

Multidimensional Stable Roommates with Master List^{*}

Robert Bredereck¹[0000–0002–6303–6276], Klaus Heeger¹[0000–0001–8779–0890] (✉),
Dušan Knop²[0000–0003–2588–5709], and Rolf Niedermeier¹[0000–0003–1703–1236]

¹ TU Berlin, Algorithmics and Computational Complexity, Germany
{`robert.bredereck, heeger, rolf.niedermeier`}@tu-berlin.de

² Czech Technical University in Prague, Prague, Czech Republic
`dusan.knop@fit.cvut.cz`

Abstract. Since the early days of research in algorithms and complexity, the computation of stable matchings is a core topic. While in the classic setting the goal is to match up two agents (either from different “gender” (this is STABLE MARRIAGE) or “unrestricted” (this is STABLE ROOMMATES)), Knuth [1976] triggered the study of three- or multidimensional cases. Here, we focus on the study of MULTIDIMENSIONAL STABLE ROOMMATES, known to be NP-hard since the early 1990’s. Many NP-hardness results, however, rely on very general input instances that do not occur in at least some of the specific application scenarios. With the quest for identifying islands of tractability, we look at the case of master lists. Here, as natural in applications where agents express their preferences based on “objective” scores, one roughly speaking assumes that all agent preferences are “derived from” a central master list, implying that the individual agent preferences shall be similar. Master lists have been frequently studied in the two-dimensional (classic) stable matching case, but seemingly almost never for the multidimensional case. This work, also relying on methods from parameterized algorithm design and complexity analysis, performs a first systematic study of MULTIDIMENSIONAL STABLE ROOMMATES under the assumption of master lists.

1 Introduction

Computing stable matchings is a core topic in the intersection of algorithm design, algorithmic game theory, and computational social choice. It has numerous applications such as higher education admission in several countries [2,5], kidney exchange [33], assignment of dormitories [30], P2P-networks [15], and wireless three-sided networks [9]. The research started in the 1960’s with the seminal work of Gale and Shapley [16], introducing the STABLE MARRIAGE problem: given two different types of agents, called “men” and “women”, each agent of one gender has preferences (i.e., linear orders aka rankings) over the agents of the opposite gender. Then, the task is to find a matching which is stable. Informally, a matching is *stable* if no pair of agents can improve by breaking up with their currently assigned partners and instead matching to each other.

^{*} Main work done while all authors were with TU Berlin.

Many variations of this problem have been studied; STABLE ROOMMATES, with only one type of agents, is among the most prominent ones. Knuth [24] asked for generalizing STABLE MARRIAGE to dimension three, i.e., having three types of agents and having to match the agents to groups of size three, where any such group contains exactly one agent of each type. Here, a matching is called *stable* if there is no group of three agents which would improve by being matched together. We focus on the MULTIDIMENSIONAL STABLE ROOMMATES PROBLEM. Here again, there is only one type of agents, now having preferences over $(d-1)$ -sets (that is, sets of size $d-1$) of (the other) agents. As this problem is NP-hard in general [27], we focus on the case where the preferences of all agents are derived from a master list. For instance, master lists naturally arise when the agent preferences are based on scores, e.g., when assigning junior doctors to medical posts in the UK [20] or when allocating students to dormitories [30]. Master lists have been frequently used in the context of (two-dimensional) stable matchings [3,20,28,30] or the related POPULAR MATCHING problem [23]. We generalize master lists to the multidimensional setting in two natural ways. First, following the above spirit of preference orders, we assume that the master list consists of sets of size $d-1$. Each agent then derives its preferences from the master list by just deleting all $(d-1)$ -sets containing the agent itself. Second, the master list orders all agents. In this case, any agent a shall prefer a $(d-1)$ -set t over a $(d-1)$ -set t' if t is “better” than t' according to the master list, where “better” means that a does not prefer the k th best agent of t' over the k th best agent from t (according to the master list). For any tuples t, t' for which neither t is “better” than t' nor t' is “better” than t , an agent may prefer t over t' or t' over t independently of the other agents. More formally, we require that any agent prefers a set of $d-1$ agents t over any set of $d-1$ agents t' dominated by $\{a_1, \dots, a_{d-1}\}$, where we say that $t = \{a_1, \dots, a_{d-1}\}$ dominates $t' = \{b_1, \dots, b_{d-1}\}$ if the master list does not prefer b_i over a_i for all $i \in [d-1]$. The agent preferences of any agent must then fulfill for any two sets $\{a_1, \dots, a_{d-1}\}$ and $\{b_1, \dots, b_{d-1}\}$ of $d-1$ agents with b_i not being before a_i that in the master list the set $\{b_1, \dots, b_{d-1}\}$ is not before $\{a_1, \dots, a_{d-1}\}$. In this case, we also relax the condition that the master list is a strict order by the condition that the master list is a partially ordered set (poset), and consider the parameterized complexity with respect to parameters measuring the similarity to a strict order. Preferences where such a parameter is small might arise if there are few similar rankings, and each agent derives its ranking from these orders, or if the objective score consists of several attributes and each agent weights these attributes slightly differently. Two agents are then incomparable in the master poset if they are ranked in different order by some agents.

Related work. STABLE ROOMMATES can be solved in linear time [19]. If the preferences are incomplete and derived from a master list, then both STABLE MARRIAGE and STABLE ROOMMATES admit a unique stable matching [20]. If the preferences are complete but contain ties, then finding a weakly stable matching in a STABLE ROOMMATES instance becomes NP-hard [32]. However, if the preferences are complete and derived from a master list, then one can decide

whether an edge of a STABLE MARRIAGE instance is contained in a stable matching in linear time [20], and a stable matching in a STABLE ROOMMATES instance always exists and can be found in linear time. For incomplete preferences with ties derived from a master list, an $O(\sqrt{nm})$ -time algorithm for finding a strongly stable matching is known [28] (where n is the number of agents and m is the number of acceptable pairs), while for general preferences, only an $O(nm)$ -time algorithm is known [25].

Several STABLE MARRIAGE problems become easier for complete preferences derived from a master list [35, Chapter 8]. STABLE ROOMMATES, however, is NP-hard if the preferences contain ties, are incomplete, and are derived from a master list [20]. There is quite some work for 3-DIMENSIONAL STABLE MARRIAGE [10,29,37,38], but less so for 3-DIMENSIONAL STABLE ROOMMATES.

While master lists are a standard setting for finding 2-dimensional stable matchings [3,20,22,28,30], we are only aware of few works combining multidimensional stable matchings with master lists. Escamocher and O’Sullivan [14] gave a recursive formula for the number of 3-dimensional stable matchings for cyclic preferences (i.e., the agents are partitioned into three sets V_1 , V_2 , and V_3 , and each agent from V_i only cares about the agent from V_{i+1} it is matched to) derived from master lists. Cui and Jia [9] showed that if the preferences are cyclic and the preferences of the agents from V_1 are derived from a master list, while each agent from V_3 is indifferent between all agents from V_1 , then a stable matching always exists and can be found in polynomial time, but it is NP-complete to find a maximum-cardinality stable matching. There is some work on d -dimensional stable matchings and cyclic preferences (without master lists) [18,26].

Deineko and Woeginger [11] showed that 3-DIMENSIONAL STABLE ROOMMATES is NP-complete for preferences derived from a metric space. For the special case of the Euclidean plane, Arkin et al. [1] showed that a stable matching does not always exist, but left the complexity of deciding existence open.

Iwama et al. [21] introduced the NP-hard STABLE ROOMMATES WITH TRIPLE ROOMS, where each agent has preferences over all other agents, and prefers a 2-set p of agents over a 2-set p' if it prefers the best-ranked agent of p over the best-ranked agent of p' , and the second-best agent of p over the second-best agent of p' .

Our scenario of MULTIDIMENSIONAL STABLE ROOMMATES can be seen as a special case of finding core-stable outcomes for hedonic games where each agent prefers size- d coalitions over singleton-coalitions which are then preferred over all other coalitions [34,36]. Notably, there are fixed-parameter tractability results for hedonic games (without fixed “coalition” size as we request) with respect to treewidth (MSO-based) [17,31]. Other research considers hedonic games with fixed coalition size [8], but aims for Pareto optimal outcomes instead of core stability which we consider.

To the best of our knowledge, the parameterized complexity of multidimensional stable matching problems has not yet been investigated.

Table 1. Results overview: six variations of MULTIDIMENSIONAL STABLE ROOMMATES.

Setting/Parameter	Complexity
Master list of 2-sets	NP-complete for $d = 3$ (Theorem 2)
Strict master list of agents	linear time (Proposition 1)
κ (max. # of incomparable agents)	$O(n^2) + (\kappa^2 2^{12\kappa})^{O(\kappa^2 2^{12\kappa})} n$ (Theorem 3)
Width of poset	W[1]-hard for $d = 3$ (Theorem 4)
Incomplete preferences, strict master list	NP-complete for $d \geq 3$ (Theorem 5)
Deletion distance to strict master list	W[1]-hard for $d = 3$ (Theorem 6)

Our contributions. For an overview of our results, we refer to Table 1. To our surprise, even if the preferences are derived from a master list of 2-sets of agents (in this case, dimension $d = 3$), a stable matching is not guaranteed to exist (Section 3). We use such an instance not admitting a stable matching to show that THREE-DIMENSIONAL STABLE ROOMMATES is NP-complete also when restricted to preferences derived from a master list of 2-sets (Theorem 2).

If the preferences are derived from a strict master list of agents, then a unique stable matching always exists and can be found by a straightforward algorithm (Proposition 1). When relaxing the condition that the master list is strict to being a poset, then the problem clearly is NP-complete, as a master list which ties all agents does not impose any condition on the preferences of the agents, and THREE-DIMENSIONAL STABLE ROOMMATES is NP-complete. Consequently, in the spirit of “distance from tractability”-parameterization, we investigate the parameterized complexity with respect to several parameters measuring the distance of the poset to a strict order. For the parameter maximum number of agents incomparable to a single agent, we show that MULTIDIMENSIONAL STABLE ROOMMATES is fixed-parameter tractable (FPT)³ (even when d is part of the input) (Theorem 3). If this parameter is bounded, then this results in one of the rare special cases of 3-dimensional stable matching problems which can be solved by an “efficient” nontrivial algorithm. Considering the stronger parameter width of the master poset, we show THREE-DIMENSIONAL STABLE ROOMMATES to be W[1]-hard⁴, and this is true also for the orthogonal parameter deletion (of agents) distance to a linear master list (Theorem 6). We also show that MULTIDIMENSIONAL STABLE ROOMMATES is NP-complete even with a linear order of the agents as a master list if each agent is allowed to declare an arbitrary set of 2-sets unacceptable (Theorem 5).

Proofs omitted due to space restrictions are marked by a star ($\star$) and can be found in the full version [6].

³ FPT with respect to a parameter k means that the problem can be solved in $f(k)\text{poly}(|\mathcal{I}|)$ time, where f is an arbitrary computable function and $|\mathcal{I}|$ denotes the size of the input instance.

⁴ Informally, W[1]-hardness with respect to a specific parameter indicates that it is very unlikely to show fixed-parameter tractability.

2 Preliminaries

Let $[n] := \{1, 2, 3, \dots, n\}$ and $[n, m] := \{n, n+1, \dots, m\}$. For a set X and an integer d , we denote by $\binom{X}{d}$ the set of size- d subsets of X . A *preference list* $\succ$ over a set X is a strict order of X . We call a set of pairwise disjoint d -subsets of V a *d -dimensional matching*. If it is clear from the context that it is a d -dimensional matching, then we may only write matching. We say that an agent v *prefers* a $(d-1)$ -set A over a $(d-1)$ -set B if $A \succ_v B$ where $\succ_v$ is the preference list of v . Any agent prefers any $(d-1)$ -set not containing itself over being unmatched. A *blocking d -set* for a d -dimensional matching M is a set of d agents $\{v_1, v_2, \dots, v_d\}$ such that for all $i \in [d]$, either v_i is unmatched in M or $\{v_1, v_2, \dots, v_d\} \setminus \{v_i\} \succ_{v_i} \{w_1^i, w_2^i, \dots, w_{d-1}^i\}$, where $\{w_j^i : j \in [d-1]\} \cup \{v_i\} \in M$. A matching is called *stable* if it does not admit a blocking d -set.

MULTIDIMENSIONAL STABLE ROOMMATES (MDSR)

Input: An integer d , a set V of agents together with a preference list $\succ_v$ over $\binom{V \setminus \{v\}}{d-1}$ for each agent $v \in V$.
Task: Decide whether a stable matching exists.

Note that we require each agent to list each size- $(d-1)$ set of other agents. We denote by ℓ -DSR the restriction of MDSR to instances with $d = \ell$. We set $n := |V|$. A 3-dimensional stable matching does not always exist, and 3-DSR is NP-complete [27].

A *master list* ML is a preference list over $\binom{V}{d-1}$. A preference list $\succ_v$ for an agent v is *derived from a master list* ML by deleting all $(d-1)$ -sets containing v .

Example 1. Let $V = \{v_1, v_2, v_3, v_4\}$ be a set of agents, $d = 3$, and let $\{v_1, v_2\} \succ \{v_2, v_4\} \succ \{v_1, v_3\} \succ \{v_3, v_4\} \succ \{v_2, v_3\} \succ \{v_1, v_4\}$ be the master list.

Then the preferences of v_1 are $\{v_2, v_4\} \succ_{v_1} \{v_3, v_4\} \succ_{v_1} \{v_2, v_3\}$, the preferences of v_2 are $\{v_1, v_3\} \succ_{v_2} \{v_3, v_4\} \succ_{v_2} \{v_1, v_4\}$, the preferences of v_3 are $\{v_1, v_2\} \succ_{v_3} \{v_2, v_4\} \succ_{v_3} \{v_1, v_4\}$, and the preferences of v_4 are $\{v_1, v_2\} \succ_{v_4} \{v_1, v_3\} \succ_{v_4} \{v_2, v_3\}$.

We now define the MULTIDIMENSIONAL STABLE ROOMMATES WITH MASTER LIST OF $(d-1)$ -SETS problem (MDSR-ML-SETS).

MDSR-ML-SETS

Input: An integer d , a set V of agents, and a master list $\succ_{ML}$ over $\binom{V}{d-1}$, from which the preference list of each agent is derived.
Task: Decide whether a stable matching exists.

Again, we denote by ℓ -DSR-ML-SETS the problem MDSR-ML-SETS restricted to instances with $d = \ell$.

We now turn to the case that the master list orders single agents instead of $(d-1)$ -sets of agents. We first need the definition of a partially ordered set.

A *partially ordered set (poset)* is a pair $(V, \succeq)$, where $\succeq$ is a binary relation over the set V such that (i) $v \succeq v$ for all $v \in V$, (ii) $v \succeq w$ and $w \succeq v$ if and only if $v = w$, and (iii) if $u \succeq v$ and $v \succeq w$, then $u \succeq w$.

If $v \succeq w$ and $v \neq w$, then we write $v \succ w$. If neither $v \succeq w$ nor $w \succeq v$, then we say that v and w are *incomparable*, and write $v \sim w$. Instead of $v \succeq w$ or $v \succ w$, we may also write $w \preceq v$ or $w \prec v$.

A *chain* is a subset $X = \{x_1, x_2, \dots, x_k\} \subseteq V$ such that $x_i \succ x_{i+1}$ for all $i \in [k-1]$. An *antichain* is a subset $X \subseteq V$ such that for all $v, w \in X$ with $v \neq w$, we have $v \sim w$. The *width* of a poset is the size of a maximum antichain.

For a poset $\succ$ over a set V , we define $\kappa_\succ(v) := |\{w \in V : v \sim w\}|$ to be the number of elements incomparable with v . We define $\kappa(\succ) := \max_{v \in V} \kappa_\succ(v)$.

Note that if $\bar{G}_\succ$ is the incomparability graph of the poset $(V, \succ)$ (i.e., the graph whose vertex set is the set V , and there is an edge between $v, w \in V$ if and only if $v \sim w$), then $\Delta(\bar{G}_\succ) = \kappa(\succ)$, where $\Delta(\bar{G}_\succ)$ is the maximum degree of a vertex in $\bar{G}_\succ$. If $\succ$ is a weak order (i.e., a linear order with ties), the parameter $\kappa(\succ)$ is equal to the maximum size of a tie.

Dilworth's Theorem [12] states that the width of a poset is the minimum number of chains such that each element of the poset is contained in one of these chains.

Having defined posets, we now show the connection to MULTIDIMENSIONAL STABLE ROOMMATES by defining preferences derived from a poset of agents.

Definition 1. Given a set of agents V , a poset $(V, \succ_{\text{ML}})$ (which we call the master poset), and an integer d , we say that a preference list $\succ_v$ on $\binom{V \setminus \{v\}}{d-1}$ is derived from $\succ_{\text{ML}}$ if whenever $a_1, \dots, a_{d-1}$ and $b_1, \dots, b_{d-1}$ with $a_i \succeq_{\text{ML}} b_i$ for all $i \in [d-1]$, then we have $\{a_1, \dots, a_{d-1}\} \succeq_v \{b_1, \dots, b_{d-1}\}$.

Example 2. Let $v_1 \succ v_2 \succ v_3 \succ v_4 \succ v_5$ be a master poset. Then v_1 has one of the two preferences: $\{v_2, v_3\} \succ_{v_1} \{v_2, v_4\} \succ_{v_1} \{v_2, v_5\} \succ_{v_1} \{v_3, v_4\} \succ_{v_1} \{v_3, v_5\} \succ_{v_1} \{v_4, v_5\}$ or $\{v_2, v_3\} \succ_{v_1} \{v_2, v_4\} \succ_{v_1} \{v_3, v_4\} \succ_{v_1} \{v_2, v_5\} \succ_{v_1} \{v_3, v_5\} \succ_{v_1} \{v_4, v_5\}$.

