitive allocations.

Keywords: housing market, respecting improvement, core, competitive allocations, integer programming, kidney exchange programmes.

1 Introduction

Shapley and Scarf [46] introduced so-called “housing markets” to model trading in commodities that are inherently indivisible. Specifically, in a housing market each agent is endowed with an object (e.g., a house or a kidney donor) and has ordinal preferences over all objects, including her own. The aim is to find plausible or desirable allocations where each agent is assigned one object. A standard approach in the literature is to discard allocations that can be blocked by a coalition of agents. Specifically, a coalition of agents blocks an allocation if they can trade their endowments so

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that each of the agents in the coalition obtains a strictly preferred allotment. Similarly, a coalition of agents weakly blocks an allocation if they can trade their endowments so that each of the agents in the coalition obtains a weakly preferred allotment and at least one of them obtains a strictly preferred allotment. Thus, an allocation is in the (strong) core if it is not (weakly) blocked. A distinct but also well-studied solution concept is obtained from competitive equilibria, each of which consists of a vector of prices for the objects and a (competitive) allocation such that each agent’s allotment is one of her most preferred objects among those that she can afford. Interestingly, the three solution concepts are entwined: the strong core is contained in the set of competitive allocations, and each competitive allocation pertains to the core.

In a separate line of research, Balinski and Sönmez [9] studied the classical two-sided college admissions model of Gale and Shapley [20] and proved that the student-optimal stable matching mechanism (SOSM) *respects improvement* of student’s quality. This means that under SOSM, an improvement of a student’s rank at a college will, *ceteris paribus*, lead to a weakly preferred match for the student. The natural transposition of this property to (one-sided) housing markets requires that an agent obtains a weakly preferred allotment whenever her object becomes more desirable for other agents. We study the following question: Do the most prominent solution concepts for Shapley and Scarf’s housing market [46] “respect improvement”? We obtain several positive answers to this question, which we describe in more detail in the next subsection.

The respecting improvement property is important in many applications where centralised clearinghouses use mechanisms to implement barter exchanges. A leading example are kidney exchange programmes (KEPs), where end-stage renal patients exchange their willing but immunologically incompatible donors [40]. In the context of KEPs, the respecting improvement property means that whenever a patient brings a “better” donor (e.g., younger or with universal blood type 0 instead of A, B, or AB) or registers an additional donor, the KEP should assign her the same or a better exchange donor.¹ In other words, the respecting improvement property incentivises each patient to bring the best possible set of donors to the market. However, in current KEPs, the typical objective is to maximise the number of transplants and their overall qualities (see, e.g., [14]) which can lead to violations of the respecting improvement property. As an illustration, consider the maximisation of the number of transplants in Figure 1, where each node represents a patient-donor pair. A directed edge, from A to B say, indicates the compatibility of the donor in node B with the patient in node A . Patients may have different levels of preference over their set of compatible donors. Initially there are only continuous edges, where a thick (thin) edge points to the most (least) preferred donor. For example, patient 3 has two compatible donors: donors 1 and 4, and donor 1 is preferred to donor 4. Obviously, the unique way to maximise the number of (compatible) transplants is obtained by picking the three-cycle (1,2,3). Suppose that patient 3 succeeds in bringing a second donor to the KEP and this donor turns out to be compatible for patient 4. Then, the discontinuous edge is included as patient-donor pair 3 “improves.” But now the unique way to

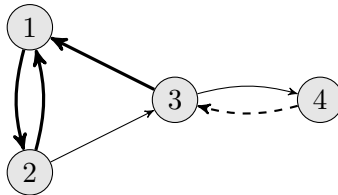


Figure 1: The maximisation of the number of transplants does not respect improvement.

(least) preferred donor. For example, patient 3 has two compatible donors: donors 1 and 4, and donor 1 is preferred to donor 4. Obviously, the unique way to maximise the number of (compatible) transplants is obtained by picking the three-cycle (1,2,3). Suppose that patient 3 succeeds in bringing a second donor to the KEP and this donor turns out to be compatible for patient 4. Then, the discontinuous edge is included as patient-donor pair 3 “improves.” But now the unique way to

¹Allowing for additional donors does not require an extension to a model where agents can be endowed with multiple objects: an agent’s set of donors can only be assigned to one other agent and this agent can only “consume” its most preferred element from the set.

maximise the number of (compatible) transplants is obtained by picking the two two-cycles (1,2) and (3,4), which means that patient 3 receives a kidney that is strictly worse than the kidney she would have received initially.

Similarly, the allocations induced by the standard objectives of KEPs need not pertain to the core. We refer to Example 1 for an illustration of this for the case of the maximisation of the number of transplants. As a consequence, blocking coalitions may exist. This is an undesirable feature because patient groups could make a potentially justified claim that the matching procedure is not in their best interest. A particular instance could occur in the organisation of international kidney exchanges if a group of patient-donor pairs, all citizens of the same country, learn that an internal (i.e., national) matching would yield a better match for all of them.

Next, we describe our contributions and review the related literature.

1.1 Contributions

Section 3 contains our theoretical results on the respecting improvement property. First, we show that for strict preferences the unique strong core allocation (which coincides with the unique competitive allocation) respects improvement (Theorem 1).

In the case of preferences with ties, since the strong core can be empty, we focus on the set of competitive allocations. Since typically multiple competitive allocations exist, we have to make setwise comparisons. We obtain a natural extension of our first result: focusing on the agent's allotments obtained at competitive allocations, we establish that her most preferred allotment in the new market is weakly preferred to her most preferred allotment in the initial market; and similarly, her least preferred allotment in the new market is weakly preferred to her least preferred allotment in the initial market (Proposition 1). Finally, we also prove that when preferences have ties the strong core respects improvement conditional on the strong core being non-empty. More precisely, under the assumption that strong core allocations exist in both the initial and new markets, we show that the agent under consideration weakly prefers each allotment in the new strong core to each allotment in the initial strong core (Theorem 2).

The literature review and the rest of the results for bounded length cycles, IP programming and simulations are relegated to the Appendix.

Next, in Appendix C, we relax an important assumption in the housing market of Shapley and Scarf, namely that allocations can contain exchange cycles of any length, i.e., cycles are unbounded. The definition of core and strong core can be naturally adjusted to the requirement that the length of exchange cycles does not exceed an exogenously given maximum. Wako [51] shows that the set of competitive allocations coincides with the core based on a antisymmetric weak domination concept. This equivalent definition, which we call the Wako-core, allows for a natural direct extension to the case of bounded exchange cycles. Unfortunately, when exchange cycles are bounded the core (and hence also the set of competitive allocations and the strong core) can be empty.² Conditional on the existence of a core, competitive, or strong core allocation, we show that even if preferences are strict, when the length of exchange cycles is limited (upper bound 3 or higher), the core, the set of competitive allocations, and the strong core do not respect improvement in terms of the most preferred allotment (Proposition 3).

In Appendix D, we provide novel integer programming (IP) formulations for finding core, competitive, and strong core allocations, which serves as a stepping-stone for our simulations in Section 3.2. Our simple sets of constraints for the three solution concepts clearly show the hierarchy

²The corresponding decision problem is NP-hard [12, 23] even for tripartite graphs (also known as the cyclic 3D stable matching problem [34]).

between them by pinpointing the additional requirements needed when moving from one solution concept to a stronger one. Our formulations are concise and useful for practical computations.

Appendix E complements our theoretical analysis and consists of computer simulations comparing core, competitive, and strong core allocations with maximum size and maximum total weight³ allocations. To carry out our simulations we draw markets from pools similar to those observed in KEPs and study both unbounded and bounded length exchange cycles.⁴ The maximisation of the size and the weight of the allocations correspond to the maximisation of the number of transplants and overall quality of the transplants, respectively. In the simulations we use our novel IP formulations for unbounded length exchanges and the IP models developed in [25] for bounded length exchanges.

2 Preliminaries

We consider housing markets as introduced by Shapley and Scarf [46]. Let $N = \{1, \dots, n\}$, $n \geq 2$, be the set of *agents*. Each agent $i \in N$ is endowed with one object, which with some abuse of notation is denoted by i . Thus, N also denotes the set of *objects*. Each agent $i \in N$ has complete and transitive (*weak*) *preferences* R_i over objects.⁵ We denote the strict part of R_i by P_i , i.e., for all $j, k \in N$, $jP_i k$ if and only if $jR_i k$ and not $kR_i j$. Similarly, we denote the indifference part of R_i by I_i , i.e., for all $j, k \in N$, $jI_i k$ if and only if $jR_i k$ and $kR_i j$. Let $R \equiv (R_i)_{i \in N}$. A (*housing*) *market* is a pair (N, R) , or if no confusion is possible, simply R .⁶ Object $j \in N$ is *acceptable* to agent $i \in N$ if $jR_i i$. Agent i 's preferences are called *strict* if they do not exhibit *ties* between acceptable objects, i.e., for all acceptable $j, k \in N$ with $j \neq k$ we have $jP_i k$ or $kP_i j$. A housing market has strict preferences if each agent has strict preferences. A housing market where agents do not necessarily have strict preferences is referred to as a housing market with weak preferences.

Given a housing market $M = (N, R)$ and a set $S \subseteq N$, the *submarket* M_S is the housing market where S is the set of agents/objects and where the preferences $(R_i)_{i \in S}$ are restricted to the objects in S .

The *acceptability graph* of a housing market $M = (N, R)$ is the directed graph $G_M = (N, E)$, or G for short, where the set of nodes is N and where (i, j) is a directed edge in E if j is an acceptable object for i , i.e., $jR_i i$. In particular, all self-loops (i, i) are in the graph (but for convenience they are omitted in all figures). Let $\tilde{N} \subseteq N$ and $\tilde{E} \subseteq E \cap (\tilde{N} \times \tilde{N})$. For each $i \in \tilde{N}$, the set of agent i 's *most preferred edges* in graph $\tilde{G} \equiv (\tilde{N}, \tilde{E})$ or simply $\tilde{E}$ is the set $\tilde{E}^{T,i} \equiv \{(i, j) : (i, j) \in \tilde{E} \text{ and for each } (i, k) \in \tilde{E}, jR_i k\}$. The most preferred edges in graph $\tilde{G}$ is the set $\cup_{i \in \tilde{N}} \tilde{E}^{T,i}$.

Let $M = (N, R)$ be a housing market. An allocation is a redistribution of the objects such that each agent receives exactly one object, i.e., an *allocation* is a vector $x = (x_i)_{i \in N} \in N^N$ such that:

- (1) for each $i \in N$, $x_i \in N$ denotes agent i 's *allotment*, i.e., the object that she receives, and

³In the literature on KEPs it is often assumed that each edge has a weight representing the fit/quality of the donor's kidney for the receiving patient. The total weight of an exchange cycle is the sum of the weights associated with the edges involved in the exchange. The total weight of an allocation is the sum of the weights of its exchange cycles. Details are in Section 3.2.

⁴In KEPs, all transplants in the same exchange cycle are usually carried out simultaneously. Obviously, if the number of surgical teams and operation rooms is small, some of the transplants have to be conducted in a non-simultaneous way. In many countries, this "risky" solution is not allowed because of possible renegeing [10]. Thus, in practice, exchange cycles are usually bounded.

⁵In other words, an agent can be indifferent between objects, including her own endowment.

⁶Therefore, when keeping the set of agents fixed, we interchangeably refer to R as the profile of preferences and the market.

- (2) no object is assigned to more than one agent, i.e., $\cup_{i \in N} \{x_i\} = N$.

We will focus on *individually rational* allocations, i.e., allocations where each agent receives an acceptable object. Then, an allocation x can equivalently be described by its corresponding *cycle cover* g^x of the acceptability graph G . Formally, $g^x = (N, E^x)$ is the subgraph of G where $(i, j) \in E^x$ if and only if $x_i = j$. Thus, the graph g^x consists of disconnected *trading cycles* or *exchange cycles*⁷ that cover G . We will often write an (individually rational) allocation in cycle-notation, i.e., as a set of exchange cycles (where we sometimes omit self-cycles). We refer to Example 1 for an illustration.

An allocation x Pareto-dominates an allocation z if for each $i \in N$, $x_i R_i z_i$, and for some $j \in N$, $x_j P_j z_j$. An allocation is *Pareto-efficient* if it is not Pareto-dominated by any allocation. Two allocations x, z are *welfare-equivalent* if for each $i \in N$, $x_i I_i z_i$.

Next, we recall the definition of solution concepts that have been studied in the literature. A non-empty coalition $S \subseteq N$ *blocks* an allocation x if there is an allocation z such that

- (1) $\{z_i : i \in S\} = S$ and
- (2) for each $i \in S$, $z_i P_i x_i$.

An allocation x is in the *core*⁸ of the market if there is no coalition that blocks x . For each market R , let $\mathcal{C}(R)$ denote its core.

A non-empty coalition $S \subseteq N$ *weakly blocks* an allocation x if there is an allocation z such that

- (1) $\{z_i : i \in S\} = S$,
- (2) for each $i \in S$, $z_i R_i x_i$, and
- (3) for some $j \in S$, $z_j P_j x_j$.

An allocation x is in the *strong core*⁹ of the market if there is no coalition that weakly blocks x . For each market R , let $\mathcal{SC}(R)$ denote its (possibly empty) strong core.

A price-vector is a vector $p = (p_i)_{i \in N} \in \mathbb{R}^N$ where p_i denotes the price of object i . A competitive equilibrium is a pair (x, p) where x is an allocation and p is a price-vector such that:

- (1) for each agent $i \in N$, object x_i is affordable, i.e., $p_{x_i} \leq p_i$ and
- (2) for each agent $i \in N$, each object she prefers to x_i is not affordable, i.e., $j P_i x_i$ implies $p_j > p_i$.

An allocation is a *competitive allocation* if it is part of some competitive equilibrium. Since there are n objects, we can assume, without loss of generality, that prices are integers in the set $\{1, 2, \dots, n\}$.

Remark 1. If (x, p) is such that

- (1) for each $i \in N$, $p_{x_i} \leq p_i$, or
- (2) for each $i \in N$, $p_i \leq p_{x_i}$,

then for each $i \in N$, $p_{x_i} = p_i$. This follows immediately by looking at each exchange cycle separately (see, e.g., the proof of Lemma 1 in [16]). Hence, at each competitive equilibrium (x, p) , for each $i \in N$, $p_{x_i} = p_i$. $\diamond$

⁷Note that each trading/exchange cycle is a non-empty directed path in which only the first and last nodes are equal. A single node is a self-cycle, i.e., a degenerate cycle.

⁸In the literature the core is sometimes called the weak core or “regular” core.

⁹In the literature the strong core is sometimes called the strict core.

[51] proved that the set of competitive allocations can be defined equivalently as a different type of core. Formally, a non-empty coalition $S \subseteq N$ *antisymmetrically weakly blocks* an allocation x if there is an allocation z such that:

- (1) $\{z_i : i \in S\} = S$,
- (2) for each $i \in S$, $z_i R_i x_i$,
- (3) for some $j \in S$, $z_j P_j x_j$, and
- (4) for each $i \in S$, if $z_i I_i x_i$ then $z_i = x_i$.

Requirements (1–3) say that coalition S weakly blocks x . The additional requirement (4) is that if an agent in S is indifferent between her allotments at x and z then she must get the very same object, i.e., $z_i = x_i$. An allocation x is in the *core defined by antisymmetric weak domination* if there is no coalition that antisymmetrically weakly blocks x . [51] proved that the set of competitive allocations coincides with the core defined by antisymmetric weak domination. Henceforth, we will often refer to the set of competitive allocations as the *Wako-core*, and for each market R we denote this set by $\mathcal{WC}(R)$. Note that when preferences are strict, requirement (4) is redundant and the equivalence of strong core and Wako-core follows immediately.

