

Popular matchings with weighted voters

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Abstract

In the POPULAR MATCHING problem, we are given a bipartite graph $G = (A \cup B, E)$ and for each vertex $v \in A \cup B$, strict preferences over the neighbors of v . Given two matchings M and M' , matching M is more popular than M' if the number of vertices preferring M to M' is larger than the number of vertices preferring M' to M . A matching M is called *popular* if there is no matching M' that is more popular than M .

We consider a natural generalization of POPULAR MATCHING where every vertex has a weight. Then, we call a matching M more popular than matching M' if the weight of vertices preferring M to M' is larger than the weight of vertices preferring M' to M . For this case, we show that it is NP-hard to find a popular matching. Our main result is a polynomial-time algorithm that delivers a popular matching or a proof for its non-existence in instances where all vertices on one side have weight c for some $c > 3$ and all vertices on the other side have weight 1.

1 Introduction

The simple majority voting rule offers a natural way of aggregating voters' preferences. Already Condorcet [9] used pairwise comparisons to calculate the winning candidate, establishing his famous paradox on the smallest voting instance not admitting a majority winner.

The concept of majority voting translates to other scenarios where voters submit preference lists. One such field is the area of two-sided matchings under preferences, where popular matchings [17, 2, 40, 33, 11] serve as a voting-based alternative to the well-known notions of stable matchings [16, 4] and Pareto optimal matchings [1]. In short, a popular matching M is a simple majority winner among all matchings, as it guarantees that no matter what alternative matching is offered on the market, a weak majority of the non-abstaining agents will opt for M . Restricted to matching instances on bipartite graphs, the popular matching problem has been studied in two models.

- *House allocation (i.e., one-sided preferences).* One side of the graph consists of agents who have strictly ordered preferences and cast votes, while the other side is formed by houses with no preferences or votes.
- *Two-sided preferences.* Vertices on both sides are agents, who all have strictly ordered preferences and cast votes. This setting is analogous to the classic stable marriage model.

In this paper, we focus on the latter model and extend it to a direction motivated by voting. More specifically, we supply the agents in the instance with weights. The vote of each agent

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then counts with multiplicity: the larger the weight is, the more influence this agent has on the outcome of the voting between two matchings. Analogous weighted voting [5] scenarios arise naturally in various real-life problems. In committees [37], the vote of persons on different posts might be weighted differently when tallying them. In liquid democracy [24], the delegates' votes also count with multiplicity, depending on the number of agents they represent. Another interpretation of weighted voters might be that prioritized voters are assigned a larger weight in order to ensure their beneficial treatment [10]. Also, coalitions often decide to vote together, in which case the decision made within the coalition will also be counted with multiplicity [3].

Weighted voting readily translates to vertex-weighted graphs in the matching context. It follows already from the Condorcet paradox that the existence of a popular matching is not guaranteed if the vertices are weighted, as Figure 1 demonstrates. However, the complexity of deciding whether an optimal solution exists in a given instance was open until now.

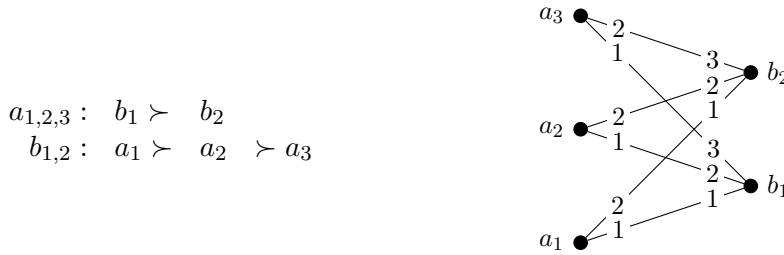


Figure 1: With certain vertex weights, no popular matching exists in this instance. The lists on the left side as well as the numbers on the edges indicate the preferences of the vertices: For every $i \in \{1, 2, 3\}$, a_i 's first choice is b_1 , while its second choice is b_2 . The Condorcet paradox with no majority winner corresponds to assigning weight 1 to a_1, a_2, a_3 and weight 0 to b_1, b_2 . The presence of 0-weight vertices is not necessary: the same example works also with weight 3 for a_1, a_2, a_3 and weight 1 for b_1, b_2 .

Another motivation for popular matchings arises from the context of stable matchings. While stable matchings provide a local stability criterium (the absence of so-called blocking pairs), popularity takes a more global perspective and does not forbid local instabilities but only global instabilities, i.e. a complete matching preferred to the current matching. Therefore, popular matchings have been proposed as an alternative to stable matchings [20] as they may be larger than stable matchings. In the context of bipartite stable matching problems, the different sides often represent fundamentally different agents (i.e. when considering the admission of students to colleges, then one side represents the students, while the other side represents the free slots at colleges); consequently, it is natural to weight the preferences of one side larger than the preferences of the other side. Recently, Kavitha [28] suggested to search for popular matchings when the vote of each vertex from one side is counted with multiplicity $c > 1$ and each vertex from the other side with multiplicity 1.