For the master poset $v_2 \succ v_3 \sim v_4 \succ v_1$, agent v_1 has one of the following preferences: $\{v_2, v_3\} \succ_{v_1} \{v_2, v_4\} \succ_{v_1} \{v_3, v_4\}$ or $\{v_2, v_4\} \succ_{v_1} \{v_2, v_3\} \succ_{v_1} \{v_3, v_4\}$.

We are now ready to formally define MDSR-POSET.

MDSR-POSET

Input: An MDSR instance $\mathcal{I} = (V, (\succ_v)_{v \in V}, d)$ and a master poset $\succeq_{\text{ML}}$ such that the preferences $\succ_v$ of each agent v are derived from $\succeq_{\text{ML}}$.

Task: Decide whether there exists a stable matching in $\mathcal{I}$.

3 Three-Dimensional Stable Roommates with Master List of 2-sets

In this section, we consider the case that the preferences are complete and derived from a master list of $(d-1)$ -sets. First, we give a small instance with six agents

Table 2. A blocking 3-set for each matching in instance $\mathcal{I}_{\text{instable}}$ from Observation 1.

Matching	Blocking 3-set	Matching	Blocking 3-set
$\{a, b, c\}, \{d, e, f\}$	$\{a, d, e\}$	$\{a, c, e\}, \{b, d, f\}$	$\{a, b, e\}$
$\{a, b, d\}, \{c, e, f\}$	$\{a, c, e\}$	$\{a, c, f\}, \{b, d, e\}$	$\{a, b, e\}$
$\{a, b, e\}, \{c, d, f\}$	$\{b, c, d\}$	$\{a, d, e\}, \{b, c, f\}$	$\{a, b, e\}$
$\{a, b, f\}, \{c, d, e\}$	$\{a, c, d\}$	$\{a, d, f\}, \{b, c, e\}$	$\{a, b, e\}$
$\{a, c, d\}, \{b, e, f\}$	$\{a, b, e\}$	$\{a, e, f\}, \{b, c, d\}$	$\{a, c, d\}$

not admitting a stable matching, and use this to show that already for $d = 3$ and preferences derived from a master list of 2-sets, deciding whether an instance admits a stable matching is NP-complete.

We first present a 3DSR-ML-SETS instance $\mathcal{I}_{\text{instable}}$ with six agents not admitting a stable matching, showing that stable matchings do not have to exist even in the presence of master lists.

The instance $\mathcal{I}_{\text{instable}}$ has six agents a, b, c, d, e , and f . The master list is: $\{a, b\} \succ \{a, c\} \succ \{a, d\} \succ \{a, f\} \succ \{b, e\} \succ \{c, d\} \succ \{a, e\} \succ \{b, f\} \succ \{c, e\} \succ \{b, d\} \succ \{d, e\} \succ \{b, c\} \succ \{c, f\} \succ \{d, f\} \succ \{e, f\}$.

Observation 1 *The instance $\mathcal{I}_{\text{instable}}$ does not admit a stable matching.*

Proof. Table 2 presents for each of the $\frac{\binom{6}{3}}{2} = 10$ matchings a blocking 3-set. $\square$

Using the instance $\mathcal{I}_{\text{instable}}$, we show NP-completeness of 3DSR-ML-SETS, reducing from 1-IN-3 POSITIVE 3-OCCURRENCE-SAT.

Note that MDSR is in NP as the size of the input is $\Omega(\binom{n}{d-1})$, where n is the number of agents, as the master preference list contains $\binom{n}{d-1}$ sets of size $d - 1$, and, thus, stability can be checked in polynomial time in the input by just checking for each d -set whether it is blocking. We arrive at the following theorem.

Theorem 2 ($\star$). *3-DSR-ML is NP-complete.*

4 Master Lists of Agents

We now consider the case when there does not exist a master list of $(d - 1)$ -sets of agents, but a master list $\succ_{\text{ML}}$ of single agents. Each agent can derive its preferences from this master list, meaning that if for two $(d - 1)$ -sets t and t' , one can find a bijection σ from the elements of t to the elements of t' such that $v \succeq_{\text{ML}} \sigma(v)$ for all $v \in t$, then any agent (not occurring in t or t') shall prefer t over t' . If the master list is an arbitrary poset, then MDSR-POSET clearly is NP-complete, as the preferences of any instance of 3-DSR are derived from the poset in which no two different agents are comparable, and 3-DIMENSIONAL STABLE MATCHING is NP-complete. We show that this problem is polynomial-time solvable if the master list is a strict order. Afterwards, we generalize this

result by showing fixed-parameter tractability for the parameter κ , the “maximum number of agents incomparable to a single agent”. On the contrary, for the stronger parameter width of the poset, we show W[1]-hardness, leaving open whether it can be solved in polynomial time for constant width (in parameterized complexity known as the question for containment in XP).

4.1 Strict Orders

We first consider the case that the master list is a strict order. In this case, an easy algorithm solves the problem: Just match the first d agents from the master list together, delete them, and recurse. Note that the preferences of any agent cannot be directly derived from the master list, as e.g. an agent may prefer either $\{v_1, v_4\}$ over $\{v_2, v_3\}$ or $\{v_2, v_3\}$ over $\{v_1, v_4\}$. Thus, the input contains the complete preferences of all agents, and the input size is $\Theta(d \binom{n}{d-1})$. Hence, the running time subsequent algorithm is sublinear.

Proposition 1 (★). *If $\succeq_{\text{ML}}$ is a strict order, then any MDSR-POSET instance admits a unique stable matching that can be found in $O(n)$ time.*

4.2 Posets

In two-dimensional stable (or popular) matching problems with master lists, the master list usually contains ties [3,20,23,28,30]. We allow the master list not only to contain ties, but to be an arbitrary poset. In this case, the problem clearly is NP-complete, as the poset where each agent is incomparable to each other agent does not pose any restrictions on the preferences of the agents. Therefore, we consider several parameters measuring the similarity of the poset to a strict order. For the parameter “maximum number of agents incomparable to a single agent”, we show fixed-parameter tractability, and for the stronger parameter width of the poset, we show W[1]-hardness.

Maximum Number of Agents Incomparable to a Single Agent In this section, we show that MDSR-POSET is fixed-parameter tractable when parameterized by $\kappa(\succ_{\text{ML}})$. As a first step of the algorithm, we show how to derive a strict order from the given poset, which guarantees that for any two elements v and w with v being “much earlier” in the strict order than w , we have that $v \succ_{\text{ML}} w$.

Lemma 1 (★). *For any poset $(V, \succeq)$, there is an order $v_1, v_2, \dots, v_n$ of V such that (i) for all $i < j$, we have that $v_i \succ v_j$ or $v_i \sim v_j$, and (ii) for all $j > i + 2\kappa(\succeq)$, we have $v_i \succ v_j$. Moreover, such an order can be found in $O(|V|^2)$ time.*

For the rest of Section 4.2, we fix an instance $\mathcal{I} = (V, (\succeq_v)_{v \in V}, \succeq_{\text{ML}})$ of MDSR-POSET, and an order $V = \{v_1, \dots, v_n\}$ of V fulfilling the conditions of Lemma 1 for the poset $(V, \succeq_{\text{ML}})$. We set $\kappa := \kappa(\succeq_{\text{ML}})$. Furthermore, we denote by $V^{\leq i} = \{v_1, \dots, v_i\}$, by $V^{[i,j]} = \{v_i, v_{i+1}, \dots, v_j\}$, and by $V^{\geq i} = \{v_i, v_{i+1}, \dots, v_n\}$.

We now show that the agents contained in a d -set of a stable matching are close to each other in the strict order derived from the master poset by Lemma 1.

Lemma 2. Let $\mathcal{I} = (V, (\succeq_v)_{v \in V}, \succeq_{ML})$ be an MDSR-POSET-instance and let $V = \{v_1, v_2, \dots, v_n\}$ such that this order fulfills Lemma 1 for the poset $(V, \succeq_{ML})$.

For any stable matching M and any d -set $\{v_{i_1}, v_{i_2}, \dots, v_{i_d}\} \in M$ with $i_1 < i_2 < \dots < i_d$, we have that $i_{j+1} - i_j \leq 2\kappa d^2 + 4\kappa + 3d + 1$ for all $j \in [d-1]$.

Proof. Let M be a stable matching, and $\{v_{i_1}, v_{i_2}, \dots, v_{i_d}\} \in M$ be a d -set contained in M . We assume $i_1 < i_2 < \dots < i_d$, and fix some $j \in [d-1]$.

Let $\mathcal{T}^+$ be the set of d -sets in M containing an agent from $V^{[i_j+2\kappa+1, i_{j+1}-2\kappa-1]}$, and an agent from $V^{\geq i_{j+1}-2\kappa}$, and let $\mathcal{T}^-$ be the set of d -sets in M containing an agent from $V^{[i_j+2\kappa+1, i_{j+1}-2\kappa-1]}$, and an agent from $V^{\leq i_j+2\kappa}$. We now give an example for the definitions of $\mathcal{T}^+$ and $\mathcal{T}^-$.

Example 3. Let $d = 4$, $\kappa = 5$, and M be a stable matching. Assume that M contains the 4-set $\{v_3, v_{14}, v_{50}, v_{157}\}$. Thus, it holds that $i_1 = 3$, $i_2 = 14$, $i_3 = 50$, and $i_4 = 157$. Taking $j = 3$ as an example, the set $\mathcal{T}^+$ contains all 4-sets containing an agent from $\{v_{61}, v_{62}, \dots, v_{146}\}$, an agent from $\{v_{147}, v_{148}, \dots, v_n\}$, and two more arbitrary agents. The set $\mathcal{T}^-$ contains all 4-sets containing an agent from $\{v_1, v_2, \dots, v_{60}\}$, an agent from $\{v_{61}, v_{62}, \dots, v_{146}\}$, and two more arbitrary agents.

Let t be a d -set from $\mathcal{T}^+$. We claim that for every d -set $t' \in \mathcal{T}^+$ other than t , there exist agents $a \in t$ and $a' \in t'$ with $a \sim_{ML} a'$. Assume that there are two d -sets t and t' such that there do not exist $a \in t$ and $a' \in t'$ with $a \sim_{ML} a'$. Let t^* contain the d agents with minimum index from $t \cup t'$. By the definition of $\mathcal{T}^+$, any d -set from $\mathcal{T}^+$ contains an agent from $V^{\leq i_{j+1}-2\kappa-1}$ and one agent from $V^{\geq i_{j+1}-2\kappa}$. Therefore, at least one agent of t^* is contained in t , and at least one agent is contained in t' . For any agent $v_p \in t \setminus t^*$ and any $v_q \in t' \cap t^*$, it holds by the definition of t^* that $q < p$. By Lemma 1, it follows that $v_q \succ_{ML} v_p$ or $v_q \sim_{ML} v_p$. However, the latter is not possible, since we assumed that there are no two agents $a \in t$ and $a' \in t'$ with $a \sim_{ML} a'$. Thus, we have that each $a \in t$ prefers t^* over t , and by symmetry, also each $a' \in t'$ prefers t^* over t' . It follows that the d -set t^* is blocking, contradicting the assumption that M is stable.

As any agent is incomparable to at most κ other agents, it follows that $|\mathcal{T}^+| \leq \kappa d + 1$. By analogous arguments, one can show that $|\mathcal{T}^-| \leq \kappa d + 1$.

Any d -set $s \in M$ consisting solely of agents from $V^{[i_j+2\kappa+1, i_{j+1}-2\kappa-1]}$ directly implies a blocking d -set $\{v_{i_1}, \dots, v_{i_j}\} \cup S_{d-j}$, where S_{d-j} contains $d-j$ arbitrary agents from s .

It follows that M contains at most $2(\kappa d + 1)$ sets containing an agent from $V^{[i_j+2\kappa+1, i_{j+1}-2\kappa-1]}$, implying that $(i_{j+1} - 2\kappa - 1) - (i_j + 2\kappa + 1) \leq d \cdot 2(\kappa d + 1) + d - 1$, where $d - 1$ is added since there can be at most $d - 1$ unmatched agents. It follows that $i_{j+1} - i_j \leq 2\kappa d^2 + 4\kappa + 3d + 1$. $\square$

We call a matching M *local* if for all $t \in M$ and any two agents $v_j, v_{j'} \in t$ it holds that $|j - j'| \leq (d-1)(2\kappa d^2 + 4\kappa + 3d + 1)$. Note that any stable matching is local due to Lemma 2. Using a dynamic program on the local matchings, we derive an FPT-algorithm for the combined parameter $\kappa + d$. This will lead to an FPT-algorithm for the parameter κ as we will later show that if κ is much smaller than d , then a stable matching always exists.

Proposition 2 ($\star$). MDSR-POSET can be solved in $O(n^2) + (\kappa d^4)^{O(\kappa d^4)} n$ time, where κ is the maximum number of agents incomparable to a single agent, d is the dimension (i.e., the group size), and n is the number of agents.

Proof (Sketch). We first apply Lemma 1 to the poset $(V, \succeq_{\text{ML}})$ to get an order $v_1, \dots, v_n$ of the agents in $O(n^2)$ time. Let $k := 2(d-1)d(2\kappa d^2 + 4\kappa + 3d + 1)$.

We store an entry $\tau[i, M]$ for each $i \in [n]$ and each local matching M such that any d -set $t \in M$ contains at least one agent of $v_i, \dots, v_{i+k}$. This entry shall be true if and only if M can be extended to a local matching M^* not admitting a blocking d -set consisting solely of agents from $v_1, \dots, v_{i+k}$.

By Lemma 2, there exists a stable matching if and only if $\tau[n-k, M] = \text{true}$ for some local matching M . It remains to show how to compute these values.

For $i = 1$, we set $\tau[1, M] := \text{true}$ if and only if M does not contain a blocking d -set inside $v_1, \dots, v_{k+1}$. For $i > 1$, given a local matching M_i fulfilling that every d -set of M_i contains an agent from $v_i, \dots, v_{i+k}$, we look up whether there exists a local matching M_{i-1} of $v_{i-1}, \dots, v_{i+k-1}$ such that for any $j \in [i, i+k-1]$, we have $M_i(v_j) = M_{i-1}(v_j)$, and such that $M_{i-1} \cup M_i$ does not admit a blocking d -set consisting of agents from $v_{i-1}, \dots, v_{i+k}$. If this is the case, then we set $\tau[i, M_i] = \text{true}$, and otherwise we set $\tau[i, M_i] = \text{false}$.

Since there are at most $k^{O(k)}$ partitions of a k -elementary set [7], the table τ contains at most $nk^{O(k)}$ entries. Each entry can be computed in $k^{O(k)}$ time, resulting in an overall running time of $k^{O(k)}n = (\kappa d^3)^{O(\kappa d^3)}n$.

We defer the correctness proof to the full version [6]. $\square$

We now extend Proposition 2 to an FPT-algorithm for the single parameter κ . To do so, we show that if κ is much smaller than d , then there always exists a stable matching. Due to space constraints, we only sketch the proof here.

Lemma 3 ($\star$). If $4\kappa 2^{4\kappa} \leq d$, then there exists a stable matching.

Proof (Sketch). Start with an empty matching $M = \emptyset$. Construct a d -set t^* such that in any matching containing t^* , no agent of t^* can be contained in a blocking d -set. Add t^* to M , delete the agents from t^* from the instance. Repeat this as long as there are at least d unmatched agents. Construct t^* as follows: For any agent $a \in V^{\leq d-2\kappa}$ and the first $(d-1)$ -set t_a in its preferences, it holds that $\{a\} \cup t_a$ contains $V^{\leq d-2\kappa}$ and 2κ agents from $V^{[d-2\kappa+1, d+2\kappa]}$. Since $d \gg \kappa$, it follows that there exists a d -set t such that $t = \{a\} \cup t_a$ for at least 4κ agents. We set $t^* := t$. $\square$

Theorem 3. MDSR-POSET can be solved in $O(n^2) + (\kappa^5 2^{16\kappa})^{O(\kappa^5 2^{16\kappa})} n$ time, where κ is the maximum number of agents an agent is incomparable to, and n is the number of agents.

Proof. If $4\kappa 2^{4\kappa} \leq d$, then we can safely answer yes by Lemma 3. Otherwise we have $d \leq 4\kappa 2^{4\kappa}$ and thus, Proposition 2 yields an algorithm running in $h(\kappa)n$ time with $h(\kappa) = f(\kappa, 4\kappa 2^{4\kappa})$ where $f(\kappa, d) = (\kappa d^4)^{O(\kappa d^4)}$. $\square$

In the natural generalization of STABLE MARRIAGE to dimension d , the set V of agents is partitioned into d sets $V^1, \dots, V^d$ of agents, and each agent of V^i has preferences over all $(d-1)$ -sets containing exactly one agent from V^j for all $j \in [d] \setminus \{i\}$. This problem is also fixed-parameter tractable parameterized by $\kappa + d$: The master list of agents can then be decomposed into d master lists of agents, one for each set V^i . Then, one can apply Lemma 1 to each of these d master lists to get a strict order for the agents from $V^i = \{v_1^i, \dots, v_n^i\}$. Similarly to Lemma 2, one can show that for any stable matching M and any d -set $\{v_{i_1}^1, \dots, v_{i_d}^d\}$ (w.l.o.g. we have $i_j \leq i_{j+1}$), it holds that $i_{j+1} \leq i_j + O(\kappa d^2)$. Now one can apply an algorithm similar to Proposition 2 (sweeping over the sets $V^1, \dots, V^d$ from top to bottom, considering any matching on $k = f(\kappa, d)$ consecutive agents) to get an FPT-algorithm parameterized by $\kappa + d$. However, Lemma 3 does not seem to generalize to this case: for $d = 3$, there exists a small instance with $|V_1| = |V_2| = |V_3| = 3$ without a stable matching. “Cloning” the agents from one of the sets, say V_3 , an arbitrary number of times will result in an instance of unbounded d but $\kappa = 3$. It is therefore unclear whether Theorem 3 generalizes to the d -partite version of MDSR-POSET.