Note that the three blocking notions introduced above are “nested”: blocking implies antisymmetrical weak blocking and antisymmetrical weak blocking implies weak blocking. Therefore, for each market R , $\mathcal{SC}(R) \subseteq \mathcal{WC}(R) \subseteq \mathcal{C}(R)$.¹⁰

The following lemma is helpful for computations and is also used in our IP formulations and simulations. It states that to check whether an allocation belongs to any of the three cores, it is sufficient to check potential blocking by coalitions that constitute cycles in the acceptability graph, i.e., it is not necessary to check all coalitions.

Lemma 1. *The strong core, the set of competitive allocations (i.e., the Wako-core), and the core consist of individually rational allocations. Moreover, each of the cores is equivalently characterised by the absence of blocking cycles in the acceptability graph $G = (N, E)$. In other words, it is sufficient to require that each coalition $\{i_1, \dots, i_k\}$ such that for each $l = 1, \dots, k \pmod k$, $(i_l, i_{l+1}) \in E$, does not block through some allocation z where for each $l = 1, \dots, k \pmod k$, $z_{i_l} = i_{l+1}$.*

Proof. Individual rationality is immediate. To prove the statement for the strong core, let x be an individually rational allocation. Suppose there is a non-empty coalition T that weakly blocks x through some allocation w . Let $j \in T$ be such that $w_j P_j x_j$. Let $S \subseteq T$ be the agents that constitute the exchange cycle, say $(i_1, \dots, i_k)$, in w that involves agent j , i.e., without loss of generality, $j = i_1$. Since w is individually rational, $S = \{i_1, \dots, i_k\}$ weakly blocks x through the allocation z defined by

$$z_i \equiv \begin{cases} w_i & \text{if } i \in S; \\ x_i & \text{if } i \notin S \end{cases}$$

and for each $l = 1, \dots, k \pmod k$, $z_{i_l} = i_{l+1}$ and $(i_l, i_{l+1}) \in E$. This proves the statement for the strong core. The statements for the core and the Wako-core follow similarly. □

¹⁰[50] showed that the strong core coincides with the set of competitive allocations if and only if any two competitive allocations are welfare-equivalent. Hence, whenever the set of competitive allocations is a singleton it coincides with the strong core.

An individually rational allocation x is a *maximum size allocation* if for each individually rational allocation z , $|\{i \in N : x_i \neq i\}| \geq |\{i \in N : z_i \neq i\}|$. Below we provide an example to illustrate the three cores and maximum size allocation.

Example 1. Let $N = \{1, \dots, 6\}$ and let preferences be given by Table 1, where each agent's own object and all her unacceptable objects are not displayed. For instance, agent 1 is indifferent between objects 2 and 3, and strictly prefers both objects to object 5.

1	2	3	4	5	6
2,3	1	2	3	2	1
5	3	4	2	6	

Table 1: Preferences

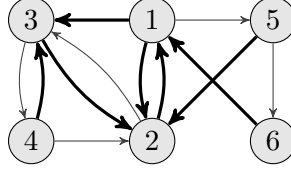


Figure 2: Acceptability graph

$$\begin{aligned}
 x^a &= \{(1, 3, 2)\} \\
 x^b &= \{(1, 2), (3, 4)\} \\
 x^c &= \{(1, 5, 2), (3, 4)\} \\
 x^d &= \{(1, 3, 4, 2)\} \\
 x^e &= \{(1, 5, 6), (2, 3, 4)\}
 \end{aligned}$$

Table 2: Allocations

Figure 2 displays the induced acceptability graph.¹¹ Here, a thick edge denotes the most preferred object(s) and a thin edge denotes the second most preferred object (if any).

Consider the allocations defined in Table 2. For instance, x^d (in cycle-notation, but without self-cycles) is the allocation $x^d = (x_1^d, x_2^d, x_3^d, x_4^d, x_5^d, x_6^d) = (3, 1, 4, 2, 5, 6)$. Using Lemma 1, it can be easily verified that x^a is the unique strong core allocation, x^a and x^b are the competitive allocations, while x^a , x^b , x^c , and x^d form the core. Hence, the strong core is a singleton and a proper subset of the set of competitive allocations, while the latter set is also a proper subset of the core. Finally, x^e is the unique maximum size allocation and does not pertain to the core. $\diamond$

Shapley and Scarf [46] (see also page 135 in [39]) showed that the set of competitive allocations is non-empty and coincides with the set of allocations that are obtained through David Gale's Top Trading Cycles algorithm,¹² which is discussed in Section 3.1. Roth and Postlewaite [39] showed that if preferences are strict, then there is a unique strong core allocation which coincides with the unique competitive allocation. In general, when preferences are not strict, the strong core can be empty (see, e.g., Footnote 13) or contain more than one allocation (see, e.g., Example 1).

If preferences are strict, the unique competitive allocation is Pareto-efficient (because it is in the strong core) and Pareto-dominates any other allocation (Lemma 1 in [39]); in particular, any other core allocation is Pareto-inefficient. If preferences are not strict, it is possible that each competitive allocation is Pareto-dominated by some allocation that is not competitive.¹³

Finally, competitive allocations need not be welfare-equivalent: in fact, different agents can strictly prefer distinct competitive allocations (see, e.g., Footnote 13). However, Wako [50] showed that all strong core allocations are welfare-equivalent. The latter result also immediately follows from Quint and Wako's algorithm [38], which is discussed in Section 3.2.

3 Respecting Improvement

Let $R, \tilde{R}$ be two preference profiles over objects N . Let $i \in N$. We say that $\tilde{R}$ is an *improvement for i with respect to R* if the only difference between R and $\tilde{R}$ is that at $\tilde{R}$ object i is ranked higher

¹¹Throughout the paper, self-loops are omitted from the acceptability graphs in the examples.

¹²If preferences are not strict, then the Top Trading Cycles algorithm is applied to the preference profiles that can be obtained by breaking ties in all possible ways.

¹³Example 1 in [48], which is attributed to Jun Wako, is illustrative: $N = \{1, 2, 3\}$ with $2P_13P_11, 1I_23P_22, 2P_31P_33$. The set of competitive allocations consists of $x = \{(1, 2), (3)\}$ and $x' = \{(1), (2, 3)\}$, which are Pareto-dominated by core allocations $\{(1, 2, 3)\}$ and $\{(1, 3, 2)\}$, respectively. Moreover, $x_1P_1x'_1$ and $x'_3P_3x_3$. The strong core is empty.

by the other agents than at R . Formally,

- (1) $\tilde{R}_i = R_i$;
- (2) for all $j \neq i$ and all k with $k R_j j$, $i I_j k \implies i \tilde{R}_j k$ and $i P_j k \implies i \tilde{P}_j k$; and
- (3) for all $j \neq i$ and all $k, l \neq i$ with $k, l R_j j$, $k R_j l \iff k \tilde{R}_j l$.

In other words, (1) only agents different from i have possibly different preferences at $\tilde{R}$ and R , (2) for each agent $j \neq i$, object i can become more preferred than some acceptable objects, and (3) for each agent $j \neq i$ and for each pair of acceptable objects different from i , preferences remain unchanged.

As a simple example with $N = \{1, 2, 3, 4, 5\}$, let R be any preference profile such that $4 P_5 1 I_5 2 I_5 3 P_5 5$. Let $\tilde{R}$ be the preference profile where agents 1, 2, 3, and 4 have the same preferences as at R and let $\tilde{R}_5$ be defined by $1 I_5 4 P_5 2 I_5 3 P_5 5$. Then, $\tilde{R}$ is an improvement for agent 1 with respect to R .

Let $\mathcal{R}$ be a domain of markets. A single-valued allocation rule (on $\mathcal{R}$) is a map ϕ that associates with each market $R \in \mathcal{R}$ an allocation $\phi(R)$. For each $i \in N$, let $\phi_i(R)$ denote agent i 's allotment at $\phi(R)$. We say that ϕ *respects improvement* (on $\mathcal{R}$) if for each $i \in N$ and each pair of markets $R, \tilde{R} \in \mathcal{R}$ such that $\tilde{R}$ is an improvement for i with respect to R , we have that $\phi_i(\tilde{R}) R_i \phi_i(R)$. Respecting improvement is a natural and important property for applications of the housing markets model such as kidney exchange programmes. If the programme employs an allocation rule that respects improvement, then it incentivises all patients to bring the best possible set of donors to the market.

We also study a (generalized) respecting improvement property for multi-valued allocation rules. A multi-valued allocation rule (on $\mathcal{R}$) is a map Φ that associates with each market $R \in \mathcal{R}$ a (possibly empty) set of allocations $\Phi(R)$.

Since multi-valued allocation rules yield sets of allocations, we will have to compare sets of allotments for individual agents. Let X be a set of allocations and $i \in N$. The *best allotments* for agent i at X are the objects that she weakly most prefers among all objects in $\{x_i : x \in X\}$, denoted by X_i^+ . Analogously, the *worst allotments* for i at X are the objects that she weakly least prefers among all objects in $\{x_i : x \in X\}$, denoted by X_i^- .

Let Φ be a non-empty¹⁴

Finally, in the case of a multi-valued allocation rule Φ that for some markets can yield the empty set, we say that it *conditionally respects improvement* if the above requirements hold conditional on $\Phi(R), \Phi(\tilde{R}) \neq \emptyset$.

3.1 Strict preferences

We consider housing markets with strict preferences and weak preferences separately. The reason is that when preferences are strict, the strong core is always a singleton (which consists of the unique competitive allocation) so that the corresponding algorithm and notation are relatively simple. Tackling first the case of strict preferences also facilitates the discussion of the (general) case of weak preferences in the next subsection.

Before we present and prove our first main result, we describe the TTC algorithm for finding strong core allocations when preferences are strict. The graphs defined in the algorithm are crucial tools for the proof of Theorem 1.

¹⁴In other words, for each $R \in \mathcal{R}$, $\Phi(R) \neq \emptyset$. multi-valued allocation rule (on $\mathcal{R}$). We say that Φ *respects improvement for the best allotments*, or satisfies the *RI-best property*, if for each $i \in N$, each pair of markets $R, \tilde{R} \in \mathcal{R}$ such that $\tilde{R}$ is an improvement for i with respect to R , and each pair of allocations $\tilde{x} \in \Phi_i^+(\tilde{R})$ and $x \in \Phi_i^+(R)$, we have $\tilde{x}_i R_i x_i$. Similarly, we say that Φ *respects improvement for the worst allotments*, or satisfies the *RI-worst property*, if for each $i \in N$, each pair of markets $R, \tilde{R} \in \mathcal{R}$ such that $\tilde{R}$ is an improvement for i with respect to R , and each pair of allocations $\tilde{x} \in \Phi_i^-(\tilde{R})$ and $x \in \Phi_i^-(R)$, we have $\tilde{x}_i R_i x_i$.

Let $M = (N, R)$ be a housing market with strict preferences. We will construct a subgraph G^{CP} of the acceptability graph G by using the Top Trading Cycles (TTC) algorithm of David Gale [46]. The node set of G^{CP} is N and its directed edges $E^{CP} = E^C \cup E^P$ are partitioned into two sets E^C and E^P , where E^C will denote the edges in the TTC cycles and E^P will denote a particular subset of edges pointing to more preferred objects.

TTC algorithm, construction of G^{CP}

Set $E^C \equiv \emptyset$, $E^P \equiv \emptyset$, and $M_1 \equiv M$. Let $G_1 = (N_1, E_1) \equiv (N, E)$ denote the acceptability graph of M_1 . We iteratively construct “shrinking” submarkets M_t ($t = 2, 3, \dots$) whose acceptability graph will be denoted by $G_t = (N_t, E_t)$. Set $t \equiv 1$.

Step 1. Let E_t^T be the set of most preferred edges in G_t .

Step 2. Let c_t be a (top trading) cycle in (N_t, E_t^T) . Let C_t and E_t denote the node set and edge set of c_t , respectively.

Step 3. Add the edges of c_t to E^C , i.e., $E^C \equiv E^C \cup E_t$.

Step 4. Let $E_t^T(\vec{C}_t)$ denote the subset of edges of E_t^T pointing to C_t from outside C_t . Formally, $E_t^T(\vec{C}_t) \equiv \{(i, j) \in E_t^T : i \in N_t \setminus C_t \text{ and } j \in C_t\}$. Add $E_t^T(\vec{C}_t)$ to E^P , i.e., $E^P \equiv E^P \cup E_t^T(\vec{C}_t)$.

Step 5. If $N_t = C_t$, stop. Otherwise, let $N_{t+1} \equiv N_t \setminus C_t$, denote the submarket $M_{N_{t+1}}$ by M_{t+1} , and go to step 1.

When the algorithm terminates the set of cycles in E^C is the unique competitive allocation and hence the unique strong core allocation (see [39]).

For each profile of strict preferences R , let $\tau(R)$ denote the unique competitive allocation (or strong core allocation). We show that the allocation rule τ respects improvement.

Theorem 1. *When preferences are strict, the competitive allocation rule (or strong core allocation rule) τ respects improvement.*

Proof. Let $i \in N$ and $R, \tilde{R}$ profiles of strict preferences such that $\tilde{R}$ is an improvement for i with respect to R . Let $x = \tau(R)$ and $\tilde{x} = \tau(\tilde{R})$. We can assume that there is a unique agent $j \neq i$ with $\tilde{R}_j \neq R_j$ and prove that $\tilde{x}_i R_i x_i$. (If there is more than one such agent, we repeatedly apply the one-agent result to obtain the result.) We can also assume that $i \tilde{R}_j x_j$. (Otherwise $x_j \tilde{P}_j i$ and hence, from the TTC algorithm, $\tilde{x} = x$.)

Consider the graph G^{CP} for the market (N, R) , i.e., the graph that is obtained in the TTC algorithm for x . We are in (exactly) one of the following four situations:

- (a). i and j are *independent*: there is no directed path from i to j or from j to i ;
- (b). i and j are *cycle-members*: there is a path from i to j that entirely consists of edges in E^C , i.e., i and j are in the same top trading cycle;¹⁵
- (c). i is a *predecessor* of j (and j is a *successor* of i): there is a path from i to j in G^{CP} using at least one edge from E^P ;¹⁶ or
- (d). j is a *predecessor* of i (and i is a *successor* of j): there is a path from j to i in G^{CP} using at least one edge from E^P .

We distinguish among three cases, depending on the relation between agents i and j (see Figure 3).

CASE I: i and j are independent or j is a predecessor of i . Let $F(i)$ be the set of *followers*¹⁷ of i in the graph G^{CP} , where we use the convention $i \in F(i)$. Then, $j \notin F(i)$. In the TTC algorithm

¹⁵Obviously, in this case there is also a path from j to i that consists of edges in E^C .

¹⁶Note that in this case j was removed from the market before i . Hence, there is no path from i to j using only edges in E^C and j is not a predecessor of i .

¹⁷We avoid the use of the usually equivalent nomenclature “successor” as the latter term has already a particular (and different) meaning.

