

Generalized Rental Harmony

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Abstract

Rental harmony is the problem of assigning rooms in a rented house to tenants with different preferences, and simultaneously splitting the rent among them, such that no tenant envies the bundle (room and price) given to another tenant. Various researchers have studied this problem mainly under two incompatible assumptions: the *miserly tenants* assumption — each tenant prefers a free room to a room with a positive price; and the *quasilinear tenants* assumption — each tenant attributes a monetary value to each room, and prefers a room of which the difference between value and price is maximum. This article shows that the main technique used for rental harmony with miserly tenants, using Sperner’s lemma, can be adapted to a much more general class of preferences, one that contains both miserly tenants and quasilinear tenants as special cases. As a corollary, some recent results derived for miserly tenants are found to be applicable to this more general class, too.

1 The problem

Several tenants would like to rent a house together. The number of tenants equals the number of rooms in the house, so each tenant should get a single room. But the rooms differ in quality, so dividing the rent equally among the tenants might make some of them — those who received lower-quality rooms — envious. To prevent such envy, it is important to determine the room prices such that each tenant believes that the bundle of room and price given to them is at least as good as the bundle of room and price given to any other agent. In: other words, we would like each tenant to feel that their room is at least as good as any other room, when taking into account their respective prices. This situation is called *rental harmony* — a term coined by Francis E. Su in a famous MONTHLY article [17].

Formally, denote by $\mathcal{N}$ the set of tenants and by $\mathcal{M}$ the set of rooms in the house, with $|\mathcal{N}| = |\mathcal{M}| = n$. The total rent is R . It is required to assign a price $p_j \in \mathbb{R}$ to each room $j \in \mathcal{M}$ such that the sum of all prices is R , and to assign each room in $\mathcal{M}$ to a unique agent in $\mathcal{N}$. The agents have different preferences on the rooms. The preferences of an agent are represented by a *demand function* — for each price vector $\mathbf{p} \in \mathbb{R}^n$, the agent has a set of one or more rooms that they consider the *best rooms* given the prices. An allocation in which each agent receives a room from the set of their best rooms is called *envy-free*.

The existence of envy-free allocations has been proved using various techniques and under various assumptions on the agents’ preferences. The first one to study this question was David Gale [10], who made the following assumption:

Quasilinear tenants assumption. For all $i \in \mathcal{N}$ and $j \in \mathcal{M}$, there is a value $v_{i,j}$ representing the value for agent i of room j . The best rooms of agent i in price $\mathbf{p}$ are the ones maximizing the difference $v_{i,j} - p_j$.

Gale proved the existence of envy-free allocations using linear programming duality. Later works provided fast algorithms for calculating an envy-free allocation with quasilinear tenants using various techniques: a compensation procedure [12], a market-based mechanism [1], and maximum matching and linear programming [18]. The latter approach was later implemented and tested in the popular website spliddit.org [9].

Su [17] studied this question under a different assumption — an agent always prefers a free room if one is available. Su considered only nonnegative prices; since we will want to consider general prices, we present a slightly generalized version of his assumption — an agent always prefers a room with a nonpositive price if one is available:

Miserly tenants assumption. Given a price vector $\mathbf{p}$ in which $p_j \leq 0$ for some $j \in \mathcal{M}$, every agent $i \in \mathcal{N}$ has a best room j_i with $p_{j_i} \leq 0$.

In addition, Su assumed that the demand functions are *continuous* in the following sense: if some room j is a best room for agent i for a convergent sequence of price vectors, then j is a best room for i in the limit price vector.

The proofs and algorithms developed for quasilinear tenants cannot be used for miserly tenants. This is because miserly tenants are not necessarily quasilinear. In fact, the miserly tenants assumption puts no restrictions on the preferences when all prices are strictly positive. The preferences may even depend on the entire price vector. For example, a miserly tenant might prefer room #1 when the price vector is (100, 200, 200, 300) and prefer room #2 when the price vector is (100, 200, 400, 100), even though the prices of both rooms have not changed. Although it seems strange, there are several situations in which such preferences may be reasonable. For example, the agent may believe that a room with a rent as high as 400 attracts wealthy neighbors and that this makes the neighboring room more attractive. As another example, if the second and third room are similar in quality, the fact that the third room costs 400 might make the second room look more attractive — a well-known behavioral bias [5].

Attaining rental harmony with miserly tenants requires a different technique, which is described next.

2 Sperner-type lemmas

Sperner's lemma and its variants consider an n -vertex simplex, which without loss of generality is the standard $(n - 1)$ -simplex

$$\Delta^{n-1} = \left\{ (x_1, \dots, x_n) \left| \sum_{j=1}^n x_j = 1, \text{ for all } j \in [n], x_j \geq 0 \right. \right\},$$

where $[n] := \{1, \dots, n\}$. For each $i \in [n]$, we denote by F_i the main vertex of the simplex in which the i th coordinate is 1 and all other coordinates are 0.