Remark 1. Until now, we assumed that the input is encoded naively, i.e., for each agent, its complete preference list is given as part of the input. However, this list is of length $\Omega(n^{d-1})$, which would result in a total input size of $\Omega(n^d)$. Thus, it may be more reasonable to assume that the input is given by an oracle, which can answer queries about the preferences. In fact, the FPT-algorithm parameterized by $\kappa + d$ only needs one type of queries, namely given two $(d-1)$ -sets t and t' and an agent a , the oracle tells whether a prefers t over t' . Thus, our FPT-algorithm parameterized only by κ also works when only using this query; however, in the case that κ is much smaller than d , it cannot compute a stable matching, but only state its existence. In order to also compute a stable matching efficiently, the algorithm would also need to be able to query what, given an agent a and a set X of agents, the first $(d-1)$ -set in a 's preference list not containing an agent from X .

Having shown that MDSR-POSET is fixed-parameter tractable for the parameter κ , we turn to a weaker parameter, the width of the master poset.

Width of the Poset Reducing from MULTICOLORED INDEPENDENT SET, we show that MDSR-POSET is W[1]-hard parameterized by the width of the poset.

Theorem 4 ($\star$). MDSR-POSET is W[1]-hard parameterized by the poset width.

4.3 Incomplete Preferences Derived from a Strict Master List

Let MDSRI be the MDSR problem with incomplete preference lists, i.e., $\succ_v$ is not a total order of $\binom{V \setminus \{v\}}{d-1}$, but a total order of a subset $X_v \subseteq \binom{V \setminus \{v\}}{d-1}$ for each $v \in V$. In this case, we define a *matching* M to be a set of disjoint d -sets such that for all $\{v_1, v_2, \dots, v_d\} \in M$, we have $\{v_1, v_2, \dots, v_d\} \setminus \{v_i\} \in X_{v_i}$ for

all $i \in [d]$. Similarly, MDSRI-ML is the MDSRI problem restricted to instances where the preferences are derived from a master list, and ℓ -DSRI is MDSRI for the special case $d = \ell$. We refer to the full version [6] for formal problem definitions.

In this section, we show that 3-DSRI-ML, the restriction of MDSRI-ML to $d = 3$, is NP-complete, even if the master list is strict. In order to do so, we reduce from PERFECT-SMTI-ML. The input of this problem is an instance of MAXIMUM STABLE MARRIAGE WITH TIES AND INCOMPLETE PREFERENCES such that the preferences of the women are derived from a strict master list, while the preference list of men is derived from a master list which may contain ties of size two. The problem asks whether there exists a perfect weakly stable matching. PERFECT-SMTI-ML is known to be NP-complete [20].

Theorem 5 ($\star$). *3-DSRI-ML is NP-complete, even if the master list is derived from a master list of agents.*

Theorem 5 also shows NP-completeness for the tripartite version of 3-DSRI-ML. By “cloning” each agent corresponding to a man $d - 3$ times (and for each “acceptable 3-set”, add the cloned men to this 3-set, and add all $d - 1$ -subsets of the resulting d -set at their corresponding place in the preferences), one can derive NP-completeness of d -DSRI-ML for any fixed $d \geq 3$.

4.4 Deletion Distance to a Strict Master List

We saw that MDSR-POSET is FPT for the maximum number of agents incomparable to a single agent but is W[1]-hard parameterized by the width of the poset. We now consider another parameter measuring the similarity to a strict order, namely the deletion distance to a strict order, i.e., the minimum number of agents which need to be deleted such that the resulting preferences are derived from a strict order. Note that this parameter is orthogonal to the two parameters investigated before: If the master list is the weak order $a_1 \sim_{\text{ML}} a_2 \succ_{\text{ML}} a_3 \sim_{\text{ML}} a_4 \succ_{\text{ML}} a_5 \sim_{\text{ML}} a_6 \succ_{\text{ML}} \dots \succ_{\text{ML}} a_{n-1} \sim_{\text{ML}} a_n$, then $\kappa(\text{ML}) = 2$, while one has to delete $\frac{n}{2}$ agents in order to arrive at a strict order. If the preferences of all but one agent are derived from a strict order, and the last agent’s preferences are derived from the inverse of this strict order, then the deletion distance is one while any master poset from which this preferences are derived from is only a single tie and thus has width n . In this section, reducing from MULTICOLORED CLIQUE we show that MDSR-POSET is W[1]-hard parameterized by the deletion distance to a strict master list.

Theorem 6 ($\star$). *3-DSR parameterized by $\lambda(\mathcal{I})$ is W[1]-hard, where $\lambda(\mathcal{I})$ denotes the minimum number of agents such that the preferences of the instance arising through the deletion of these agents are derived from a strict master list.*

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A Missing Proofs of Section 3

1-IN-3 POSITIVE 3-OCCURRENCE-SAT

Input: A boolean formula in conjunctive normal form, where each clause contains *exactly* three *pairwise different* variables, and each variable appears exactly three times and only non-negated in the formula.

Task: Decide whether there exists an assignment satisfying exactly one literal from each clause.

The basic idea of the reduction is the following. For each clause C_j , we have two agents c_j and d_j , and for each variable x_i , we have three agents x_i^1 , x_i^2 , and x_i^3 , one for each occurrence of the variable. Additionally, there are six agents $z_i^{k,\ell}$ for each literal of a clause. In any stable matching the agents c_j and d_j are matched to an agent x_i^ℓ corresponding to a variable occurring in this clause. Now consider a variable x_i . If the agent x_i^k (corresponding to the k -th occurrence of x_i) is matched to a 2-set $\{c_j, d_j\}$ (corresponding to the clause C_j) for all $k \in [3]$, then a stable matching can match $\{z_i^{k,1}, z_i^{k,2}, z_i^{k,3}\}$ and $\{z_i^{k,4}, z_i^{k,5}, z_i^{k,6}\}$. If the agent x_i^k (corresponding to occurrences of this variable) is not matched to a 2-set $\{c_j, d_j\}$ (corresponding to the clause C_j) for all $k \in [3]$, then we can match $\{x_i^1, x_i^2, x_i^3\}$ and then again match $\{z_i^{k,1}, z_i^{k,2}, z_i^{k,3}\}$ and $\{z_i^{k,4}, z_i^{k,5}, z_i^{k,6}\}$. If however the agents x_i^k are matched to a 2-set $\{c_j, d_j\}$ for one or two values of k , then an agent x_i^j which is not matched to a 2-set $\{c_j, d_j\}$ will together with $z_i^{k,1}, z_i^{k,2}, z_i^{k,3}, z_i^{k,4},$ and $z_i^{k,5}$ form a subinstance of six agents which does not admit a stable matching, and, thus, the resulting matching will not be stable.

Thus, any stable matching matches c_j and d_j to exactly one variable occurring in the clause C_j , and for each variable x_i , either all or none of the agents x_i^k are matched to such 2-sets $\{c_j, d_j\}$. In other words, setting the variables x_i such that x_i^k is matched to a 2-set $\{c_j, d_j\}$ to 1 and all other variables to 0, we get a solution of the 1-IN-3 POSITIVE 3-OCCURRENCE-SAT instance $\mathcal{I}$ from each stable matching.

“Inverting” this process (i.e., matching each clause to its true variable, and then matching the variable gadgets as described above), allows to construct a stable matching from a solution to the 1-IN-3 POSITIVE 3-OCCURRENCE-SAT instance $\mathcal{I}$.

A.1 Proof of Theorem 2

The reduction Let $x_1, \dots, x_n$ be the variables and $C_1, \dots, C_m$ the clauses of a 1-IN-3 POSITIVE 3-OCCURRENCE-SAT instance $\mathcal{I}$. We construct a 3-DSR-ML instance $\mathcal{I}' = (V, (\succ_v)_{v \in V})$ as follows.

For each clause C_j , we add two agents c_j and d_j to V . For the k -th occurrence ($k \in [3]$) of a variable x_i in a clause, we add an agent x_i^k . We refer to the agent corresponding to the ℓ -th literal of clause C_j as y_j^ℓ . The k -th occurrence of x_i is

also the ℓ -th literal of a clause C_j , and we will denote the agent x_i^k also by y_j^ℓ , i.e., $x_i^k = y_j^\ell$. For each agent x_i^k , we add six agents $z_i^{k,1}, \dots, z_i^{k,6}$ to V .

For each $j \in [m]$, we define $\mathcal{A}_j$ to be the following part of the master list:

$$\{c_j, d_j\} \succ \{y_j^1, d_j\} \succ \{y_j^3, c_j\} \succ \{y_j^2, d_j\} \succ \{y_j^2, c_j\} \succ \{y_j^3, d_j\} \succ \{y_j^1, c_j\}.$$

For each agent x_i^k , we define $\mathcal{B}_i^k$ to be the following part of the master list (note that (by renaming x_i^k to a , agent $z_i^{k,1}$ to b , agent $z_i^{k,2}$ to c , $\dots$, and $z_i^{k,5}$ to f) this contains the instance $\mathcal{I}_{\text{instable}}$); this ensures that x_i^k has to be matched to a 2-set which is before $\mathcal{B}_i^k$ in the master list):

$$\begin{aligned} \{x_i^k, z_i^{k,1}\} &\succ \{x_i^k, z_i^{k,2}\} \succ \{x_i^k, z_i^{k,3}\} \succ \{x_i^k, z_i^{k,5}\} \succ \{z_i^{k,1}, z_i^{k,4}\} \succ \{z_i^{k,2}, z_i^{k,3}\} \\ &\succ \{x_i^k, z_i^{k,4}\} \succ \{z_i^{k,1}, z_i^{k,5}\} \succ \{z_i^{k,2}, z_i^{k,4}\} \succ \{z_i^{k,1}, z_i^{k,3}\} \succ \{z_i^{k,3}, z_i^{k,4}\} \\ &\succ \{z_i^{k,1}, z_i^{k,2}\} \succ \{z_i^{k,2}, z_i^{k,5}\} \succ \{z_i^{k,3}, z_i^{k,5}\} \succ \{z_i^{k,4}, z_i^{k,5}\} \succ \{x_j^i, z_i^{k,6}\} \\ &\succ \{z_i^{k,1}, z_i^{k,6}\} \succ \{z_i^{k,2}, z_i^{k,6}\} \succ \{z_i^{k,3}, z_i^{k,6}\} \succ \{z_i^{k,4}, z_i^{k,6}\} \succ \{z_i^{k,5}, z_i^{k,6}\}. \end{aligned}$$

Note that by renaming x_i^k to a , agent $z_i^{k,1}$ to b , agent $z_i^{k,2}$ to c , agent $z_i^{k,3}$ to d , agent $z_i^{k,4}$ to e , and agent $z_i^{k,5}$ to f , one sees that the master list restricted to these six agents is identical to $\mathcal{I}_{\text{instable}}$ from Observation 1.

We extend this to a sublist $\mathcal{C}_i$ as follows:

$$\{x_i^1, x_i^2\} \succ \{x_i^2, x_i^3\} \succ \{x_i^1, x_i^3\} \succ \mathcal{B}_i^1 \succ \mathcal{B}_i^2 \succ \mathcal{B}_i^3.$$

The complete master list looks as follows.

$$\mathcal{A}_1 \succ \dots \mathcal{A}_m \succ \mathcal{C}_1 \succ \dots \mathcal{C}_n \succ \dots, \text{ where rest is in arbitrary order.}$$

We call the constructed 3-DSR-ML instance $\mathcal{I}'$.

Proof of the forward direction We show how to construct a stable matching from a satisfying truth assignment.

Lemma 4. *Let $f : \{x_i\} \rightarrow \{1, 0\}$ be a solution to the 1-IN-3 POSITIVE 3-OCCURRENCE-SAT instance $\mathcal{I}$. Then $\mathcal{I}'$ admits a stable matching.*

Proof. We construct a stable matching M as follows.

Denote by $I := \{i \in [n] : f(x_i) = 1\}$ the set of indices such that the corresponding variables are set to 1 by f .

For each $i \in I$ and $k \in [3]$, let $j(i, k)$ be the index of the clause containing the k -th occurrence of x_i . We add $\{x_i^k, c_{j(i, k)}, d_{j(i, k)}\}$ to M . For all $i \in [n] \setminus I$, we add the 3-sets $\{x_i^1, x_i^2, x_i^3\}$ to the matching. Finally, for each 2-set $(i, k) \in [n] \times [3]$, we add the 3-sets $\{z_i^{k,1}, z_i^{k,2}, z_i^{k,3}\}$ and $\{z_i^{k,4}, z_i^{k,5}, z_i^{k,6}\}$.

It remains to show that the resulting matching is stable.

Claim. For all $j \leq m$, neither c_j nor d_j is contained in a blocking 3-set.

Proof of Claim: We prove the claim by induction over j . For $j = 0$, there is nothing to show, as there are no agents c_0 or d_0 .

For the induction step, first note that we can ignore all 2-sets containing an agent c_p or d_p for some $p < j$, as we already know that they are not contained in a blocking 3-set. Thus, we consider the sublist ML' of ML arising by deleting all such 2-sets. The first 2-set of ML' is $\{c_j, d_j\}$. Let $y_j^\ell = x_i^k$ such that $\{c_j, d_j, y_j^\ell\} \in M$. The variable y_j^ℓ is not contained in any blocking 3-set, as it is matched to the first 2-set of sublist ML' . If c_j is contained in a blocking 3-set, then the blocking 3-set is $\{c_j, y_j^p, d_j\}$ for some $p < \ell$, as $\{y_j^p, d_j\}$ for $p < \ell$ are the only 2-sets c_j prefers over $\{y_j^\ell, d_j\}$. However, d_j does not prefer $\{y_j^p, c_j\}$ over $\{y_j^\ell, c_j\}$, and thus, $\{c_j, y_j^p, d_j\}$ is not a blocking 3-set. By symmetric arguments, d_j also cannot be contained in a blocking 3-set.

Claim. No agent x_i^k or $z_i^{k,p}$ for $i \leq n$, $k \in [3]$ and $p \in [6]$ is part of a blocking 3-set.

Proof of Claim: We prove the claim by induction over i . For $i = 0$ there is nothing to show.

Note that all 2-sets from $\bigcup_{\ell \in [m]} \mathcal{A}_\ell$ contain an agent of the form c_ℓ or d_ℓ , and thus, no blocking 3-set contains a 2-set from $\bigcup_{\ell \in [m]} \mathcal{A}_\ell$. Furthermore, by the induction hypothesis, no 2-set from $\mathcal{C}_q$ for $q < i$ can be contained in a blocking 3-set. Thus, it is enough to consider the sublist ML_i arising through the deletion of $\mathcal{A}_j$ for all $j \in [m]$ and $\mathcal{C}_q$ for $q < i$.

If x_i is set to 1, then all agents x_i^k are matched better than any 2-set from ML_i , and thus, cannot be part of a blocking 3-set. Otherwise, all of x_i^1 , x_i^2 , and x_i^3 are matched to the first 2-set not containing themselves in ML_i , and thus not contained in a blocking 3-set.

Considering the sublist ML'_i arising from ML_i by deleting all 2-sets containing an agent x_i^k , one sees that the first 15 sets of two agents of this sublist only consider agents of $z_i^{k,p}$ for $p \in [6]$, and all agents $z_i^{k,p}$ are matched to one of these 15 sets of size two. Thus, any blocking 2-set containing an agents $z_i^{k,p}$ consists only of agents from $\{z_i^{k,q} : q \in [6]\}$. By enumerating all 20 such 3-sets one easily verifies that none of them is blocking.

This concludes the proof of the lemma. $\square$

Proof of the backward direction Now we show how to construct a solution to $\mathcal{I}$ from a stable matching. First, we identify small subinstances of $\mathcal{I}'$ not admitting a stable matching, implying that at least one vertex of these subinstances must be matched outside the subinstance.

Lemma 5. *The subinstance of $\mathcal{I}'$, which arises through the deletion of all but the agents x_i^k and $\{z_i^{k,p} : p \in [6]\}$ for a fixed $i \in [n]$ and a fixed $k \in [3]$ does not admit a stable matching.*

Proof. This can be seen in the same way as Observation 1; note that all 2-sets containing $z_i^{k,6}$ appear on the end of the preference lists. $\square$

We now show that the two agents c_j and d_j created for the clause C_j have to be matched to an agent corresponding to a literal in this clause; indeed, we will later see that this literal satisfies the clause C_j in the constructed solution.

Lemma 6. *In any stable matching M , for each $j \in [m]$, there exists an $\ell \in [3]$ such that $\{c_j, d_j, y_j^\ell\} \in M$.*

Proof. By induction over j .

Base case: If M does not contain a 2-set $\{c_1, d_1, y_1^\ell\}$, then $\{c_1, d_1, y_1^\ell\}$ is a blocking 2-set for all $\ell \in [3]$.

Induction step: By the induction hypothesis, no agent c_p or d_p for $p < j$ is matched to c_j , d_j , or some y_j^ℓ . Hence, neither c_j , d_j nor y_j^ℓ is matched to a 2-set which comes before $\mathcal{A}_j$ in the preference list. Thus, if the lemma does not hold, then $\{c_j, d_j, y_j^\ell\}$ is a blocking 3-set for all $\ell \in [3]$, contradicting the stability of M . $\square$

Before proving the backward direction, we need the following structural statement about the agents x_i^1 , x_i^2 , and x_i^3 , essentially stating that if one of these agents is matched to a 2-set $\{c_j, d_j\}$ (corresponding to setting this literal to 1), then all three of them are.

Lemma 7. *Let M be any stable matching. Then for all $i \in [n+1]$ and all $i^* < i$ either $\{x_{i^*}^1, x_{i^*}^2, x_{i^*}^3\} \in M$ or for each $k \in [3]$, there exists some $j \in [m]$ such that $\{x_{i^*}^k, c_j, d_j\}$. Furthermore, for each $k \in [3]$, the matching M contains two 3-sets t_1 and t_2 with $t_1, t_2 \subseteq \{z_{i^*}^{k,p} : p \in [6]\}$.*

Proof. We prove the lemma by induction over i . For $i = 1$, there is nothing to show.