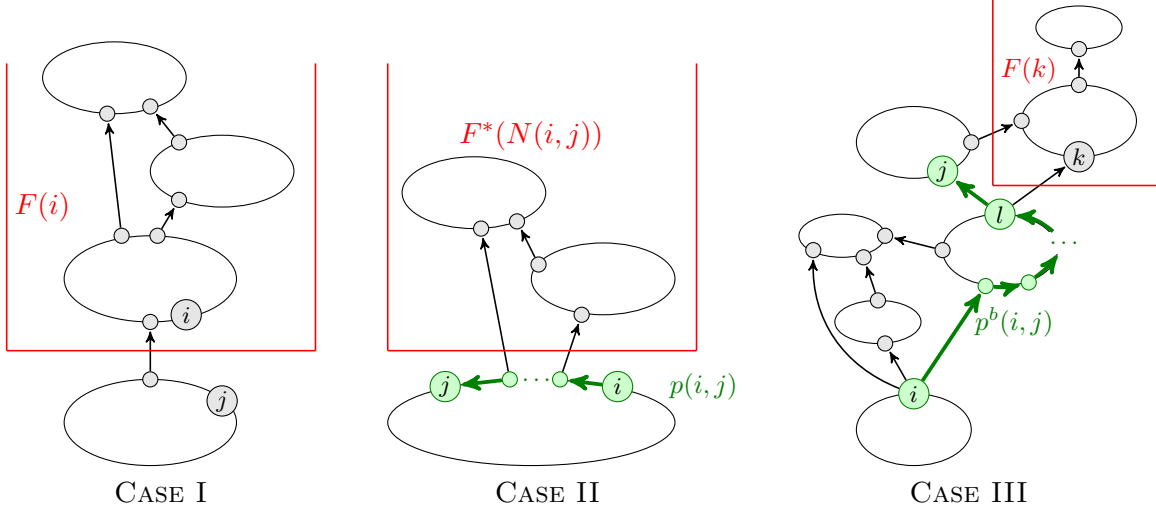


Figure 3: Graph G^{CP} (simplified) in the proof of Theorem 1. Each ellipse represents a top trading cycle.

for R , the agents in $F(i)$ form, among themselves, trading cycles. Since for each agent $k \in F(i)$, $\tilde{R}_k = R_k$, it follows that the trading cycles formed by $F(i)$ in the TTC algorithm for R are also formed in the TTC algorithm for $\tilde{R}$. Hence, $\tilde{x}_i = x_i$.

CASE II: i and j are cycle-members. Let C be the cycle in the graph G^{CP} that contains i and j . Let $p(i, j)$ be the unique path from i to j in the graph G^{CP} . Obviously, $p(i, j)$ is part of C . Let $N(i, j)$ be the nodes on $p(i, j)$. (So, $i, j \in N(i, j)$.) Let $F^*(N(i, j))$ be the followers outside of $N(i, j)$ that can be reached by some path in G^{CP} that (1) starts from a node in $N(i, j)$ and (2) does not contain edges in C . Then, the nodes in $F^*(N(i, j))$ constitute trading cycles at x . Moreover, the nodes in $F^*(N(i, j))$ are neither predecessors of j nor cycle-members with j . Hence, by the same arguments as in Case I, at $\tilde{x}$ the nodes in $F^*(N(i, j))$ constitute the same trading cycles as at x . Since $i\tilde{R}_jx_j$, the trading cycle of agent i at $\tilde{x}$ is the cycle that consists of the path $p(i, j)$ and the edge (j, i) . Since $p(i, j)$ is part of C , it follows that $\tilde{x}_i = x_i$.

CASE III: i is a predecessor of j .

We define the *best path* from i to j to be the path from i to j in G^{CP} where at each node $k \neq j$ on the path, the path follows agent k 's (unique) most preferred edge in

$$\{(k, l) \in E^{CP} : \text{there is a path from } l \text{ to } j \text{ using edges in } E^{CP}\}.$$

Let $p^b(i, j)$ denote the unique best path from i to j in G^{CP} . For each node $k \neq j$ on $p^b(i, j)$, if there are multiple paths from k to j , then $p^b(i, j)$ follows the edge that points to the object that is part of the earliest top trading cycle.

Since for each $k \neq j$, $\tilde{R}_k = R_k$, $i\tilde{R}_jx_j$, and $\tilde{R}_j$ is obtained from R_j by shifting i up, it follows that at some step in the TTC algorithm for $\tilde{R}$, agent j will start pointing to agent i and will keep doing so if and as long as agent i is present. Next, consider the predecessor of j on the path $p^b(i, j)$ (in G^{CP} for the market (N, R)), say agent l . Let $k \in N$ with kP_lj . By definition of $p^b(i, j)$, k and j are not cycle-members nor is k a predecessor of j . From the same arguments as in Case I, the nodes in $F(k)$ (the followers of k , where $k \in F(k)$) form among themselves the same trading cycles at x and $\tilde{x}$. Hence, at some step in the TTC algorithm for $\tilde{R}$, agent l will start pointing to agent j and will keep doing so if and as long as agent j is present. We can repeat the same arguments until we conclude that each node in the cycle formed by $p^b(i, j)$ and the edge (j, i) will, at some

step, start pointing to its follower and will keep doing so if and as long as the follower is present. Thus, the cycle is a trading cycle at $\tilde{x}$. Let i' be the follower of i in this cycle. Note that in the graph G^{CP} , $(i, i') \in E^C$ or $(i, i') \in E^P$. If $(i, i') \in E^C$, then $i' = x_i$, in which case $\tilde{x}_i = i' = x_i$. If $(i, i') \in E^P$, then by definition of E^P , $\tilde{x}_i = i' P_i x_i$. □

3.2 Weak preferences

For each market R , let $\mathcal{T}(R)$ denote the set of competitive allocations. We show that the multi-valued allocation rule $\mathcal{T}$ satisfies the RI-best and RI-worst properties.

Proposition 1. *Let $i \in N$. Let $R, \tilde{R}$ be a pair of profiles of preferences such that $\tilde{R}$ is an improvement for i with respect to R . Then,*

- *there is $\tilde{x} \in \mathcal{T}(\tilde{R})$ such that for each $x \in \mathcal{T}(R)$, $\tilde{x}_i R_i x_i$; and*
- *there is $x \in \mathcal{T}(R)$ such that for each $\tilde{x} \in \mathcal{T}(\tilde{R})$, $\tilde{x}_i R_i x_i$.*

Note that in general there is no competitive allocation where each agent receives her most preferred allotment (among those that are obtained at competitive allocations), i.e., agents do not unanimously agree on the “best” competitive allocation (see, e.g., agents 3 and 4 and competitive allocations x^a and x^b in Example 1). Nonetheless, Proposition 1 implies that any optimistic agent who believes that she will always receive the best possible allotment subscribes to the thesis that “the competitive correspondence” will respect any of her potential improvements. A similar statement holds for any pessimistic agent who believes that she will always receive the worst possible allotment.

Proof. Let $R^1, R^2, \dots, R^m$ be the profiles of strict preferences that are obtained from R by breaking ties between acceptable objects in each possible way. Similarly, let $\tilde{R}^1, \tilde{R}^2, \dots, \tilde{R}^p$ be the profiles of strict preferences that are obtained from $\tilde{R}$ by breaking ties between acceptable objects in each possible way, where possibly $p \neq m$. Then, from the results of [46] (see also page 306 in [50]), $\mathcal{T}(R) = \{\tau(R^1), \tau(R^2), \dots, \tau(R^m)\}$ and $\mathcal{T}(\tilde{R}) = \{\tau(\tilde{R}^1), \tau(\tilde{R}^2), \dots, \tau(\tilde{R}^p)\}$. It is not difficult to see that for each R^k , there is some $\tilde{R}^l$ such that $\tilde{R}^l$ is an improvement for i with respect to R^k . Similarly, for each $\tilde{R}^l$, there is some R^k such that $\tilde{R}^l$ is an improvement for i with respect to R^k .

We can assume, without loss of generality, that for each $k = 1, \dots, m$, $\tau_i(R^1) R_i \tau_i(R^k)$. Let $\tilde{R}^l$ be an improvement for i with respect to R^1 . Then, from Theorem 1, $\tau_i(\tilde{R}^l) R_i^1 \tau_i(R^1)$. Since $R_i^1 = R_i$, $\tau_i(\tilde{R}^l) R_i \tau_i(R^1)$. This proves the first statement.

We can also assume, without loss of generality, that for each $l = 1, \dots, p$, $\tau_i(\tilde{R}^l) R_i \tau_i(\tilde{R}^p)$. Let R^k be such that $\tilde{R}^p$ is an improvement for i with respect to R^k . Then, from Theorem 1, $\tau_i(\tilde{R}^p) R_i^k \tau_i(R^k)$. Since $R_i^k = R_i$, $\tau_i(\tilde{R}^p) R_i \tau_i(R^k)$. This proves the second statement. □

Next, we turn to the strong core, which in the case of weak preferences is a (possibly strict) subset of the set of competitive allocations.

Similarly to the case of strict preferences, before we state and prove our generalization of Theorem 1 to the domain of weak preferences, we first describe the efficient algorithm of Quint and Wako [38] for finding a strong core allocation whenever there exists one. The graphs defined in the algorithm are crucial tools for the proof of Theorem 2.

Let $M = (N, R)$ be a housing with weak preferences. We use the simplified interpretation of [16] and construct a subgraph G^{SP} of G with node set N and edge set $E^{SP} \equiv E^S \cup E^P$, which will be useful for our later analysis.

A *strongly connected component* of a directed graph is a subgraph where there is a directed path from each node to every other node. An *absorbing set* is a strongly connected component with no outgoing edge.¹⁸ Note that each directed graph has at least one absorbing set.

Quint-Wako algorithm, construction of G^{SP}

Set $E^S \equiv \emptyset$, $E^P \equiv \emptyset$, and $M_1 = M$. Let $G_1 = (N_1, E_1) \equiv (N, E)$ denote the acceptability graph of M_1 . We iteratively construct “shrinking” submarkets M_t ($t = 2, 3, \dots$) whose acceptability graph will be denoted by $G_t = (N_t, E_t)$. Set $t \equiv 1$.

Step 1. Let E_t^T be the set of most preferred edges in G_t .

Step 2. Let S_t be an absorbing set in (N_t, E_t^T) . Let $N_t(S_t)$ and $E_t^T(S_t)$ denote the node set and edge set of S_t .

Step 3. Add the edges of S_t to E^S , i.e., $E^S \equiv E^S \cup E_t^T(S_t)$.

Step 4. Let $E_t^T(\vec{S}_t)$ denote the subset of edges of E_t^T pointing to $N_t(S_t)$ from outside $N_t(S_t)$. Formally, $E_t^T(\vec{S}_t) \equiv \{(i, j) \in E_t^T : i \in N_t \setminus N_t(S_t) \text{ and } j \in N_t(S_t)\}$. Add $E_t^T(\vec{S}_t)$ to E^P , i.e., $E^P \equiv E^P \cup E_t^T(\vec{S}_t)$.

Step 5. If $N_t = N_t(S_t)$, stop. Otherwise, let $N_{t+1} \equiv N_t \setminus N_t(S_t)$, denote the submarket $M_{N_{t+1}}$ by M_{t+1} , and go to step 1.

Quint and Wako [38] proved that there is a strong core allocation for M if and only if for each absorbing set S_t defined in the above algorithm there exists a cycle cover, i.e., a set of cycles covering all the nodes of S_t . Finding a cycle cover, if one exists, can be done with the classical Hungarian method [29] for finding a perfect matching for the corresponding bipartite graph where the objects are on one side, the agents are on the other side, and there is an undirected arc between an object-agent pair if the object is among the agent’s most preferred objects (which might include her own object). We refer to [2], [38], and [16] for further details on this reduction.

We can now show that the multi-valued allocation rule $\mathcal{SC}$ conditionally respects improvement.

Theorem 2. *Let $i \in N$. Let $R, \tilde{R}$ be a pair of profiles of preferences such that $\tilde{R}$ is an improvement for i with respect to R . If $\mathcal{SC}(R), \mathcal{SC}(\tilde{R}) \neq \emptyset$, then for each $\tilde{x} \in \mathcal{SC}(\tilde{R})$ and each $x \in \mathcal{SC}(R)$, $\tilde{x}_i R_i x_i$.*

The proof of Theorem 2 is in Appendix B.

Conclusion

We summarise our main findings, together with the additional results from a follow-up paper by Schlotter et al. [45] in Table 3 for strict and weak preferences.

	housing market ($k = \infty$)	roommates problem ($k = 2$)	$k = 3$
Strict preferences			
Core	RI-best, no RI-worst ([45])	cond. RI-best ([45]), no RI-worst (Ex 3)	no RI (Ex 5)
CE/Wako-core	RI (Thm 1)	cond. RI-best ([45]), no RI-worst (Ex 3)	no RI (Ex 5)
Strong core	RI (Thm 1)	cond. RI-best ([45]), no RI-worst (Ex 3)	no RI (Ex 5)
Weak preferences			
Core	RI-best, no RI-worst ([45])	no RI-best/worst (Ex 4/3)	no RI (Ex 5)
CE/Wako-core	RI-best/worst (Prop 1)	no RI-best/worst (Ex 4/3)	no RI (Ex 5)
Strong core	cond. RI (Thm 2)	no RI-best/worst (Ex 4/3)	no RI (Ex 5)

Table 3: Respects improvement.

¹⁸In other words, there is no edge from a node in the absorbing set to a node outside the absorbing set.

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Appendix A: Literature review

Housing markets. The non-emptiness of the core was proved in [46] by showing the balancedness of the corresponding NTU-game, and also in a constructive way, by showing that David Gale’s famous Top Trading Cycles algorithm (TTC) always yields competitive allocations. [39] later showed that for strict preferences the TTC results in the unique strong core allocation, which coincides with the unique competitive allocation in this case. However, if preferences are not strict (i.e., ties are present), the strong core can be empty or contain more than one allocation, but the TTC still produces all competitive allocations. Wako [49] showed that the strong core is always a subset of the set of competitive allocations. Quint and Wako [38] provided an efficient algorithm for finding a strong core allocation whenever there exists one. Their work was further generalised and simplified by Ceclárová and Fleiner [16] who used graph models. Wako [51] showed that the set of competitive allocations coincides with the core based on an antisymmetric weak domination concept, which we refer to as Wako-core in this paper. This equivalence is key for our extension of the definition of competitive allocations to the case of bounded exchange cycles.

Respecting improvement. For Gale and Shapley’s college admissions model [20], Balinski and Sönmez [9] proved that the student-optimal stable matching mechanism (SOSM) respects improvement of student’s quality. Kominers [28] generalised this result to more general settings. Balinski and Sönmez [9] also showed that SOSM is the unique stable mechanism that respects improvement of student quality. Abdulkadiroglu and Sönmez [1] proposed and discussed the use of TTC in a model of school choice, which is closely related to the college admissions model. Klaus and Klijn [24] proved that TTC respects improvement of student quality.

Focusing on the other side of the market, Hatfield et al. [22] studied the existence of mechanisms that respect improvement of a *college’s* quality. The fact that colleges can match with multiple students leads to a strong impossibility result: they proved that there is no stable nor Pareto-efficient mechanism that respects improvement of a college’s quality. In particular, the (Pareto-efficient) TTC mechanism does not respect improvement of a college’s quality.

In the context of KEPs with pairwise exchanges, the incentives for bringing an additional donor to the exchange pool was first studied by Roth et al. [41]. In the model of housing markets their *donor-monotonicity* property boils down to the respecting improvement property. They showed that so-called priority mechanisms are donor-monotonic if each agent’s preferences are dichotomous, i.e., she is indifferent between all acceptable donors. However, if agents have non-dichotomous preferences, then any mechanism that maximises the number of pairwise exchanges (so, in particular any priority mechanism) does not respect improvement. This can be easily seen by means of Example 2 in Section 3.

IP formulations for matching. Quint and Wako [38] already gave IP formulations for finding core and strong core allocations, but the number of constraints in their paper is highly exponential, as their formulations contain a no-blocking condition for each set of agents and any possible exchanges among these agents. Other studies provided IP formulations for other matching problems. In particular, for Gale and Shapley’s college admissions model [20], Baïou and Balinski [7] already described the stable admissions polytope, which can be used as a basic IP formulation. Further recent papers in this line of research focused on college admissions with special features [3], stable project allocation under distributional constraints [4], the hospital–resident problem with couples [11], and ties [30, 18].