The simplex is triangulated, and each vertex of the triangulation is labeled with a label in $[n]$. The goal of all Sperner-type lemmas is to identify conditions that guarantee the existence of a *fully-labeled sub-simplex* — a sub-simplex of the triangulation whose vertices are labeled with n different labels.

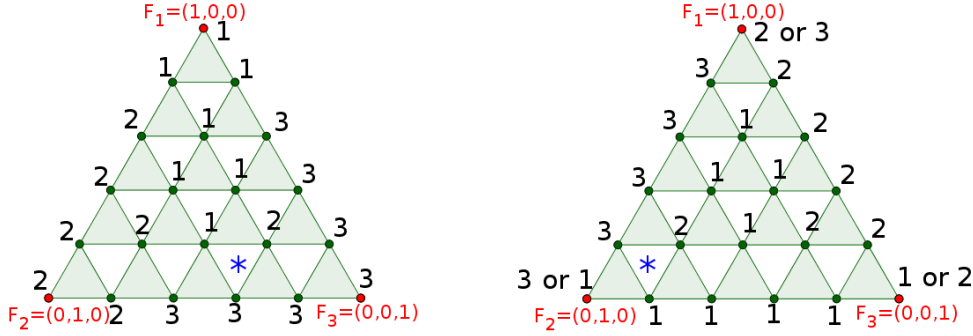


Figure 1: Two labelings of a triangulation of a 3-simplex. **Left:** the labeling satisfies the conditions of Sperner's lemma. **Right:** the labeling satisfies the conditions of Scarf's lemma. In both cases, * denotes a fully-labeled simplex (there may be more than one).

We say that a labeling satisfies *Sperner's boundary condition* if the label on each vertex $(x_1, \dots, x_n)$ is some j for which $x_j > 0$. In particular, the label of F_1 is 1; the label of each vertex on the line between F_1 and F_2 is either 1 or 2; and so on. Sperner's lemma says that *every labeling that satisfies Sperner's boundary condition admits a fully-labeled simplex*. An example is shown on the left in Figure 1.

Sperner's lemma has many variants; of particular interest here is the variant proved by Scarf [16]. We say that a labeling satisfies *Scarf's boundary condition* if the label on each vertex $(x_1, \dots, x_n)$ on the boundary of Δ^{n-1} is some j for which $x_j = 0$. In particular, when $n = 3$, the label of F_1 is either 2 or 3; the label of all vertices in the interior of the line between F_1 and F_2 is 3; and so on. Scarf's lemma says that, in any sufficiently fine triangulation (specifically, in any triangulation in which no sub-simplex touches all n faces of the large simplex), *every labeling that satisfies Scarf's boundary condition admits a fully-labeled simplex*. An example is shown on the right in Figure 1.

3 Miserly tenants and Scarf's lemma

Su's proof of rental harmony with miserly tenants has several components.

First, associate each vertex of the triangulation with one of the n agents, such that in each sub-simplex, all n agents are represented. While this is not possible for every triangulation, there exist arbitrarily fine triangulations that can be labeled in this way, for example, repeated barycentric subdivisions, or the triangulations illustrated in Figure 1. Next, associate each point $\mathbf{x}$ in Δ^{n-1} with a price vector $\mathbf{p}$ such that, for all $j \in \mathcal{M}$,

$$p_j = R x_j. \quad (1)$$

Label each vertex $\mathbf{x}$ of the triangulation with the index of a best room of the agent who owns $\mathbf{x}$, in the price $\mathbf{p}$ associated with $\mathbf{x}$. If that agent has two or more best rooms in $\mathbf{p}$, then choose one of them arbitrarily, as long as the choice is consistent with the miserly tenants assumption. That is, if there are free rooms in $\mathbf{p}$, then the label of $\mathbf{x}$ must be the index of a free room.¹

¹We cannot just assume that the agent always prefers a single room, since this assumption is incompatible with continuity. For example, suppose there are two rooms, and Alice prefers room 1 when $p_1 < 500$ and room 2 when $p_1 > 500$. Then the continuity assumption requires that when $p_1 = 500$ she prefers both rooms.

Each vertex on the boundary of Δ^{n-1} corresponds to a price vector in which one or more rooms are free. Therefore, the labeling described above satisfies Scarf's boundary condition. By Scarf's lemma, there is a fully-labeled simplex. Consider a sequence of finer and finer triangulations. The sequence of fully-labeled simplices has a subsequence that converges to a point. At the limit point, by the continuity of the preferences, each agent has a different best room. Hence there is an envy-free allocation with prices determined by the coordinates of the limit point.

Su's theorem has been extended in various ways:

1. *Rental harmony with roommates*, when each room can accommodate several tenants [5];
2. *Rental harmony with a secretive agent*, when only $n - 1$ agents are present, and they have to determine a price vector such that, when the n th agent comes and picks a room he prefers, the other $n - 1$ agents can allocate the remaining $n - 1$ rooms among them such that the resulting allocation is envy-free [4, 8].
3. *Rental harmony with an extra agent*, when $n + 1$ agents are present, and they have to determine a price vector such that, when any agent leaves, the remaining n agents can allocate the n rooms among them in an envy-free way [14].
4. *Multi-house rental harmony*, when there are several n -room houses (say, a bedroom house and an office building that are being rented to a set of agents together), and each agent should receive a room in each house [15].