By the induction hypothesis, no agent x_q^k or $z_q^{k,p}$ is matched to a 2-set containing an agent x_q^r or $z_q^{r,s}$ for $q < i$, and by Lemma 6, no agent $z_{i^*}^{k,p}$ is matched to a 2-set containing an agent c_j or d_j .

Let $I := \{k : \exists j \in [m] \text{ s.t. } \{x_{i^*}^k, c_j, d_j\} \in M\}$.

Case 1: $I = [3]$.

Then M must match $\{z_{i^*}^{jk} : k \in [6]\}$ into two 3-sets for each $j \in [3]$, as all 2-sets before $\mathcal{B}_{i^*}^k$ contain an agent which is not matched to an agent $z_{i^*}^{k,p}$ by M .

Case 2: $I = \emptyset$.

Then M contains $\{x_{i^*}^1, x_{i^*}^2, x_{i^*}^3\}$, as all 2-sets before $\mathcal{C}_{i^*}$ contain an agent which is not matched to an agent $x_{i^*}^k$ in M . By the same argument as in Case 1, M contains two 3-sets t_1 and t_2 with $t_1, t_2 \subseteq \{z_{i^*}^{k,p} : p \in [6]\}$.

Case 3: $|I| \in \{1, 2\}$.

We show that this case leads to a contradiction and therefore cannot occur.

Case 3 (a): No 3-set in M contains two agents of the form $x_{i^*}^k$ for $k \in [3]$.

If there does not exist some $j \in [m]$ such that $\{x_{i^*}^1, c_j, d_j\} \in M$, then the agents $x_{i^*}^1, z_{i^*}^{1,1}, \dots, z_{i^*}^{1,5}$ have to be matched to two 3-sets, implying by Lemma 5 that M contains a blocking 3-set inside $x_{i^*}^1, z_{i^*}^{1,1}, \dots, z_{i^*}^{1,5}$. Thus, $x_{i^*}^1$ is matched to a clause, then M matches $z_{i^*}^{11}, \dots, z_{i^*}^{16}$ together to two 3-sets by the same argument as in Case 1.

We can conclude then by the same argument that x_i^2 is matched to a 2-set $\{c_j, d_j\}$ for some $j \in [m]$, and from this that x_i^3 is matched to a 2-set $\{c_j, d_j\}$ for some $j \in [m]$, a contradiction.

Case 3 (b): There is a 3-set $\{x_i^{k_1}, x_i^{k_2}, z\} \in M$ with $z \notin \{x_i^k\}$.

Then $x_i^{k_1}$ prefers to be matched to the 2-set $\{z_i^{11}, z_i^{12}\}$. However, z_i^{11} and z_i^{12} also prefer to be matched by $\{x_i^{k_1}, z_i^{11}, z_i^{12}\}$ (all 2-sets preferred over $\{x_i^{k_1}, z_i^{11}\}$ contain a vertex x_k^j with $j < i$ or a clause c_j or d_j). Thus, we have a blocking 3-set, contradicting the stability of M . $\square$

The backward direction now easily follows.

Lemma 8. *If there exists a stable matching M , then there is a satisfying truth assignment for $\mathcal{I}$.*

Proof. Consider the assignment $f : \{x_i\} \rightarrow \{1, 0\}$, where $f(x_i) = 1$ if and only if all x_i^k are matched to a 2-set of the form $\{c_j, d_j\}$.

Assume that f is not a solution to $\mathcal{I}$.

Case 1: There is a clause C_j which is not satisfied by f .

By Lemma 6, for each $j \in [m]$, there exists some $\ell \in [3]$ such that $\{c_j, d_j, y_j^\ell\} \in M$. Let $y_j^\ell = x_i^k$. By Lemma 7, all three agents x_i^1 , x_i^2 , and x_i^3 are matched to a 2-set $\{c_p, d_p\}$, and thus, the clause C_j is satisfied by f , a contradiction.

Case 2: There is a clause C_j which is satisfied by at least two variables x_i and $x_{i'}$.

The matching M can only contain one 2-set containing c_j or d_j , and so without loss of generality x_i^k is not matched to $\{c_j, d_j, x_i^k\}$ for all $k \in [3]$. By Lemma 6, literal $x_i^{k'}$ contained in C_j does not match to any 2-set $\{c_q, d_q\}$ for $q \in [m]$, and thus, we have $f(x_i) = 0$, a contradiction.

Thus, f is a solution to $\mathcal{I}$. $\square$

Finally, observe that 3-DSR-ML is in NP (the stability of a matching can be checked in $O(n^3)$ by enumerating all 3-sets, where n is the number of agents).

The reduction described in Appendix A.1 can be performed in linear time. Thus, NP-completeness follows directly from Lemma 4 and Lemma 8.

Thus, we have now proven the Theorem 2.

B Missing Proofs of Section 4

B.1 Proof of Proposition 1

Proof. We number the agents in such a way that $v_1 \succ_{\text{ML}} v_2 \succ_{\text{ML}} \dots \succ_{\text{ML}} v_n$.

We claim that $M := \{\{v_{d(i-1)+1}, v_{d(i-1)+2}, \dots, v_{di}\} : 1 \leq i \leq \lfloor \frac{n}{d} \rfloor\}$ is a stable matching. Assume that there is a blocking d -set $\{v_{i_1}, v_{i_2}, \dots, v_{i_d}\}$ with $i_1 < i_2 < \dots < i_d$. Let $\{v_{i_1}, w_2, w_3, \dots, w_{d-1}\} \in M$ be the d -set containing v_{i_1} . Note that such a d -set exists as M leaves at most the $d - 1$ last agents of the master list unmatched, and the agents v_{i_j} are after v_{i_1} in the master list for

$j \in \{2, 3, \dots, d\}$. Thus, v_{i_1} is matched, and cannot prefer $\{v_{i_2}, \dots, v_{i_d}\}$ over M as $w_j \succ_{\text{ML}} v_{i_j}$ for all $j \in \{2, 3, \dots, d\}$.

It remains to show that M is the unique stable matching. Assume that there is a stable matching $M' \neq M$. Let i be the smallest index such that v_i is matched in M' , but to another $(d-1)$ -set than in M . Since only the at most $d-1$ agents with highest index are unmatched in M , we get that v_i is matched in M . Thus, we have that $i = d(j-1) + 1$ for some $j \in \mathbb{N}$.

We claim that $T := \{v_i, v_{i+1}, v_{i+2}, \dots, v_{i+d-1}\}$ is a blocking d -set for M' . By the definition of i , no agent with index smaller than i is matched to one of the agents $v_i, v_{i+1}, \dots, v_{i+d-1}$, and we have that $T \notin M$. Thus, for any $v \in T$, matched to a $(d-1)$ -set T'_v in M' , the bijection $\sigma_v : T \setminus \{v\} \rightarrow T'_v \setminus \{v\}$ matching the agent with the j th-lowest index in $T \setminus \{v\}$ to the agent with the j th-lowest index in $T'_v \setminus \{v\}$ satisfies $w \succeq_{\text{ML}} \sigma(w)$ for all $w \in T \setminus \{v\}$. Since the preferences are derived from $\succeq_{\text{ML}}$, it follows that T is a blocking d -set. $\square$

B.2 Proof of Lemma 1

Proof. Let $v_1 \in V$ be an element such that $v_1 \succeq v$ or $v_1 \sim v$ for all $v \in V$. Such an element v_1 has to exist in any poset.

By induction, we can find such an order of $V \setminus \{v_1\}$.

We now add v_1 at the beginning. Let V' be the set of elements incomparable with v_1 . It remains to show that the vertices from V' are among the $2\kappa(\succeq)$ first vertices.

Note that there is no $v' \in V'$ and $v \in V \setminus V'$ with $v \succ v'$, as otherwise $v_1 \succeq v \succ v'$ and thus $v_1 \succ v'$, but by the definition of V' we have $v_1 \sim v'$.

Thus, for any $v' \in V'$, there are at most $|V'| + \kappa_{\succ}(v') \leq 2\kappa(\succeq)$ elements before v' , and thus, the new order satisfies the lemma.

Since a maximal element in a poset can be found in linear time, the order can be found in quadratic time. $\square$

B.3 Proof of Proposition 2

Proof. We first apply Lemma 1 to the poset $(V, \succeq_{\text{ML}})$ to get an order $v_1, \dots, v_n$ of the agents in $O(n^2)$ time. Let $k := 2d(d-1)(2\kappa d^2 + 4\kappa + 3d + 1)$.

We will store an entry $\tau[i, M]$ for each $i \in [n]$ and each local matching M such that any d -set $t \in M$ contains at least one agent of $v_i, \dots, v_{i+k}$. This entry shall be **true** if and only if M can be extended to a local matching M^* not admitting a blocking d -set consisting solely of agents from $v_1, \dots, v_{i+k}$.

By Lemma 2, there exists a stable matching if and only if $\tau[n-k, M] = \text{true}$ for some local matching M . Thus, it remains to show how to compute these values.

For $i = 1$, we set $\tau[1, M] := \text{true}$ if and only if M does not contain a blocking d -set inside $v_1, \dots, v_{k+1}$.

For $i > 1$, given a local matching M_i fulfilling that every d -set of M_i contains an agent from $v_i, \dots, v_{i+k}$, we look up whether there exists a local matching M_{i-1}

of $v_{i-1}, \dots, v_{i+k-1}$ such that for any $j \in [i, i+k-1]$, we have $M_i(v_j) = M_{i-1}(v_j)$, and such that $M_{i-1} \cup M_i$ does not admit a blocking d -set consisting of agents from $v_{i-1}, \dots, v_{i+k}$. If this is the case, then we set $\tau[i, M_i] = \text{true}$, and otherwise we set $\tau[i, M_i] = \text{false}$.

Each entry can be computed in $k^{O(k)}$ time, resulting in an overall running time of $k^{O(k)}n = (\kappa d^3)^{O(\kappa d^3)}n$.

It remains to show the correctness of this dynamic program, which we do by induction. For $i = 1$, the values $\tau[i, M]$ are computed correctly by definition. Let $i > 1$. First assume that $\tau[i, M_i] = \text{true}$. Let M_{i-1} be the local matching with $\tau[i-1, M_{i-1}] = \text{true}$ and $M_{i-1} \cup M_i$ not admitting a blocking d -set consisting of agents from $v_{i-1}, \dots, v_{i+k}$. By the induction hypothesis, $\tau[i-1, M_{i-1}]$ was computed correctly, implying that there exists a local matching M_{i-1}^* on $v_1, \dots, v_{i+k-1}$ such that for each $j \in [i-1, i+k-1]$, we have $M_{i-1}(v_j) = M_{i-1}^*(v_j)$, and no blocking d -set consists solely of agents from $v_1, \dots, v_{i+k-1}$. We define $M_i^* := M_{i-1}^* \cup M_i$. Clearly, it holds that $M_i(v_j) = M_i^*(v_j)$ for all $j \in [i, i+k]$. Next, we show that M_i^* contains no blocking d -set consisting solely of agents from $V^{\leq k+i}$. Any such d -set t must contain v_{k+i} , since M_{i-1}^* does not admit blocking d -sets consisting solely of agents from $v_1, \dots, v_{k+i-1}$. Furthermore, it must contain at least one agent v_j with $j < i-1$ since we checked for all d -sets of agents from $v_{i-1}, \dots, v_{i+k}$ whether they are blocking. Since $k-i > 2d(d-1)2\kappa d^2 + 4\kappa + 3d + 1$, it follows that there exists some $\ell \in [i + (d-1)2\kappa d^2 + 4\kappa + 3d + 1, k - (d-1)2\kappa d^2 + 4\kappa + 3d + 1]$ such that t contains no agent from $V^{\ell-2\kappa d^2+4\kappa+3d+1, \ell+2\kappa d^2+4\kappa+3d+1}$, and there exists some $t' \in M_i^*$ with $v_\ell \in t'$ (note that such an ℓ exists as there can be at most $d-1$ unmatched agents). Then the d agents with minimum index from $t \cup t'$ form a blocking d -set.

Now assume that the dynamic program computed $\tau[i, M_i]$ to be **false**. We assume that the correct value of $\tau[i, M_i]$ is **true**, and will reach a contradiction. Let M_i^* be a local matching witnessing that the correct value of $\tau[i, M_i]$ is **true**, i.e., we have $M_i^*(v_j) = M_i(v_j)$ for all $j \in [i, i+k]$, and M_i^* does not admit a blocking d -set consisting solely of agents from $v_1, \dots, v_{i+k}$. Let M_{i-1} be the restriction of M_i^* to the d -sets containing $v_{i-1}, \dots, v_{i+k-1}$. The matching M_i^* also witnesses that $\tau[i-1, M_{i-1}] = 1$, and this value is correctly computed by induction. Extending M_{i-1} with $M_i(v_{i+k})$ does not lead to a blocking d -set containing only agents from $v_{i-1}, \dots, v_{i+k}$, as M_i^* does not contain such a blocking 3-set. It follows that the algorithm computed $\tau[i, M_i]$ to be **true**, a contradiction. $\square$

B.4 Proof of Lemma 3

Proof. We apply Lemma 1 to the poset $(V, \succeq_{\text{ML}})$, getting an order $V = \{v_1, \dots, v_n\}$ of the agents.

Claim. There exists a tuple t^* such that for any matching containing t^* , no blocking d -set contains an agent from t^* .

Proof of Claim: For each $i \in [d - 2\kappa]$, let $t_i := \{v_i\} \cup t$, where t is the first $(d - 1)$ -tuple in v_i 's preferences. By Lemma 1, it holds that t_i contains v_j for each $j \in [d - 2\kappa]$, and all agents of t_i are from $V^{\leq d+2\kappa}$. Thus, there are at most $\binom{2\kappa}{4\kappa} \leq 2^{4\kappa}$ different classes. Since $d \geq 4\kappa 2^{4\kappa}$, there exists a d -set t^* with $t^* = t_i$ for at least 4κ agents v_i .

Consider any matching M containing t^* , and assume that there is a blocking d -set t containing an agent $a^* \in t^*$. Let $t \setminus t^* = \{v_{i_1}, \dots, v_{i_r}\}$ and $t^* \setminus t = \{v_{j_1}, \dots, v_{j_r}\}$. Note that for $p \in [4\kappa]$, it holds that $j_p \leq (d - 2\kappa) - (4\kappa - p) < (d - 2\kappa + p) - 2\kappa \leq i_p - 2\kappa$. For $p > 4\kappa$, it holds that $j_p \leq d + 2\kappa < d + 4\kappa + 1 - 2\kappa \leq i_p - 2\kappa$. By Lemma 1 it follows that $v_{j_p} \succ_{\text{ML}} v_{i_p}$. Thus, a^* prefers t^* over t , contradicting that t is a blocking d -set.

From the claim, the lemma follows easily: We start with an empty matching $M = \emptyset$ and as long as there are at least d unmatched agents, we successively compute such a d -set t^* , add t^* to M , delete the agents from t^* , and repeat. The resulting matching is clearly stable, as the agents from the d -sets added to M are not part of a blocking d -set. $\square$

B.5 Proof of Theorem 4

In this section, at some points it does not matter how the preferences between a set of 2-sets look, as long as the preferences are derived from the poset (which will be described in Section B.5). Thus, whenever we describe the preferences of an agent, and these preferences contain a set of 2-sets, then this means that the preferences of the agent arise through replacing this set by a strict order of the elements of the set which is derived from the master list poset.

The reduction We provide a parameterized reduction from MULTICOLORED INDEPENDENT SET.

MULTICOLORED INDEPENDENT SET

Input: A k -partite graph $G = (V^1 \dot{\cup} V^2 \dot{\cup} \dots \dot{\cup} V^k, E)$ with $|V^i| = n$ for all $i \in [k]$.

Task: Decide whether G contains an independent set I such that $I \cap V^i \neq \emptyset$ for all $i \in [k]$.

Let $V^i = \{v_1^i, \dots, v_n^i\}$.

When describing the reduction, we only describe the beginning of the preferences of an agent. The remaining acceptable 2-sets can be added in an arbitrary way obeying the master poset.

“Cut-off” gadgets.

For each agent v (except for those belonging to cut-off gadgets), the reduction contains a cut-off gadget. This gadget “cuts off” the preference list of agent v after a specific 2-set p_v , and enforces v to be matched to a 2-set p with $p \succeq_v p_v$.

Given an instance $\mathcal{I}$ of MDSR-POSET with master poset $\succ_{\text{ML}}$, the cut-off gadget for an agent v and a 2-set p_v , denoted by CO_v , contains six agents $z_v^1, \dots, z_v^6$.

The preferences of the agents from the cut-off gadget look as follows (note that the preferences of z_v^r are very similar to those of $z_i^{k,r}$ in the NP-hardness reduction of 3DSR-ML-SETS, see Appendix A.1). The preferences of any agent z_v^r start with all 2-sets in $\binom{W}{2} \cup (W \times \{z_v^\ell : \ell \in [6]\})$, where $W := \{w, z_w^\ell : w \succ_{\text{ML}} v, \ell \in [6]\}$, and are then followed by (ignoring all 2-sets containing the agent itself):

$$\begin{aligned} \{v, z_v^1\} &\succ \{v, z_v^2\} \succ \{v, z_v^3\} \succ \{v, z_v^5\} \succ \{z_v^1, z_v^4\} \succ \{z_v^2, z_v^3\} \succ \{v, z_v^4\} \succ \{z_v^1, z_v^5\} \\ &\succ \{z_v^2, z_v^4\} \succ \{z_v^1, z_v^3\} \succ \{z_v^3, z_v^4\} \succ \{z_v^1, z_v^2\} \succ \{z_v^2, z_v^5\} \succ \{z_v^3, z_v^5\} \succ \{z_v^4, z_v^5\} \\ &\succ \{v, z_v^6\} \succ \{z_v^1, z_v^6\} \succ \{z_v^2, z_v^6\} \succ \{z_v^3, z_v^6\} \succ \{z_v^4, z_v^6\} \succ \{z_v^5, z_v^6\}. \end{aligned}$$

The preferences of agent v are modified as follows: After the 2-set p , we insert all 2-sets containing agents from the cut-off gadget in the same order as in the above list. All agents w with $v \succ_{\text{ML}} w$ add all 2-sets containing an agent from the cut-off gadget at the beginning of their preference, while all other agents add all 2-sets containing an agent z_v^r at the end of their preferences in an arbitrary order.