1.1 Related work

Besides two-sided matchings, majority voting has also been defined for the roommates problem [15, 18], spanning trees [12], permutations [41, 32], the ordinal group activity selection problem [13], and branchings [29]. Matchings in bipartite graphs are nevertheless the most actively researched area [8, 15, 14, 31, 18, 22] of the majority voting rule outside of the usual voting scenarios. In the context of matchings it was first introduced by Gärdenfors [19] for matching markets with two-sided preferences, and then studied by Abraham et al. [2] in the house allocation model. Polynomial time algorithms to find a popular matching were given in both settings. In the two-sided preferences model, it was already noticed by Gärdenfors [19]

that all stable matchings are popular, which implies that in standard bipartite stable matching instances, popular matchings always exist. In fact stable matchings are the smallest size popular matchings, as shown by Biró et al. [7], while maximum size popular matchings can be found in polynomial time as well [20, 25]. Notice that as soon as there are vertices with different weights, for example in the house allocation model, the existence of a popular matching is not guaranteed any more.

Arguably the most natural extension of various matching problems is to consider graphs with edge or vertex weights. Notice however that the two weighted extensions define inherently different problems for popularity. If vertices are weighted, as in our setting, then this weight influences the voting power of an agent and redefines the more popular relation between matchings. If edge weights are given, the vertices vote exactly as in the non-weighted version, but the matchings carry a different total weight, and the goal is to find a maximum weight matching, subject to popularity.

For house allocation instances with edge weights and strict preferences, a maximum weight one among popular matchings can be found in $O(n + m)$ time [35], where n is the number of vertices and m is the number of edges in the graph. For instances with two-sided preferences, Faenza et al. [14] showed that it is NP-complete to decide if there exists a popular matching that contains two given edges, which is a very restricted case of popular matchings with edge weights. The same authors provided a 2-approximation algorithm for non-negative edge weights. NP-hardness was established for edge-weighted non-bipartite instances a couple of years earlier already [22].

The vertex-weighted case has been studied extensively in the house allocation setting. Mestre [36] presented algorithms to find a popular matching, or a proof for its non-existence. For instances with strict preference lists, he gave an $O(n + m)$ time algorithm, while for preference lists with ties, he solved the problem in $O(\min\{k\sqrt{n}, n\}m)$ time, where k is the number of distinct weights the agents are given. The algorithm for strict lists was later extended to capacitated house allocation instances by Sng and Manlove [40], who posed the complexity of the case with weakly ordered preferences as an open problem. Itoh and Watanabe [23] studied the existence probability of a popular matching in random house allocation instances with vertex weights. Ruangwises and Itoh [38] designed an algorithm to compute the approximability measure introduced by McCutchen [34] and called unpopularity factor for a given matching. As a byproduct, they also developed a polynomial algorithm to verify the popularity of a given matching, even in non-bipartite instances.

Kavitha [28] gave a polynomial-time algorithm for finding a matching which is popular when each vertex has weight 1 and simultaneously is popular in the house allocation setting, i.e. when each vertex from A has weight 1 while each vertex from B has weight 0.

1.2 Structure of the paper and our contribution

We define our notation and main problem in Section 2. There we also sketch three relevant known techniques. We extend one of these, the concept of a witness, to the vertex-weighted case in Section 3. Then in Section 4 we give two different hardness reductions to show that finding a popular matching is NP-hard even if the vertex weight function is very restricted. A strong inapproximability result is also provided there for the problem variant with edge costs. We complement these findings in Section 5 with our main result, an algorithm to find a popular matching in polynomial time (if any exists) for instances where the weight of every vertex in A is a constant $c > 3$, while the weight of every vertex in B is 1. Since A and B model two different kinds of vertices, it is natural to assign each vertex from A a different weight to the weight of vertices from B (while each vertex from A (respectively B) has the same weight). Notably, Kavitha [28] recently observed that almost nothing is known when all agents from one side have weight $c > 1$ while all vertices from the other side have weight 1.

We pose open questions in Section 6. Results marked with (★) are proven in the appendix.

2 Preliminaries

In this section we first introduce the notation we use, and then describe three known techniques we rely on in our proofs.

2.1 Notation, input, and the popularity criterion

In our input, a simple bipartite graph $G = (A \cup B, E)$ on n vertices and m edges is given. We denote the set of vertices by $V = A \cup B$, the set of vertices adjacent to $v \in V$ in G by $N_G(v)$, and the set of edges incident to $v \in V$ in G by $\delta_G(v)$. The vertices of G are equipped with weights determined by the weight function $w : V \rightarrow \mathbb{Q}_{\geq 0}$. Furthermore, each vertex $v \in V$ has a strictly ordered preference list $\succ_v$ over the vertices in $N_G(v)$, where $u \succ_v z$ means that v prefers u to z . We assume that being matched to any vertex in $N_G(v)$ is preferred to staying unmatched. We may drop the subscript G if it is clear from context.