All these extensions use appropriate generalizations of Scarf's lemma, and they all make an assumption similar to the miserly tenants assumption.

Quasilinear tenants do not satisfy the miserly tenants assumption. For example, consider a house with total rent $R = 1000$ and three rooms: a spacious living room and two basements. Consider a quasilinear tenant who values the living room at 800 and each basement at 100. If the price vector is $(600, 400, 0)$, then the quasilinear tenant (quite understandably) strictly prefers the living room to both basements. To further illustrate the difficulty with the miserly tenants assumption, note that this assumption combined with the continuity of the demand function implies that every agent is indifferent among all free rooms [8], which is clearly unrealistic.

Su [17] notes, in the "Comments and Discussion" section, that his proof can be adapted to use a weaker assumption: whenever a free room is available, each agent has a best room that is not the most expensive room. Formally, we have:

Weak miserly tenants assumption. Given a price vector $\mathbf{p}$ in which $p_j \leq 0$ for some $j \in \mathcal{M}$, every agent $i \in \mathcal{N}$ has a best room j_i with $p_{j_i} < \max_j p_j$.

The above example shows that quasilinear tenants do not satisfy even this weaker assumption: the quasilinear agent prefers the living room although it is the most expensive one.

We could strengthen the miserly tenants assumption and require that, when there are free rooms, *all* the best rooms of every agent must be free rooms. But this would make the theorem weaker. For example, the theorem would not be applicable to indifferent agents, who prefer all rooms regardless of their prices, even though an envy-free allocation obviously exists in this case.

This raises the question of whether or not the above extensions are valid for quasilinear tenants.² The next section answers this question in the affirmative, by describing a new proof of the existence of rental harmony with quasilinear tenants — a proof using Sperner’s lemma.

4 Quasilinear tenants and Sperner’s lemma

To handle quasilinear tenants, all we need to change is the interpretation of the points in the $(n - 1)$ -simplex: Associate each point $\mathbf{x}$ in Δ^{n-1} with a price vector $\mathbf{p}$ such that, for all $j \in \mathcal{M}$,

$$p_j = 1/x_j. \quad (2)$$

Then, the boundary points correspond to price vectors in which some rooms cost infinity. This means that each price is an element of the (positive) extended real number line $\mathbb{R}_+ \cup \{\infty\}$, rather than an element of $\mathbb{R}_+$. Note that, since $\sum_{j=1}^n x_j = 1$ and $x_j \geq 0$ for all $j \in [n]$, all prices are positive and at least one price is finite.

As in Su’s proof, each vertex of the triangulation is labeled with a best room of the vertex owner. A quasilinear agent always prefers a room with a finite price to a room with an infinite price. Hence, the labeling satisfies Sperner’s boundary condition: the label on a boundary vertex is always the index of room j for which $x_j > 0$. By Sperner’s lemma, the labeling admits a fully-labeled sub-simplex. Continuity of preferences is preserved, too. Hence there exists an envy-free allocation with some price vector $\mathbf{p}$.

One problem remains: the sum of prices in $\mathbf{p}$ may be unequal to R . However, with quasilinear agents this is easy to resolve. First, note that all prices in $\mathbf{p}$ are finite — otherwise no allocation would have been envy-free. Let $S := \sum_{j=1}^n p_j$. Let $\mathbf{q}$ be a new price vector defined by: $q_j := p_j + (R - S)/n$. For a quasilinear agent, adding a fixed amount to the price of each room does not change the relative preference-ordering between the rooms. Hence, the same allocation is envy-free with price vector $\mathbf{q}$, and the sum of prices is $S + (R - S) = R$.

Note that adding $(R - S)/n$ to all prices might make some prices negative. This means that some tenants are paid to live in their room.³ This is inevitable: in some situations with quasilinear tenants, all envy-free allocations have negative prices. For example, suppose $n = 2$, tenant 1 values the bedroom at 150 and the basement at 0, tenant 2 values the bedroom at 140 and the basement at 10, and $R = 100$. Then in any envy-free allocation, tenant 1 gets the bedroom and tenant 2 gets the basement, and to avoid envy the difference in prices must be at least 130. Since the sum of prices is 100, the price of the basement must be at most -15 .⁴

²An affirmative answer has recently been given to two of these questions: roommates [11] and a secretive agent [3]

³Negative prices may make sense in some situations. For example, if one of the rooms requires constant maintenance in order to prevent nuisances to the other rooms, then the tenants might agree to pay anyone who will take this room and do the maintenance job.

⁴Brams and Kilgour [6] show an example with $n = 4$ agents and rooms, in which for each agent the sum of values of all rooms equals R , and still there are negative prices in any envy-free allocation.