Lemma 9. *Let $\mathcal{I}_0$ be a MDSR-POSET-instance, and for each $v \in V$, let p_v be a 2-set not containing v . Let $V = \{v_1, \dots, v_n\}$ with $v_i \succsim v_j$ for all $i < j$. Let $\mathcal{I}_k$ arise from $\mathcal{I}$ by adding for $i = 1, \dots, k$ a cut-off gadget for agent v_i and 2-set p_{v_i} . Let $\mathcal{M}^k$ be the set of stable matchings M in $\mathcal{I}_0$ with $M(v) \succ_v p_v$ for all $v \in \{v_1, \dots, v_k\}$. For any matching $M \in \mathcal{M}^k$, the matching $M^k := M \cup \{\{z_v^1, z_v^2, z_v^3\}, \{z_v^4, z_v^5, z_v^6\} : v \in \{v_1, \dots, v_k\}\}$ is stable in $\mathcal{I}_k$. A matching M' in $\mathcal{I}_k$ is stable if and only if $M' \cap \binom{V}{3}$ is stable in $\mathcal{I}_0$.*

Proof. We prove the lemma by induction over k . For $k = 0$, there is nothing to show. Fix $k > 0$.

First, we show that for any stable matching $M \in \mathcal{I}_0$ with $M(v) \preceq_v p_v$ for all $v \in \{v_1, \dots, v_k\}$, it holds that $M_k := M \cup \{\{z_v^1, z_v^2, z_v^3\}, \{z_v^4, z_v^5, z_v^6\} : v \in \{v_1, \dots, v_k\}\}$ is a stable matching in $\mathcal{I}_k$. Let $v := v_k$. No blocking 3-sets consists solely of agents from $\{z_v^r : r \in [6]\}$; this can be seen by checking all 20 such 3-sets. Any blocking 3-set needs to contain at least one agent z_v^r (else it would already be a blocking 3-set in $\mathcal{I}_{k-1}$). Agent v is not part of such a blocking 3-set, as it ranks all 2-sets containing an agent z_v^r after $M(v)$. All other 2-sets preferred by the agent z_v^r contain an agent v' or $z_{v'}^q$ with $v' \succ_{\text{ML}} v$. However, v' has to be matched, as else $v' \cup M(v)$ would be a blocking 3-set. It follows that v' and $z_{v'}^q$ do not prefer any 2-set containing z_v^r over $M(v')$, and is thus not part of a blocking 3-set. Thus, the blocking 3-set has to be inside a single cut-off gadget, a contradiction.

Vice versa, let M_k be a stable matching in $\mathcal{I}_k$. Let M be the matching arising from M_k by deleting all 3-sets containing an agent $z_{v_i}^r$. We need to show that M is stable and $M(v) \preceq_v p$ for all $v \in \{v_1, \dots, v_k\}$. If $M(v) \preceq_v p$ does not hold for all $v \in \{v_1, \dots, v_k\}$, then let v_i be the agent of minimal index for which $p_v \succ M(v_i)$. Then for all $j < i$, the matching M matches the agents from $\{z_{v_j}^r : r \in [6]\}$ in two 3-sets. By a similar case distinction as in Lemma 5, it follows that there is a

blocking 3-set in $\{v_i\} \cup \{z_{v_i}^r : r \in [5]\}$, contradicting the stability of M . Assume that M is not stable, and let t be a blocking 3-set. Then one agent of t has to be matched in M_k but not in M , implying that t contains an agent from the cut-off gadgets, a contradiction. $\square$

Vertex-selection gadget.

A vertex-selection gadget has $6n$ agents a_j ($j \in [n]$), b_k ($k \in [n]$), c_i , $\bar{c}_i$ ($i \in [n-1]$), d_p , and $\bar{d}_p$ ($p \in [n+1]$). The intuitive idea is the following. The agents a_i and b_i want to be matched to 2-sets $\{c_i, \bar{c}_i\}$. As c_i prefers $\{a_j, \bar{c}_i\}$ over $\{b_k, \bar{c}_i\}$ while $\bar{c}_i$ prefers $\{b_k, c_i\}$ over $\{a_j, c_i\}$, we can match the $n-1$ sets of size two of the form $\{c_i, \bar{c}_i\}$ to the agents $\{a_j : j < v\} \cup \{b_k : k < n+1-v\}$ for any $v \in V$, corresponding to selecting the vertex v to be part of the independent set. The agents $\{a_j : j \geq v\} \cup \{b_k : k \geq n+1-v\}$ are then matched to the 2-sets $\{d_i, \bar{d}_i\}$, and can form blocking 3-sets with the edge gadget (which are described later).

Formally, the preferences look as follows.

$$\begin{aligned} a_j, b_k &: \{\{c_p, \bar{c}_q\} : p, q \in [n-1]\} \succ \{\{d_p, \bar{d}_q\} : p, q \in [n+1]\} \succ \text{CO} \\ c_i &: \{\{a_j, \bar{c}_q\} : j \in [n], q \in [n-1]\} \succ \{\{b_k, \bar{c}_q\} : k \in [n], q \in [n-1]\} \succ \text{CO}_{c_i} \\ \bar{c}_i &: \{\{b_k, c_q\} : k \in [n], q \in [n-1]\} \succ \{\{a_j, c_q\} : j \in [n], q \in [n-1]\} \succ \text{CO}_{\bar{c}_i} \\ d_i &: \{\{a_j, \bar{d}_q\} : j \in [n], q \in [n+1]\} \succ \{\{b_k, \bar{d}_q\} : k \in [n], q \in [n+1]\} \succ \text{CO}_{d_i} \\ \bar{d}_i &: \{\{b_k, d_q\} : k \in [n], q \in [n-1]\} \succ \{\{a_j, d_q\} : j \in [n], q \in [n+1]\} \succ \text{CO}_{\bar{d}_i} \end{aligned}$$

See Figure 1 for an example of the 3-sets before the cut-off gadgets.

Lemma 10. *Let $\mathcal{I}$ be a MDSR-POSET-instance arising from a vertex-selection gadget by adding agents such that for every agent a in the vertex-selection gadget except from $\{a_i, b_i : i \in [3n'+1]\}$, the preferences of a all 2-sets consisting of two agents in the vertex-selection gadget are preferred to a 2-set containing an agent outside the vertex-selection gadget.*

For every stable matching M in $\mathcal{I}$, there exists some $i^ \in [n]$ such that $\{a_i, c_j, \bar{c}_k\} \in M$ for all $i < i^*$, $\{a_i, d_j, \bar{d}_k\} \in M$ for $i \geq i^*$, $\{b_i, c_j, \bar{c}_k\} \in M$ for $i \leq n - i^*$, and $\{b_i, d_j, \bar{d}_k\} \in M$ for $i > n - i^*$.*

Vice versa, for any $i^ \in [n]$, the matching $M_{i^*} = \{\{a_i, c_i, \bar{c}_{n-i^*+i}\} : i < i^*\} \cup \{\{a_i, d_{i-i^*+1}, \bar{d}_{i+1}\} : i \geq i^*\} \cup \{\{b_i, c_{i+i^*-1}, \bar{c}_i\} : i \leq n-i^*\} \cup \{\{b_i, d_{i+1}, \bar{d}_{i-i^*-1}\} : i > n-i^*\}$ contains no blocking 3-set solely consisting from agents in the vertex-selection gadget.*

Proof. Consider any stable matching M . Due to the cut-off gadgets, any agent c_i is matched to a 2-set $\{a_j, \bar{c}_p\}$ or $\{b_k, \bar{c}_p\}$, and any agent d_i is matched to a 2-set $\{a_j, \bar{d}_p\}$ or $\{b_k, \bar{d}_p\}$. Since $|\{a_i, b_i : i \in [n]\}| = 2n = |\{c_i : i \in [n-1]\}| + |\{d_i : i \in [n+1]\}|$, we get that any agent a_i or b_i is matched to a 2-set $\{c_p, \bar{c}_q\}$ or $\{d_p, \bar{d}_q\}$. Assume that there exists some $i \in [n]$ with $\{a_i, c_p, \bar{c}_q\} \notin M$ but $\{a_{i+1}, c_r, \bar{c}_s\} \in M$ (the case that there exists such b_i is symmetric). Then $\{a_i, c_r, \bar{c}_s\}$ is a blocking 3-set. Thus, the first statement holds.

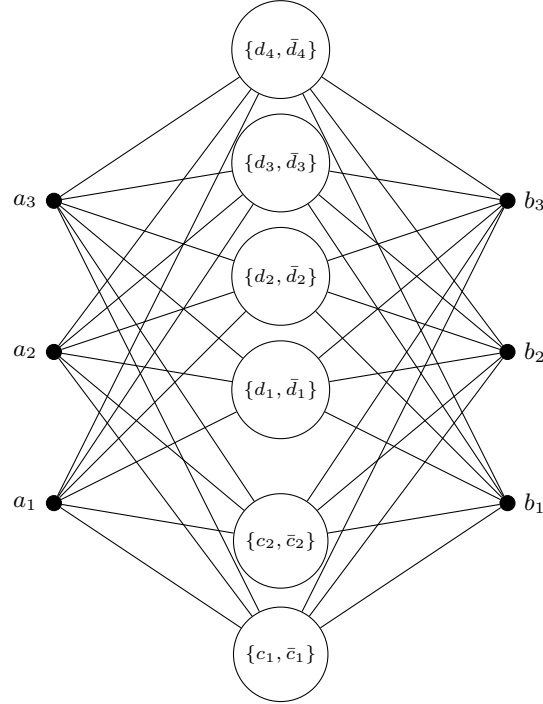


Fig. 1. The acceptable 3-sets (i.e., 3-sets which are preferred over the cut-off gadget by all agents they contain) of a vertex-selection gadget. For the sake of readability, we ignored 3-sets containing the agents c_i and $\bar{c}_j$ or d_i and $\bar{d}_j$ for $i \neq j$. Each edge corresponds to a 3-set containing the one endpoint a_i or b_i and the two vertices contained in the circle of the other endpoint of the edge. For example, the edge between a_1 and $\{c_1, \bar{c}_1\}$ indicates that $\{a_1, c_1, \bar{c}_1\}$ is an acceptable 3-set.

Let $i^* \in [n]$ and assume that M_{i^*} contains a blocking 3-set inside the vertex-selection gadget. Then this 3-set contains two agents c_p and $\bar{c}_q$ or d_p and $\bar{d}_q$, and an agent a_i or b_i .

Case 1: $t = \{a_i, c_p, \bar{c}_q\}$. If $i \leq \min\{p, q + i^* - n\}$, then a_i does not prefer t over M . If $p \leq \min\{i, q + i^* - n\}$, then c_p does not prefer t over M . If $q \leq \min\{i, p\} + n - i^*$, then $\bar{c}_q$ does not prefer t over M .

Case 2: $t = \{a_i, d_p, \bar{d}_q\}$. If $i \leq \min\{p + i^* - 1, q - 1\}$, then a_i does not prefer t over M . If $p \leq \min\{i, q - 1\} - i^* + 1$, then d_p does not prefer t over M . If $q \leq \min\{i, p + i^* - 1\} + 1$, then $\bar{d}_q$ does not prefer t over M .

Case 3: $t = \{b_i, c_p, \bar{c}_q\}$. If $i \leq \min\{p - i^* + 1, q\}$, then b_i does not prefer t over M . If $p \leq \min\{i, q\} + i^* - 1$, then c_p does not prefer t over M . If $q \leq \min\{i, p - i^* + 1\}$, then $\bar{c}_q$ does not prefer t over M .

Case 4: $t = \{b_i, d_p, \bar{d}_q\}$. If $i \leq \min\{p - 1, q + i^* + 1\}$, then b_i does not prefer t over M . If $p \leq \min\{i, q + i^* + 1\} + 1$, then d_p does not prefer t over M . If $q \leq \min\{i, p - 1\} - i^* - 1$, then $\bar{d}_q$ does not prefer t over M .

In each of the four cases, we have a contradiction, finishing the proof of the lemma. $\square$

The reduction will create k such vertex-selection gadgets. We number these gadgets $A^1, \dots, A^k$, and refer to agent a_j (respectively $b_j, c_j, \bar{c}_j, d_j$, or $\bar{d}_j$) from the i -th vertex-selection gadget via a_j^i (respectively $b_j^i, c_j^i, \bar{c}_j^i, d_j^i$, or $\bar{d}_j^i$).

We say that a vertex-selection gadget A^i *selects* the vertex $v_{i^*}^i$ if the matching inside the vertex-selection gadget contains the 3-sets $\{a_j^i, c_j^i, \bar{c}_j^i\} : j < i^*\}$.

Edge gadget.

Fix a pair of vertex-selection gadgets A^i and A^j , and let $E^{i,j} := \{\{v, w\} \in E(G) : v \in V^i, w \in V^j\}$. For each edge $e = \{v_{r_i}^i, v_{r_j}^j\}$ in $E^{i,j}$, our reduction contains an edge gadget between A^i and A^j , containing the 15 agents $h_a^{e,i}, h_b^{e,i}, h_a^{e,j}, h_b^{e,j}, g_1^e, g_2^e, g_3^e, \bar{g}_1^e, \bar{g}_2^e, \bar{g}_3^e, f^e, \bar{f}^e, \alpha_1^e, \alpha_2^e$, and α_3^e .

The intuitive function of the edge gadget is the following. By Lemma 10, in any stable matching there exists some i^* such that A^i selects $v_{i^*}^i$, i.e., a_k^i is matched to $\{d_p, \bar{d}_q\}$ for all $k \geq i^*$, and thus prefers being matched to the edge gadget (more specifically, to the 2-set $\{h_a^{e,i}, \alpha_1^e\}$). Similarly, b_k for $k > n - i^*$ prefers to be matched to the edge gadget, namely to the 2-set $\{h_b^{e,i}, \alpha_1^e\}$. The agent $h_a^{e,i}$ [$h_b^{e,i}$] prefers the 2-set $\{a_{i^*}^i, \alpha_1^e\}$ [$\{b_{n+1-i^*}^i, \alpha_1^e\}$] over the 2-set $\{f^e, \bar{f}^e\}$ if $r_i \leq i^*$ [$r_i \geq i^*$]. Thus, if $v_{r_i}^i = v_{i^*}^i$, i.e., if the vertex selected by A^i is an endpoint of e , then both $h_a^{e,i}$ and $h_b^{e,i}$ cannot be matched to $\{f^e, \bar{f}^e\}$. The edge gadget is now designed in such a way that an arbitrary of these vertices $h_x^{e,y}$ ($x \in \{a, b\}$ and $y \in \{i, j\}$) has to be matched to $\{f^e, \bar{f}^e\}$, which is possible if and only if e is not an edge between the two vertices selected by the vertex-selection gadgets, i.e., $e \neq \{v_{i^*}^i, v_{j^*}^j\}$.

We now give a formal description of the edge gadget. The preferences of the agents look as follows.

$$\begin{aligned}
h_a^{e,i} &: \{\{g_p^e, \bar{g}_q^e\} : p, q \in [3]\} \succ \{\{a_p^i, \alpha_1^e\} : p \leq r_i\} \succ \{f^e, \bar{f}^e\} \succ \text{CO}_{h_a^{e,i}}, \\
h_a^{e,j} &: \{\{g_p^e, \bar{g}_q^e\} : p, q \in [3]\} \succ \{\{a_p^j, \alpha_1^e\} : p \leq r_j\} \succ \{f^e, \bar{f}^e\} \succ \text{CO}_{h_a^{e,j}}, \\
h_b^{e,i} &: \{\{g_p^e, \bar{g}_q^e\} : p, q \in [3]\} \succ \{\{b_p^i, \alpha_1^e\} : p < n + 1 - r_i\} \succ \{f^e, \bar{f}_2^e\} \succ \text{CO}_{h_b^{e,i}}, \\
h_b^{e,j} &: \{\{g_p^e, \bar{g}_q^e\} : p, q \in [3]\} \succ \{\{b_p^j, \alpha_1^e\} : p < n + 1 - r_j\} \succ \{f^e, \bar{f}_2^e\} \succ \text{CO}_{h_b^{e,j}}, \\
g_{1/2/3}^e &: \{\{h_a^{e,i}, \bar{g}_p^e\} : p \in [3]\} \succ \{\{h_b^{e,i}, \bar{g}_p^e\} : p \in [3]\} \succ \{\{h_a^{e,j}, \bar{g}_p^e\} : p \in [3]\} \succ \{\{h_b^{e,j}, \bar{g}_p^e\} : p \in [3]\} \succ \text{CO}_{g_{1/2/3}^e}, \\
\bar{g}_{1/2/3}^e &: \{\{h_b^{e,j}, g_p^e\} : p \in [3]\} \succ \{\{h_a^{e,j}, g_p^e\} : p \in [3]\} \succ \{\{h_b^{e,i}, g_p^e\} : p \in [3]\} \succ \{\{h_a^{e,i}, g_p^e\} : p \in [3]\} \succ \text{CO}_{\bar{g}_{1/2/3}^e}, \\
f^e &: \{h_a^{e,i}, \bar{f}^e\} \succ \{h_b^{e,i}, \bar{f}^e\} \succ \{h_a^{e,j}, \bar{f}^e\} \succ \{h_b^{e,j}, \bar{f}^e\} \succ \text{CO}_{f^e}, \\
\bar{f}^e &: \{h_b^{e,j}, f^e\} \succ \{h_a^{e,j}, f^e\} \succ \{h_b^{e,i}, f^e\} \succ \{h_a^{e,i}, f^e\} \succ \text{CO}_{\bar{f}^e}, \\
\alpha_1^e &: \{\{a_p^i, h_a^{e,i}\}, \{b_p^i, h_a^{e,i}\}, \{a_p^j, h_a^{e,j}\}, \{b_p^j, h_a^{e,j}\} : p \in [n]\} \succ \{\alpha_2^e, \alpha_3^e\} \succ \text{CO}_{\alpha_1^e}, \\
\alpha_2^e &: \{\alpha_1^e, \alpha_3^e\} \succ \text{CO}_{\alpha_2^e}, \\
\alpha_3^e &: \{\alpha_1^e, \alpha_2^e\} \succ \text{CO}_{\alpha_3^e}.
\end{aligned}$$

See Figure 2 for the 3-sets which are before the cut-off gadget for all agents they contain.

