

# Improving Efficiency and Equality in School Choice

JOSUÉ ORTEGA AND THILO KLEIN

How should students be assigned to schools? Two mechanisms have been suggested and implemented around the world: deferred acceptance (DA) and top trading cycles (TTC). These two mechanisms are widely considered excellent choices owing to their outstanding stability and incentive properties. We show theoretically and empirically that both mechanisms perform poorly with regard to two key desiderata such as efficiency and equality, even in large markets. In contrast, the rank-minimizing mechanism is significantly more efficient and egalitarian. It is also Pareto optimal for the students, unlike DA, and generates less justified envy than TTC.

## 1 INTRODUCTION

School choice is a common way to assign students to schools based on the students' and schools' preferences. Students and schools rank their potential matches and submit this information to a centralized clearinghouse. Afterwards, an algorithm (also known as a mechanism) is applied to the submitted data and an allocation of students to schools is generated.

But which mechanism should we use to assign students to schools? [11] convincingly argue that: *“the selection must be based on some measure of aggregate welfare of participants. For instance, if one Pareto efficient mechanism yields a significantly higher utilitarian welfare level or a much more equal payoff distribution than others, that would constitute an important rationale for favoring such a mechanism”*.

We show that the mechanism that minimizes the sum of ranks for the students (henceforth RM, 15) outperforms two of the most popular mechanisms used in school choice with respect to the two desiderata named above, i.e. utilitarian welfare and equality. RM is superior to Gale's and Shapley's deferred acceptance (DA) mechanism and to Gale's top trading cycles (TTC) mechanism in that: i) RM assigns the average student to a school they prefer more (i.e. it is more efficient), and ii) RM assigns the worst-off student to a school that they prefer *much* more (i.e. it is more egalitarian).

In particular, if there are  $n$  students and  $n$  schools with one seat each, and preferences for both sides are drawn uniformly at random, TTC and DA asymptotically assign the average student to approximately their  $\log(n)$  most preferred school, whereas RM assigns them to a school better than their second choice. If we focus on the worst placement, rather than the average, the difference is even bigger: RM assigns them to their  $\log_2(n)$  most preferred school, whereas DA assigns them to their  $\log^2(n)$  and TTC to a school in the bottom half of their rank list (see Fig. 1 for the rank distribution). And because TTC is equivalent to a random serial dictatorship, RM also outperforms this and any other Pareto optimal and strategy-proof mechanism in terms of efficiency and fairness.

Furthermore, RM is Pareto optimal for the students, unlike DA, and generates justified envy for fewer students than TTC, which is surprising because RM does not use schools' priorities but TTC does. We prove these properties for random markets where preferences and priorities are drawn independently and uniformly at random (see Table 1), and document them by analyzing real data from the student assignment system in Budapest (see Fig. 2).

## 2 RELATED LITERATURE

The closest paper to ours, by [11], also studies the expected efficiency of TTC. Using a random market approach, in which preferences have both an uncorrelated and a correlated component, and assuming that the ranking of objects is bounded (an assumption that we do not impose), they show that the *normalized* payoff distribution generated by any Pareto optimal mechanism uniformly converges to the best possible payoff in large markets. Furthermore, they compare the rank

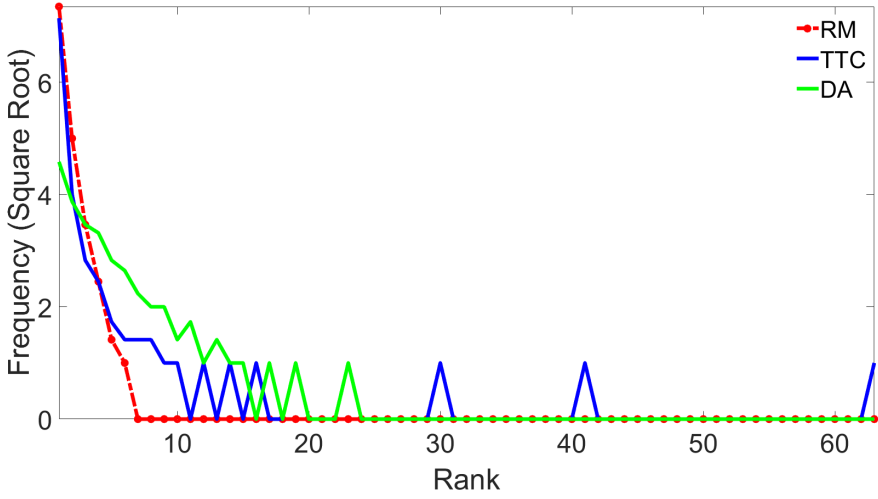


Fig. 1. Rounded average rank distribution in 1,000 random markets with  $n = 100$ . Preferences are drawn independently and uniformly at random. The x-axis is truncated at the highest value with positive density. See Appendix B for details.