5 Nonlinear tenants

Svensson [19] and Alkan [2] generalized the quasilinear tenants model by assuming that each agent i has a transitive preference relation $\succeq_i$ on (room, price) pairs. For each price vector $\mathbf{p}$, the best rooms of agent i are the rooms j for which the pairs (j, p_j) are maximal (by $\succeq_i$). They assume that the preference relation is continuous and monotonic in the price, i.e., $(j, p) \succeq_i (j, p + \delta)$ whenever $\delta \geq 0$. Without further assumptions, an envy-free allocation might not exist. For example, if $(1, p) \succ_i (2, q)$ for all agents i and prices p, q , then the agent who receives room 2 always envies the agent who receives room 1. Therefore they make assumptions whose thrust is that every agent can be convinced to select any room, if its price is sufficiently low relative to the other rooms. The following assumption is made before Theorem 1 of [19], and in the proof of Theorem 2 of [2].⁵

Archimedean tenants assumption. There exists a number $T \geq R$ such that an agent always prefers any free room to any room that costs T . Formally, for any agent i and for any two rooms j and j' , the following holds: $(j, 0) \succeq_i (j', T)$.

Note that, if the assumption is satisfied with some T , then it is satisfied with any $T' > T$ by the price-monotonicity. Therefore, the requirement $T \geq R$ does not restrict the generality of the assumption.

The Archimedean tenants assumption is more general than the quasilinear tenants assumption: every quasilinear tenant is Archimedean with any T that is at least as large as the largest value-difference between two rooms, i.e.,

$$T \geq \max_{i \in \mathcal{N}} \left(\max_j v_{i,j} - \min_j v_{i,j} \right).$$

In particular, if the tenant assigns a nonnegative value to each room, and the sum of all values is R , then $T = R$ will do.

However, the Archimedean tenants assumption is still not sufficiently general to handle miserly tenants. For example, the tenant in the example at the end of Section 1 cannot be represented by a preference relation on room-rent pairs, since the preference between the first two rooms depends on the rents of the other rooms.

To handle such *externalities* in the preferences, Velez [20] presents the *compensation assumption* (Definition 2). The following is a slightly simplified version of his assumption, adapted to the language of rooms and rent.

Compensable tenants assumption. There exists a number $T \geq R$ such that, if there is a room that costs at most 0 and the most expensive room costs T , then each agent prefers a room that costs less than T . Formally, given a price vector $\mathbf{p}$ in which $\min_j p_j \leq 0$ and $\max_j p_j = T$, every agent $i \in \mathcal{N}$ has a best room j_i with $p_{j_i} < T$.

The compensable tenants assumption is more general than all previous assumptions:

- Every *Archimedean tenant* (with some T) is compensable with the same T . Suppose $p_j \leq 0$ for some $j \in \mathcal{M}$, and $p_{j'} = T$ for the most expensive room $j' \in \mathcal{M}$. Every tenant weakly prefers (j, p_j) to $(j, 0)$ by price-monotonicity and $(j, 0)$ to (j', T) by the Archimedean assumption. Hence, each agent has a best room that costs less than T .

⁵The term “Archimedean” was coined by Rodrigo Velez. A similar assumption, called “Assumption A1,” is presented in his recent survey paper [21].

- Every (weakly) miserly tenant is compensable with any $T > 0$. Again suppose $p_j \leq 0$ for some $j \in \mathcal{M}$, and $p_{j'} = T$ for the most expensive room $j' \in \mathcal{M}$. By the weak miserly tenants assumption, every agent has a best room that is not the most expensive, so it costs less than T .

Fortunately, even this most general case can be handled by Sperner's lemma: all that is needed is a different interpretation of the points in the $(n - 1)$ -simplex.

Theorem 1. *An envy-free allocation among compensable tenants always exists.*

Proof. Associate each point $\mathbf{x}$ in Δ^{n-1} with a price vector $\mathbf{p}$ such that, for all $j \in \mathcal{M}$,

$$p_j = T - (Tn - R)x_j. \quad (3)$$

In particular, when $T = R$ this gives $p_j = R \cdot (1 - (n - 1)x_j)$. Note that $\sum_{j \in \mathcal{M}} p_j = Tn - (Tn - R) = R$. If $x_{j'} = 0$ for some $j \in \mathcal{M}$, then $p_{j'} = T$. Moreover, when $x_{j'} = 0$, there is at least one other room j for which $x_j \geq 1/(n - 1)$, which implies $p_j \leq (R - T)/(n - 1) \leq 0$ (here the assumption $T \geq R$ is used).