A *matching* $M \subseteq E$ is a set of pairwise disjoint edges. A matching M is *perfect* if every vertex is covered by an edge of M . For each matching M and vertex $v \in V$, let $M(v)$ denote the vertex M matches v to. If v remains unmatched in M , then we write $M(v) := \perp$. The preference list $\succ_v$ naturally defines a preference relation between any two matchings M and M' for each $v \in V$: vertex v prefers M to M' if $M(v) \succ_v M'(v)$. If $M(v) = M'(v)$, then v is indifferent between M and M' .

Given two matchings M and M' , let $V^+(M, M')$ be the set of vertices preferring M to M' . We define $\Delta_w(M, M') := \sum_{v \in V^+(M, M')} w(v) - \sum_{v \in V^+(M', M)} w(v)$. A matching M is *popular* if $\Delta_w(M, M') \geq 0$ for every matching M' . In words, M is popular if it is never beaten in a pairwise comparison with another matching, where each vertex casts a vote for its preferred matching with the multiplicity of its weight. We call an edge $e \in E(G)$ *popular* if it is contained in some popular matching. We now formally define the main problem studied in this paper.

POPULAR MATCHING WITH WEIGHTED VOTERS

Input: A bipartite graph $G = (A \cup B, E)$, $(\succ_v)_{v \in V}$, and vertex weights $w : V \rightarrow \mathbb{Q}_{\geq 0}$.
Question: Is there a popular matching M ?

2.2 Popular matching characterizations

We now sketch useful known techniques for characterizing popular matchings in house allocation instances, instances with two-sided lists, and finally, instances with weighted voters.

2.2.1 House allocation instances

We first focus on house allocation instances, which correspond to the subcase of our problem definition with $w(a) = 1$ for each $a \in A$ and $w(b) = 0$ for each $b \in B$. The characterization of Abraham et al. [2] uses the notion of an *f-post* and *s-post* of a vertex $a \in A$.

Definition 1. The *f-post* $f(a)$ of $a \in A$ is the first vertex in the preferences of a . The *s-post* $s(a)$ of $a \in A$ is the vertex that is best-ranked among all vertices that are not the *f-post* of any vertex, i.e. the best-ranked vertex among $B \setminus \{f(a') : a' \in A\}$. If no such vertex exists, then we set $s(a) := \perp$.

The following characterization of popular matchings leads to an $O(n + m)$ algorithm that outputs either a largest cardinality popular matching or a proof for its nonexistence.

Theorem 1 (Abraham et al. [2]). *In a house allocation instance, a matching M is popular if and only if the following conditions are both fulfilled.*

- Each vertex b such that $b = f(a)$ for some $a \in A$ is matched in M to some $a' \in A$ with $b = f(a')$.
- For each vertex $a \in A$, it holds that $M(a) \in \{f(a), s(a)\}$.

2.2.2 Instances with two-sided lists

Now we turn to instances with two-sided preferences and unit weights, i.e. $w(v) = 1$ for each $v \in V$. Let $\tilde{G}$ be the graph G augmented with a loop at each vertex, such that each vertex is its own last choice. This allows us to regard any matching M in G as a perfect matching $\tilde{M}$ in $\tilde{G}$ by including loops at all vertices left unmatched in M . First we define the vote of $u \in V$ for $v \in N_G(u)$ as

$$\text{vote}_u^M(v) := \begin{cases} 0 & \text{if } \{u, v\} \in M, \\ 1 & \text{if } M(u) = \perp \text{ or } v \succ_u M(u), \text{ and} \\ -1 & \text{otherwise (i.e. } M(u) \succ_u v). \end{cases}$$

This function allows us to express the total votes of the two end vertices if edge e is added to M as follows. For an edge $e = \{u, v\}$, we define $\text{vote}^M(e) := \text{vote}_u^M(v) + \text{vote}_v^M(u)$. For loops, $\text{vote}^M(\{u, u\}) = 0$ if u is matched to itself in $\tilde{M}$, else $\text{vote}^M(\{u, u\}) = -1$. For any matching M' in G , it is easy to see that $\text{vote}^M(\tilde{M}') := \sum_{e \in \tilde{M}'} \text{vote}^M(e) = \Delta_{w=1}(M', M)$. This delivers the first characterization of popular matchings in instances with two-sided preferences.

Theorem 2 (Kavitha et al. [30]). *M is popular in G if and only if $\text{vote}^M(\tilde{M}') \leq 0$ holds for every perfect matching $\tilde{M}'$ in $\tilde{G}$.*

The second characterization follows from LP-duality and the fact that G is a bipartite graph.