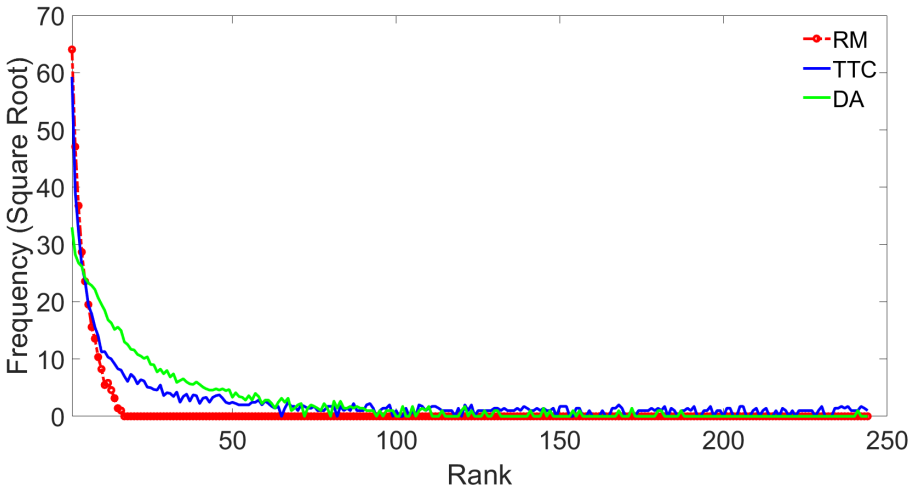


Fig. 2. Rank distribution generated for 10,131 students in the secondary school admissions in Budapest. The maximum rank is 244. See Section 5 for details.

distribution generated by DA and TTC (but not RM) using data from the New York City school choice program. The main lesson from their paper is that all Pareto optimal mechanisms are equivalent in large markets, and therefore there is no reason to prefer any Pareto optimal mechanism

Table 1. Theoretical properties of school choice mechanisms in large random markets.

|                            | RM          | TTC       | DA          |
|----------------------------|-------------|-----------|-------------|
| Average rank               | $\leq 2$    | $\log(n)$ | $\log(n)$   |
| Maximum rank               | $\log_2(n)$ | $> 0.5 n$ | $\log^2(n)$ |
| Students w. justified envy | $0.33 n$    | $0.39 n$  | 0           |
| Pareto optimal             | Yes         | Yes       | No          |
| Strategy-proof             | No          | Yes       | Yes         |

over another.<sup>1</sup> Conversely, the message of this paper is that Pareto optimal mechanisms are not equivalent, as can be observed in Figs. 1 and 2.<sup>2</sup>

To show that RM is more efficient than DA and TTC, we connect the school choice problem to that of assigning one of  $n$  jobs to each of  $n$  workers so to minimize costs.<sup>3</sup> Worker  $i$  incurs in a cost  $c_{ij}$  when completing job  $j$ . The matrix  $C$  contains all such costs. When each row of  $C$  is an independent random permutation of  $\{1, \dots, n\}$ , this problem is equivalent to that of finding the rank-minimizing allocation of students to schools, ignoring schools' priorities. Each entry  $c_{ij}$  denotes the rank (cost) of school (job)  $j$  for student (worker)  $i$ . To show that the RM is more efficient than TTC and DA, we invoke a result in [29] which shows that the cost-minimizing allocation has an average cost smaller than 2, and compare it with the well-known average rank in TTC and DM, which is around  $\log(n)$ . Obtaining the maximum rank lower bound and the fraction of students with justified envy is easy using the limit distribution of ranks in RM, which is also provided by Parviainen.

The result of average rank being bounded by 2 in RM was recently independently discovered by [26]. His proof uses random graph arguments and is different (and significantly more involved) than ours. [37] shows that Nikzad's result can be proved using the cost assignment problem with costs distributed in  $(0, 1)$  [3], without referring to Parviainen's work. [26] and [37] do not study the maximum rank and justified envy in RM, TTC and DA, and do not analyse the performance of these three mechanisms using real-life data in which preferences are correlated.<sup>4</sup>

[32] obtains a similar equivalence result to that of Che and Tercieux: he shows that any anonymous statistics, such as rank distribution, generated by Pareto efficient and strategy-proof mechanisms are equivalent (note that RM is not strategy-proof). This implies that all of our results for TTC's poor performance with regards to efficiency and equality also apply to the random serial dictatorship mechanism (RSD), which in the words of [33]: *"has a long history and is used in a wide variety of practical allocation problems, including school choice, worker assignment, course allocation, and the allocation of public housing"*. [24] shows that the number of preferences profiles for which the random serial dictatorship mechanism is ordinally efficient (and thus rank efficient) vanishes when the number of agents grows. Our result complement his by showing that RSD (and TTC) not only rarely produces a rank efficient allocation, but also the size of its inefficiency does not vanish in large markets.

The RM mechanism has only been studied in economics by [15]. He documents that RM has been used in practice to assign teachers to schools in the US, and shows that any selection of the RM mechanism cannot be strategy-proof. Nonetheless, he shows that truth-telling is a best response in

<sup>1</sup>See also Theorem 1 in [12].

<sup>2</sup>A two-sample Kolmogorov-Smirnov test rejects the null-hypothesis that any two distributions in Figures 1 or 2 are the same at the 1% significance level.

<sup>3</sup>A large literature in mathematics, uncited in economics, has studied this problem. See [27] and [22] for a summary of it.

<sup>4</sup>Here is the proof that our work is independent.

RM when students have little information about other students' preferences and do not truncate their preference list. He shows that a rank efficient allocation must be ordinally efficient (and thus ex-post efficient), but the converse is not necessarily true. He also shows that an inefficient assignment can converge to the RM outcome by performing local swaps.