The compensable tenants assumption then implies that for each agent $i \in \mathcal{N}$ there is a best room $j_i^* \in \mathcal{M}$ for which $p_{j_i^*} < T$, which implies $x_{j_i^*} > 0$. Label each vertex owned by agent i by j_i^* . These labelings satisfy Sperner's boundary condition, hence a fully-labeled simplex exists. Taking finer and finer triangulations leads to an infinite sequence of fully-labeled simplices. It has a convergent subsequence, and its limit point corresponds to an envy-free allocation. $\square$

Remark. In general, the prices in the envy-free allocation of Theorem 1 might be negative. As noted in Section 4, this may be inevitable since the theorem covers quasilinear tenants as a special case.

But if the tenants, in addition to being compensable, are also (weakly) miserly, and label their vertices accordingly, then the process described at the end of the proof of Theorem 1 converges to an envy-free allocation in which all prices are nonnegative. To see this, assume by contradiction that the limit point has a negative price, e.g. $p_j < 0$. This means that, in the convergent subsequence of fully-labeled simplices, there are infinitely many simplices in which $p_j < 0$ in all vertices. But in such simplices, weakly miserly tenants would not use the label of the most expensive room, so these simplices cannot be fully-labeled — a contradiction.

Thus, the proof combines advantages of previous proofs: it works both with and without the miserly tenants assumption, and with this assumption it guarantees an allocation with nonnegative prices.

The proof of rental harmony existence using Sperner's lemma has two advantages over previous proofs regarding quasilinear agents.

First, it is arguably simpler. Due to the discrete nature of Sperner's lemma, and thanks to the beautiful and simple proofs available for it [17], it is easily understood even by students who are not mathematicians.⁶

Second, the new proof enables us to extend the results on rental harmony existence in all the settings listed in Section 3 (roommates, secretive agents, extra agents, or multiple houses) from miserly tenants to compensable tenants. In particular, all these results hold for quasilinear tenants. The following section presents these results briefly.

⁶This observation is based on the author's experience teaching fair division to computer programmers.

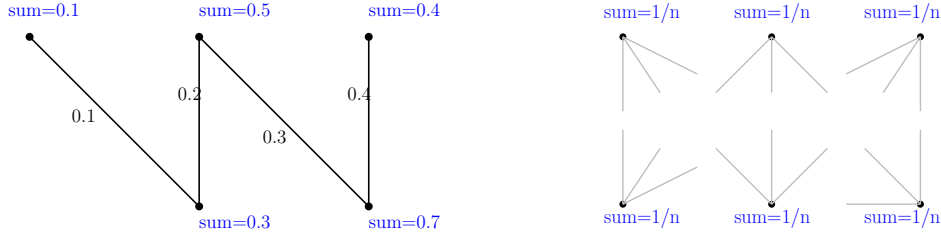


Figure 2: **Left:** A possible bipartite graph $G(\sigma)$ in a setting with $n = 3$ agents and $m = 2$ labels. This graph corresponds to a sub-simplex $\sigma \in \Delta^{2-1}$ in which agent 1 labels both vertices by 1, agent 2 labels one vertex by 1 and one vertex by 2, and agent 3 labels both vertices by 2. The weights on the edges show a possible weight function w . Here $\mathbf{b}(\sigma, w) = [0.1, 0.5, 0.4]$ and $\mathbf{a}(\sigma, w) = [0.3, 0.7]$. **Right:** A possible bipartite graph $G(\mathbf{p})$ in which $\mathbf{a}(w, \mathbf{p}) = \mathbf{b}(w, \mathbf{p}) = (1/n, \dots, 1/n)$.

6 Extensions

To prove the extensions, we will use a generalization of Sperner's lemma, which was proved recently by Meunier and Su [14]. It considers a triangulation of the standard simplex Δ^{m-1} , which is labeled with labels from the set $\{1, \dots, m\}$ by n different agents. For each sub-simplex σ of the triangulation, denote by $G(\sigma)$ the bipartite graph with the n agents on one side, the m labels on the other side, and with an edge between an agent and a label if and only if the agent uses that label anywhere on that simplex.

Let $G(\sigma)$ be the bipartite graph, and let w be a function that assigns a nonnegative weight to each edge of $G(\sigma)$, such that the sum of all weights is 1. For each agent $i \in \{1, \dots, n\}$, let $b(\sigma, w)_i$ be the sum of weights of the edges adjacent to i in $G(\sigma)$. Similarly, for each label $j \in \{1, \dots, m\}$, let $a(\sigma, w)_j$ be the sum of weights of the edges adjacent to j in $G(\sigma)$. Note that the vector $\mathbf{b}(\sigma, w)$ is a point in Δ^{n-1} and the vector $\mathbf{a}(\sigma, w)$ is a point in Δ^{m-1} . An illustration is shown on the left in Figure 2.

The following is proved by Meunier and Su [14] as a crucial step before the proof of their Theorem 2.3.2.