Kidney exchange programmes. Starting with the seminal works [43] and [42], initial research on KEPs focused on integer programming (IP) models for selecting pairs for transplantation in such a way that maximum (social) welfare, generally measured by the number of patients transplanted, is achieved. Authors in [17, 19, 31] proposed new, compact formulations that, besides extending the models in [43] and [42] to accommodate non-directed donors and patients with multiple donors, also aimed to efficiently solve problems of larger size. The reader is referred to [6] for a recent operations perspective on KEPs.

In Europe at least ten countries have active national kidney exchange programmes. Details of current practices and optimisation aspects are summarised in [10] and [14], respectively. Furthermore, there are already several international collaborations between European countries [26], which motivated a new line of research on group-fairness, [15, 27, 33] where agents (e.g. hospitals, regional and national programmes) can collaborate. Allowing agents to control their internal exchanges, Carvalho et al. [15] studied strategic interaction using non-cooperative game theory. Specifically, for the two-agent case, they designed a game such that some Nash equilibrium maximises the overall social welfare. Considering multiple matching periods, Klimentova et al. [27] assumed agents to be non-strategic. Taking into account that at each period there can be multiple optimal allocations, each of which can benefit different agents, the authors proposed an integer programming model to achieve an overall fair allocation. Finally, Mincu et al. [33] proposed integer programming formulations for the case where optimisation goals and constraints can be distinct for different agents.

A recent line of research acknowledges the importance of considering patients' preferences (associated with e.g. graft quality) over matches, raising the question of individual fairness. In the computer science and OR literature, Biró and Cechlárová [13] considered a model for unbounded length kidney exchanges, where patients most care about the quality of the graft they receive, but as a secondary factor they prefer to be involved in an exchange cycle that is as short as possible. The authors showed that although core allocations can still be found by the TTC algorithm, finding a core allocation with maximum number of transplants is a computationally hard problem (inapproximable, unless $P = NP$). In two independent papers [12] and [23] stable exchanges were studied for bounded length cycles with NP-hardness results for the case of 3-cycles. Recently, Klimentova et al. [25] provided integer programming formulations for the case where each patient has preferences over the organs that she can receive. The authors focused on allocations that among all (strong) core allocations have maximum cardinality. Moving away from the (strong) core, they also analysed the trade-off between maximum cardinality and the number of blocking cycles. As we show through simulations in this paper, core allocations do not create a substantial number of violations of the respecting improvement property (for best allotments), and thus incentivise the participation in KEPs.

In the economics and game theory literature, the preferences of the recipients were dichotomous (i.e., either acceptable or unacceptable) in the classical papers, starting with [41]. The first departure from this literature was by Nicolò and Rodríguez-Álvarez [35], who considered a setting where the quality of potential transplants is given and each recipient can set an acceptability threshold. They proved an impossibility result for pairwise exchanges (that they later generalised in [36]) and they studied conditions under which truth-telling is the unique protective strategy for the recipients. In a follow-up paper [37] they considered pairwise exchanges with age-based preferences and ties, and they proposed a deterministic sequential priority rule that satisfies efficiency, strategy-proofness, and non-bossiness. Andersson and Kratz [5] considered a model motivated by the Swedish application, where (1) ABO-incompatible transplants are allowed in the exchanges and (2) each ABO-incompatible recipient-donor pair only accepts a fully compatible donor. They studied a priority matching rule on their trichotomous preference domain. Another proposal for

giving incentives to compatible pairs by prioritising patients in case of future graft failure was given by Sönmez, Ünver, and Yenmez [47]. In a recent paper, Balbuzanov [8] considered bounded length exchange problems under strict preferences. He showed that there is no deterministic mechanism that satisfies individual rationality, ex-post efficiency, and weak strategy-proofness. He also provided a random mechanism for pairwise exchanges that is individually rational, ordinally efficient, and anonymous.

Appendix B: Proof of Theorem 2

Remark 2. If for each absorbing set S_t defined in the above algorithm there exists a cycle cover, then the set of cycle covers (one cycle cover for each absorbing set) constitutes a strong core allocation. Conversely, as shown in the proof of Theorem 5.5 in [38], each strong core allocation can be written as a set of cycle covers (one for each absorbing set S_t). Therefore, if the strong core is non-empty, all its allocations can be obtained by selecting all possible cycle covers in the algorithm. $\diamond$

Remark 3. In the Quint-Wako algorithm, each agent obtains the same welfare at any two cycle covers in which she is involved (because the agent is indifferent between any two of her outgoing edges in an absorbing set). Together with Remark 2, this immediately proves Theorem 2(2) in [50], which states that all strong core allocations are welfare-equivalent. $\diamond$

We can now prove that the multi-valued allocation rule $\mathcal{SC}$ conditionally respects improvement, as stated in Theorem 2.

Proof. (of Theorem 2) Let $x \in \mathcal{SC}(R)$. From Remark 3 it follows that it is sufficient to show that there exists $\tilde{x} \in \mathcal{SC}(\tilde{R})$ with $\tilde{x}_i R_i x_i$. We can assume that there is a unique agent $j \neq i$ with $\tilde{R}_j \neq R_j$. (If there is more than one such agent, we repeatedly apply the one-agent result to obtain the result.) We can also assume that $i \tilde{R}_j x_j$. (Otherwise $x_j \tilde{P}_j i$ and hence, from the Quint-Wako algorithm, $x \in \mathcal{SC}(R) = \mathcal{SC}(\tilde{R})$.)

Consider the graph G^{SP} for the market (N, R) , i.e., the graph that is generated in the Quint-Wako algorithm to obtain x . We are in (exactly) one of the following four situations:

- (a). i and j are *independent*: there is no directed path from i to j or from j to i ;
- (b). i and j are *absorbing set members*: there is a path from i to j that entirely consists of edges in E^S , i.e., i and j are in the same absorbing set;¹⁹
- (c). i is a *predecessor* of j (and j is a *successor* of i): there is a path from i to j in G^{SP} using at least one edge from E^P ;²⁰ or
- (d). j is a *predecessor* of i (and i is a *successor* of j): there is a path from j to i in G^{SP} using at least one edge from E^P .

We distinguish among three cases, depending on the relation between agents i and j .

CASE I: i and j are independent or j is a predecessor of i . Let $F(i)$ be the followers of i in the graph G^{SP} , where we use the convention $i \in F(i)$. Then, $j \notin F(i)$. In the Quint-Wako algorithm for R , the nodes in $F(i)$ are exactly the nodes of a collection of absorbing sets. Since for each agent $k \in F(i)$, $\tilde{R}_k = R_k$, it follows that in the Quint-Wako algorithm for $\tilde{R}$ the nodes in $F(i)$ are exactly the nodes of the same collection of absorbing sets. Since $\mathcal{SC}(\tilde{R}) \neq \emptyset$, it follows from Remark 2 that there exists $\tilde{x} \in \mathcal{SC}(\tilde{R})$ such that for each agent $k \in F(i)$, $\tilde{x}_k = x_k$. In particular, $\tilde{x}_i = x_i$.

CASE II: i and j are absorbing set members. Let S_t be the absorbing set that contains i and j in the Quint-Wako algorithm for R . Let (i, l) be an edge in S_t , where possibly $l = i$. Let $F^*(N_t(S_t))$ be the followers outside of $N_t(S_t)$ that can be reached by some path in G^{SP} that starts from a node in $N_t(S_t)$. Then, the nodes in $F^*(N_t(S_t))$ are exactly the nodes of a collection of absorbing sets in the Quint-Wako algorithm for R . Moreover, the nodes in $F^*(N_t(S_t))$ are neither predecessors of j nor absorbing set members with j . Hence, by the same arguments as in Case I, in the Quint-Wako

¹⁹Obviously, in this case there is also a path from j to i that consists of edges in E^S .

²⁰Note that in this case j was removed from the market before i . Hence, there is no path from i to j using only edges in E^S and j is not a predecessor of i .

algorithm for $\tilde{R}$ the nodes in $F^*(N_t(S_t))$ are again the nodes of the same collection of absorbing sets. Therefore, when the Quint-Wako algorithm is applied to $\tilde{R}$, the absorbing set that contains i will again contain l . Since $\mathcal{SC}(\tilde{R}) \neq \emptyset$, at each $\tilde{x} \in \mathcal{SC}(\tilde{R})$, agent i will receive an object $\tilde{x}_i$ such that $\tilde{x}_i I_i l$. Since also $x_i I_i l$, we obtain $\tilde{x}_i I_i x_i$.

CASE III: i is a predecessor of j .

A path from i to j in G^{SP} is said to be a best path from i to j if at each node $k \neq j$ on the path, the path follows one of agent k 's most preferred edges in

$$\{(k, l) \in E^{SP} : \text{there is a path from } l \text{ to } j \text{ using edges in } E^{SP}\}.$$

Let $P^b(i, j)$ denote the *set of best paths* from i to j in G^{SP} .

Since for each $k \neq j$, $\tilde{R}_k = R_k$, $i\tilde{R}_j x_j$, and $\tilde{R}_j$ is obtained from R_j by shifting i up, it follows that at some step in the Quint-Wako algorithm for $\tilde{R}$, agent j will start pointing to agent i and will keep doing so if and as long as agent i is present. Next, consider the predecessor of j on a best path $p^b(i, j) \in P^b(i, j)$ (in G^{SP} for the market (N, R)), say agent l . Let $k \in N$ with $k P_l j$. By definition of $p^b(i, j)$, k and j are not absorbing set members nor is k a predecessor of j . From the same arguments as in Case I, the nodes in $F(k)$ (the followers of k , where $k \in F(k)$) form among themselves the same absorbing sets in the Quint-Wako algorithm for both R and $\tilde{R}$. Hence, during the Quint-Wako algorithm for $\tilde{R}$, at some step agent l will start pointing to agent j and will keep doing so if and as long as agent j is present. We can repeat the same arguments until we conclude that each node in the cycle formed by $p^b(i, j)$ and the edge (j, i) will, at some step, start pointing to its follower and will keep doing so if and as long as the follower is present. Hence, at some step of the algorithm the cycle formed by $p^b(i, j)$ and the edge (j, i) is part of an absorbing set. Let i^b denote the follower of agent i in path $p^b(i, j)$. Since $\mathcal{SC}(\tilde{R}) \neq \emptyset$, at each $\tilde{x} \in \mathcal{SC}(\tilde{R})$, agent i will receive an object $\tilde{x}_i$ such that $\tilde{x}_i I_i i^b$. Note that in the graph G^{SP} , $(i, i^b) \in E^S$ or $(i, i^b) \in E^P$. If $(i, i^b) \in E^S$, then $i^b I_i x_i$, in which case $\tilde{x}_i I_i x_i$. If $(i, i^b) \in E^P$, then by definition of E^P , $i^b R_i x_i$, in which case $\tilde{x}_i R_i x_i$. $\square$

Corollary 1. *For each $i \in N$ and each pair of profiles of strict preferences $R, \tilde{R}$ such that $\mathcal{SC}(R), \mathcal{SC}(\tilde{R}) \neq \emptyset$ and $\tilde{R}$ is an improvement for i with respect to R ,*

- *there is $\tilde{x} \in \mathcal{SC}(\tilde{R})$ such that for each $x \in \mathcal{SC}(R)$, $\tilde{x}_i R_i x_i$; and*
- *there is $x \in \mathcal{SC}(R)$ such that for each $\tilde{x} \in \mathcal{SC}(\tilde{R})$, $\tilde{x}_i R_i x_i$.*

Appendix C: Bounded length exchange cycles

Motivated by kidney exchange programmes, we consider housing markets where the length of allowed exchange cycles in allocations is limited. We provide several examples to demonstrate the possible violations of the respecting improvement property (or variants/extensions of the property) in the setting of bounded length exchanges.

Definitions

Assuming that blocking coalitions are subject to the same length of allowed exchange cycles, the core and strong core can be adjusted straightforwardly (see also [12]). In view of Wako’s result [51], we similarly adjust the set of competitive allocations by using the (equivalent) Wako-core.

For a housing market $M = (N, R)$, let k be an integer that indicates the maximal allowed length of exchange cycles. An allocation is a k -allocation if each exchange cycle has length at most k . Formally, an allocation x is a k -allocation if there exists a partition of $N = S_1 \cup S_2 \cup \dots \cup S_q$ such that for each $p \in \{1, \dots, q\}$, $|S_p| \leq k$ and $\{x_i : i \in S_p\} = S_p$. The definition of the three cores can be adjusted accordingly as well. Specifically, the k -core consists of the k -allocations for which there is no blocking coalition of size at most k ; the *strong* k -core consists of the k -allocations for which there is no weakly blocking coalition of size at most k ; the *Wako- k -core* consists of the k -allocations that are not antisymmetrically weakly dominated through a coalition of size at most k .

Similarly to the unbounded case, the three blocking notions are “nested.” Hence, the strong k -core is a subset of the Wako- k -core and the Wako- k -core is a subset of the k -core. Similarly to the unbounded case, it follows easily that for strict preferences the strong k -core coincides with the Wako- k -core.

To keep notation as simple as possible, whenever the context is clear, we will omit “ k ” from k -allocation, k -core, etc. and instead refer to k -housing markets to invoke the above restriction on exchange cycles, blocking coalitions, allocations, and cores.

The case of *pairwise exchanges*, i.e., $k = 2$, yields the well-known *stable roommates problem* and, if the graph is bipartite, the *stable marriage problem* as introduced in [20]. For strict preferences the core, Wako-core, and strong core are all equivalent and they correspond to the set of *stable matchings*. For weak preferences the core and Wako-core are the same, and correspond to *weakly stable matchings*, whilst the strong core corresponds to *strongly stable matchings*. We refer to [32] for more details.

Pairwise exchanges

As mentioned in Section 1, the maximisation of the number of pairwise exchanges does not respect improvement. Example 2 below proves this formally. A consequence is that the priority mechanisms studied by Roth et al. [41] need not be donor-monotonic if agents’ preferences can be non-dichotomous.

Example 2. Let $N = \{1, \dots, 4\}$ and let preferences R be given by Table 4 and the new preferences $\tilde{R}$ by Table 5, where the only improvement is that agent 1 becomes acceptable for agent 3.

1	2	3	4
2	1	3	2
3	4		4
1	2		

Table 4: R

1	2	3	4
2	1	1	2
3	4	3	4
1	2		

Table 5: $\tilde{R}$

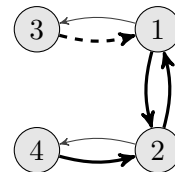


Figure 4: Acceptability graph

Initially, at R , there are two ways to maximise the number of pairwise exchanges, namely by picking either of the two-cycles $(1, 2)$ and $(2, 4)$. Assume, without loss of generality, that $(1, 2)$ is selected. (In case $(2, 4)$ is selected, similar arguments can be employed.) Suppose the discontinuous edge (in Figure 4) is included so that agent 1 “improves” and we obtain $\tilde{R}$. Then, the unique way to maximise the number of pairwise exchanges is obtained by picking the 2 two-cycles $(1, 3)$ and $(2, 4)$, which means that agent 1 is strictly worse off than in the initial situation. $\diamond$

We illustrate that the RI-worst property can be violated under strict preferences for (strong, Wako-) core. In particular, we give an example where the best allotment remains the same for the improving agent, but a new core allocation emerges in the new market such that her allotment there is worse than in the unique core allocation in the initial market.

Example 3. Let $N = \{1, \dots, 4\}$ and let preferences R be given by Table 6 and the new preferences $\tilde{R}$ by Table 7, where the only improvement is that agent 1 becomes acceptable for agent 3.