Furthermore, we extend the preferences of a_k^ℓ by inserting the 2-set $\{h_a^{e,\ell}, \alpha_1^e\}$ directly after $\{c_{n-1}^\ell, \bar{c}_{n-1}^\ell\}$, and the preferences of b_k^ℓ by the 2-set $\{h_b^{e,\ell}, \alpha_1^e\}$ directly after $\{c_{n-1}^\ell, \bar{c}_{n-1}^\ell\}$.

Lemma 11. *If for a matching M and an edge $e = \{v_r^i, v_s^j\}$ the vertex-selection gadgets A^i and A^j select the vertices v_r^i and v_s^j , then this matching M contains a blocking 3-set.*

Vice versa, given a matching inside the vertex-selection gadgets such that for no edge, both endpoints are selected by a vertex-selection gadget, this matching can be extended to the edge gadget without introducing a blocking 3-set.

Proof. Let M be a matching such that A^i selects the vertex v_r^i and A^j selects the vertex v_s^j , and let $e = \{v_r^i, v_s^j\}$. We need to show that any extension M' of M to the edge gadget contains a blocking 3-set.

By the cut-off gadgets for f^e and $\bar{f}^e$, we know that M' must contain a 2-set $\{h_x^{e,k}, f^e, \bar{f}^e\}$ for some $k \in \{i, j\}$ and $x \in \{a, b\}$. We assume $k = i$ and $x = a$, the other cases are symmetric. We claim that $\{a_r^i, h_a^{e,i}, \alpha_1^e\}$ is a blocking 3-set. Agent a_r^i prefers $\{h_a^{e,i}, \alpha_1^e\}$ over $M(a_r^i) = \{d_1^i, \bar{d}_1^i\}$. Also agent $h_a^{e,i}$ prefers $\{a_r^i, \alpha_1^e\}$ over $M'(h_a^{e,i}) = \{f^e, \bar{f}^e\}$ as v_r^i is an endpoint of e . Note that agents $h_x^{e,k}$ are matched to $\{g_p^e, \bar{g}_q^e\}$ for $(k, x) \neq (i, a)$ due to the cut-off gadgets for g_p^e . Thus, we have $M'(\alpha_1^e) = \{\alpha_2^e, \alpha_3^e\}$, and therefore, α_1^e prefers $\{a_r^i, h_a^{e,i}\}$ over M' .

To see the second part of the lemma, let M be a matching such that the vertices selected by the vertex selection gadgets are $v_{i^*}^i$ and $v_{j^*}^j$. Let the edge gadget correspond to the edge $\{v_r^i, v_s^j\}$.

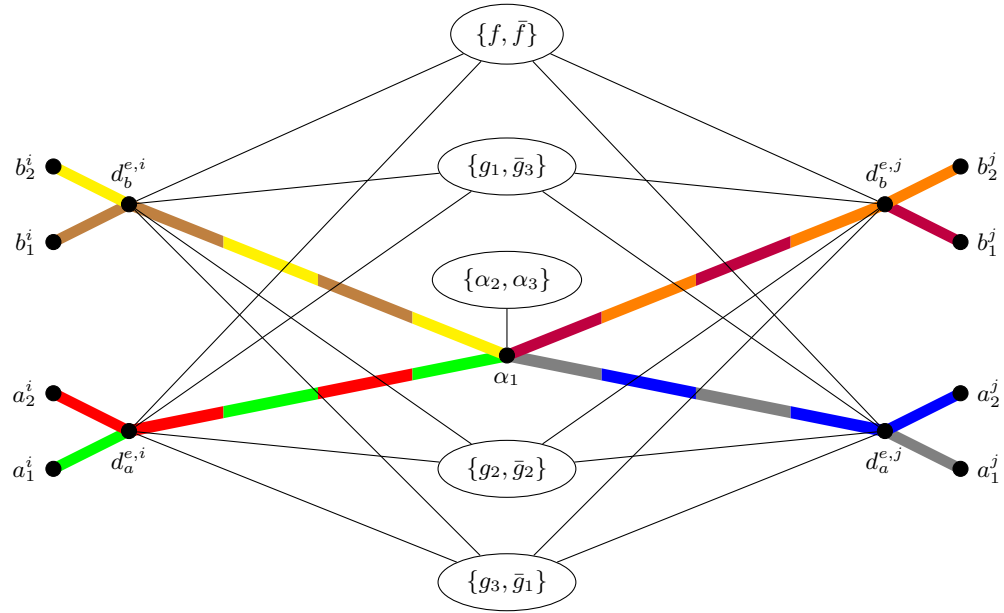


Fig. 2. The acceptable 3-sets (i.e., 3-sets which are preferred over the cut-off gadget by all agents they contain) of the edge gadget for the edge $\{v_1^i, v_2^j\}$. Acceptable 3-sets are drawn in two ways: If they contain one of the circles or the ellipse, then they consist of the two agents inside the circle and the other endpoint of an edge incident to the cycle. Otherwise, they are marked by the bold colored paths of length two; if a 2-set of agents is contained in two different edges (all such 3-sets contain α_1), then the corresponding edge is colored alternating in the two colors.

We assume that $r < i^*$ ($i^* > r$ is symmetric by switching the roles of a and b), and $s \leq j^*$ (again $s \geq j^*$ is symmetric by switching the roles of a and b).

We extend M to the edge gadget by the 3-sets $\{h_a^{e,i}, f^e, \bar{f}^e\}$, $\{h_b^{e,i}, g_1^e, \bar{g}_3^e\}$, $\{h_a^{e,j}, g_2^e, \bar{g}_2^e\}$, $\{h_b^{e,j}, g_3^e, \bar{g}_1^e\}$, and $\{\alpha_1^e, \alpha_2^e, \alpha_3^e\}$. It remains to show that this does not lead to a blocking 3-set. First, note that none of the agents g_p^e , $\bar{g}_p^e$, f^e , and $\bar{f}^e$ is contained in a blocking 3-set. Therefore, the agents $h_b^{e,i}$, $h_a^{e,j}$, and $h_b^{e,j}$ are not part of a blocking 3-set. The only remaining possible blocking 3-set is $\{a_p^i, \alpha_1^e, h_a^{e,i}\}$ for $p \leq r < i^*$. However, a_p^i prefers M over $\{\alpha_1^e, h_a^{e,i}\}$. $\square$

Proof of the Forward Direction

Lemma 12. *If G contains a multicolored independent set I , then there exists a stable matching.*

Proof. Each vertex-selection gadget A^i selects a vertex from $I \cap V^i$. This matching is extended as described in Lemmas 9 and 11. Due to Lemmas 9 to 11 we immediately get the stability of the constructed matching. $\square$

Proof of the Backward Direction

Lemma 13. *If there exists a stable matching, then G contains a multicolored independent set.*

Proof. By Lemma 10, every vertex-selection gadget selects a vertex. By Lemma 11, no two selected vertices are adjacent. Thus, the selected vertices form a multicolored independent set. $\square$

The parameter It remains to show that the preferences of the constructed MDSR-POSET-instance can be derived from a poset of bounded width.

The poset looks as follows. For each vertex-selection gadget A^i , there are six chains of the poset. One chain contains all agents $\{a_j^i : j \in [n]\}$, one the agents $\{b_j^i : j \in [n]\}$, one the agents $\{c_j^i : j \in [n-1]\}$, one the agents $\{\bar{c}_j^i : j \in [n-1]\}$, one the agents $\{d_j^i : j \in [n+1]\}$, and one the agents $\{\bar{d}_j^i : j \in [n+1]\}$. All of these six chains are ordered by j .

For each $i \neq j \in [k]$, the agents from the edge gadgets for all edges from $E^{i,j}$ are contained in 15 posets, each one containing all agents of the form $d_a^{e,i}$, one of the form $d_b^{e,i}$, and so on. We fix an arbitrary order of the edges from $E^{i,j}$, and order the agents inside the chains by the order of the edges in which edge gadget they belong.

Finally, we add the agents from the cut-off gadgets to the master poset. We group the agents from the cut-off gadgets into six chains, where again one chain contains the agents of one form. The master poset orders the agents from one chain by the order the corresponding cut-off gadgets have been added to the instance.

We arrive at the following observation.

Observation 7 *The preferences are derived from a poset of width at most $O(k^2)$.*

We now have proven Theorem 4.

B.6 Proof of Theorem 5

MDSRI

Input: A set V of agents together with preference lists $\succ_v$ over X_v for a subset $X_v \subseteq \binom{V \setminus \{v\}}{d-1}$ for each $v \in V$.

Task: Decide whether a stable matching exists.

MDSRI-ML

Input: An MDSRI instance, and a total order $\succ_{\text{ML}}$ of the agents (called *master list*) such that for each agent, $\succ_v$ arises from $\succ_{\text{ML}}$ through the deletion of some $(d-1)$ -sets.

Task: Decide whether there exists a stable matching.

PERFECT-SMTI-ML

Input: A STABLE MARRIAGE WITH TIES AND INCOMPLETE PREFERENCES instance, where the preferences are derived from two master lists $\succ_m$ and $\succ_w$ (one for the men, and one for the women), in which the women's master list is strictly ordered and the maximum length of a tie in the men's master list is two.

Task: Decide whether there exists a perfect stable matching.

For the rest of this section, we fix a PERFECT-SMTI-ML instance $\mathcal{I} = (G, \succ_m, \succ_w)$, where G is the acceptability graph (i.e., the graph where each agent is a vertex, and two agents are connected by an edge if and only if they find each other mutually acceptable), and $\succ_m$ and $\succ_w$ are the master list of men and women, respectively. We denote the set of men by U , and the set of women by W . Let $W = \{w_1, \dots, w_{|W|}\}$ such that $w_i \succ_m w_{i+1}$ or $w_i \sim_m w_{i+1}$ for all $i \in [|W| - 1]$ and let $U = \{m_1, \dots, m_{|U|}\}$ such that $m_i \succ_w m_{i+1}$ for all $i \in [|U| - 1]$.

The reduction adds an agent a_i for each m_i , and an agent b_j for each w_j . If (m_i, w_j) is an acceptable pair, then we also add an agent $c_{i,j}$, and the 3-set $\{a_i, b_j, c_{i,j}\}$ is acceptable. The preferences of a_i and b_j correspond to those of m_i and w_j , i.e., a_i prefers $\{b_j, c_{i,j}\}$ over $\{b_{j'}, c_{i,j'}\}$ if m_i prefers w_j over $w_{j'}$, and b_j prefers $\{a_i, c_{i,j}\}$ over $\{a_{i'}, c_{i',j}\}$ if and only if w_j prefers m_i over $m_{i'}$. If m_i ties two women w_j and w_{j+1} , then this is modeled by a tie gadget. Finally, there

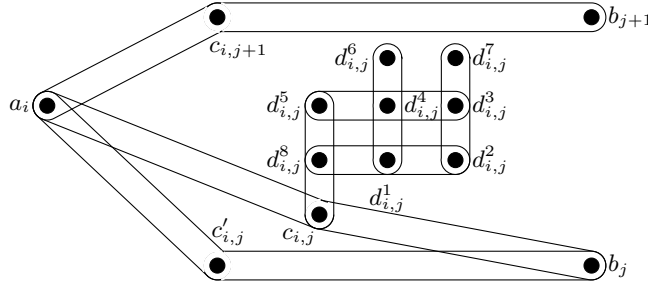


Fig. 3. The acceptable 3-sets of a tie gadget T_i^j . For example, the line around a_i , $c_{i,j+1}$, and b_{j+1} indicates that the 3-set $\{a_i, c_{i,j+1}, b_{j+1}\}$ is acceptable.

is a cut-off gadget, which ensures that any agent a_i is matched in any stable matching.

Tie gadget. Given a man $m_i \in M$ who ties two women w_j and w_{j+1} , we construct a tie gadget T_i^j as follows. This gadget models this tie, i.e., it allows m_i to be matched to w_j or w_{j+1} . The idea is the following: There are two stable matchings inside the gadget, one leaving $c_{i,j}$ unmatched while the other matches $c_{i,j}$. The first one allows to match m_i to w_j via the 3-set $\{a_i, b_j, c_{i,j}\}$, while the second allows to match m_i to w_{j+1} via $\{a_i, b_{j+1}, c_{i,j+1}\}$ (note that in this case $c_{i,j}$ prevents the 3-set $\{a_i, b_j, c_{i,j}\}$ from being blocking). In this case, the 3-set $\{a_i, b_j, c'_{i,j}\}$ ensures if $\{a_i, b_{j+1}, c_{i,j+1}\}$ is not part of the matching, then the 3-set $\{a_i, b_j, c'_{i,j}\}$ can be blocking to represent the possibly blocking pair (m_i, w_j) .

We add nine agents $c'_{i,j}$ and $d_{i,j}^1, \dots, d_{i,j}^8$, together with the acceptable 3-sets $\{a_i, c_{i,j}, b_j\}$, $\{a_i, c_{i,j}, b_{j+1}\}$, $\{a_i, c'_{i,j}, b_{j+1}\}$, $\{c_{i,j}, d_{i,j}^5, d_{i,j}^8\}$, $\{d_{i,j}^1, d_{i,j}^2, d_{i,j}^8\}$, $\{d_{i,j}^1, d_{i,j}^4, d_{i,j}^6\}$, $\{d_{i,j}^2, d_{i,j}^3, d_{i,j}^7\}$, and $\{d_{i,j}^3, d_{i,j}^4, d_{i,j}^5\}$. See Figure 3 for an example.

The preferences of any agent arise from the following preferences through the deletion of all 2-sets which are not acceptable for an agent. $\{d_{i,j}^1, d_{i,j}^2\} \succ \{d_{i,j}^1, d_{i,j}^4\} \succ \{d_{i,j}^2, d_{i,j}^3\} \succ \{d_{i,j}^3, d_{i,j}^4\} \succ \{d_{i,j}^1, d_{i,j}^6\} \succ \{d_{i,j}^3, d_{i,j}^5\} \succ \{d_{i,j}^4, d_{i,j}^5\} \succ \{d_{i,j}^2, d_{i,j}^7\} \succ \{d_{i,j}^3, d_{i,j}^7\} \succ \{d_{i,j}^1, d_{i,j}^8\} \succ \{d_{i,j}^2, d_{i,j}^8\} \succ \{d_{i,j}^4, d_{i,j}^6\} \succ \{d_{i,j}^5, d_{i,j}^8\} \succ \{d_{i,j}^5, c_{i,j}\} \succ \{d_{i,j}^8, c_{i,j}\} \succ \{a_i, c_{i,j}\} \succ \{a_i, c_{i,j+1}\} \succ \{a_i, c'_{i,j}\} \succ \{b_j, c_{i,j}\} \succ \{b_{j+1}, c_{i,j+1}\} \succ \{b_j, c'_{i,j}\}$, which can be derived from the following strict order of agents: $d_{i,j}^1 \succ d_{i,j}^2 \succ \dots \succ d_{i,j}^8 \succ a_i \succ b_j \succ b_{j+1} \succ c_{i,j} \succ c_{i,j+1} \succ c'_{i,j}$.

We now observe that the tie gadget indeed models ties, i.e., it contains a stable matching which matches m_i to w_j and one which matches m_i to w_{j+1} . Furthermore, it is also possible to match m_i or w_j and w_{j+1} better than w_j or m_i .

Observation 8 Let T_i^j be a tie gadget. The matchings $M_1 = \{\{a_i, c_{i,j+1}, b_{j+1}\}, \{c_{i,j}, d_{i,j}^5, d_{i,j}^8\}, \{d_{i,j}^2, d_{i,j}^3, d_{i,j}^7\}, \{d_{i,j}^1, d_{i,j}^4, d_{i,j}^6\}\}$ and $M_2 = \{\{a_i, c_{i,j}, b_j\}, \{d_{i,j}^1, d_{i,j}^2, d_{i,j}^8\}, \{d_{i,j}^3, d_{i,j}^4, d_{i,j}^5\}\}$ are stable. In $T_i^j - \{a_i\}$ or $T_i^j - \{b_j, b_{j+1}\}$, also the matching $M = \{\{c_{i,j}, d_{i,j}^5, d_{i,j}^8\}, \{d_{i,j}^2, d_{i,j}^3, d_{i,j}^7\}, \{d_{i,j}^1, d_{i,j}^4, d_{i,j}^6\}\}$ is stable.

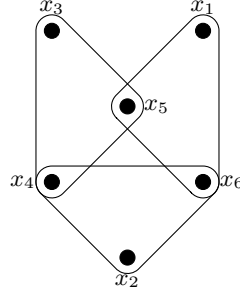


Fig. 4. The acceptable 3-sets of a cut-off gadget.

Cut-off gadget A cut-off gadget has six agents $x_1, \dots, x_6$. The only acceptable 3-sets are $\{x_1, x_5, x_6\}$, $\{x_2, x_4, x_6\}$, and $\{x_3, x_4, x_5\}$. See Figure 4 for an example.