Finally, [1] show that TTC minimizes justified envy among all Pareto optimal and strategy-proof mechanisms. Neither DA nor RM are in this class of mechanisms. We find theoretically that fewer students experience justified envy in RM than in TTC. In practice RM and TTC generate roughly the same amount of justified envy.

### 3 MODEL

We study a standard one-to-one school choice market [1], which consists of:

- (1) A set of students  $T = \{1, \dots, n\}$ ,
- (2) A set of schools  $S = \{s_1, \dots, s_n\}$ , with each school having space for one student only,
- (3) Strict students' preferences over schools  $\succ := (\succ_1, \dots, \succ_n)$ , and
- (4) Strict schools' priorities over students  $\triangleright = (\triangleright_{s_1}, \dots, \triangleright_{s_n})$ .

An allocation  $x$  is a perfect matching between  $T$  and  $S$ . We will denote by  $x_t$  the school to which student  $t$  is assigned, and by  $x_s$  the student that school  $s$  is assigned to. Student  $i$  experiences justified envy in allocation  $x$  if there exists a school  $s$  such that  $s \succ_t x_t$  and  $t \triangleright_s x_s$ .

The function  $rk_t(x_t)$  returns an integer between 1 and  $n$  corresponding to the ranking of  $x_t$  in the preference list of student  $t$ , i.e. the most desirable option gets a ranking of 1, whereas the least desirable one gets a ranking of  $n$ . A mechanism is a map from school choice markets to (a probability distribution over) allocations. An allocation  $x$  Pareto dominates a different allocation  $y$  if, for every student  $t$ ,  $rk_t(x_t) \leq rk_t(y_t)$  and for some student  $j$ ,  $rk_j(x_j) < rk_j(y_j)$ . An allocation is Pareto optimal if it is not Pareto dominated. A Pareto optimal mechanism returns a Pareto optimal allocation in every school choice problem.

We use  $x^*$  to denote one of the (possibly many) allocations that minimizes the sum of ranks for students, which we henceforth call rank efficient or rank minimizing.  $X^*$  denotes the set of all rank efficient allocations. The rank-minimizing mechanism (henceforth RM) is one that returns a rank efficient allocation for every matching market.<sup>5</sup>

Two other mechanisms are of interest. The first one is top trading cycles (TTC), in which the following two steps are repeated until all agents have been assigned an object.

- (1) Construct a graph with one vertex per student or school. Each student (resp. school) points to his top-ranked school (resp. student) among the remaining ones. At least one cycle must exist and no two cycles overlap. Select the cycles in this graph.
- (2) Permanently assign each student in a cycle to the school he points to. Remove all students and schools involved in a cycle.

The second mechanism of interest is student-proposing deferred acceptance (DA). It works as follows.

- (1) All unmatched students apply to their most preferred school that has not rejected them. Each school that has received a proposal puts the one sent by the highest priority student in a waiting list and permanently rejects all other received applications (if any).
- (2) Repeat step 1 until all schools have received at least one application. Assign each student to the school which has him on a waiting list.

<sup>5</sup>Rank efficiency is a stronger efficiency notion than ordinal efficiency and Pareto optimality [15]. We simply write efficiency to refer to rank efficiency.

We use  $x^{\text{TTC}}$  and  $x^{\text{DA}}$  to denote the allocation obtained by the TTC and DA mechanisms, respectively. Schools' priorities are used to compute TTC and DA, but are irrelevant in RM.

#### 4 RESULTS

Our theoretical results relate to the properties of the expected allocation generated by RM, TTC and DA when students' preferences and schools' priorities are drawn independently and uniformly at random. This assumption is commonly used to analyze matching markets.<sup>6</sup> We study the asymptotic behavior of: i) expected average rank (efficiency), ii) expected maximum rank (inequality), and iii) expected number of students with justified envy generated by RM, TTC and DA in the next subsections.

*Efficiency.* We first study the expected average rank generated by RM, TTC and DA in random markets. To do so, we define  $\bar{x} := \frac{1}{n} \sum_{i=1}^n \text{rk}_i(x_i)$ , which denotes the average rank of the school to which students are assigned in allocation  $x$ .

Proposition 1 shows that the expected average ranking in RM is smaller (i.e. better) than that in TTC and DA. It follows directly from a result by [29] that has not yet been cited in the economics literature. In contrast, the results for DA and TTC are well-known and we simply restate them for completeness.

**Proposition 1.** *The expected average rank in RM, TTC and DA is:*

$$\lim_{n \rightarrow \infty} \mathbb{E}[\bar{x}^*] \leq 2 \quad (1)$$

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[\bar{x}^{\text{TTC}}]}{\log n} = 1 \quad (2)$$

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[\bar{x}^{\text{DA}}]}{\log n} = 1 \quad (3)$$

PROOF. Statement 1 is proven by [29, Theorem 2.1, p. 105].<sup>7</sup> Statement 2 is proven by [21, equation 4, p. 439].<sup>8</sup> Statement 3 is proven by [30, Theorem 2, p. 538]. □

Proposition 1 shows that the rank inefficiency of DA and TTC does not vanish as the market grows large because, even if the average rank obtained by DA and TTC grows slowly with size of the market, the average rank obtained by RM is constant and does not grow with  $n$ . This is in sharp contrast with [11], who argue that the rank distribution generated by any Pareto optimal mechanism is the same and attains the utilitarian upper bound, i.e. that generated by RM, even when preferences have no common component (Proposition 1 in their paper).