Theorem 3 (Kavitha et al. [30, 26]). *A matching M in $G = (A \cup B, E)$, $(\succ_v)_{v \in V}$ is popular if and only if there exists a vector $\mathbf{y} \in \{0, \pm 1\}^n$ such that $\sum_{v \in V} \mathbf{y}_v = 0$,*

$$\mathbf{y}_a + \mathbf{y}_b \geq \text{vote}^M(\{a, b\}) \quad \forall \{a, b\} \in E \quad \text{and} \quad \mathbf{y}_v \geq \text{vote}^M(\{v, v\}) \quad \forall v \in V.$$

2.2.3 Instances with weighted voters

Ruangwises and Itoh [38] extended Theorem 2 to instances with weighted voters. They redefined the vote of $u \in V$ for $v \in N_G(u)$ as

$$\widetilde{\text{vote}}_u^M(v) := \begin{cases} 0 & \text{if } \{u, v\} \in M, \\ w(u) & \text{if } M(u) = \perp \text{ or } v \succ_u M(u), \text{ and} \\ -w(u) & \text{otherwise.} \end{cases}$$

For an edge $e = \{u, v\}$, Ruangwises and Itoh [38] defined $\widetilde{\text{vote}}^M(e) := \widetilde{\text{vote}}_u^M(v) + \widetilde{\text{vote}}_v^M(u)$. For a loop $e = \{v, v\}$, they set $\widetilde{\text{vote}}^M(e) = -w(v)$ if v is matched in M and $\widetilde{\text{vote}}^M(e) = 0$ otherwise. Note that for any matching M' in G , we have $\widetilde{\text{vote}}^M(\tilde{M}') := \sum_{e \in \tilde{M}'} \widetilde{\text{vote}}^M(e) = \Delta_w(M', M)$.

Theorem 4 (Ruangwises and Itoh [38]). *M is popular if and only if $\widetilde{\text{vote}}^M(\tilde{M}') \leq 0$ for every matching $\tilde{M}'$ in $\tilde{G}$, where $\tilde{G}$ is the input graph G augmented with a loop $\{v, v\}$ for every vertex $v \in V$, as defined in Section 2.2.2.*

3 Witness of popularity

We now extend Theorem 3 to two-sided instances with weighted voters. The proof is essentially analogous to the proof of Theorem 3: Start with the characterization of popular matchings from Theorem 4, formulate this as an LP using the standard bipartite matching LP and then dualize the LP.

Theorem 5 (★). *For POPULAR MATCHING WITH WEIGHTED VOTERS, matching M is popular if and only if there exists a vector $\mathbf{y} \in \mathbb{Q}^V$ with $\sum_{v \in V} \mathbf{y}_v = 0$ such that $\mathbf{y}_v + \mathbf{y}_u \geq \widetilde{\text{vote}}^M(e)$ for every edge $e = \{u, v\} \in E(G)$, $\mathbf{y}_v \geq 0$ for every vertex unmatched by M , and $\mathbf{y}_v \geq -w(v)$ for every $v \in V$.*

Proof. Using the standard linear program formulation for PERFECT BIPARTITE MATCHING [6] (see also [39, Theorem 18.1]) for the instance $\tilde{G}$ constructed in Section 2.2.2, we derive the equivalent linear program (1). Dualizing LP (1) results in LP (2).

$$\max \left\{ \sum_{e \in E(\tilde{G})} \widetilde{\text{vote}}^M(e) x_e \mid \sum_{e \in \delta_{\tilde{G}}(v)} x_e = 1 \ \forall v \in V, x_e \geq 0 \ \forall e \in E(\tilde{G}) \right\} \quad (1)$$

$$\min \left\{ \sum_{v \in V} y_v \mid y_v + y_u \geq \widetilde{\text{vote}}^M(\{u, v\}) \ \forall \{u, v\} \in E(G), y_v \geq \widetilde{\text{vote}}^M(\{v, v\}) \right\} \quad (2)$$

By the strong duality for linear programs (see e.g. [39, Theorem 5.4]), this LP admits a solution of cost 0 if and only if there exists a perfect matching of cost 0 in $\tilde{G}$, which is by Theorem 4 equivalent to M being popular. $\square$

Theorem 5 motivates the following definition of a witness:

Definition 2 (Witness). *Let $G = (V = A \cup B, E)$ together with preferences for each $v \in V$ and vertex weights $w : V \rightarrow \mathbb{Q}_{\geq 0}$ be an instance of POPULAR MATCHING WITH WEIGHTED VOTERS. Given a popular matching M , a witness of M is a vector $\mathbf{y} \in \mathbb{Q}^V$ such that $\sum_{v \in V} \mathbf{y}_v = 0$, $\mathbf{y}_a + \mathbf{y}_b \geq \widetilde{\text{vote}}^M(\{a, b\})$ for every edge $\{a, b\} \in E(G)$, $\mathbf{y}_v \geq 0$ if v is unmatched in M and $\mathbf{y}_v \geq -w(v)$ otherwise for every $v \in V$.*

For matching M and vector $\mathbf{y} \in \mathbb{Q}^V$, we call an edge $\{a, b\}$ conflicting if $\mathbf{y}_a + \mathbf{y}_b < \widetilde{\text{vote}}^M(\{a, b\})$.

By Theorem 5, a matching M is popular if and only if there exists a witness of M .