The master list is $\{x_2, x_4\} \succ \{x_1, x_5\} \succ \{x_1, x_6\} \succ \{x_3, x_4\} \succ \{x_3, x_5\} \succ \{x_2, x_6\} \succ \{x_4, x_5\} \succ \{x_4, x_6\} \succ \{x_5, x_6\}$. Note that this list can be derived from $x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5 \succ x_6$.

We now observe that a cut-off gadget does not admit a stable matching, implying that one of the agents has to be matched outside the gadget. Thus, if we add the cut-off gadget for an agent v , then this means that we add the agents $x_2, \dots, x_6$ and identify v with x_1 , ensuring that any stable matching matches v .

Lemma 14. *A cut-off gadget does not admit a stable matching.*

Proof. Note that any acceptable 3-set contains two agents from $\{x_4, x_5, x_6\}$. Thus, any matching contains only one 3-set.

For each of the three possible matchings, we give a blocking 3-set in Table 3.

□

Matching	Blocking 3-set
$\{x_1, x_5, x_6\}$	$\{x_2, x_4, x_6\}$
$\{x_2, x_4, x_6\}$	$\{x_3, x_4, x_5\}$
$\{x_3, x_4, x_5\}$	$\{x_1, x_5, x_6\}$

Table 3. The blocking 2-sets in the subinstance from Lemma 5.

We observe that if the agent v is matched, then the cut-off gadget does not contain a blocking 3-set.

Observation 9 *The cut-off gadget without x_1 admits a stable matching, namely $\{x_3, x_4, x_5\}$.*

The reduction Our reduction is structured similarly to the NP-completeness proof of 3-DIMENSIONAL STABLE MARRIAGE WITH INCOMPLETE CYCLIC PREFERENCES by Biró and McDermid [4]. In both reductions, there is one agent for each man and each woman. Each such agent is forced to be matched in any stable matching by a gadget based on a small unsolvable instance. However, modelling the ties in the preferences is a bit more complicated in our case, and is done by the tie gadget described in Section 4.4.

We construct a 3-DSRI-ML instance $\mathcal{I}'$ with a strict master list from $\mathcal{I}$ as follows.

For each men m_i , we add an agent a_i , and for each women w_j , we add an agent b_j . For each man m_i , and each woman w_j who is not tied with another woman in m_i ' preference list, we add an agent $c_{i,j}$. For each man m_i , and each tie $w_j \sim w_{j+1}$ in m_i 's preference list, we add a tie gadget T_i^j (which is described in Section 4.4).

For each agent a_i , we define a sublist $\mathcal{A}_i$ as follows. Process all woman w_j adjacent to m_i by increasing j . If the woman is not tied with another woman adjacent to m_i , then add the 2-set $\{a_i, c_{i,j}\}$, followed by $\{b_j, c_{i,j}\}$. Otherwise, w_j is tied with w_{j+1} in the preference list of m_i . Then add the 2-sets $\{a_i, c_{i,j}\} \succ_{\text{ML}} \{a_i, c_{i,j+1}\} \succ_{\text{ML}} \{a_i, c'_{i,j}\} \succ_{\text{ML}} \{b_j, c_{i,j}\} \succ_{\text{ML}} \{b_j, c_{i,j+1}\} \succ_{\text{ML}} \{b_j, c'_{i,j}\}$ to $\mathcal{A}_i$ (see Section 4.4).

For each man m_i , we add a cut-off gadget I_i (described in Section 4.4), where the vertex x_1 of the cut-off gadget is identified with a_i .

The master list looks as follows. First, we add for each tie gadget T_i^j (ordered by $i|U| + j$) the 2-sets $\{d_{i,j}^1, d_{i,j}^6\} \succ_{\text{ML}} \{d_{i,j}^3, d_{i,j}^5\} \succ_{\text{ML}} \{d_{i,j}^4, d_{i,j}^2\} \succ_{\text{ML}} \{d_{i,j}^7, d_{i,j}^8\} \succ_{\text{ML}} \{d_{i,j}^1, d_{i,j}^8\} \succ_{\text{ML}} \{d_{i,j}^2, d_{i,j}^8\} \succ_{\text{ML}} \{d_{i,j}^4, d_{i,j}^6\} \succ_{\text{ML}} \{d_{i,j}^5, c_{i,j}\} \succ_{\text{ML}} \{d_{i,j}^8, c_{i,j}\}$. This is followed by the sublists $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_{|U|}$. Then, all 2-sets containing agents from cut-off gadgets follow, in a way such that the relative order between two 2-sets containing agents from the same cut-off gadget coincides with the order given in Section 4.4.

Observation 10 *The master list $\succ_{\text{ML}}$ is derived from a strict order.*

Proof. The master list is derived from the following order. At the beginning are the agents $d_{i,j}^k$ from the tie gadgets, in an arbitrary order satisfying $d_{i,j}^k \succ d_{i,j}^{k'}$ if $k < k'$. Then, the agents a_i follow, ordered increasingly by i . Then, the agents b_i follow, ordered increasingly by i . Afterwards, the agents $c_{i,j}$ come, ordered in such a way that $c_{i,j}$ is before $c_{i',j'}$ if and only if either $i < i'$ or $i = i'$ and $j < j'$. The agents $c'_{i,j}$ are inserted directly after $c_{i,j+1}$. Finally, the agents from the cut-off gadgets I_i for $1 \leq i \leq n$ come, in the order specified in Section 4.4. $\square$

Proof of the forward direction

Lemma 15. *If the PERFECT-SMTI-ML instance $\mathcal{I}$ admits a perfect stable matching M , then the MDSRI-ML instance $\mathcal{I}'$ admits a stable matching.*

Proof. We construct a stable matching M' as follows.

For each edge $(m_i, w_j) \in M$, we add the 3-set $\{a_i, b_j, c_{i,j}\}$ to M' . If w_j is tied with another woman w_ℓ , then we add the 3-sets $\{c_{i,\ell}, d_{i,j}^5, d_{i,j}^8\}$, $\{d_{i,j}^2, d_{i,j}^3, d_{i,j}^7\}$, and $\{d_{i,j}^1, d_{i,j}^4, d_{i,j}^6\}$ if $\ell = j - 1$, and the 3-sets $\{d_{i,\ell}^1, d_{i,\ell}^2, d_{i,\ell}^8\}$, and $\{d_{i,\ell}^3, d_{i,\ell}^4, d_{i,\ell}^5\}$ if $\ell = j + 1$.

For each tie gadget between m_i , w_j and w_{j+1} such that both $(m_i, w_j) \notin M$ and $(m_i, w_{j+1}) \notin M$ hold, we add the 3-sets $\{c_{i,j}, d_{i,j}^5, d_{i,j}^8\}$, $\{d_{i,j}^2, d_{i,j}^3, d_{i,j}^7\}$, and $\{d_{i,j}^1, d_{i,j}^4, d_{i,j}^6\}$.

For each cut-off gadget, we add the edge $\{x_3, x_4, x_5\}$ to M' .

We claim that M' is a stable matching.

Since M is perfect, every agent a_i is matched to a 2-set it prefers over any 2-set of agents from its cut-off gadget. By Observation 9, we get that no agent from $I_i \setminus \{a_i\}$ participates in a blocking 3-set. For all tie gadgets T_i^j , Observation 8 tells us that no blocking 3-set contains only agents of the form $d_{i,j}^k$. Any other acceptable 3-sets contains an agent a_i as well as an agent b_j , and a_i prefers such a 3-set over $M(a_i) = \{w_\ell, c_{i,\ell}\}$ if and only if m_i prefers w_j over $M(m_i)$ or if $\ell = j - 1$. However, if $\ell = j - 1$ and $(m_i, w_j) \notin M$, then we have $\{c_{i,\ell}, d_{i,j}^5, d_{i,j}^8\} \in M'$, and thus $c_{i,\ell}$ does not prefer $\{a_i, b_\ell\}$ over M' . The agent b_j prefers this 3-set over $M'(b_j)$ if and only if w_j prefers a_i over $M(w_j)$. Thus, by the stability of M , no blocking 3-set contains an agent a_i , and therefore, no blocking 3-set exists.

Altogether, M' is stable. $\square$

Proof of the backward direction

Lemma 16. *If the MDSRI-ML instance $\mathcal{I}'$ admits a stable matching M' , then the PERFECT-SMTI-ML instance $\mathcal{I}$ admits a perfect stable matching.*

Proof. Let M' be such a stable matching in $\mathcal{I}'$. By Lemma 14, each agent a_i has to be matched to a 2-set outside its cut-off gadget. Any such 2-set involves an agent b_j . Thus, this defines a perfect matching $M := \{(m_i, w_j) : \exists v \text{ s.t. } \{a_i, b_j, v\} \in M'\}$.

We claim that M is stable. Assume that M admits a blocking pair (m_i, w_j) . If w_j is not tied with $M(m_i)$ by m_i , then $\{a_i, b_j, c_{i,j}\}$ is a blocking 3-set (a_i and b_j prefer this 3-set as (m_i, w_j) is blocking, and $c_{i,j}$ as it is the only acceptable 3-set for $c_{i,j}$). If m_i ties w_j with a woman w_ℓ , then M cannot contain one of the edges (m_i, w_j) or (m_i, w_ℓ) (as else (m_i, w_j) was not blocking). Thus, M' does not contain the 3-set $t := \begin{cases} \{a_i, b_j, c_{i,j}\} & \text{if } \ell = j - 1 \\ \{a_i, b_j, c'_{i,j}\} & \text{if } \ell = j + 1 \end{cases}$. Since $c_{i,j}$ if $\ell = j - 1$ or $c'_{i,j}$ if $\ell = j + 1$ is unmatched, and (m_i, w_j) is a blocking pair, we get that t is a blocking 3-set, contradicting the stability of M' . $\square$

The problem is clearly in **NP**, as we can check the stability of a matching in $O(n^3)$ time. The reduction can clearly be performed in linear time, and its correctness is proven in Lemmas 15 and 16. Observation 10 shows that the master list is strict.

C Missing Proofs of Section 4.4

C.1 MDSR-Poset Example

Example 4. Consider a MDSR-POSET instance $\mathcal{I}$ with the following preferences.

$$\begin{aligned} v_1 &: \{v_2, v_3\} \succ \{v_3, v_4\} \succ \{v_2, v_5\} \succ \{v_2, v_4\} \succ \{v_3, v_5\} \succ \{v_4, v_5\}, \\ v_2 &: \{v_1, v_3\} \succ \{v_3, v_4\} \succ \{v_1, v_5\} \succ \{v_1, v_4\} \succ \{v_3, v_5\} \succ \{v_4, v_5\}, \\ v_3 &: \{v_1, v_2\} \succ \{v_1, v_4\} \succ \{v_2, v_5\} \succ \{v_2, v_4\} \succ \{v_1, v_5\} \succ \{v_4, v_5\}, \\ v_4 &: \{v_1, v_5\} \succ \{v_1, v_2\} \succ \{v_2, v_5\} \succ \{v_1, v_3\} \succ \{v_3, v_5\} \succ \{v_2, v_3\}, \\ v_5 &: \{v_2, v_3\} \succ \{v_3, v_4\} \succ \{v_1, v_2\} \succ \{v_2, v_4\} \succ \{v_1, v_3\} \succ \{v_1, v_4\}. \end{aligned}$$

After the deletion of v_5 , the preferences are

$$\begin{aligned} v_1 &: \{v_2, v_3\} \succ \{v_3, v_4\} \succ \{v_2, v_4\}, \\ v_2 &: \{v_1, v_3\} \succ \{v_3, v_4\} \succ \{v_1, v_4\}, \\ v_3 &: \{v_1, v_2\} \succ \{v_1, v_4\} \succ \{v_2, v_4\}, \\ v_4 &: \{v_1, v_2\} \succ \{v_1, v_3\} \succ \{v_2, v_3\}. \end{aligned}$$

which is derived from $v_1 \succ v_2 \succ v_3 \succ v_4$. Hence, it holds that $\lambda(\mathcal{I}) = 1$.

C.2 Proof of Theorem 6

In this section, we show that 3-DSR parameterized by $\lambda(\mathcal{I})$ is W[1]-hard. We give a parameterized reduction from MULTICOLORED CLIQUE, which is known to be W[1]-complete [13].

MULTICOLORED CLIQUE

Input: A k -partite graph $G = (V^1 \dot{\cup} V^2 \dot{\cup} \dots \dot{\cup} V^k, E)$.

Task: Decide whether G contains a clique C with $C \cap V^i \neq \emptyset$ for all $i \in [k]$.

The sets $V^1, \dots, V^k$ are called *color classes*. By adding vertices and edges, we may assume without loss of generality that there exists some $n' \in \mathbb{N}$ such that $|V^i| = 3n' + 1$ for all $i \in [k]$, and that there exists some $m' \in \mathbb{N}$ such that $|E(V^i, V^j)| = 3m' + 1$ for all $i, j \in [k]$. For each color class V^i , we fix an arbitrary order of the vertices, i.e., $V^i = \{v_1^i, v_2^i, \dots, v_{3n'+1}^i\}$. For a vertex $v \in V(G)$, we denote by $\delta(v)$ the set of all edges incident to v .

When describing the reduction, we only describe the beginning of the preferences of an agent. The remaining acceptable 2-sets can be added in an arbitrary way obeying the strict master list (extended to the $\lambda(\mathcal{I})$ agents not contained in this list).

We begin by describing the gadgets used in the reduction.

Cut-off gadget In Section 3, we have seen

If we add a cut-off gadget for an agent v and a 2-set $\{x, y\}$ (with $v \notin \{x, y\}$), then we add six agents b^v, c^v, d^v, e^v, f^v , and g^v . The preferences of b^v, c^v, d^v, e^v, f^v begin with the same preferences as b, c, d, e , and f in Section 3 (where a is identified with v), i.e., the start of the preferences is derived from

$$\begin{aligned} \{v, b^v\} &\succ \{v, c^v\} \succ \{v, d^v\} \succ \{v, f^v\} \succ \{b^v, e^v\} \succ \{c^v, d^v\} \succ \{v, e^v\} \succ \{b^v, f^v\} \\ &\succ \{c^v, e^v\} \succ \{b^v, d^v\} \succ \{d^v, e^v\} \succ \{b^v, c^v\} \succ \{c^v, f^v\} \succ \{d^v, f^v\} \succ \{e^v, f^v\} \\ &\succ \{b^v, g^v\} \succ \{c^v, g^v\} \succ \{d^v, g^v\} \succ \{e^v, g^v\} \succ \{f^v, g^v\}. \end{aligned}$$

The preferences of g^v first contain all 2-sets of agents from $\{b^v, c^v, d^v, e^v, f^v\}$, and then all other 2-sets of agents in an arbitrary order.

The preferences of v look as follows.

$$\begin{aligned} v : & \text{old preferences until } \{x, y\} \\ & \succ \{b^v, e^v\} \succ \{c^v, d^v\} \succ \{b^v, f^v\} \succ \{c^v, e^v\} \succ \{b^v, d^v\} \\ & \succ \{d^v, e^v\} \succ \{b^v, c^v\} \succ \{c^v, f^v\} \succ \{d^v, f^v\} \succ \{e^v, f^v\} \\ & \succ \{b^v, g^v\} \succ \{c^v, g^v\} \succ \{d^v, g^v\} \succ \{e^v, g^v\} \succ \{f^v, g^v\} \\ & \succ \text{old preferences after } \{x, y\}. \end{aligned}$$

We now formally show that the cut-off gadget works as desired, i.e., forces v to be matched to a 2-set it does not prefer over $\{x, y\}$.

Lemma 17. *In any stable matching M , agent v does not prefer $\{x, y\}$ over $M(v)$.*

Proof. Let M be a stable matching. Note that the agents b^v, c^v, d^v, e^v , and f^v can only be matched to v or agents from $b^v, c^v, d^v, e^v, f^v, g^v$, as the first 2-sets of their preferences and the preferences of g^v only contain these agents.

Note that the subinstance on the agents v and b^v, c^v, d^v, e^v, f^v is the instance from Lemma 5 not admitting a stable matching.

Therefore, in any stable matching, v is not matched to agents from $b^v, c^v, d^v, e^v, f^v, g^v$, which can only be achieved if v does not prefer $\{x, y\}$ over $M(v)$. $\square$

Lemma 18. *Any matching M in which v does not prefer $\{x, y\}$ over $M(v)$ can be extended to $b^v, c^v, d^v, e^v, f^v, g^v$ without introducing a new blocking 3-set.*

Proof. We extend M by $\{b^v, c^v, d^v\}$ and $\{e^v, f^v, g^v\}$. Since v does not prefer $\{x, y\}$ over $M(v)$, this agent is not contained in any new blocking 3-set.

The agents b^v, c^v, d^v, e^v, f^v , and g^v can thus only be contained in a blocking 3-set if this 3-set consists of three of these agents.

However, b^v is matched to its most preferred 2-set not containing v and thus is not part of a blocking 3-set. Agent c^v only prefers only $\{b^v, f^v\}$ over $\{b^v, d^v\}$; however since b^v is not part of a blocking 3-set, c^v is also not part of a blocking 3-set. All 2-sets d^v prefers over $\{b^v, c^v\}$ contain v, b^v , or c^v , and thus, d^v is not part of a blocking 3-set. Thus, no agent from b^v, c^v , and d^v is blocking, and thus, no new blocking 2-sets are introduced. $\square$

If we add a cut-off gadget for an agent v and a 2-set $\{x, y\}$, then we write its preference list just until $\{x, y\}$, and then write CO to represent an arbitrary order obeying the master list of the remaining 2-sets (the exact order of these 2-sets is irrelevant for the stability of the matching).