1	2	3	4
2	4	4	3
3	1	3	2
1	2		4

Table 6: R

1	2	3	4
2	4	1	3
3	1	4	2
1	2	3	4

Table 7: $\tilde{R}$

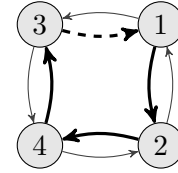


Figure 5: Acceptability graph

Initially, at R , the unique (strong) core allocation is $x^a = \{(1, 2), (3, 4)\}$. Suppose the discontinuous edge (in Figure 5) is included so that agent 1 “improves” and we obtain $\tilde{R}$. Then, another (strong) core allocation is created, $x^b = \{(1, 3), (2, 4)\}$, which is strictly worse for 1. $\diamond$

The following example illustrates violation of the RI-best property for the core and Wako-core allocations in case of weak preferences and pairwise exchanges.

Example 4. Let $N = \{1, \dots, 4\}$ and let preferences R be given by Table 8. In the new preferences $\tilde{R}$, Table 9, agent 4 makes an improvement and becomes acceptable for agent 1.

1	2	3	4
3	4	1,4	1
			3
			4

Table 8: R

1	2	3	4
3	4	1,4	1
4			3
			4

Table 9: $\tilde{R}$

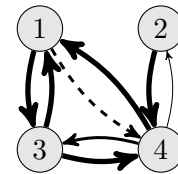


Figure 6: Acceptability graph

For initial preferences R there exist two (Wako-) core allocations $x^a = \{(3, 4)\}$ and $x^b = \{(1, 3), (2, 4)\}$. The best allotment for agent 4 is object 3, obtained in allocation x^a . In preferences $\tilde{R}$ we introduce another acceptable object for agent 1, represented with a discontinuous edge in Figure 6, and the newly formed cycle $(1, 4)$ is blocking for allocation x^a , while $(1, 3)$ is blocking for the new feasible allocation $x^c = \{(1, 4)\}$. Thus, the (Wako-) core consists of the unique allocation x^b . Hence, the improvement for the best allotment is not respected. $\diamond$

We summarise the above finding in the following statement.

Proposition 2. *The conditional RI-best property can be violated for the core and Wako-core in the case of pairwise exchanges under weak preferences.*

Three-way and longer bounded exchanges

In the following example we exhibit two housing markets and we prove that for each housing market the three cores coincide (and are non-empty). Subsequently, we will use the example to show that the three cores do not conditionally respect improvement for the best allotments when the maximal allowed length of exchange cycles is 3. A simple extension of this example can be used to show the same result for larger upper bounds.

Example 5. Throughout the example we focus on the core. However, since all blocking arguments can be replaced by weak blocking arguments, all statements also hold for the strong core, and hence also for the Wako-core. Let $N = \{1, \dots, 10\}$ be the set of agents. We consider two housing markets that only differ in preferences. First, consider the housing market (N, R) , or simply R for short, with “cyclic” strict preferences (where unacceptable objects are not displayed) given in Table 10.

1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	1
10	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	10

Table 10: preferences R

Since only directly neighbouring objects (and one’s own object) are acceptable, it follows that the only exchange cycles where each agent is assigned an acceptable object are the 10 self-cycles and the 10 two-cycles $(i, i + 1) \pmod{10}$ where agents i and $i + 1$ swap their objects.²¹ The core $\mathcal{C}(R) = \{x^a, x^b\}$ consists of the following two allocations:

$$\begin{aligned} x^a &= \{(1, 2), (3, 4), (5, 6), (7, 8), (9, 10)\} \text{ and} \\ x^b &= \{(10, 1), (2, 3), (4, 5), (6, 7), (8, 9)\}. \end{aligned}$$

Next, we create an extended housing markets R^b by inserting one three-cycle in R . Preferences R^b are provided in Table 11, where the changes with respect to R are bold-faced and depicted in Figure 7.

1	2	3	4	5	6	7	8	9	10
4	3	4	5	6	7	8	1	10	1
2	1	2	8	4	5	6	9	8	9
10	2	3	3	5	6	7	7	9	10
1			4				8		

Table 11: Preferences R^b

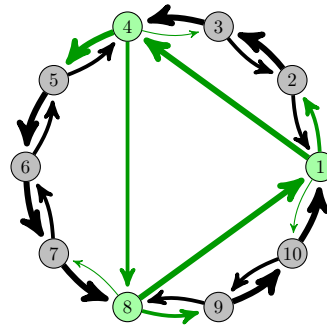


Figure 7: Acceptability graph for R^b

Apart from the earlier mentioned self-cycles and two-cycles, the only additional exchange cycle with only acceptable objects in R^b is $c^b = (1, 4, 8)$. Allocation x^b is in the core of R^b because c^b does not block x^b : agent 4 obtains object 8 in c^b , which is strictly less preferred to her assigned object at x^b . In fact, x^b is the unique core allocation of R^b . To see this, note first that x^a is not

²¹So, the core coincides with the set of stable matchings of the corresponding “roommate problem” (Gale and Shapley, 1962).

in the core of R^b as c^b blocks it. And second, the only new exchange cycle created in R^b , i.e., c^b , cannot be part of a core allocation, because if it were, then to avoid blocking cycle (4, 5), the next two-cycle (5, 6) would have to be part of the allocation, in which case 7 would remain unmatched (i.e., be a self-cycle) and cycle (6, 7) would block the allocation. Therefore, x^b is the unique core allocation of R^b , i.e., $\mathcal{C}(R^b) = \{x^b\}$. $\diamond$

Using the above example we can easily prove that the conditional RI-best property is violated for the core, Wako-core, and strong core as well.

Proposition 3. *Suppose the maximum allowed length of exchange cycles is 3. Then, there are 3-housing markets with strict preferences (N, R) and $(N, \tilde{R})$ with*

- $X(R) \equiv \mathcal{SC}(R) = \mathcal{WC}(R) = \mathcal{C}(R) \neq \emptyset$ and
- $X(\tilde{R}) \equiv \mathcal{SC}(\tilde{R}) = \mathcal{WC}(\tilde{R}) = \mathcal{C}(\tilde{R}) \neq \emptyset$

such that for some $i \in N$, $\tilde{R}$ is an improvement for i with respect to R but

- $X(\tilde{R}) \subseteq X(R)$,
- *for the unique $x \in X(R) \setminus X(\tilde{R})$ and for each $\tilde{x} \in X(\tilde{R}) \setminus X(R)$, $x_i P_i \tilde{x}_i$.*

Proof. Let $(N, \tilde{R})$ be the 3-housing market with $N = \{1, \dots, 10\}$ and $\tilde{R} = R^b$ from Example 5. Let (N, R) be the 3-housing market that is obtained from $(N, \tilde{R})$ by making object 1 unacceptable for agent 8. Obviously, $\tilde{R}$ is an improvement for agent 1 with respect to R . As shown in Example 5, $\mathcal{SC}(\tilde{R}) = \mathcal{WC}(\tilde{R}) = \mathcal{C}(\tilde{R}) = \{x^b\} \neq \emptyset$. One also easily verifies that $\mathcal{SC}(R) = \mathcal{WC}(R) = \mathcal{C}(R) = \{x^a, x^b\} \neq \emptyset$. Finally, agent 1's most preferred allotment in $\mathcal{SC}(R) = \mathcal{WC}(R) = \mathcal{C}(R)$ is object 2, while agent 1's unique (hence, most preferred) allotment in $\mathcal{SC}(\tilde{R}) = \mathcal{WC}(\tilde{R}) = \mathcal{C}(\tilde{R})$ is object 10. Since agent 1 strictly prefers object 2 to object 10, the result follows. $\square$

Finally, the above example can be adjusted to demonstrate the violation of the RI-best property for any larger upper bound k on the length of the exchange cycles as follows. We keep the structure of the example with the double outer cycle and the embedded 3-cycle, but we extend the length of the outer cycle, so that we do not create any new cycle of length at most k , while also keeping the parity of the highest two nodes of the embedded 3-cycle. That is, if 1 is the starting node and i, j are the other two nodes of the embedded 3-cycle, then both i and j remain even. For example, for $k = 4$, we extend the double outer cycle to consist of 14 vertices and we have an inner 3-cycle that consists of agents 1, 6, and 10.

Appendix D: Integer Programming Formulations

In this section we propose new integer programming (IP) formulations for the core, the set of competitive allocations (i.e., the Wako-core), and the strong core. First, we propose novel edge-formulations for the unbounded case for all three solution concepts. Second, we improve the formulations in [38] by giving alternative cycle-formulations for the core and strong core. Third, we provide a new formulation for the Wako-core for the case of bounded length exchange cycles. The novel IP formulations serve as a stepping-stone for our computational experiments in Section 3.2.

Novel edge-formulations

Let (N, R) be a housing market and $G \equiv G(N, R) = (N, E)$ its acceptability graph. Since all three cores only contain individually rational allocations, we can restrict attention to the edges of the acceptability graph. Specifically, with each edge $(i, j) \in E$ we associate a variable y_{ij} as follows:

$$y_{ij} = \begin{cases} 1 & \text{if agent } i \text{ receives object } j; \\ 0 & \text{otherwise.} \end{cases}$$

Then, the base model reads as follows:

$$\sum_{j:(i,j) \in E} y_{ij} = 1 \quad \forall i \in N \quad (1)$$

$$\sum_{j:(j,i) \in E} y_{ji} = 1 \quad \forall i \in N \quad (2)$$

$$y_{ij} \in \{0, 1\} \quad \forall (i, j) \in E \quad (3)$$

Constraints (1) guarantee that agent i receives exactly one (acceptable) object (possibly her own). Constraints (2) guarantee that object i is given to exactly one agent. Each vector $(y_{ij})_{(i,j) \in E}$ that satisfies (1), (2), and (3) yields an allocation x defined by $x_i = j$ if and only if $y_{ij} = 1$. Moreover, each allocation can be obtained in this way. So, there is a one-to-one correspondence between allocations and vectors that satisfy (1), (2), and (3).

We introduce for each $i \in N$ an additional integer variable p_i that represents the price of object i .

$$p_i \in \{1, \dots, n\} \quad \forall i \in N \quad (4)$$

In what follows we give our IP formulations for the general case of weak preferences and explain how they can be simplified for strict preferences. We tackle the core, the set of competitive allocations, and the strong core (in this order), by subsequently adding constraints. Given an allocation x , we say that x *dominates* an edge (i, j) in the acceptability graph G if agent i weakly prefers her allotment x_i to object j , i.e., $x_i R_i j$.

IP for the core

It follows from Lemma 1 that an individually rational allocation x is in the core if and only if each cycle in G contains an edge that is dominated by x . Or equivalently, there exists no cycle in G that consists of undominated edges. Note that the undominated edges form a cycle-free subgraph if and only if there is a topological order of the objects. The existence of this topological order is equivalent to the existence of prices of the objects such that for each undominated edge (i, j) , $p_i < p_j$. Therefore, an allocation x is in the core if and only if there exist prices $(p_i)_{i \in N}$ such that

$$(i, j) \in E \text{ is not dominated by } x \implies p_i < p_j. \quad (*)$$

Thus, core allocations are characterised by constraints (1)–(4) together with (5) below:

$$p_i + 1 \leq p_j + n \cdot \sum_{k: k R_i j} y_{ik} \quad \forall (i, j) \in E \quad (5)$$

Proposition 4. *Let x be an allocation. Let y be the corresponding vector that satisfies (1), (2), and (3). Allocation x is in the core if and only if there are prices $(p_i)_{i \in N}$ such that (4) and (5) hold.*

Proof. First observe that for each $(i, j) \in E$,

$$\begin{aligned} (i, j) \text{ is dominated by } x &\iff x_i R_i j \\ &\iff \text{there is } k \in N \text{ with } k R_i j \text{ and } y_{ik} = 1 \\ &\iff \sum_{k: k R_i j} y_{ik} = 1. \end{aligned} \quad (**)$$

Suppose x is in the core. Then, there exist prices $(p_i)_{i \in N}$ that satisfy (4) and (*). We verify that (5) holds. Let $(i, j) \in E$. If (i, j) is not dominated by x , then (5) follows immediately from (*). Suppose (i, j) is dominated by x . From (**), $\sum_{k: k R_i j} y_{ik} = 1$. Hence,

$$p_i + 1 \leq n + 1 \leq p_j + n = p_j + n \cdot \sum_{k: k R_i j} y_{ik}.$$

Suppose that there exist prices $(p_i)_{i \in N}$ such that (4) and (5) hold. We verify that (*) holds. Let $(i, j) \in E$ and suppose it is not dominated by x . From (**), $\sum_{k: k R_i j} y_{ik} = 0$. Hence, from (5), $p_i + 1 \leq p_j + n \cdot 0$, i.e., $p_i < p_j$. $\square$

IP for the set of competitive allocations

The set of competitive allocations is characterised by constraints (1)–(5) together with (6) below:

$$p_i \leq p_j + n \cdot (1 - y_{ij}) \quad \forall (i, j) \in E \quad (6)$$

Proposition 5. *Let x be an allocation. Let y be the corresponding vector that satisfies (1), (2), and (3). Allocation x is competitive if and only if there exist prices $(p_i)_{i \in N}$ such that (4), (5), and (6) hold. Moreover, if such prices exist, then together with x they constitute a competitive equilibrium.*

Proof. Suppose x is competitive. Let $(p_i)_{i \in N}$ be prices such that (x, p) is a competitive equilibrium. Then, (4) and (*) hold. From the first part of the proof of Proposition 4 it follows that (5) holds. We now prove that (6) holds as well. Let $(i, j) \in E$. If $y_{ij} = 0$, then immediately $p_i \leq p_j + n = p_j + n \cdot (1 - y_{ij})$. If $y_{ij} = 1$, then $x_i = j$, and since (x, p) is a competitive equilibrium it follows from Remark 1 that $p_i = p_{x_i} = p_j$.

Suppose that there exist prices $(p_i)_{i \in N}$ such that (4), (5), and (6) hold. We verify that (x, p) is a competitive equilibrium. First, it follows from (6) that for each $i \in N$, taking $j = x_i$ yields $p_i \leq p_{x_i} + n \cdot (1 - 1) = p_{x_i}$, i.e., $p_i \leq p_{x_i}$. Hence, from Remark 1, for each $i \in N$, $p_i = p_{x_i}$. Second, let $j \in N$ be an object such that $j P_i x_i$. Then, $(i, j) \in E$ is not dominated by x . From the second part of the proof of Proposition 4 it follows that (*) holds. Hence, we obtain $p_i < p_j$. $\square$

IP for the strong core

The strong core is characterised by constraints (1)–(6) together with (7) below:

$$p_i \leq p_j + n \cdot \left(\sum_{k:kP_{ij}} y_{ik} \right) \quad \forall (i, j) \in E \quad (7)$$

Proposition 6. *Let x be an allocation. Let y be the corresponding vector that satisfies (1), (2), and (3). Allocation x is in the strong core if and only if there exist prices $(p_i)_{i \in N}$ such that (4), (5), (6), and (7) hold. Moreover, if such prices exist, then together with x they constitute a competitive equilibrium.*

Proof. Suppose x is in the strong core. By Remark 2, x can be obtained in the Quint-Wako algorithm by choosing for each absorbing set in the algorithm a particular cycle cover. Hence, there exist price $(p_i)_{i \in N}$ such that (i) constraints (4) are satisfied, (ii) all objects in the same absorbing set have the same price, and (iii) an absorbing set that is processed earlier by the algorithm has a strictly higher associated price (of its objects). It is easy to verify that (x, p) is a competitive allocation. Hence, from the first part of the proof of Proposition 5 it follows that (5) and (6) hold. Finally, to see that (7) holds note that from the definition of the prices it follows that (i) if jR_ix_i then $p_i \leq p_j$ and (ii) if x_iP_{ij} then $p_i \leq n \cdot (\sum_{k:kP_{ij}} y_{ik})$.