Lemma 2 ([14]). *Let $\mathbf{a}_0$ be an arbitrary vector in Δ^{m-1} and $\mathbf{b}_0$ an arbitrary vector in Δ^{n-1} . If n agents label the vertices of a triangulation of Δ^{m-1} by labels from $[m]$, and all labelings satisfy Sperner's boundary condition, then there exists a sub-simplex σ and a weight function w on $G(\sigma)$ such that*

$$\mathbf{b}(\sigma, w) = \mathbf{b}_0 \quad \mathbf{a}(\sigma, w) = \mathbf{a}_0.$$

We will apply this lemma to labelings generated as in Theorem 1, with n agents and m rooms. For each i , agent i labels each vertex $\mathbf{x}$ with the index of a best room of i given the price vector $\mathbf{p}$ calculated from $\mathbf{x}$ by equation (3). As explained in the proof of Theorem 1, all these labelings satisfy Sperner's boundary condition. For each price vector $\mathbf{p} \in \mathbb{R}^m$, let $G(\mathbf{p})$ be the bipartite graph defined like $G(\sigma)$ above, with the n agents on one side, the m rooms on the other side, and each agent is adjacent to all their best rooms given the price $\mathbf{p}$. By a standard continuity argument, we get the following corollary of Lemma 2.

Corollary 3. *Suppose there are m rooms and n compensable agents. Let $\mathbf{a}_0$ be an arbitrary vector in Δ^{m-1} and $\mathbf{b}_0$ be an arbitrary vector in Δ^{n-1} . There exists a price vector*

$\mathbf{p} \in \mathbb{R}^m$ and a weight function w on $G(\mathbf{p})$ such that

$$\mathbf{b}(\mathbf{p}, w) = \mathbf{b}_0 \qquad \mathbf{a}(\mathbf{p}, w) = \mathbf{a}_0.$$

To illustrate the usefulness of Corollary 3, let us use it to re-prove the existence of an envy-free room allocation in the standard setting in which $m = n$. Apply Corollary 3 with $\mathbf{a}_0 = \mathbf{b}_0 = (1/n, \dots, 1/n)$. This implies the existence of a price vector $\mathbf{p}$ and a weight function w on $G(\mathbf{p})$ as illustrated on the right in Figure 2: all weights are weakly positive, and the sum of weights near each agent and near each label is $1/n$. This implies that each agent is adjacent to at least one room; every two agents are adjacent together to at least two rooms; and so on. In general, the total weight near each subset of k agents is k/n , so they must be adjacent to at least k different rooms. This means that the graph $G(\mathbf{p})$ satisfies the conditions of Hall’s marriage theorem [13]. Hence, there exists a perfect matching of agents to rooms, where each agent is matched to one of their best rooms given $\mathbf{p}$.

Next, consider the problem of *rental harmony with roommates*. There are $|\mathcal{M}| = m$ rooms with $m \leq n$, and each room $j \in \mathcal{M}$ has a fixed capacity c_j , where $\sum_{j \in \mathcal{M}} c_j = n$. The goal is to assign to each room j a subset c_j of tenants and a price p_j (which is shared equally among the c_j tenants) such that no tenant is envious. Azrieli and Shmaya [5] proved the existence of an envy-free allocation assuming all tenants are miserly. Ghodsi, Latifian, Mohammadi, Moradian, and Seddighin [11] proved the same assuming all tenants are quasilinear.⁷ The following theorem generalizes both results.

Theorem 4. *When all agents are compensable, an envy-free allocation with roommates exists.*

Proof. Apply Corollary 3 with $\mathbf{b}_0 = (1/n, \dots, 1/n)$ and $\mathbf{a}_0 = (c_1/n, \dots, c_m/n)$. This implies the existence of a price vector $\mathbf{p}$ such that, in the bipartite graph $G(\mathbf{p})$, each room j is adjacent to at least c_j agents. Moreover, every two rooms j_1, j_2 are adjacent to at least $c_{j_1} + c_{j_2}$ agents, and so on. By a straightforward generalization of Hall’s marriage theorem (see Theorem 3 in Appendix B of [5]), the graph $G(\mathbf{p})$ admits a one-to-many matching in which each room j is matched to exactly c_j agents. This corresponds to an envy-free room allocation with roommates. $\square$

Next, consider the problem of *rental harmony with a secretive agent*. There are $n \geq 2$ rooms and n agents, but only $n - 1$ agents are present. They need to decide on a price vector $\mathbf{p} \in \mathbb{R}^n$ such that, when the n th agent comes and picks a room, the remaining agents can allocate the remaining rooms among them (without changing the prices) such that there is no envy. Asada et al. [4] and Frick et al. [8] proved that it is possible with miserly tenants. Arunachaleswaran, Barman, and Rathi [3] proved that it is possible with Archimedean tenants.⁸ The paper of Velez [20] implies a slightly weaker result for

⁷They used a reduction to the standard setting: For each room j , construct c_j sub-rooms with capacity 1, and let each agent value all sub-rooms of room j by v_j . Note that this reduction cannot be used for miserly tenants, since it does not preserve the miserly tenants assumption. For example, suppose the living room has capacity 2 and the basement has capacity 1. Suppose the two sub-rooms of the living room are priced at 0 and 200 and the basement is priced at 200. Then the living room and the basement have the same positive price, so a tenant who prefers the basement satisfies the miserly tenants assumption in the original problem, but not in the modified problem.