4 NP-hardness

In this section, we show that two highly restricted variants of POPULAR MATCHING WITH WEIGHTED VOTERS are NP-complete. The first variant assumes a set of non-unit weight vertices of constant size and only 2 kinds of weights for them. We reduce from the NP-complete problem of deciding, given an instance of POPULAR MATCHING (with two-sided preferences and unit weights) and two edges e_1 and e_2 , whether there is a popular matching containing both e_1 and e_2 [15]. The reduction consists of replacing each of e_1 and e_2 by the gadget from Figure 2, ensuring that every popular matching “contains” e_1 and e_2 (a formal.

Theorem 6 (★). *POPULAR MATCHING WITH WEIGHTED VOTERS is NP-complete, even if all but 14 vertices have weight one, and for each vertex, its weight is one, two, or four.*

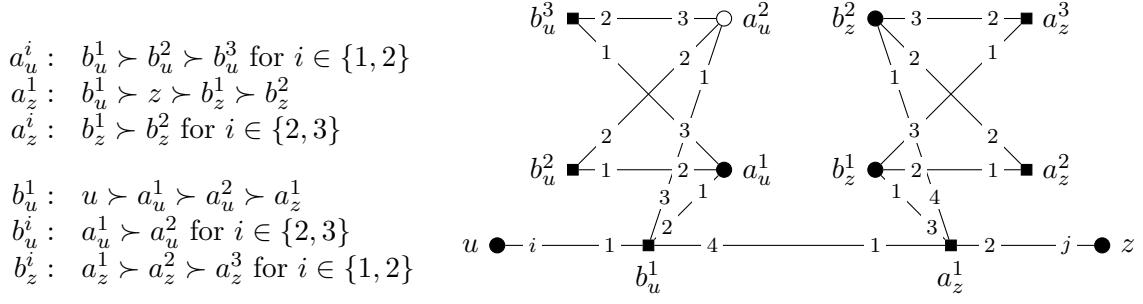


Figure 2: An example of the edge gadget for a forced edge $\{u, z\}$, where u ranks z at the i -th position, and z ranks u at the j -th position. Squared vertices have weight four, white vertices have weight two, and round black vertices have weight one.

Proof Sketch. Given an instance $\mathcal{I}$ of POPULAR MATCHING WITH TWO FORCED EDGES, we replace each forced edge $\{u, z\}$ by the gadget depicted in Figure 2. All agents not contained in such a gadget have weight one. We call the resulting instance $\mathcal{J}$.

Given a popular matching $M^\mathcal{I}$ in $\mathcal{I}$ containing the two forced edges, we construct a popular matching $M^\mathcal{J}$ in $\mathcal{J}$ by replacing each forced edge $\{u, z\}$ by the edges $\{a_u^2, b_u^2\}$, $\{a_u^1, b_u^3\}$, $\{a_z^2, b_z^2\}$, $\{a_z^3, b_z^1\}$, $\{u, b_u^1\}$, and $\{a_z^1, z\}$. To show that $M^\mathcal{J}$ is popular, one first shows that when comparing $M^\mathcal{J}$ to any other matching M' in $\mathcal{J}$, then summed weighted vote of the agents added in the two edge gadgets will never be in favor of M' . As a second step, we show that if there is a matching M' which is more popular than $M^\mathcal{J}$, then we may assume that M' contains either edges $\{u, b_u^1\}$ and $\{z, a_z^1\}$ or neither edge $\{u, b_u^1\}$ nor $\{z, a_z^1\}$ for every forced edge $\{u, z\}$. Consequently, if matching M' is more popular than $M^\mathcal{J}$, then also the matching M'' arising from M' by “inverting” the edge gadgets (i.e., removing the edges in the edge gadgets and adding edge $\{u, z\}$ whenever $\{u, b_u^1\}$ and $\{a_z^1, z\}$ are contained in M') is more popular than $M^\mathcal{I}$. Thus, the popularity of $M^\mathcal{I}$ implies the popularity of $M^\mathcal{J}$.

For the reverse direction, given a popular matching $M^\mathcal{J}$ in $\mathcal{J}$, the first step is to show that $M^\mathcal{J}$ contains $\{u, b_u^1\}$ and $\{a_z^1, z\}$ for every forced edge $\{u, z\}$. The matching $M^\mathcal{I}$ arising from $M^\mathcal{J}$ by inverting the edge gadgets is then a popular matching in $\mathcal{I}$ which contains every forced edge. $\square$

Our second result is valid on graphs where all vertices on one side have identical weights. This time, we reduce from 3-SAT. As usual, the reduction is based on variable and clause gadgets. However, the reduction also uses another gadget which we call “6-path gadget” (one might view this gadget also as part of the variable gadgets). The idea behind the 6-path gadget is to ensure that the two solutions from the variable gadgets do not weakly dominate each other (where weak dominance is defined as in Section 5.2.2).