Suppose that there exist prices $(p_i)_{i \in N}$ such that (4), (5), (6), and (7) hold. It follows from Proposition 5 that (x, p) is a competitive equilibrium. We prove that x is a strong core allocation. Suppose there is a coalition S that weakly blocks x through an allocation z . From Lemma 1 it follows that we can assume, without loss of generality, that $S = \{1, \dots, r\}$ and that for each $i = 1, \dots, r-1$, $z_i = i+1$, $z_r = 1$, and $z_1P_1x_1$. Since x is individually rational, $r > 1$. Since (x, p) is a competitive equilibrium, $p_1 < p_2$. Since $3 = z_2R_2x_2$, we have $\sum_{k:kP_{23}} y_{2k} = 0$. Hence, from (7),

$$p_2 \leq p_3 + n \cdot \left(\sum_{k:kP_{23}} y_{2k} \right) = p_3.$$

So, $p_2 \leq p_3$. By repeatedly applying the same arguments we find $p_2 \leq p_3 \leq \dots \leq p_r \leq p_1$. Since $p_1 < p_2$, we obtain a contradiction. Therefore, there is no coalition that weakly blocks x . Hence, x is a strong core allocation. $\square$

Remark 4. We note that in the case of strict preferences, constraints (7) are satisfied by any competitive equilibrium (x, p) . To see this note that if $y_{ij} = 1$ then (6) implies (7), since $1 - y_{ij} = 0$, and hence

$$p_i \leq p_j + n \cdot (1 - y_{ij}) = p_j \leq p_j + n \cdot \left(\sum_{k:kP_{ij}} y_{ik} \right).$$

Otherwise, if $y_{ij} = 0$ then (5) implies (7), since for strict preferences $\sum_{k:kP_{ij}} y_{ik} = \sum_{k:kP_{ij}} y_{ik} + y_{ij} = \sum_{k:kR_{ij}} y_{ik}$, and hence

$$p_i < p_i + 1 \leq p_j + n \cdot \left(\sum_{k:kR_{ij}} y_{ik} \right) = p_j + n \cdot \left(\sum_{k:kP_{ij}} y_{ik} \right).$$

Therefore, in either case, constraints (7) are satisfied. This reflects the fact that for strict preferences the strong core is a singleton that consists of the unique competitive allocation. $\diamond$

Quint and Wako's IP formulations

To compare our IP formulations with the IP formulations for the core and the strong core given by Quint and Wako [38], we describe the latter IP formulations using our notation.

First, for both the core and the strong core, Quint and Wako [38] used the “basic” constraints (1), (2), and (3). We refer to their formulas (9.2), (9.3), (9.4), as well as (8.2), (8.3), (8.4), together with an integrality condition.

Next, to obtain the core, Quint and Wako [38] imposed the following additional no-blocking condition (see (9.1) in [38]):

$$\sum_{i \in S} \left(\sum_{j: j R_i \pi_i} y_{ij} \right) \geq 1 \quad \forall S \subseteq N, \pi \in \Pi_S \quad (8)$$

Finally, to obtain the strong core, Quint and Wako [38] imposed the following additional no-blocking condition (see (8.1) in [38]):

$$\sum_{i \in S} \left(\sum_{j: j P_i \pi_i} y_{ij} + \frac{1}{|S|} \sum_{j: j I_i \pi_i} y_{ij} \right) \geq 1 \quad \forall S \subseteq N, \pi \in \Pi_S, \quad (9)$$

where Π_S is the set of allocations in the submarket M_S (so that π is an allocation in M_S).

Constraints (8) and (9) directly describe that no coalition S can block / weakly block through an allocation π , respectively. Both sets of constraints are highly exponential (in the number of agents), since they are required not only for all subsets S of N , but also for all possible redistributions within each S .

Alternative cycle-formulations

In view of Lemma 1, it is sufficient to impose constraints (8) and (9) for the cycles of the acceptability graph G . Based on this observation and results in [25], we will describe alternative cycle-formulations for the core and the strong core. Furthermore, we will provide a new proposition and IP formulation for the Wako-core.

Let $M = (N, R)$ be a housing market. Let $\mathcal{K}$ denote the set of exchange cycles in $G(N, R)$. For a cycle $c \in \mathcal{K}$, let $N(c)$ and $A(c)$ denote the set of nodes and edges in c , respectively, and let $|c|$ denote the size/length of c . We write $c_i = j$ if agent i receives object j in the exchange cycle c , i.e., $(i, j) \in A(c)$.

Proposition 7 ([25]). *An allocation x is in the core if and only if for each cycle $c \in \mathcal{K}$, for some agent $i \in N(c)$, $x_i R_i c_i$.*

The corresponding IP constraints, which reduce the constraints (8) to cycles, are as follows:

$$\sum_{(i,j) \in A(c)} \sum_{k: k R_i j} y_{ik} \geq 1 \quad \forall c \in \mathcal{K} \quad (10)$$

Next, we describe the alternative cycle-formulation for the strong core. First we focus on the special case of strict preferences.

Proposition 8 ([25]). *Suppose preferences are strict. Then, an allocation x is in the strong core if and only if for each cycle $c \in \mathcal{K}$, c is an exchange cycle in x or for some agent $i \in N(c)$, $x_i P_i c_i$.*

Proposition 8 leads to the following constraints:

$$\sum_{(i,j) \in A(c)} y_{ij} + |c| \cdot \left[\sum_{(i,j) \in A(c)} \sum_{k: kP_i j} y_{ik} \right] \geq |c| \quad \forall c \in \mathcal{K} \quad (11)$$

The alternative cycle-formulation for the strong core in the general case (where preferences can have ties) is as follows.

Proposition 9 ([25]). *An allocation x is in the strong core if and only if for each cycle $c \in \mathcal{K}$,*

- (i) c is an exchange cycle in x , or
- (ii) for some agent $i \in N(c)$, $x_i P_i c_i$, or
- (iii) for each agent $i \in N(c)$, $c_i I_i x_i$.

The corresponding IP constraints, which reduce the constraints (9) to cycles, are as follows:

$$\sum_{(i,j) \in A(c)} \sum_{k: kI_i j} y_{ik} + |c| \cdot \left[\sum_{(i,j) \in A(c)} \sum_{k: kP_i j} y_{ik} \right] \geq |c| \quad \forall c \in \mathcal{K} \quad (12)$$

Finally, similarly to the core and strong core, we provide a new alternative characterisation for the Wako-core.

Proposition 10. *An allocation x is in the Wako-core if and only if for each cycle $c \in \mathcal{K}$,*

- (i) c is an exchange cycle in x , or
- (ii) for some agent $i \in N(c)$, $x_i P_i c_i$, or
- (iii) for some agent $i \in N(c)$, $c_i I_i x_i$ and $c_i \neq x_i$.

The proof of Proposition 10 is omitted as it can be shown in a similar way as Proposition 9 (see [25]).

Proposition 10 leads to the following constraints, which can be used to find competitive allocations (i.e., allocations in the Wako-core):

$$\sum_{(i,j) \in A(c)} y_{ij} + |c| \cdot \left[\sum_{(i,j) \in A(c)} \sum_{k: kR_i j, k \neq j} y_{ik} \right] \geq |c| \quad \forall c \in \mathcal{K} \quad (13)$$

To see the correctness of this new formulation, observe that the first term of (13) is equal to $|c|$ if condition (i) of Proposition 10 is satisfied and less than $|c|$ otherwise; and the second term has value at least $|c|$ if condition (ii) or (iii) of Proposition 10 is satisfied and 0 otherwise. Therefore, constraint (13) is satisfied if and only if at least one of the three conditions of Proposition 10 holds.

Bounded length exchange cycles

Note that the above cycle-formulations are not very practical due to the exponentially large number of cycles. In fact, this justified the novel IP formulations proposed in Section 3.2. However, the cycle-formulations are practical for the case of *bounded* length exchange cycles.

One easily verifies that Lemma 1 can be extended to bounded length exchange cycles in a natural way: the strong core, Wako-core, and core of a k -housing market can be defined equivalently by the absence of corresponding blocking *cycles* of size at most k . In fact, Klimentova et al. [25] provided IP formulations for the core and the strong core by adapting constraints (10) and (12) to bounded exchange cycles. One can similarly adapt constraints (13) to obtain an IP formulation for the Wako-core of a k -housing market. In our simulations we used the most efficient cycle-edge formulations by Klimentova et al. (see the detailed description in Section 3.3 of [25]).

Appendix E: Computational Experiments

In this section, we describe our computational analysis of the models proposed in the previous section and compare these models with the models for bounded length exchange cycles that were proposed in [25]. Furthermore, using computer simulations we estimate for each solution concept how often the respecting improvement property is violated. The simulations have been conducted for both strict and weak preferences and considering two objective functions, namely (1) maximisation of the size of the allocation (corresponding to the maximisation of the number of transplants in the context of KEPs) and (2) maximisation of total weight (where weights of edges can be interpreted as the scores given to the corresponding transplants in a KEP, i.e., reflecting the quality of the transplants).

In Section 3.2, we overview the test instances used for computational analysis and discuss some implementation details.

First, we compare the size/weight of maximum size/weight allocations with the size/weight when stability requirements are imposed, i.e., the size/weight of core, competitive (Wako-core), and strong core allocations. For the unbounded case, we further analyse the price of fairness: the difference in percentage in the number of transplants of maximum total weight allocations and core, competitive, and strong core allocations for both objectives, when compared to maximum size allocations.

Next, we compute the average number of weakly blocking cycles for allocations provided by each formulation. Thus, we obtain an estimation of how far allocations are from the strong core. We also analyse the average number of nodes of an instance that can obtain a strictly better allotment in at least one weakly blocking cycle.

Average CPU times required to solve an instance of a given size for each of the formulations in Section 3.2 are presented in Section 3.2.

Finally, we provide our results on the frequency of violations of the respecting improvement property for the best allotments with respect to all of our models.

Test instances and implementation details

Test instances were generated with the generator proposed in [44, 27] to mimic the pools observed in KEPs and are available from <https://doi.org/10.25747/xh4y-2r05>. The generator creates compatibility (acceptability) graphs for KEPs, with the set of agents N consisting of incompatible pairs and non-directed-donors (NDDs), i.e., donors with no associated recipient. Dummy edges were created from a NDD to each node to handle chains initiated by NDDs in the same way as cycles are operated. The number of pairs of an instance ranges from 20 to 150; 50 instances of each size were generated. The weights associated with the edges of the graph were generated randomly within the interval $(0, 1)$, and preferences were assigned in accordance with the weights: the higher the weight of an outgoing edge of a given node, the more preferred the corresponding object for that agent. We assume that the preferences for the dummy edges from NDDs to all pairs are defined by the randomly generated weights, representing the quality of the donors in these pairs who will donate to the waiting lists if the dummy edge is selected. Thus, the preferences of the NDDs actually represent the interest of the patients in the waiting list. When generating instances with weak preferences, outgoing edges with weights within each interval of length $\frac{1}{|N|}$ were considered equally preferable.

For unbounded length exchange cycles, in order to speed up the running time of the IP formulations we implemented the TTC algorithm and used its output as a starting allocation for all models. Even if this starting allocation is infeasible (which can happen for the strong core formulations) it

is accepted by the solver.

For the bounded case, if the core, Wako-core, or the strong core turned out to be empty, then we computed an allocation that minimises the number of blocking or weakly blocking cycles in the same way as described in [25]. Note that in the IP formulation the stability requirements are written for each cycle; hence, the minimisation of the number of blocking cycles is equivalent to the minimisation of the number of violated constraints.

For the unbounded case, the strong core can be empty for weak preferences. In this case it is difficult to relax the stability requirements in the IP in a meaningful way, so we decided to ignore these instances in the analysis.

All programs were implemented using Python programming language and tested using Gurobi as optimisation solver [21]. The code is available at https://gitlab.com/xenia.klimentova/housemarket_pub. Tests were executed on a MacMini 8 running macOS version 10.14.3 in a Intel Core i7 CPU with 6 cores at 3.2 GHz with 8GB of RAM.

Impact of stability on the number of transplants

Figure 8 depicts average maximum size and maximum weight in the case of weak preferences and increasingly stringent stability requirements: no stability requirements (Max), core, competitive and strong core allocations. We refrain ourselves from presenting the results for the case of strict preferences as all curves are similar (in fact, for strict preferences the competitive and strong core allocations coincide).

As expected, both the number of transplants and total weight decrease by increasing the number of constraints: from Max to Core, then to Competitive, and then to Strong Core. The Strong Core curve is non-monotonic, which is explained by the non-existence strong core allocations for several instances. Next to the curve we present the number of instances out of the total 50 where a strong

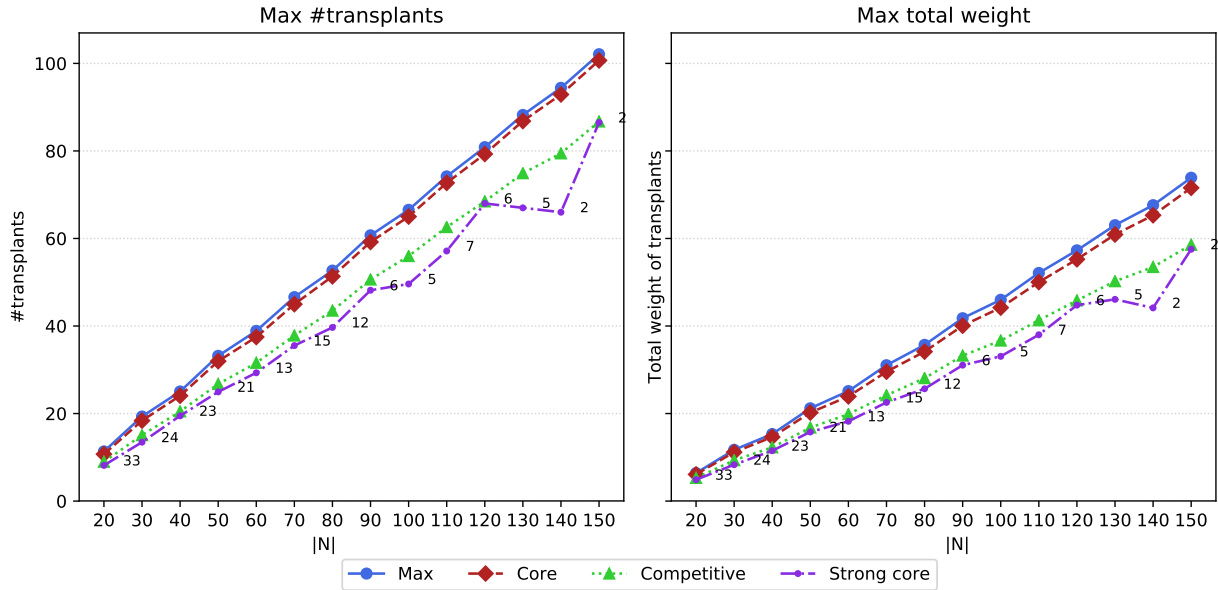


Figure 8: Number of transplants (left) and total weight of transplants (right) for unbounded length and weak preferences. The numbers in the chart reflect the number of instances where a strong allocation

existed.

Figure 9 makes a similar analysis for the bounded case, when $k = 2$ and $k = 3$, indicated as $[k]$ next to the name of a curve. $\text{Max}[\infty]$ refers to the problem with unbounded length exchange cycles, whilst $\text{Max}[2]$ and $\text{Max}[3]$ correspond to the bounded problem for $k = 2$ and $k = 3$, respectively. Similar notation is used for the core, Wako-core²² (W.-Core), and strong core (S.Core). To facilitate the comparison between the bounded and unbounded cases, we plot again the two curves from Figure 8 associated with maximum size/weight (Max) which, in both cases, represent an upper bound for our allocations. Unsurprisingly, the curves associated with $k = 2$ are located below those associated with $k = 3$. We can observe that the maximum number of transplants for $k = 3$ and for unbounded k are very similar (see Figure 9 (left)). Notice also that even though some curves overlap and seem identical, there are minor differences among them, except for the case $k = 2$ where core and Wako-core allocations coincide. As before, we present results for weak preferences only, as this is the more general case. In the case of strict preferences, for $k = 3$ the curves are similar, whilst for $k = 2$, core, competitive and strong core allocations coincide, and the latter two are also the same for unbounded exchange cycles by definition.