*Inequality.* We measure inequality as the rank of the object obtained by the worst-off agent in the market, i.e. the maximum rank in the rank distribution. This measure follows John Rawls' idea that the welfare of a society is that of its worst-off member.<sup>9</sup> To do so, we define  $\underline{x} := \max_i \text{rk}_i(x_i)$ , which denotes the rank of the object obtained by the worst-off agent in allocation  $x$ .

<sup>6</sup>See [11] and references therein.

<sup>7</sup>[29] also provides a lower bound, and thus the expected average rank in RM is such that  $\pi^2/6 \leq \lim_{n \rightarrow \infty} \mathbb{E}[\bar{x}^*] \leq 2$ .

<sup>8</sup>Knuth shows that  $\mathbb{E}[\sum_{i=1}^n \text{rk}_i(x_i^{\text{TTC}})] = (n+1)H_n - n$ , where  $H_n$  is the  $n$ -th harmonic number and, therefore,  $\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[\sum \text{rk}(x_i^{\text{TTC}})] = \log(n)$ . See also the note after the acknowledgements in [16], p. 807.

<sup>9</sup>Alternatively, one could define inequality as the difference in ranks between the worst- and best-off agent. Because the rank of the object obtained by the best-off agent is 1 in any Pareto optimal allocation [2], both measures are equivalent.

Proposition 2 shows that RM generates a significantly more egalitarian allocation than DA and TTC. In particular, TTC generates an allocation so unequal that the worst-off student is assigned to a highly undesirable school in the lower half of his preference list. Such rank is much higher than the corresponding value for RM ( $\log_2(n)$ ) and DA ( $\log^2(n)$ ).

**Proposition 2.** *The expected maximum rank of RM, TTC and DA is:*

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[x^*]}{\log_2(n)} = 1 \quad (4)$$

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[x^{TTC}]}{n} > 0.5 \quad (5)$$

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[x^{DA}]}{\log^2(n)} = 1 \quad (6)$$

PROOF. Statement 5 was proven by [21, p. 440]. Statement 6 was proven by [31], theorem 6.1, p. 382 and note before references, p. 400.

To prove statement 4 we use the asymptotic rank distribution in RM. The probability that a student is assigned to his  $i$ -th choice is asymptotically equal to  $\frac{1}{2^i}$  (Theorem 1.3 in [29]), so that a student is assigned to a school with rank 1 with probability  $1/2$ , to a school with rank 2 with probability  $1/4$ , and so on. This probability distribution is very similar to the number of consecutive heads in  $n$  independent coin tosses, in which 0 heads obtains with probability  $1/2$ , 1 heads with probability  $1/4$  and so on (the distribution of ranks in RM is shifted by  $+1$ ). Finding the longest run of heads is a known problem, in which the longest run is equal to  $\frac{\log(n/2)}{\log(2)}$  [36]. Therefore, the maximum rank in RM is equal to  $\frac{\log(n/2)}{\log(2)} + 1 = \log_2(n)$ .<sup>10</sup>  $\square$

Although we only provide a lower bound for the maximum rank in TTC (of  $0.5 n$ ), simulations suggest that the maximum rank in TTC converges to  $0.63 n$ .

*Justified Envy.* We use  $e^*$ ,  $e^{TTC}$  and  $e^{DA}$  to denote the fraction of students who experience justified envy in the allocation obtained in RM, TTC and DA, respectively. Proposition 3 shows that RM generates fewer cases of expected envy than TTC, which is interesting since TTC is envy minimal in the class of strategy-proof and Pareto optimal mechanisms [1].

**Proposition 3.** *The expected fraction of students with justified envy in RM, TTC and DA is:*

$$\lim_{n \rightarrow \infty} \mathbb{E}[e^*] = 0.33 \quad (7)$$

$$\lim_{n \rightarrow \infty} \mathbb{E}[e^{TTC}] = 0.3863 \quad (8)$$

$$\lim_{n \rightarrow \infty} \mathbb{E}[e^{DA}] = 0 \quad (9)$$

PROOF. Statement 9 is well-known, as DA does not generate justified envy [18].

For the remainder of the proof we use the fact that the number of students with justified envy in TTC and RSD ( $e^{RSD}$ ) is asymptotically equivalent [10]. Since schools' priorities are irrelevant in both RM and RSD, a student who is assigned to his  $i$ -th most preferred school does *not* experience justified envy with probability  $\frac{1}{2^{i-1}}$ . To see this, notice that students placed into their 1st choice

<sup>10</sup>[17, Theorem 2, p. 1436] proves that the maximum cost in the cost assignment problem when costs are uniformly distributed in  $[0, n]$  is  $\Theta(\log(n))$ .

trivially do not experience justified envy with probability 1; students placed into their second best choice do not experience justified envy if the student who is accepted at his most preferred school has a higher priority than him, which occurs with probability  $1/2$ ; for students who are assigned to their third choice, they do not experience justified envy if their first and second most preferred school rank their assigned student above them, i.e. with probability  $1/4$ , and so on.