Theorem 7 (★). *For any constant $1 < c \leq 2$, POPULAR MATCHING WITH WEIGHTED VOTERS is NP-complete even if $w(a) = c$ for all $a \in A$ and $w(b) = 1$ for all $b \in B$.*

In Section 5, we will show that POPULAR MATCHING WITH WEIGHTED VOTERS can be solved in linear time if all agents from A have weight c for some $c > 3$ while all agents from B have weight 1. For such weight functions, we have a strong inapproximability result for finding an optimal popular matching. In this problem, the input additionally contains a function $\omega : E \rightarrow \mathbb{Q}_{\geq 0}$ on the edges, and the goal is to find a popular matching maximizing $\sum_{e \in M} \omega(e)$ among all popular matchings. We show that the resulting problem is hard even to approximate by a reduction from INDEPENDENT SET. Note that this is in sharp contrast to the 2-approximation for the unit-vertex-weight case [15] and also with the optimal house allocation case with weighted voters, for which the results of Mestre [36] and McDermid and Irving [35] imply that an optimal popular matching can be computed in polynomial time.

Theorem 8 (★). *Unless $P=NP$, no $\varepsilon > 0$ exists such that there is a $O(n^{1-\varepsilon})$ -approximation for OPTIMAL POPULAR MATCHING WITH WEIGHTED VOTERS even if $w(a) = c > 3$ for all $a \in A$, $w(b) = 1$ for all $b \in B$, and every edge $e \in E$ has $\omega(e) \in \{0, 1\}$.*

5 Algorithm for different weights for A and B

Following up on Theorems 7 and 8, we now consider the case that all vertices from one side of the bipartition have a weight $c > 3$ while all vertices from the other side of the bipartition have weight 1. The main result of this section is that this restricted variant of POPULAR MATCHING WITH WEIGHTED VOTERS is solvable in polynomial time if $c > 3$. Before we describe the algorithm, we show some structural properties for the witnesses of popular matchings.

5.1 Structural properties of witnesses

We now restrict the value domain of possible witnesses. This technique has been used by Kavitha [27] upon defining truly popular matchings for non-bipartite instances.

Given an edge $e = \{a, b\} \in E$, we call a witness $\mathbf{y}$ of a popular matching M *tight on e* if and only if $\mathbf{y}_a + \mathbf{y}_b = \widehat{\text{vote}}^M(e)$. It follows from complementary slackness that every witness is tight on every edge contained in some popular matching.

Lemma 1 (★). *Let e be an edge which is contained in a popular matching M^* . For any popular matching M with witness $\mathbf{y}$, we have that $\mathbf{y}$ is tight on e .*

Next, we prove that for each popular matching, there exists a witness taking one of only six possible values at each vertex. More precisely, we can find a witness $\mathbf{y}$ with $\mathbf{y}_a \in R^a := \{-c, 1-c, 2-c, -1, 0, 1\}$ for all $a \in A$ and $\mathbf{y}_b \in R^b := \{-1, 0, 1, c-2, c-1, c\}$ for all $b \in B$. We call such a witness *nice*. The idea behind the proof of the existence of such a nice witness is that, given a witness $\mathbf{y}$, we can either increase $\mathbf{y}_a$ by some $\varepsilon > 0$ for all $a \in A$ with $\mathbf{y}_a$ being not nice (i.e. $\mathbf{y}_a \notin R^a$) and decrease $\mathbf{y}_b$ by some $\varepsilon > 0$ for all $b \in B$ with $\mathbf{y}_b \notin R^b$, or decrease $\mathbf{y}_a$ and increase $\mathbf{y}_b$, which results in another feasible witness that takes “nice” values at more vertices.

Lemma 2 (★). *Let M be a popular matching. Then there exists a nice witness of M .*

We call a witness containing only the values $-c, 2-c, -1, 1, c-2$, and c *odd*, while we call all witnesses containing only the values $1-c, 0, c-1$ *even*.

Lemma 3 (★). *Let M_1 and M_2 be popular matchings and C a connected component of $M_1 \triangle M_2$. Then every nice witness $\mathbf{y}$ of M_1 or M_2 on C is either even or odd.*

5.2 The algorithm

Having shown structural properties of witnesses, we can now describe the algorithm.

Our algorithm consists of the following three stages. Each of these stages is illustrated on an example in Appendix B.6.

1. Pruning non-popular edges

We reduce the set of edges that may appear in a popular matching to two incident edges per vertex. This allows us to decompose the graph containing the edges that may appear in a popular matching into a disjoint union of paths and cycles.

2. Computing “local” witnesses

For each path or cycle component, we show that we can restrict ourselves to up to four possible “local” witnesses of popularity.

3. Constructing a “global” witness

Our algorithm greedily recognizes local witnesses that cannot be part of a global witness. At termination, we have either constructed a global witness and thus found a popular matching, or dismissed all local witnesses for a path or cycle component. In the latter case, we conclude that no popular matching exists.