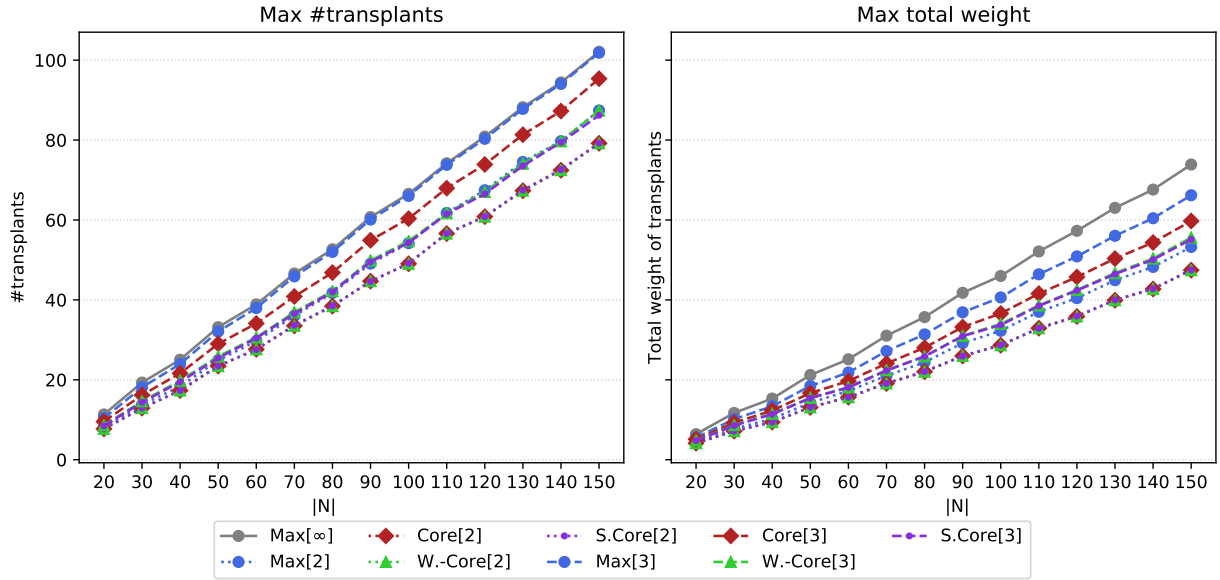


Figure 9: Comparison of the number of transplants (left) and the total weight of transplants (right) for bounded length exchange cycles ($k = 2, 3$, indicated as $[k]$ next to the name of a curve) and weak preferences. A solid line is used for the unbounded case ($[\infty]$), dotted lines for $k = 2$, and dashed lines for $k = 3$.

From a practical point of view, an interesting question to explore is the impact of (core) stability requirements on the achievable number of transplants. Although KEPs have many other key performance indicators, the achievable number of transplants is unarguably the most relevant one, as this criterion is optimised as a first objective in all European KEPs [14]. Figure 10 depicts our findings on the *price of fairness* for strict and weak preferences. The price of fairness is calculated as the difference in percentage in the number of transplants for maximal weight (Max-w) allocations as well as for core, competitive and strong core allocations under both objectives, when compared to the maximum number of transplants achievable (Max-t). As can be observed, the price of fairness for competitive and strong core allocations is significantly higher when compared to the core. It

²²As mentioned before, for the unbounded case the set of competitive allocations is equivalent to the Wako-core, and for the bounded case we only work with the Wako-core, as competitive equilibrium is not well-defined.

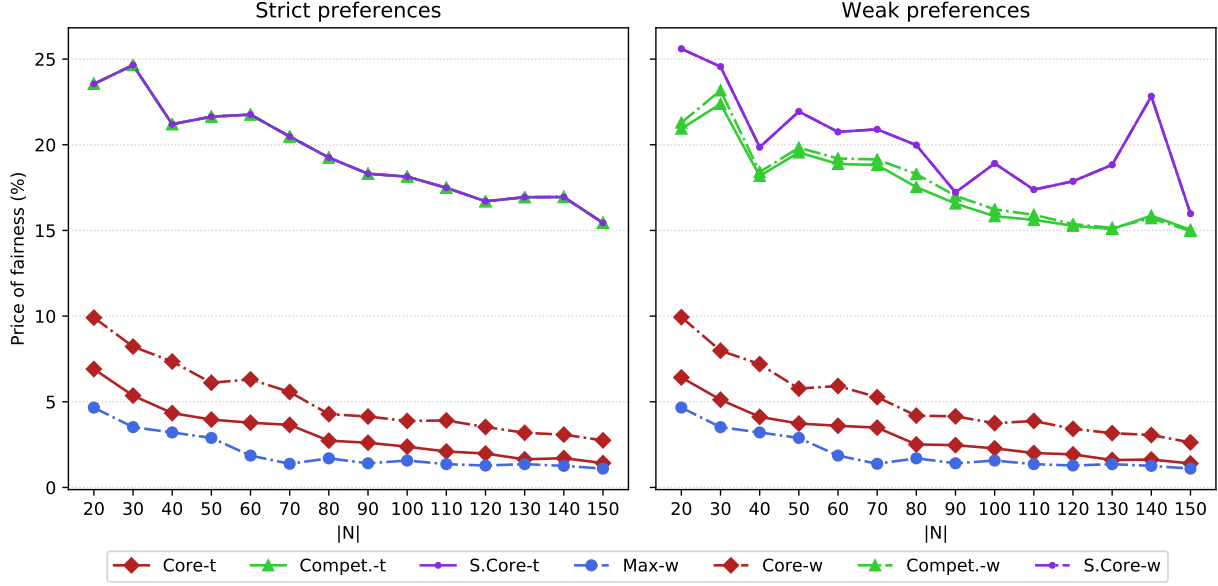


Figure 10: Price of fairness with respect to maximum number of transplants for core, competitive and strong core allocations, with maximum number of transplants (-t) and maximum total weight (-w) allocations; for strict (left) and weak (right) preferences. Solid lines are used for models that maximise the number of transplants and dash-dotted lines for those that maximise the weight.

decreases with problem size for both objective functions and for all allocation models, being slightly higher for the core for the total weight objective (see curve *Core-w*). For the maximisation of the number of transplants (curve *Core-t*), for instances with more than 50 nodes the reduction is of less than 3%, decreasing to 1% for the largest instance. This finding is of major practical relevance as it indicates that with increasing size of the kidney exchange programmes one can take into account preferences without a significant reduction in the number of pairs that are transplanted.

Analysis of the number of blocking cycles

In this analysis we calculate the average number of (weakly) blocking cycles for different allocations, considering different bounds on the maximum length of these cycles: $l = 2, 3, 4, 5$. When the exchange cycles (of the allocations) are bounded by k , the analysis is naturally restricted to the cases where $l \leq k$. For the sake of clarity, we focus on the more general case and discuss only the results for weakly blocking cycles. Our results for blocking cycles are very similar. Figure 11 shows the average number of weakly blocking cycles of size 2 for Max, core, and competitive (Wako-core) allocations. For the bounded case, following the same reasoning as in [25], the figure also reports the minimum average number of weakly blocking cycles for the cases where the strong core is empty, i.e., for the maximum number of transplants/total weight allocations with minimum number of weakly blocking cycles. For the unbounded case, the strong core can also be empty for weak preferences, but we do not depict that case, since we cannot count efficiently the number of weakly blocking cycles of unbounded length. Interestingly, the “unstability” of the allocations that maximise the number of transplants (curves *Max*[∞], *Max*[2], *Max*[3], Figure 11 (left)) barely depends on the maximal allowed size of exchange cycles. This is not true for the *Core*: the number of blocking cycles is considerably smaller for $k = 2$. For this and all the remaining cases, the average number of weakly blocking cycles is very low, in most cases below 1. It is worth noting that the average

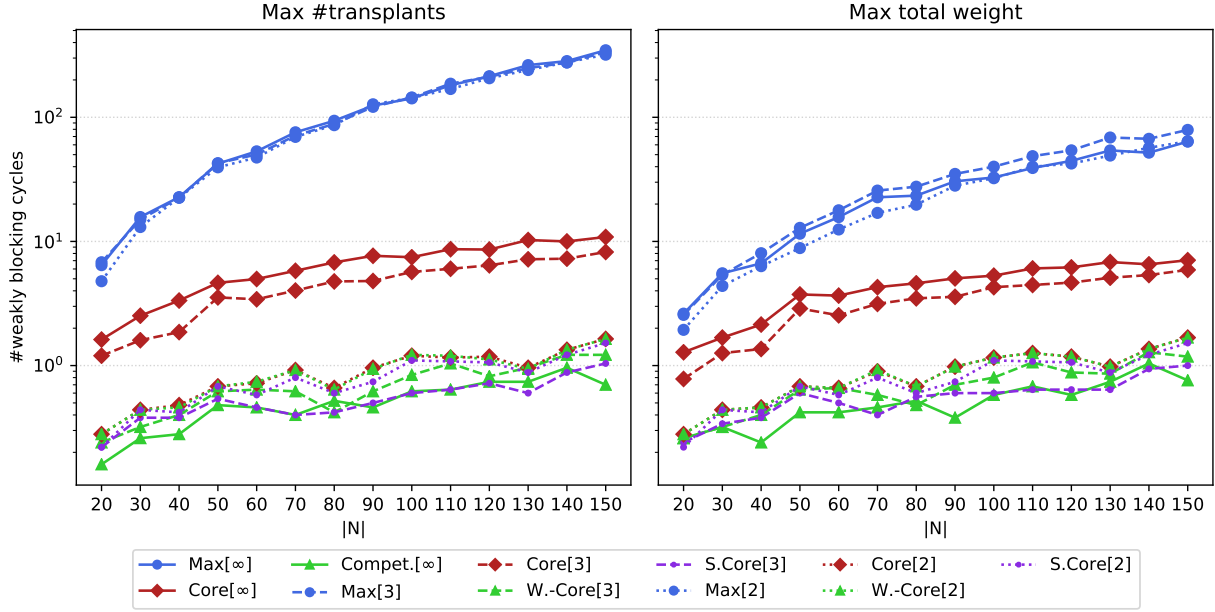


Figure 11: Number of weakly blocking cycles of size $l = 2$ for allocations with maximum number of transplants (left) and maximum total weight of transplants (right), for unbounded exchange cycles and exchange cycles of size up to $k = 2$ and $k = 3$ for weak preferences. A solid line is used for the unbounded case, dotted lines for $k = 2$, and dashed lines for $k = 3$.

number of blocking cycles tends to be smaller when the objective is to maximise the total weight (Figure 11 (right)). A possible explanation for this is that the weights reflect patients' preferences and therefore allocations that take into account preferences will tend to have less blocking cycles.

Figure 12 presents the same analysis, now considering weakly blocking cycles of size up to 3. The conclusions drawn for $l = 2$ remain valid for this case.

For the unbounded case, the number of blocking cycles is larger, since one must consider also the cases when $l > 3$. Figure 13 provides information on the number of weakly blocking cycles of size up to 4 and up to 5 (indicated by suffixes $_4$ and $_5$). We do not present results for $l > 5$ as searching for these larger blocking cycles would lead to excessively long CPU time.

Although the findings above are already insightful, Figure 14 complements the analysis by focusing on the average number of nodes of an instance that strictly prefer their allotments in at least one weakly blocking cycle (i.e., the number of patients that can receive a strictly better kidney). An important conclusion can be drawn from the results in the figure: the maximisation of total weight decreases the number of agents that can get a better allotment in a blocking cycle when compared to the maximum size allocations (compare curves **Max** in Figure 14 (left) and (right)). Comparing Figure 14 with Figure 10 gives insights into the reduction of the total number of transplants that would be necessary to meet a certain level of patients' preferences.

CPU time for unbounded models

In Table 12, we present the average CPU time for solving an instance of a given size with one of the three newly proposed IP models for the unbounded case. Recall that the allocation obtained by TTC algorithm was provided as a starting allocation for all formulations.

In the case of weak preferences CPU time is much longer for the core and, especially, the competitive allocations. However, finding strong core allocations for weak preferences is faster

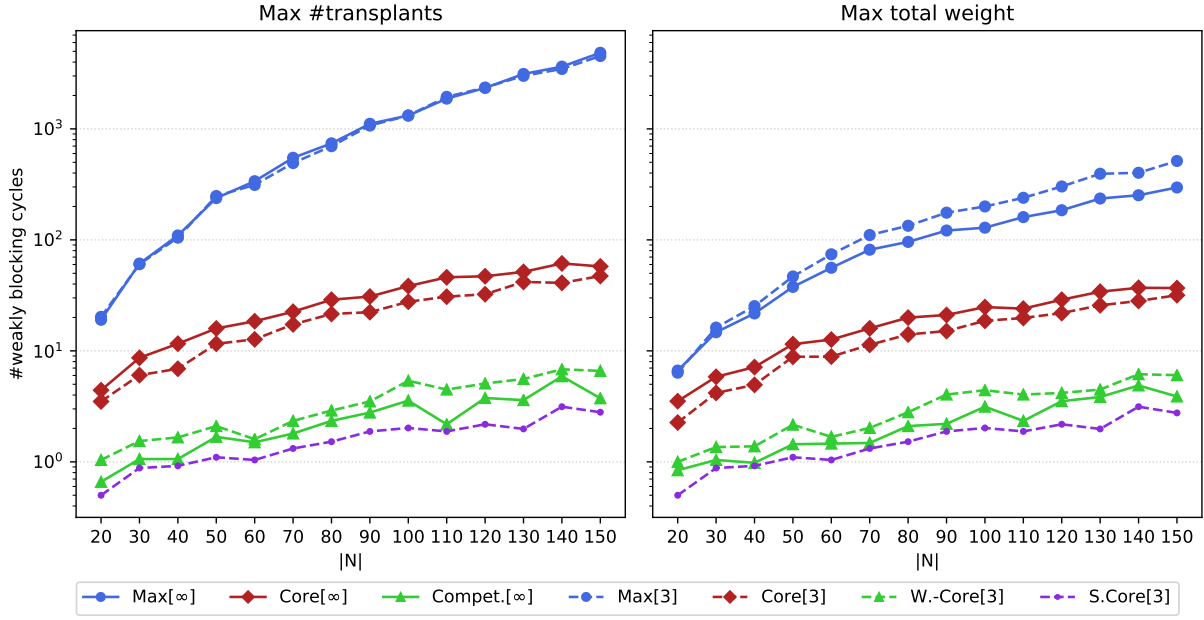


Figure 12: Number of weakly blocking cycles of size $l = 3$ for allocations with maximum number of transplants (left) and maximum total weight of transplants (right), for unbounded exchange cycles and exchange cycles of size up to $k = 3$ for weak preferences.