Thus, to obtain the total fraction of students who do not experience justified envy in RM and RSD (TTC), we just need to multiply i) the probability that a student matched to their  $i$ -th most preferred school does not experience justified envy, times ii) the fraction of students who are assigned to such a choice in RSD and RM. The fraction of students assigned to their  $i$ -th choice in RM asymptotically equals  $\frac{1}{2^i}$  (Theorem 1.3 in [29]), whereas in RSD the probability that the  $k$ -th dictator is assigned to his  $j$ -th most preferred school is given by  $p_{k,j} = \frac{1}{n!} \binom{k-1}{j-1} (j-1)!(n-j)!(n+1-k)$  [21].<sup>11</sup> Putting these expressions together, and after some algebra detailed in Appendix A (for RSD only), we obtain:

$$e^* = 1 - \sum_{i=1}^n \frac{1}{2^i} \times \frac{1}{2^{i-1}} = 1 - \sum_{i=1}^n \frac{1}{2^{2i-1}} \rightarrow 0.33 \quad (10)$$

$$e^{\text{RSD}} = 1 - \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^i \frac{1}{n!} \binom{k-1}{j-1} (j-1)!(n-j)!(n+1-k) \frac{1}{2^{i-1}} \rightarrow 0.3863 \quad (11)$$

which finalizes the proof, since  $\lim_{n \rightarrow \infty} \mathbb{E}[e^{\text{TTC}}] = \lim_{n \rightarrow \infty} \mathbb{E}[e^{\text{RSD}}]$ .  $\square$

## 5 DATA

One critique that can be made to our random market results is that they assume that students' preferences are independent, whereas students' preferences tend to be correlated, and such correlation may improve the performance of DA and TTC with regards to efficiency and equality. We show that this is not the case by using real-life data from secondary school admissions in Hungary in 2015. In summary, we find that TTC and DA perform even worse than when we assumed independent uniform preferences.

Our data contains the preferences and priorities of 10,131 students and 244 schools in Budapest. Because students only rank a few schools and schools only rank students who apply to them, we apply RM, DA and TTC to i) the actual reported preferences and priorities (Hungary assigns students to schools using DA [6], so using the reported preferences as real preferences is a sensible strategy), and ii) the estimated, complete preferences and priorities.<sup>12</sup> Figures 2 and 3 present the distribution of the ranks realized after applying RM, DA and TTC to the estimated and reported preferences, respectively. Table 2 presents summary statistics.

The lessons we learn from computing the rank distributions in Budapest are similar to those we learned from looking at random markets. Table 2 shows that RM performs better than TTC and the currently used DA with regards to efficiency and equality with reported and estimated preferences. When full, estimated rank lists are used, the average student substantially improves their placement (average rank in RM is 2.7, compared to 8.9 in TTC and 12.3 in DA). RM still generates a better average rank when we use stated preferences, but the difference with the average

<sup>11</sup>For example, if  $k = 1$ , then  $p_{1,1} = 1$  and  $p_{1,j} = 0$  for any  $j > 1$ . Similarly, when  $j = 1$ , then  $p_{k,1} = \frac{n+1-k}{n}$ . Note that  $\binom{n}{0} = 1$  and  $\binom{n}{m} = 0$  for any  $m > n$ .

<sup>12</sup>The complete preferences are estimated using the original students' reported preferences assuming that i) students do not use dominated strategies, and ii) the realized assignment is stable. For the complete estimation procedure see [5]. In both cases, we balanced the demand and supply for seats by adjusting the schools capacities (this is because there are substantially more school seats than students, repeating the exercise with the unbalanced market does not change the comparison between mechanisms). When a student only ranks  $k$  schools, we use  $k + 1$  as the rank of being unassigned. RM chooses the rank minimizing assignment randomly among all rank efficient allocations.

Table 2. Rank descriptive statistics for Budapest.

| Preferences<br>Variable \ Mechanism    | Estimated Preferences |       |       | Reported Preferences |       |       |
|----------------------------------------|-----------------------|-------|-------|----------------------|-------|-------|
|                                        | RM                    | TTC   | DA    | RM                   | TTC   | DA    |
| Mean                                   | 2.7                   | 8.9   | 12.3  | 1.5                  | 1.9   | 2.1   |
| Maximum                                | 16                    | 244   | 241   | 6                    | 14    | 13    |
| Variance                               | 4.7                   | 607.4 | 220.1 | 0.6                  | 1.7   | 1.9   |
| Share of students<br>w. justified envy | 0.58                  | 0.64  | 0     | 0.46                 | 0.44  | 0     |
| Unassigned students                    | -                     | -     | -     | 2,555                | 2,508 | 2,704 |

For the reported preferences, the mean is computed dividing by the number of assigned students.

rank generated by TTC and DA is smaller (this is because the average student only ranks 4.1 schools on average). RM assigns the average student to a school in their 16 percentile of their preference lists, whereas the corresponding percentile for DA and TTC are 35 and 29, respectively.