⁸They assume that each agent has a utility function that assigns a utility to each combination of room and price. They assume that all utility functions are continuous, monotonically decreasing in the price, and bounded. The boundedness assumption is equivalent to the Archimedean tenants assumption.

compensable tenants: an envy-free allocation exists even when only $n - 1$ agents are compensable (but the preferences of the n th agent must still be known). The following theorem generalizes these results.

Theorem 5. *When at least $n - 1$ agents are compensable, an envy-free allocation can be found even before the preferences of the n th agent are known.*

Proof. Apply Corollary 3 with the $n - 1$ present agents and the n rooms, with $\mathbf{b}_0 = (\frac{1}{n-1}, \dots, \frac{1}{n-1})$ and $\mathbf{a}_0 = (\frac{1}{n}, \dots, \frac{1}{n})$. This implies the existence of a price vector $\mathbf{p}$ and a weight function on the bipartite graph $G(\mathbf{p})$, such that the sum of weights near each present agent is $\frac{1}{n-1}$, and the sum of weights near each room is $\frac{1}{n}$. Suppose that the n th agent now comes and picks a room. There are $n - 1$ remaining rooms. The weight near each subset of $k \leq n - 1$ rooms is at least $\frac{k}{n}$. This fraction is larger than $\frac{k-1}{n-1}$, so every k rooms are adjacent together to more than $k - 1$ agents, which means at least k agents. Hence, the remaining graph satisfies Hall's marriage condition, and there is a perfect matching between the $n - 1$ agents and the $n - 1$ remaining rooms. $\square$

Remark. Frick et al. [8] asked whether there always exists an envy-free allocation with both roommates and a secretive agent. The answer is yes, and it can be proved by combining the proofs of Theorems 4 and 5.

Next, consider the problem of *rental harmony with an extra agent*. There are n rooms and $n + 1$ agents. They need to decide on a price vector $\mathbf{p} \in \mathbb{R}^n$ such that, when any agent leaves, the remaining agents can allocate the rooms among them (without changing the prices) such that there is no envy.

Theorem 6. *When all agents are compensable, an envy-free allocation with an extra agent exists. Particularly, such an allocation exists when the agents are quasilinear or Archimedean.*

Proof. Apply Corollary 3 with the $n + 1$ agents and the n rooms, with $\mathbf{b}_0 = (1/(n + 1), \dots, 1/(n + 1))$ and $\mathbf{a}_0 = (1/n, \dots, 1/n)$. This implies the existence of a price vector $\mathbf{p}$ and a weight function on the bipartite graph $G(\mathbf{p})$, such that the sum of weights near each agent is $1/(n + 1)$, and the sum of weights near each room is $1/n$. Suppose that agent i leaves. Remove from $G(\mathbf{p})$ the vertex representing i and all its adjacent edges. In the remaining graph, the weight near each subset of k rooms is at least $k/n - 1/(n + 1) > (k - 1)/(n + 1)$, so every k rooms are adjacent together to at least k remaining agents. Hence, the remaining graph satisfies Hall's marriage condition, and there is a perfect matching between the n remaining agents and the n rooms. $\square$

7 Multiple houses

In this section, we consider the case where there are two houses with n rooms in each, e.g., a bedroom house and an office house. The total rent is R for each house.⁹ The prices of bedrooms are given by $\mathbf{p} \in \mathbb{R}^n$ and the prices of offices are given by $\mathbf{q} \in \mathbb{R}^n$; the sum of each vector is R . The rooms are rented in pairs — a “rent package” consists of a bedroom, an office, and a price for each room. There are $m \geq n$ potential tenants, each

⁹For the sake of readability, we focus on a setting of two houses with equal rents, but the results can be easily extended to more than two houses, and to houses with different total rents.

of whom has a demand function on room pairs: for each two price vectors $\mathbf{p}, \mathbf{q}$, the agent has a set of *best room-pairs*. An allocation in which each agent receives a best room-pair is called *envy-free*.

We first have to extend the assumptions on the tenants to this setting. We denote the sets of rooms in the two houses by $\mathcal{M}_1$ and $\mathcal{M}_2$.