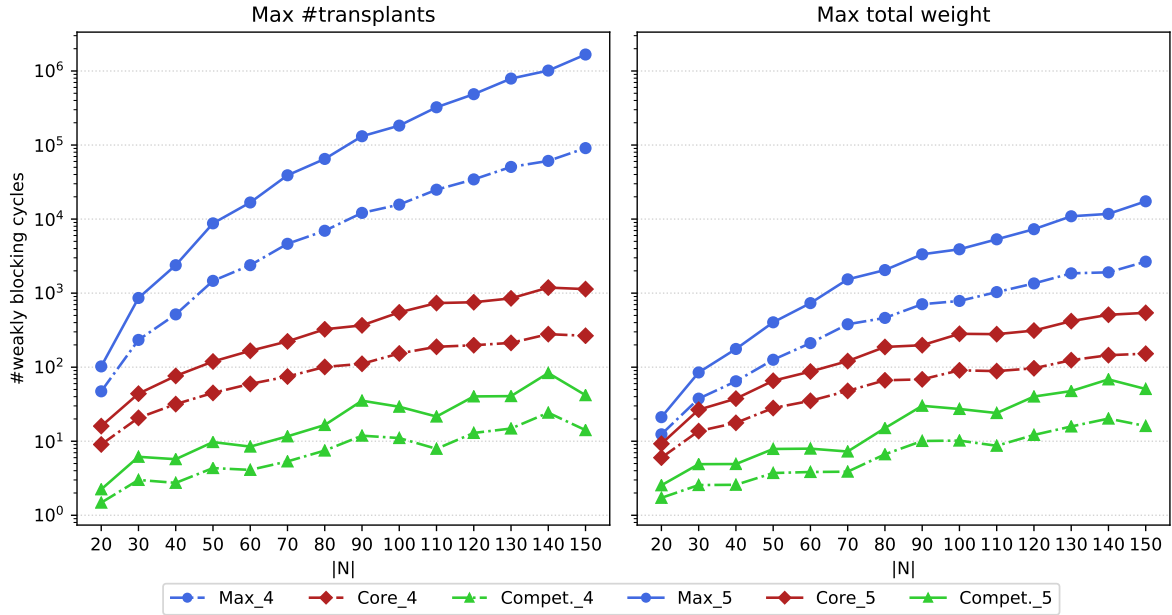


Figure 13: Number of weakly blocking cycles of size $l = 4$ and $l = 5$, indicated as l next to the name of a curve, for allocations with maximum number of transplants (left) and maximum total weight of transplants (right), for unbounded exchange cycles and weak preferences.

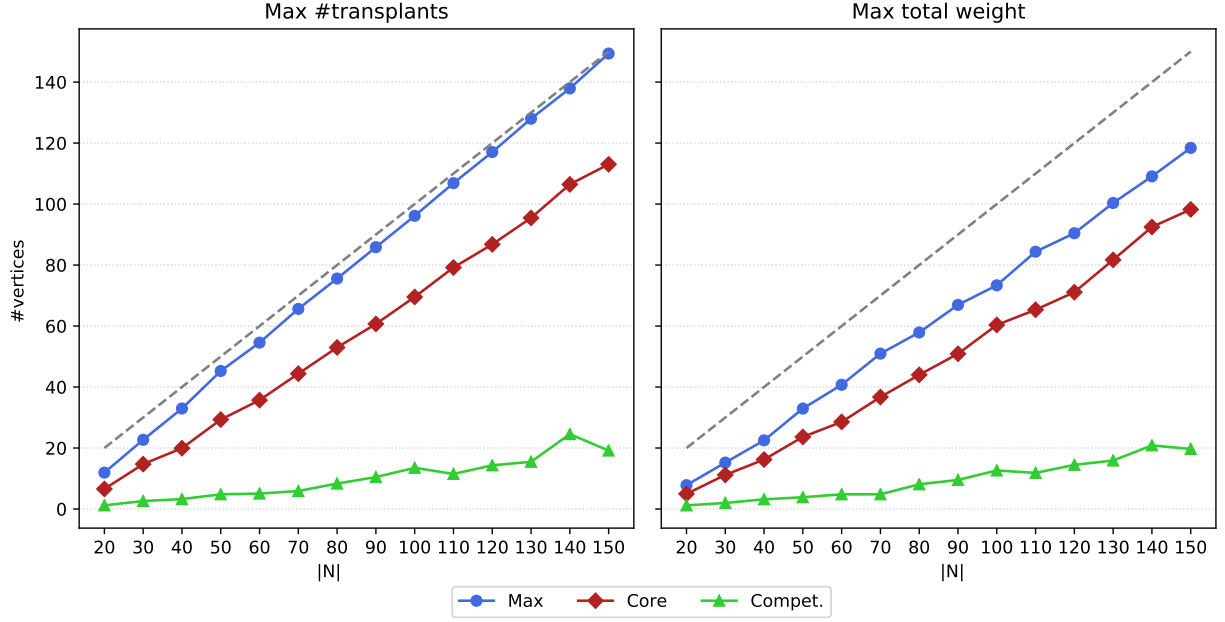


Figure 14: Weak preferences. Conditional on the existence of at least one weakly blocking cycle, average number of agents that receive a strictly better allotment at some weakly blocking cycle. The grey line is a reference line showing the number of nodes in an instance.

$ V $	Max # transplants			Max total weight			Max # transplants			Max total weight		
	Core	Compet.	S.Core	Core	Compet.	S.Core	Core	Compet.	S.Core	Core	Compet.	S.Core
Strict preferences							Weak preferences					
20	0.00	0.03	0.01	0.00	0.02	0.01	0.00	0.04	0.01	0.00	0.03	0.01
30	0.03	0.13	0.04	0.02	0.11	0.03	0.02	0.28	0.04	0.02	0.17	0.03
40	0.08	0.48	0.12	0.06	0.25	0.11	0.09	0.63	0.10	0.06	0.44	0.08
50	0.24	1.74	0.38	0.16	0.58	0.34	0.20	2.15	0.25	0.17	1.06	0.21
60	0.47	2.39	0.87	0.28	0.91	0.79	0.52	6.03	0.44	0.26	2.87	0.39
70	1.06	3.91	1.94	0.66	2.29	1.50	0.84	16.99	1.09	0.53	7.35	0.77
80	1.62	6.54	3.26	0.82	3.39	2.32	1.41	32.21	1.63	0.76	17.47	1.01
90	3.14	36.34	5.31	3.27	5.38	3.59	3.29	167.15	2.36	1.82	80.88	1.49
100	3.53	16.19	19.26	2.43	6.15	9.81	4.51	188.35	8.87	3.08	95.39	4.62
110	8.73	21.42	28.26	4.97	9.01	13.79	6.68	331.64	16.40	5.92	159.12	7.24
120	17.84	72.87	57.36	6.81	15.36	24.32	20.14	392.88	19.60	6.79	218.58	10.87
130	14.34	46.92	84.49	14.24	22.68	34.11	14.78	586.27	21.75	12.32	438.23	10.42
140	29.50	61.99	110.82	21.51	34.33	46.67	41.59	708.92	40.97	16.43	539.56	14.89
150	41.99	161.10	214.32	30.66	52.61	70.77	57.13	786.43	61.79	27.82	682.99	23.91

Table 12: Average CPU time (in seconds) for solving an instance of a given size with the proposed formulation.

than doing so for strict preferences. Moreover, surprisingly, finding the strong core is the most time consuming task for strict preferences, while it is the least time consuming task for weak preferences. Finally, we notice that the models for finding core and strong core allocations are performing (with respect to CPU time) within the same ranges of magnitude compared with the corresponding models for the bounded case, analysed in [25].

Violations of respecting improvement property

In this section, we conduct a computational analysis on how often the respecting improvement (RI) property is violated for the best allotments under different models, for both unbounded and bounded exchange cycles. For the unbounded case we considered only the **Max** and **Core** models under both objectives: Theorem 1 and Proposition 1 show that competitive allocations respect improvement for the best allotments; Theorem 1 and Corollary 1 show that strong core allocations (conditionally) respect improvement for the best allotments.

For each model and for instances with 20 and 30 nodes we run the procedure presented in Algorithm 1 below. For each agent i , given some preferences R_i , define for each object k a rank $r_k^i \in \{1, \dots, |N|\}$ such that for all objects j, k we have $r_k^i \leq (<, =) r_j^i$ if and only if $kR_i(P_i, I_i)j$.

Algorithm 1 Procedure for checking the RI-best property

Ensure: M number of violations of the RI-best property

```

1:  $M \leftarrow 0$ 
2: for  $i \in N, j \in N, i \neq j$  do
3:   Let  $R$  be the current preferences of agents;
4:   Find allocation with the best allotment for  $i$  with respect to  $R$ , denote the allocation by  $\bar{y}$ ;
5:   For  $\bar{y}_{il} = 1$  denote  $\bar{r} \leftarrow r_l^i$ ;
6:   while  $\exists k P_j i$  do
7:     Let  $k$  be the first strictly preferred object for  $j$  that precedes  $i$  in  $R$ ;
8:     if strict preferences then
9:       Swap  $i$  with  $k$  in the list of preferences of  $j$ ;
10:    end if
11:    if weak preferences then
12:      Let  $i$  become equally preferred for  $j$  as  $k$  (i.e.  $r_i^j \leftarrow r_k^j$ );
13:    end if
14:    Denote modified preferences by  $\tilde{R}$ ;
15:    Find allocation with the best allotment for  $i$  with respect to  $\tilde{R}$ , denote allocation by  $\tilde{y}$ ;
16:    if core/Wako-core/strong core is empty then
17:      continue;
18:    end if
19:    For  $\tilde{y}_{it} = 1$ , denote  $\tilde{r} = r_t^i$ ;
20:    if  $\bar{r} < \tilde{r}$  then
21:      The RI-best property is violated:  $M \leftarrow M + 1$ ;
22:    end if
23:     $\bar{r} \leftarrow \tilde{r}; R \leftarrow \tilde{R}$ 
24:  end while
25: end for
```

For each pair of agents i and j , agent i is consecutively making improvements, moving up in the preference list of agent j until its top. In each step (see **while** loop in the algorithm) for the case of strict preferences, i is swapped with k , who is the first strictly preferred agent by j to i .

For the case of ties, agent i first becomes equally preferred for j as k . After the improvements, the allocations that provides the best allotments for i for the original (R) and improved ($\tilde{R}$) preferences are compared for i . If such an allocation does not exist for $\tilde{R}$, the algorithm continues with the next iteration of the **while** loop. It is considered that there is a violation of the RI-best property if i obtains a strictly worse allotment in allocation for $\tilde{R}$.

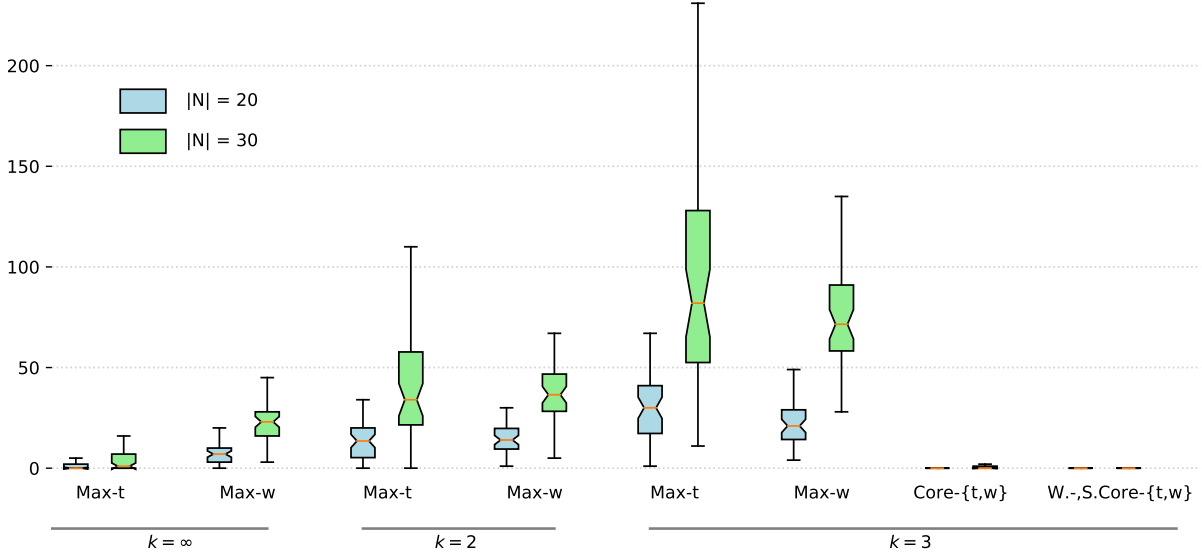


Figure 15: Number of violations of the RI-best property for all instances in total of a given size, $|N| = 20, 30$, for strict preferences.

Figures 15 and 16 present box plots, with outliers omitted, for the number of violations of the RI-best property for instances of a given size for strict and weak preferences, respectively, for models where the RI-best property is violated at least once. Models whose results are the same, independently of the objective considered, are plotted together. This is the case, for example, of Core-t and Core-w, for $k = 3$ and strict preferences (see Figure 15), or of W.-Core and Core, weak preferences and $k = 2$ and $k = 3$ (see Figure 16).

For (Wako-, strong) core models there were few cases of violations of the RI-best property, as reflected in the figures. To give an indication, the total number of violations for all instances with 30 nodes, weak preferences, and $k = 3$ was 4549 for Max-t, 3145 for Max-w, 10 for Core-{t,w}, 20 for W.-Core-{t,w}, and 2 for S.Core-{t,w}. For maximum size and maximum weight allocations (Max-t and Max-w, respectively), both for the unbounded and the bounded cases, one can observe a significant number of violations. These numbers increase with instance size. Interestingly, for the unbounded case the number of violations for Max-t was lower than that for Max-w. This can be explained by the fact that the former problem has a larger number of alternative allocations that provide the same allotment for an agent, while for the weighted objective problem the allocation is usually unique. On the contrary, for the bounded case maximum weight allocations violated the RI-best property less often, compared to maximum size allocations.

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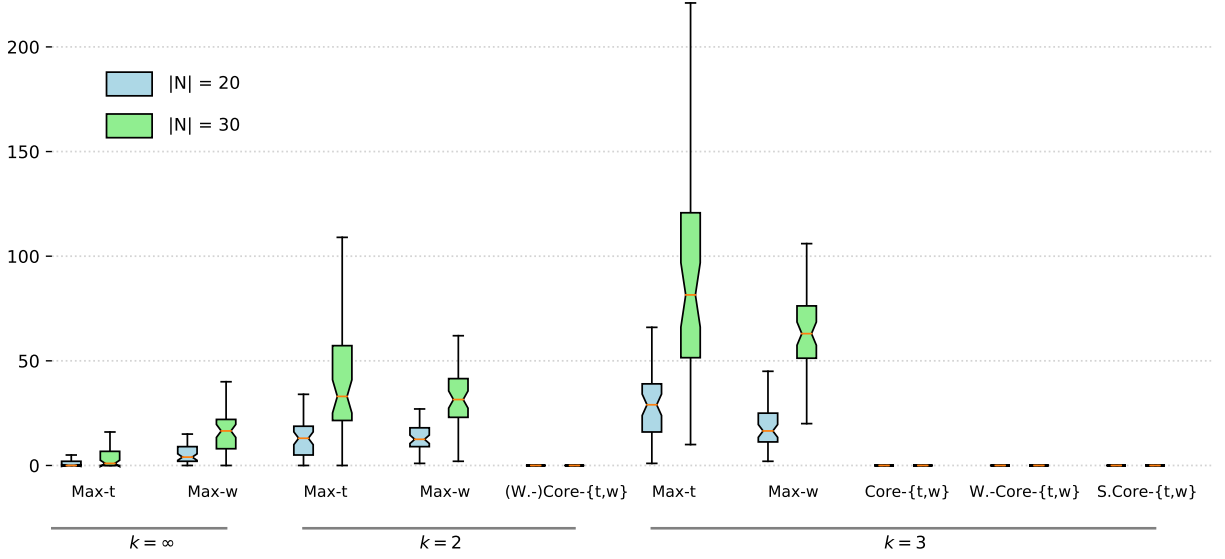


Figure 16: Number of violations of the RI-best property for all instances in total of a given size, $|N| = 20, 30$, for weak preferences.

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