5.2.1 Pruning non-popular edges

We start by computing a subgraph of degree at most two at each vertex that contains every popular edge. First, we show that for every vertex $a \in A$, there are at most two other vertices to which a can be matched in a popular matching, namely $f(a)$ and $s(a)$. To do so, given an instance $\mathcal{I}$, we show that each popular matching is also popular in the instance $\mathcal{I}'$ where the weight of each vertex in B is reduced to 0, while the rest of $\mathcal{I}'$ equals $\mathcal{I}$. Since $w(a) = c$ for each $a \in A$, this modified instance is equivalent to a house allocation instance, for which there is a characterization of popular matchings that relies on $f(a)$ and $s(a)$ vertices, as presented in Theorem 1. We remark that this key lemma does not hold anymore when vertices from A may have different weights or weight smaller or equal to three.

Lemma 4. *Let $\mathcal{I}$ be an instance of POPULAR MATCHING WITH WEIGHTED VOTERS such that $w(a) = c > 3$ for all $a \in A$ and $w(b) = 1$ for all $b \in B$. Any popular matching M in the instance $\mathcal{I}$ is also popular in the instance $\mathcal{I}'$ that arises from $\mathcal{I}$ by setting $w(b) = 0$ for all $b \in B$.*

Proof. Let M be a popular matching in $\mathcal{I}$, and assume for a contradiction that there is a matching that is more popular than M in $\mathcal{I}'$. Then, by Theorem 1, there exists a vertex $a \in A$ not matched to its f -post while its f -post is unmatched, a vertex $a \in A$ not matched to its f -post $f(a)$ while $f(a)$ is matched to a vertex a' with $f(a') \neq f(a)$, or a vertex $a \in A$ matched neither to its f -post nor its s -post. In the first case, matching a to its f -post clearly results in a more popular matching. In the second case, matching a to its f -post $f(a)$ and a' to its f -post $f(a')$ (possibly breaking up the matching of $f(a')$) results in a more popular matching M' since a and a' prefer M' while at most one vertex in A , namely $M(f(a'))$, and three vertices in B , namely $M(a)$, $f(a)$, and $f(a')$, can prefer M . So assume that the third case applies and neither of the first two cases applies for any vertex. Note that this implies that a prefers $s(a)$ to $M(a)$. We construct a matching M^* more popular than M in $\mathcal{I}$ from M by matching a to its s -post $s(a)$, and if $s(a)$ is matched by M to a vertex a' , then also matching a' to its f -post $f(a')$ (possibly breaking up the matching of $f(a')$). Note that $f(a') \neq s(a)$ by the definition of an s -post. Matching M^* is more popular matching M since a and a' prefer M while at most one vertex in A , namely $M(f(a'))$, and three vertices in B , namely $M(a)$, $s(a)$, and $f(a')$, can prefer M . $\square$

We now define a subgraph $H^{\deg(A) \leq 2}$ of G that contains all edges of the form $\{a, f(a)\}$ or $\{a, s(a)\}$ for some $a \in A$ —no other edge can appear in a popular matching by Theorem 1 and Lemma 4. Next we also restrict the degree of each vertex in B to at most two. For every cycle C in $H^{\deg(A) \leq 2}$, Lemma 4 implies that every vertex in C has to be matched along an edge in C and consequently, we can delete every edge incident to a vertex in C if this edge is not in $E(C)$. This reduces $H^{\deg(A) \leq 2}$ to a graph $H^{C+\mathcal{T}}$ that is a disjoint union of cycles and trees. For every tree T , one can show that for every vertex $b \in B$, there are at most two edges incident to b that appear in a popular matching (see Claim 16 in the appendix for the proof). For every component of $H^{C+\mathcal{T}}$ that is a tree, we can compute the set of edges contained in a popular matching (in the instance restricted to solely this tree) via bottom-up induction, resulting in a graph $H^{C+\mathcal{P}}$, which contains every popular edge and whose maximum degree is two by Claim 16. Therefore, $H^{C+\mathcal{P}}$ is a disjoint union of paths and cycles, and furthermore, every connected component of $H^{C+\mathcal{P}}$ consists of two disjoint popular matchings.

Lemma 5 (★). *We can compute in $O(n + m)$ time a subgraph H^{C+P} of G such that H^{C+P} contains every popular edge, H^{C+P} is a disjoint union of paths and cycles, and every edge in H^{C+P} is contained in a popular matching in H^{C+P} .*

While restricting the set of popular edges to disjoint paths and cycles as in Lemma 5 is a severe restriction, we remark that this is not sufficient to gain tractability: For the instance constructed in the reduction in Theorem 7, we can compute such a subgraph, but the instance is anyway hard.

5.2.2 Computing “local” witnesses

We first restrict for each connected component of H^{C+P} the set of possible nice witnesses, and afterwards dismiss possible witnesses one by one, until we either found a popular matching or have no witness remaining for a connected component and conclude that no popular matching exists. The idea here is to order the nice witnesses (and the matching whose popularity the witness proves) by how “good” they are on the vertices in B . Given such an order, we can apply the following algorithm: At the beginning, we assign to each connected component C the witness that is optimal for $B \cap V(C)$. This induces a matching M by taking for every connected component the matching whose popularity is verified by the corresponding witness. Whenever a conflicting edge, i.e. an edge $\{a, b\}$ with $\widetilde{\text{vote}}^M(a) + \widetilde{\text{vote}}^M(b) > \mathbf{y}_a + \mathbf{y}_b$, occurs, we know that the only way to eliminate this conflict is to change the witness (and possibly also the matching) on the connected component C_a containing a , or, in one special case, the witness on the connected component containing b . Therefore, we dismiss the current witness for C_a and replace it with the next witness in the order we set up on them. This procedure eventually terminates with a feasible witness (and thus also a popular matching), or dismisses all witnesses for a connected component of H^{C+P} . In the latter case, we know that no popular matching exists.