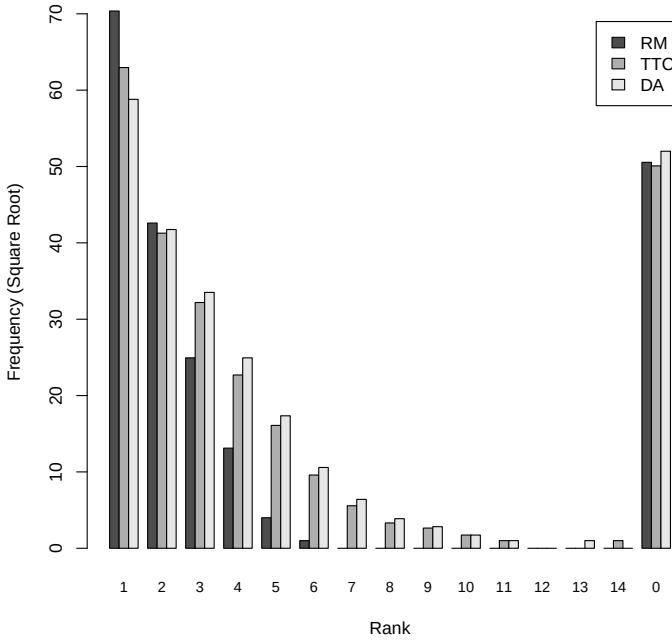


Fig. 3. Rank distribution generated for students in the secondary school admissions in Budapest using reported preferences. The last bar (0) denotes unassigned students.

With regards to inequality, RM performs much better than DA and TTC with complete preferences, assigning the worst-off student to the 16th best choice rather than to their 241th and 244th, respectively (out of 244). It also assigns less than 2% of the student population to their 10th ranked

school or worse, whereas TTC and DA assign 16% and 41% of the student population to such school, respectively. RM also generates a significantly more egalitarian allocation with reported (incomplete) preferences, assigning the worst-off student to the 6th best choice rather than their 13th or 14th best. With estimated and reported preferences, we find that DA and TTC are incomparable in terms of equality, since TTC assigns more students to a really undesirable school, but also assigns more students to a top 3 school. When reported (incomplete) preferences are used, the number of students unassigned in RM, TTC and DA are equivalent.

Our rank distributions are similar to those documented in other studies. [11] and [1] also document that TTC assigns more students to their first choice than DA. Both studies also find that DA and TTC generate a similar number of unassigned students.

Our empirical analysis has used the preferences that students submit in DA (arguably their true ones) to generate the TTC and RM allocations. One potential concern is that students would submit different preferences when allocations are determined by RM, which is not strategy-proof. This concern is mitigated because the potential gains from misrepresentation are small (the average student can only improve by less than one rank place in his preference list with iid preferences, and by less than 2 rank places in the data). Furthermore, truth-telling is a best-response in the RM mechanism if the student has symmetric beliefs about others [15]. Moreover, there is evidence that truthful behavior can depend more on the complexity of the mechanism rather than actual strategy-proofness [9], and RM is arguably more complex to explain and predict than DA and TTC.<sup>13</sup> Thus, we argue that computing the RM rank distribution using the preferences revealed in DA (and those estimated based on this information) is reasonable, particularly because we do not make any individual comparisons, but are only interested in the shape of the entire rank distribution which is unlikely to change if a small fraction of students submit reasonable untruthful preferences, such as not ranking their first choice [13] or truncating their preference lists.<sup>14</sup>

## 6 CONCLUSION

Our paper highlights the efficiency loss and inequality generated by the celebrated deferred acceptance and top trading cycles mechanisms. We are also the first to compute the asymptotic fraction of students with justified envy in the rank minimizing and the top trading mechanisms. Our findings are confirmed by real school choice data and simulations.<sup>15</sup>

The rank minimizing mechanism is superior than deferred acceptance and top trading cycles in terms of efficiency and equality, two key desiderata in economic theory and distributive justice [25]. The outstanding efficiency and equality of the rank minimizing mechanism in theory and practice are strong arguments for its use in some situations, particularly when schools' priorities are random lotteries (as in Brighton and Hove in England [4] and Amsterdam [28]), or when schools' priorities are not allowed by law to depend on academic selection (as in the case of secondary schools in Northern Ireland [8]).

Nonetheless, deferred acceptance and top trading cycles are better than RM in two fronts. Firstly, the rank minimizing mechanism fails to be strategy-proof. But while strategy-proofness was seen in the past as a key property, recent results have shown that theoretical strategy-proofness does not imply truthful behavior in practice. Substantial strategic behavior in strategy-proof mechanisms

<sup>13</sup>[9] find that almost twice as many people (70% versus 40%) behave truthfully in the efficiency adjusted deferred acceptance (EADA) mechanism versus standard DA, even though DA is strategy-proof and EADA is not. Truthful behavior is also more common in complex mechanisms in other setups, like cake-cutting [23].

<sup>14</sup>If students truncate their preferences the RM allocation remains largely unchanged, because most students are assigned to a school in their top three choices, and nobody is assigned to a school worse than their sixth choice.

<sup>15</sup>RM generates justified envy in fewer students than TTC only with complete preferences. When reported preferences are used, they generate approximately the same justified envy.

such as DA and TTC has been documented in laboratory experiments [9, 14, 19, 35] and in real life [20, 34, 39].<sup>16</sup> Thus, it is not clear that RM would in practice be more manipulated than DA or TTC, because as we have argued in the previous subsection, the expected gains from manipulation are small and RM is arguably a more complex mechanism to understand. Whether students actually misrepresent their preferences in RM more than they do in DA or TTC is an open question that could be tackled using lab experiments.