- **Quasilinear tenants:** For all $i \in \mathcal{N}$ and $j_1 \in \mathcal{M}_1$ and $j_2 \in \mathcal{M}_2$, there is a value v_{i,j_1,j_2} representing the value for agent i of the room-pair (j_1, j_2) . The best pairs of agent i in prices $\mathbf{p}, \mathbf{q}$ are the ones maximizing the difference $v_{i,j_1,j_2} - p_{j_1} - q_{j_2}$.
- **Miserly tenants:** In each house, if there is a free room, then all agents prefer a pair that contains a free room in that house. Formally, given price vectors $\mathbf{p}, \mathbf{q}$, if $p_{j_1} \leq 0$ for some $j_1 \in \mathcal{M}_1$, then every agent $i \in \mathcal{N}$ has a best pair (j_1^i, j_2^i) with $p_{j_1^i} \leq 0$, and if $q_{j_2} \leq 0$ for some $j_2 \in \mathcal{M}_2$, then every agent $i \in \mathcal{N}$ has a best pair (j_1^i, j_2^i) with $q_{j_2^i} \leq 0$.¹⁰
- **Compensable tenants:** There exists a number $T \geq R$ such that the following holds. In each house, if there is a free room and the most expensive room costs T , then all agents prefer a pair that contains a room in that house whose price is less than T . Formally, given price vectors $\mathbf{p}, \mathbf{q}$, if $\min_j p_j \leq 0$ and $\max_j p_j = T$, then every agent $i \in \mathcal{N}$ has a best pair (j_1^i, j_2^i) with $p_{j_1^i} < T$, and if $\min_j q_j \leq 0$ and $\max_j q_j = T$, then every agent $i \in \mathcal{N}$ has a best pair (j_1^i, j_2^i) with $q_{j_2^i} < T$.

As before, both the quasilinear tenants assumption and the miserly tenants assumption imply the compensable tenants assumption.

Previous works in this setting considered only miserly tenants. Cloutier et al. [7] showed that when $m = n$ an envy-free allocation may not exist.¹¹ Nyman et al. [15] showed that, if there are $m \geq 2n - 1$ potential tenants, all of whom are miserly, then there exist two price vectors and a subset of n tenants such that each tenant in this subset can get a best room-pair (so that the n selected tenants do not envy each other).

Their proof uses a variant of Sperner's lemma for polytopes.¹² Let $P(n, 2) := (\Delta^{n-1})^2 =$ the Cartesian product of two standard $(n - 1)$ -simplices. We consider labelings that assign to each triangulation vertex $\mathbf{x} := (\mathbf{x}^1, \mathbf{x}^2)$ a pair $\mathbf{L}(\mathbf{x}) := (L^1(\mathbf{x}), L^2(\mathbf{x}))$, where $L^i(\mathbf{x}) \in [n]$. We say that such a labeling satisfies Sperner's boundary condition if, for all $i \in [2]$, the label $L^i(\mathbf{x})$ is some $j \in [n]$ for which $x_j^i > 0$.

Lemma 7. *[[15], Proposition 5.3] If a triangulation of $P(n, 2)$ is labeled by labels from $[n]^2$, and the labeling satisfies Sperner's boundary condition, then there exists a sub-simplex σ and a subset M of its vertices such that $|M| = n$ and the pairs $\mathbf{L}(\mathbf{x})$ for $\mathbf{x} \in M$ are disjoint.*

¹⁰If both houses have free rooms, then every agent has a best pair in which both rooms are free.

¹¹As an example, suppose $m = n = 2$ and consider two agents, Alice and George, with the following demand functions. The best pairs of Alice are the pairs in which both rooms cost at most $R/4$ (if any), and the cheaper among the two pairs $(1, 1)$ and $(2, 2)$. The best pairs of George are the pairs in which both rooms cost at most $R/4$ (if any), and the cheaper among the two pairs $(1, 2)$ and $(2, 1)$. It can be verified that these demand functions are continuous and satisfy the miserly tenants assumption, and that an envy-free allocation does not exist for any pair of price vectors. See Theorem 3 and Figure 4 of [7].

¹²For simplicity we state their argument only for the special case $k = 2$, which is sufficient for our purpose.

Using this lemma, the result of [15] can be generalized to compensable tenants (in particular, it holds when all tenants are quasilinear or Archimedean).

Theorem 8. *When there are two n -room houses and $m \geq 2n - 1$ compensable agents, there exist two price vectors and a subset of n agents such that each agent in the subset can be allocated a best room-pair.*

Proof. Triangulate the polytope $P(n, 2)$. This polytope is $2(n - 1)$ -dimensional; hence, each sub-simplex of the triangulation has $2n - 1$ vertices. As usual, assign each vertex of the triangulation to an agent such that the owners are unique in each sub-simplex.¹³ Associate each point $(\mathbf{x}^1, \mathbf{x}^2)$ in Δ^{n-1} with price vectors $\mathbf{p}, \mathbf{q}$ such that, for all $j_1 \in \mathcal{M}_1$ and $j_2 \in \mathcal{M}_2$,

$$p_{j_1} = T - (Tn - R)x_{j_1}^1 \qquad q_{j_2} = T - (Tn - R)x_{j_2}^2. \quad (4)$$

Label each triangulation vertex by the best room-pair of the vertex owner. The compensable tenants assumption guarantees that the labelings satisfy Sperner's boundary condition. Hence, by Lemma 7, there exists a sub-simplex and a subset of n vertices, such that the label-pairs on these vertices are disjoint. Assign each room-pair to the vertex owner to get the desired envy-free allocation. $\square$

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¹³This is why we need $2n - 1$ agents to start with.

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