Vertex-selection gadget For each color class V_i , we add a vertex selection gadget A_i . This vertex-selection gadget contains an agent for each vertex v_i^p with $p \in [3n' + 1]$; we identify the agent and the vertex and call both v_i^p . Furthermore, the vertex-selection gadget contains two agents s_i and s'_i . The vertex-selection gadget also contains a cut-off gadget for each of the agents s_i , s'_i , and $v_i^{3n'+1}$.

The preferences of an agent $v_i^p \in V_i$ start with $\{s_i, s'_i\} \succ (\{v_i^1\} \times \{v_i^q : q \in [3n' + 1]\}) \succ (\{v_i^2\} \times \{v_i^q : q \in [3n' + 1]\}) \succ \dots (\{v_i^{3n'+1}\} \times \{v_i^q : q \in [3n' + 1]\})$. These preferences need to be extended to the agents not contained in the vertex-selection gadget. This extension is done by adding all 2-sets containing an agent v_j^p with $j < i$ in front of the list, and adding all other 2-sets, in an arbitrary way obeying the master list described in Lemma 19.

For the agent $v_i^{3n'+1}$, a cut-off gadget follows; for the other agents, the preferences are extended in an arbitrary strict way.

The preferences of s_i and s'_i are as follows (the agents $x_{i,j}$, $x'_{i,j}$, a_e , and b_e will be introduced in the incidence-checking gadgets in Appendix C.2).

$$\begin{aligned} s_i: \{s'_i, v_i^1\} &\succ (\{a_e, b_e\} \times \{x_{i,j}\})_{j \neq i, e \in E(V_i, V_j) \cap \delta(v_i^1)} \succ \{s'_i, v_2\} \\ &\succ (\{a_e, b_e\} \times \{x_{i,j}\})_{j \neq i, e \in E(V_i, V_j) \cap \delta(v_i^2)} \succ (s'_i, v_3) \succ \dots \succ \{s'_i, v_{3n'+1}\} \succ \text{CO} \\ s'_i: \{s_i, v_{3n'+1}\} &\succ (\{a_e, b_e\} \times \{x_{i,j}\})_{j \neq i, e \in E(V_i, V_j) \cap \delta(v_i^{3n'+1})} \succ \{s_i, v_{3n'}\} \\ &\succ (\{a_e, b_e\} \times \{x_{i,j}\})_{j \neq i, e \in E(V_i, V_j) \cap \delta(v_i^{3n'})} \succ \{s_i, v_{3n'-1}\} \succ \dots \succ \{s_i, v_1\} \succ \text{CO} \end{aligned}$$

Incidence-checking gadget For each 2-set of color classes V_i and V_j with $i < j$, we add an incidence-checking gadget $B_{i,j}$. Let $E(V_i, V_j) = \{e_1, \dots, e_{3m'+1}\}$. For each edge $e_\ell \in E(V_i, V_j)$, it contains two agents a_ℓ and b_ℓ . Furthermore, there are $3m'$ agents c_ℓ . The gadget also contains seven agents $x_{i,j}$, $x'_{i,j}$, $x_{j,i}$, $x'_{j,i}$, $z_{i,j}$, $z'_{i,j}$ and $z''_{i,j}$, which do not have the master preferences, and an cut-off gadget for $z'_{i,j}$. Let $X := \{x_{i,j}, x'_{i,j}, x_{j,i}, x'_{j,i}\}$.

The preferences of a_ℓ (resp. b_ℓ) are as follows (deleting all 2-sets containing a_ℓ (resp. b_ℓ) and ignoring agents outside the incidence-checking gadget; the preferences can be extended in an arbitrary way complying with the master list). Any set of 2-sets in the preferences shall be replaced by an arbitrary order of the elements in this set which is consistent with the master list.

$$\begin{aligned} (\{s_i, s'_i, s_j, s'_j\} \times X) &\succ \binom{X}{2} \succ (X \times \{a_q : q \in [3m' + 1]\}) \succ (X \times \{b_q : q \in [3m' + 1]\}) \\ &\succ (\{a_q, b_q : q \in [\ell - 1]\} \times \{z_{i,j}\}) \succ \binom{\{a_q, b_q, c_q : q \in [\ell]\}}{2} \\ &\succ (\{a_q, b_q : q \in [\ell, 3m' + 1]\} \times \{z_{i,j}\}) \end{aligned}$$

Similarly as for the agents v_i^p , we presented only the relevant part of the preferences of a_ℓ and b_ℓ . These preferences need to be extended outside the incidence-checking gadget by adding all 2-sets involving agents from vertex-selection gadgets or incidence-checking gadgets $B_{i',j'}$ with $i' < i$ or $i' = i$ and $j' < j$ at the beginning of the preferences, and all other 2-sets in an arbitrary order obeying the master list described in Lemma 19.

The preferences of $z_{i,j}$ start as follows. $(\{a_\ell\} \times \{b_\ell\}) \succ \{z'_{i,j}, z''_{i,j}\}$.

The preferences of $z'_{i,j}$ start as follows. $\{z_{i,j}, z''_{i,j}\} \succ \text{CO}$.

The preferences of $z''_{i,j}$ start with the 2-set $\{z_{i,j}, z'_{i,j}\}$.

To describe the preferences of $x_{i,j}$ and $x'_{i,j}$, we define sublists C_ℓ and $\overleftarrow{C}_\ell$. The sublist C_ℓ contains the 2-sets $\{x'_{i,j}, a_e\}$ for all $e \in E(V_i, V_j) \cap \delta(v_i^\ell)$, ordered by increasing index of the endpoint of e contained in V_j . The sublist $\overleftarrow{C}_\ell$ contains the 2-sets $\{x_{i,j}, a_e\}$ for $e \in E(V_i, V_j) \cap \delta(v_i^\ell)$, but is ordered by **decreasing** index of the endpoint of e contained in V_j . The preferences of $x_{i,j}$ and $x'_{i,j}$ look as follows.

$x_{i,j}: \{s_i, v_i^{3n'+1}\} \succ C_{3n'+1} \succ \{s_i, v_i^{3n'}\} \succ C_{3n'} \succ \dots \succ \{s_i, v_i^1\} \succ C_1 \succ \text{rest in arbitrary order.}$

$x'_{i,j}: \{s'_i, v_i^1\} \succ \overleftarrow{C}_1 \succ \{s'_i, v_i^2\} \succ \overleftarrow{C}_2 \succ \dots \succ \{s'_i, v_i^{3n'+1}\} \succ \overleftarrow{C}_{3n'+1} \succ \text{rest in arbitrary order.}$

For the vertices $x_{j,i}$ and $x'_{j,i}$, the preferences are identical to those of $x_{i,j}$ and $x'_{i,j}$, except that in each 2-set containing an agent a_ℓ , this agent is replaced by b_ℓ , and in each 2-set containing an agent v_i^ℓ , this agent is replaced by v_j^ℓ .

The reduction Given an instance (G, k) of MULTICOLORED CLIQUE, we design an MDSR-instance $\mathcal{I}'$ as follows.

The instance contains k vertex-selection gadgets A_i , one for each color class V_i . Between each pair $\{A_i, A_j\}$ of vertex-selection gadgets, there is an incidence-checking gadget $B_{i,j}$.

We now show that our parameter λ is indeed small for the constructed instance $\mathcal{I}'$.

Lemma 19. *We have $\lambda(\mathcal{I}') = O(k^2)$.*

Proof. We delete the agents $s_i, x_{i,j}, x_{i,j'}, z_{i,j}, z'_{i,j}, z''_{i,j}$ for $i \neq j \in [k]$, and all agents contained in cut-off gadgets. Note that there are $O(k^2)$ such agents.

It remains to show that the remaining preferences are derived from a strict master list. The master list looks as follows. It starts with all agents v_i^p , ordered in such a way that agent v_i^p is before agent $v_{i'}^{p'}$ if and only if either $i < i'$ or $i = i'$ and $p < p'$. Then, the agents from the edge-selection gadgets follow. We add all agents from one edge-selection gadget at once, and the edge-selection gadgets $B_{i,j}$, ordered in such a way that $B_{i,j}$ is before $B_{i',j'}$ if and only if either $i < i'$ or $i = i'$ and $j < j'$. The agents inside an edge-selection gadget are ordered as follows. $a_1 \succ b_1 \succ c_1 \succ a_2 \succ b_2 \succ c_2 \succ \dots \succ c_{3m'} \succ a_{3m'+1} \succ b_{3m'+1}$. $\square$

Proof of the forward direction We prove that if G contains a clique of size k , then $\mathcal{I}'$ admits a stable matching. So let $\{v_1^{p_1}, \dots, v_k^{p_k}\}$ be a multicolored clique. We construct a stable matching M as follows.

For the vertex-selection gadget A_i , we add the 3-set $\{v_i^{p_i}, s_i, s'_i\}$. All other vertex agents v_i^q are matched to each other, according to their index q (i.e., matching the three agents with lowest q together, then the next three agents, and so on). For each edge $e_\alpha = \{v_i^{p_i}, v_j^{p_j}\}$ with $i < j$ from the clique, we add $\{x_{i,j}, x'_{i,j}, a_\alpha\}$ and $\{x_{j,i}, x'_{j,i}, b_\alpha\}$ to M . Furthermore, we add the edges $\{a_\ell, b_\ell, c_\ell\}$ for $\ell < \alpha$, and $\{a_\ell, b_\ell, c_{\ell-1}\}$ for $\ell > \alpha$. The agents from the cut-off gadgets are matched as described in Lemma 18. We call the resulting matching M .

It remains to show that M is stable. In order to do so, we will show that no agent is part of a blocking 3-set.

Lemma 20. *No agent s'_i is part of a blocking 3-set.*

Proof. All 2-sets which s'_i ranks better than $\{s_i, v_i^{p_i}\}$ are of the form $\{s_i, v_i^\ell\}$ for $\ell > p_i$, or $\{x'_{i,j}, a_\ell\}$, or $\{x'_{i,j}, b_\ell\}$ for an edge $e_\ell \in E(V_i, V_j)$ whose endpoint in V_i is v_i^q with $q > r$.

For the 2-sets $\{s_i, v_i^\ell\}$, note that s_i does not prefer $\{s'_i, v_i^\ell\}$ over $\{s'_i, v_i^{p_i}\}$.

For the 2-sets $\{x'_{i,j}, a_\ell\}$ and $\{x'_{i,j}, b_\ell\}$, note that $x'_{i,j}$ does not prefer $\{a_\ell, s'_i\}$ or $\{b_\ell, s'_i\}$ over $\{x'_{i,j}, a_{e^*}\}$ for the edge $e^* = \{v_i^{p_i}, v_j^{p_j}\}$ as $q > r$.

Thus, s'_i is not part of a blocking 3-set. $\square$

Lemma 21. *No vertex s_i is part of a blocking 3-set.*

Proof. The proof works in complete analogy to the proof of Lemma 20. $\square$

Lemma 22. *No agent $x_{i,j}$ or $x'_{i,j}$ is part of a blocking 3-set.*

Proof. A blocking 3-set cannot contain s_i, s'_i, s_j , or s'_j by Lemmas 20 and 21.

Thus, it is of the form $\{x_{i,j}, x'_{i,j}, a_e\}$ or $\{x_{i,j}, x'_{i,j}, b_e\}$ for some edge $e \in E(V_i, V_j)$. Since $x_{i,j}$ prefers $\{x'_{i,j}, a_e\}$ over $\{x'_{i,j}, a_{\{v_i^{p_i}, v_j^{p_j}\}}\}$, the agent $x'_{i,j}$ does not prefer $\{x_{i,j}, a_e\}$ over $\{x_{i,j}, a_{\{v_i^{p_i}, v_j^{p_j}\}}\}$. $\square$

Lemma 23. *For each $i \leq k$, no vertex from V_i is part of a blocking 3-set.*

Proof. We prove the statement by induction. For $i = 0$ there is nothing to show.

Fix $i \in [k]$. Note that all agents $v_j^{p_j}$ are matched to their first choice, and thus not part of a blocking 2-set. By Lemmas 20 and 21, no agent s_j or s'_j is involved in a blocking 3-set. Thus, by induction over p , one easily sees that all 2-sets, which v_i^p prefers over the 2-set it is matched to in M , contain an agent about which we already know that it is not contained in a blocking 3-set, implying that also v_i^p is not contained in a blocking 3-set.

Thus, no agent of V_i is part of a blocking 3-set. $\square$

Lemma 24. *No agent a_e, b_e , or c_ℓ is part of a blocking 3-set.*

Proof. The argument is basically the same as in the proof of Lemma 23.

We do an induction over the incidence-checking gadgets, note that the agents matched to $\{x_{i,j}, x'_{i,j}\}$ cannot be contained in a blocking 3-set, and then note that for every agent, all 2-sets which it prefers over the 2-set it is matched to in M contain an agent which is not contained in a blocking 3-set (using Lemma 22 for some of these 2-sets). $\square$

Lemma 25. *Matching M is stable.*

Proof. By Lemmas 20 to 24, no agent outside a cut-off gadget is contained in a blocking 3-set. Thus, by Lemma 18, there is no blocking 3-set. $\square$

Proof of the backward direction We prove that if $\mathcal{I}'$ admits a stable matching, then G contains a clique of size k . Let M be a stable matching.

Lemma 26. *Any stable matching M contains a 3-set $\{s_i, s'_i, v_i^p\}$ for some $p \in [3n' + 1]$, and all other agents v_i^q are matched to each other for $q \neq p$.*

Proof. Consider a stable matching M . We prove the statement by induction over i . Thus, we know that no agent from a former gadget is matched to the current gadget. By the preferences, the agents v_i^p prefer most to be matched to other agents of the form v_i^q , or to $\{s_i, s'_i\}$. Due to the cut-off gadget for $v_i^{3n'+1}$, Lemma 17 implies that exactly one agent v_i^p is matched to $\{s, s'\}$. $\square$

We say that a matching selects a vertex v_i^p if it contains the 3-set $\{s_i, s'_i, v_i^p\}$.

Lemma 27. *A stable matching on $A_i \cup A_j$ cannot be extended to the incidence-checking gadget $B_{i,j}$ without introducing a blocking 3-set if G does not contain an edge between the two vertices selected by the vertex-selection gadgets A_i and A_j .*

Proof. Let M be a stable matching on the vertex-selection gadgets. By Lemma 26, the vertex-selection gadgets A_i and A_j select a vertex, say v_i^p and v_j^q .

Now assume that the edge $\{v_i^p, v_j^q\}$ is not contained in G . We need to show that any extension of M contains a blocking 3-set. So fix such an extension M' , and assume for a contradiction that M' is stable.

Note that the cut-off gadgets for $x_{i,j}$ and $x'_{i,j}$ and the observation that the agent s_i from the vertex-selection gadget is matched to the 2-set $\{s'_i, v_i^p\}$ implies that $x_{i,j}$ and $x'_{i,j}$ are matched to an agent a_r , i.e., $\{x_{i,j}, x'_{i,j}, a_r\} \in M'$. Similarly, $x_{j,i}$ and $x'_{j,i}$ are matched to an agent b_s .

Let $e_r = \{v_i^{r_i}, v_j^{r_j}\}$. If $r_i < p$, then $\{e_{r_i}, s_i, x_{i,j}\}$ is a blocking 3-set. If $r_i > p$, then $\{e_{r_i}, s'_i, x'_{i,j}\}$ is a blocking 3-set. Thus, v_i^p is an endpoint of e_r . By symmetric arguments, we get that v_j^q is an endpoint of e_s .

Since $\{v_i^p, v_j^q\} \notin E(V_i, V_j)$, it follows that $s \neq r$.

First assume $r < s$. By the cut-off gadget for $z'_{i,j}$, the matching M' contains the 3-set $\{z_{i,j}, z'_{i,j}, z''_{i,j}\}$. The stability of M' implies that any stable matching contains $\{a_\ell, b_\ell, c_\ell\}$ for $\ell < r$ (by induction over ℓ , all 2-sets preferred over

this 3-set contain an agent which is already matched). However, the 3-set $t = \{z_{i,j}, b_r, a_{r+1}\}$ is blocking, as $z_{i,j}$ prefers this 3-set over $M'(z_{i,j}) = \{z'_{i,j}, z''_{i,j}\}$, every 2-set the agent b_r or a_{r+1} prefers over t contains an agent from $A := \{s_i, s'_i, s_j, s'_j, x_{i,j}, x'_{i,j}, x_{j,i}, x'_{j,i}\} \cup \{a_q, b_q, c_q : q \in [r]\}$, but all agents from A are matched neither to a_{r+1} nor b_r .

If $s < r$, then symmetric arguments show $\{z_{i,j}, a_s, b_{s+1}\}$ is a blocking 3-set for M' . $\square$

Lemma 28. *If $\mathcal{I}'$ admits a stable matching, then G admits a clique of size k .*

Proof. The vertices selected by the vertex-selection gadgets form a clique, which follows directly from Lemma 27. $\square$

Theorem 6 now directly follows from Lemmas 25, 28, and 19.

D Conclusion

Being a fundamental problem within the field of stable matching and the analysis of hedonic games, our work provides a seemingly first systematic study on the parameterized complexity of MULTIDIMENSIONAL STABLE ROOMMATES. Focusing on the concept of master lists with the goal to identify efficiently solvable special cases, we could only report partial success. While we have one main algorithmically positive result, namely fixed-parameter tractability for the parameter “maximum number of agents incomparable to a single agent”, all other (single) parameterizations led again to (often surprising) hardness results (see Table 1).

As to challenges for future research, first, it remained open whether our fixed-parameter tractability result mentioned above also transfers to the setting of MULTIDIMENSIONAL STABLE MARRIAGE. Second, further following the quest for identifying islands of tractability, the study of further, perhaps also combined parameters might be a worth-while goal.

Acknowledgements

RB is now with Humboldt-Universität Berlin (Insitut für Informatik, Algorithm Engineering, robert.bredereck@hu-berlin.de). KH was supported by DFG Research Training Group 2434 “Facets of Complexity”. DK was partially supported by DFG project NI 369/19 while at TU Berlin.