Secondly, DA and TTC are superior to RM in that they actually take schools' priorities into account, which is a desirable property in some situations. But although TTC uses schools' priorities and RM does not, TTC generates more justified envy than RM in theory and about the same in practice, suggesting that TTC does not use priorities *in the right way* – a point also argued by [10]. This concern, added to the fact that TTC assigns some students to a school in the bottom half of their preference list, are objections that our paper raises against using TTC in applications. Pycia's equivalence result implies that these serious concerns extend to the well-known random serial dictatorship mechanism and to any other Pareto optimal and strategy-proof mechanism. Furthermore, these concerns extend to the application of TTC and random serial dictatorship to one-sided matching problems with or without ownership (sometimes called assignment problems) [7, 38], to where our results carry over.

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<sup>16</sup>[19] write: “In a school-allocation setting, we find that roughly half of the observed truth-telling under TTC and DA is the result of naïve (non-strategic) behaviour. Only 14–31% of the participants choose actions that are compatible with rational behaviour. We argue that the use of a default option, confusion and other behavioural biases account for the vast majority of truthful play in both TTC and DA in laboratory experiments”.

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### APPENDIX A - PROOF OF PROPOSITION 3

In RSD the probability that the  $k$ -th dictator is assigned to his  $j$ -th most preferred school is given by

$$p_{k,j} = \frac{1}{n!} \binom{k-1}{j-1} (j-1)! (n-j)! (n+1-k) \quad (12)$$

$$= (n+1-k) \frac{(k-1)! (j-1)! (n-j)!}{n! (j-1)! (k-j)!} \quad (13)$$

$$= (n+1-k) \frac{(k-1)! (j-1)! (n-j)!}{n! (j-1)! (k-j)!} \frac{k}{k} \frac{j}{j} \quad (14)$$

$$= (n+1-k) \frac{k! j! (n-j)!}{n! j! (k-j)!} \frac{1}{k} \quad (15)$$

$$p_{k,j} = \frac{(n+1-k)}{k} \frac{\binom{k}{j}}{\binom{n}{j}} \quad (16)$$

Since RSD is independent of schools' priorities, a student placed in their  $j$ -th most preferred school does *not* experience envy with probability  $\frac{1}{2^{j-1}}$ .

Therefore, the total number of students without justified envy in RSD ( $NE^{\text{RSD}}$ ) equals

$$NE^{\text{RSD}} = \sum_{k=1}^n \sum_{j=k}^n \frac{(n+1-k)}{k} \frac{\binom{k}{j}}{\binom{n}{j}} \frac{1}{2^{j-1}} \quad (17)$$

$$= \sum_{j=1}^n \sum_{k=j}^n \frac{(n+1-k)}{k} \frac{\binom{k}{j}}{\binom{n}{j}} \frac{1}{2^{j-1}} \quad (18)$$

$$= \sum_{j=1}^n \frac{1}{\binom{n}{j} 2^{j-1}} \sum_{k=j}^n \frac{(n+1-k)}{k} \binom{k}{j} \quad (19)$$

$$= \sum_{j=1}^n \frac{1}{\binom{n}{j} 2^{j-1}} \left[ (n+1) \underbrace{\sum_{k=j}^n \frac{1}{k} \binom{k}{j}}_A - \sum_{k=j}^n \binom{k}{j} \right] \quad (20)$$

Where

$$A = \sum_{k=j}^n \frac{1}{k} \binom{k}{j} = \sum_{k=j}^n \frac{1}{k} \frac{k!}{j! (k-j)!} = \sum_{k=j}^n \frac{(k-1)!}{j (j-1)! (k-j)!} \quad (21)$$

$$= \frac{1}{j} \sum_{k=j}^n \binom{k-1}{j-1} = \frac{1}{j} \sum_{k=j-1}^{n-1} \binom{k}{j-1} \quad (22)$$

Plugging this in our expression for  $NE^{\text{RSD}}$ , we have

$$NE^{\text{RSD}} = \sum_{j=1}^n \frac{1}{\binom{n}{j} 2^{j-1}} \left[ \frac{n+1}{j} \sum_{k=j-1}^{n-1} \binom{k}{j-1} - \sum_{k=j}^n \binom{k}{j} \right] \quad (23)$$

Using the hockey stick identity, which establishes that  $\sum_{k=j}^n \binom{n}{k} = \binom{n+1}{j+1}$ , we can simplify the expression to

$$NE^{\text{RSD}} = \sum_{j=1}^n \frac{1}{\binom{n}{j} 2^{j-1}} \left[ \frac{n+1}{j} \binom{n}{j} - \binom{n+1}{j+1} \right] \quad (24)$$

$$= \sum_{j=1}^n \frac{1}{\binom{n}{j} 2^{j-1}} \left[ \frac{n+1}{j} \binom{n}{j} - \frac{(n+1) n!}{(j+1) j! (n-j)!} \right] \quad (25)$$

$$= \sum_{j=1}^n \frac{1}{\binom{n}{j} 2^{j-1}} \left[ (n+1) \left( \frac{1}{j} - \frac{1}{j+1} \right) \binom{n}{j} \right] \quad (26)$$

$$= (n+1) \sum_{j=1}^n \left( \frac{1}{2} \right)^{j-1} \left( \frac{1}{j} - \frac{1}{j+1} \right) \quad (27)$$

We divide both sides by  $n+1$ , and continue simplifying the expression to obtain

$$\frac{NE^{\text{RSD}}}{n+1} = \sum_{j=1}^n \left( \frac{1}{2} \right)^{j-1} \frac{1}{j} - \sum_{j=1}^n \left( \frac{1}{2} \right)^{j-1} \frac{1}{j+1} \quad (28)$$

$$= 2 \sum_{j=1}^n \left( \frac{1}{2} \right)^j \frac{1}{j} - 4 \sum_{j=1}^n \left( \frac{1}{2} \right)^{j+1} \frac{1}{j+1} \quad (29)$$

$$= 2 \sum_{j=1}^n \left( \frac{1}{2} \right)^j \frac{1}{j} - 4 \sum_{j=2}^{n+1} \left( \frac{1}{2} \right)^j \frac{1}{j} \quad (30)$$

$$= -2 \sum_{j=2}^n \left( \frac{1}{2} \right)^j \frac{1}{j} + 1 - \frac{4}{(n+1)} \left( \frac{1}{2} \right)^{n+1} \quad (31)$$

$$= -2 \sum_{j=2}^n \left( \frac{1}{2} \right)^j \frac{1}{j} + 1 - \frac{4}{(n+1)} \left( \frac{1}{2} \right)^{n+1} - 1 + 1 \quad (32)$$

$$= -2 \sum_{j=1}^n \left( \frac{1}{2} \right)^j \frac{1}{j} + 2 - \frac{4}{(n+1)} \left( \frac{1}{2} \right)^{n+1} \quad (33)$$

$$= 2 - \frac{1}{(n+1)} \left( \frac{1}{2} \right)^{n-1} - \underbrace{2 \sum_{j=1}^n \left( \frac{1}{2} \right)^j \frac{1}{j}}_B \quad (34)$$

Now take limits as  $n$  goes to infinity, and use the fact that  $B$  is a truncated form of the Taylor expansion for  $-\ln(1/2)$  to obtain

$$\lim_{n \rightarrow \infty} \frac{NE^{\text{RSD}}}{n} = \lim_{n \rightarrow \infty} \frac{NE^{\text{RSD}}}{n+1} = \lim_{n \rightarrow \infty} 2 + 2 \ln\left(\frac{1}{2}\right) = 0.6137 \quad (35)$$

Thus, the fraction of students who experience justified envy in RSD tends to  $e^{\text{RSD}} = 1 - 0.6137 = 0.3863$ , which is what we wanted to prove.

## APPENDIX B - SIMULATIONS

In simulated markets (see Tables 3 and 4), we clearly see that RM dominates TTC and DA in efficiency (average rank) and inequality (max rank). Given the large ranks that realize in TTC, it is unsurprising that the variance of the rank distribution is large too. The variance of RM is much smaller, which shows that the ranks are heavily concentrated among the first four top choices. Table 3 also allows us to assess the accuracy of the random market results presented in section 4. For TTC, the mean rank is surprisingly close to the theoretical prediction ( $\pm 1$  of  $\log(n)$ ). In RM, the upper bound provided of 2 for the mean is quite tight, and the approximation  $\log_2(n)$  for the max rank is also remarkably accurate.

Table 3. Rank descriptive statistics. Average over 1,000 simulations.

| Variable | $n = 100$ |      |      | $n = 500$ |       |      |
|----------|-----------|------|------|-----------|-------|------|
|          | RM        | TTC  | DA   | RM        | TTC   | DA   |
| Mean     | 1.8       | 4.3  | 5.0  | 1.8       | 5.8   | 6.7  |
| Max      | 6         | 64   | 23.2 | 8         | 315   | 42.6 |
| Variance | 1.3       | 73.3 | 18.5 | 1.34      | 421.0 | 37.6 |

The severity of the inequality generated by TTC is fully exposed in Table 4. TTC not only makes someone really worse off, assigning them a really bad object ( $0.63n$ ), but it assigns an object in the bottom 90% (not top 10%) of their preferences to over 1.5% of the agents. In contrast, RM does not assign such a poor option to any agent. RM also assigns more agents to their top choice than TTC.

Table 4. Percentage of agents who receive an object with rank higher (worse) than  $m$ . Average over 1,000 simulations.

| $m$       | $n = 100$ |     |    | $n = 500$ |      |      |
|-----------|-----------|-----|----|-----------|------|------|
|           | RM        | TTC | DA | RM        | TTC  | DA   |
| 1         | 46        | 50  | 96 | 46.6      | 49.8 | 93.2 |
| 2         | 21        | 33  | 72 | 21.0      | 33.2 | 79.6 |
| $\log(n)$ | 3         | 20  | 40 | 0.6       | 14.2 | 39.8 |
| $0.1n$    | 0         | 8   | 12 | 0         | 1.8  | 0    |
| $0.25n$   | 0         | 3   | 0  | 0         | 0.6  | 0    |
| $0.5n$    | 0         | 1   | 0  | 0         | 0.2  | 0    |