We now describe how we order the witnesses for one connected component of H^{C+P} . Consider a connected component C of H^{C+P} , two popular matchings M_1 and M_2 , and two witnesses $\mathbf{y}^1$ and $\mathbf{y}^2$ of them. If we have that (i) $\mathbf{y}_b^1 \geq \mathbf{y}_b^2 + 2$ or (ii) $\mathbf{y}_b^1 \geq \mathbf{y}_b^2$ and $M_1 \succeq_b M_2$, then $\mathbf{y}^1$ is clearly better than $\mathbf{y}^2$ for b , i.e. any conflicting edge $\{a, b\}$ for $\mathbf{y}^1$ will also be conflicting for $\mathbf{y}^2$. In this case, we also say that $\mathbf{y}^1$ *dominates* $\mathbf{y}^2$ at b . If $\mathbf{y}^1$ dominates $\mathbf{y}^2$ at every $b \in B \cap C$, then we say that $\mathbf{y}^1$ *dominates* $\mathbf{y}^2$. Note that we show in the appendix (Claims 20 and 21) that domination at a vertex $b \in B$ already implies domination on the whole connected component of H^{C+P} . We will show that, using this dominance relationship, we can order all odd witnesses. However, there may be pairs of odd and even witnesses that do not dominate each other, e.g. it may happen that $\mathbf{y}^1$ is even, $\mathbf{y}^2$ is odd, $\mathbf{y}_b^1 = \mathbf{y}_b^2 + 1$ and $M_2 \succ_b M_1$. When considering only conflicting edges $\{a, b\}$ (note that these conflicting edges may be between different components of H^{C+P}) for which the witness in the component containing a is odd, it is irrelevant whether $\mathbf{y}_b = \mathbf{y}_b^1$ or $\mathbf{y}_b = \mathbf{y}_b^2$ holds because $\widetilde{\text{vote}}_a^M(b) + \widetilde{\text{vote}}_b^M(a) \in \{-1 - c, 1 - c, c - 1, 1 + c\}$ for every matching M . Therefore, if $\mathbf{y}^1$ is even and $\mathbf{y}^2$ dominates $\mathbf{y}^1 - \mathbf{1}$, then we say that $\mathbf{y}^2$ *weakly dominates* $\mathbf{y}^1$ at b . We now formalize the implications of the (weak) dominance relation.

Lemma 6. *Let C and D be two connected components of H^{C+P} , and let $\mathbf{y}^C$ and $\mathbf{y}^D$ be nice witnesses for two popular matchings M^C and M^D on C and D . Assume that there is a conflicting edge $\{a, b\}$ for matching $M^C \cup M^D$ with $a \in V(C) \cap A$ and $b \in V(D) \cap B$.*

If $\mathbf{y}^C$ is odd, then $\{a, b\}$ is also conflicting when exchanging $\mathbf{y}^D$ for a witness that $\mathbf{y}^D$ weakly dominates.

Proof. Let $\mathbf{y}$ be a witness of a popular matching M on D that is dominated by $\mathbf{y}^D$. If $\mathbf{y}_b^D \geq \mathbf{y}_b + 2$, then $\mathbf{y}_a^C + \mathbf{y}_b \leq \mathbf{y}_a + \mathbf{y}_b^D - 2 < \widetilde{\text{vote}}_a^{M^C}(b) + \widetilde{\text{vote}}_b^{M^D}(a) - 2 \leq \widetilde{\text{vote}}_a^{M^C}(b) + \widetilde{\text{vote}}_b^M(a) + 2 - 2 = \widetilde{\text{vote}}_a^{M^C}(b) + \widetilde{\text{vote}}_b^M(a)$ and consequently $\{a, b\}$ is also conflicting when exchanging $\mathbf{y}^D$ for $\mathbf{y}$.

If $\mathbf{y}_b^D \geq \mathbf{y}_b$ and b does not prefer $M(b)$ to $M^D(b)$, then $\mathbf{y}_a^C + \mathbf{y}_b \leq \mathbf{y}_a + \mathbf{y}_b^D < \widetilde{\text{vote}}_a^{M^C}(b) + \widetilde{\text{vote}}_b^{M^D}(a) \leq \widetilde{\text{vote}}_a^{M^C}(b) + \widetilde{\text{vote}}_b^M(a)$ and consequently $\{a, b\}$ is also conflicting when exchanging $\mathbf{y}^D$ for $\mathbf{y}$.

If $\mathbf{y}^D$ weakly dominates $\mathbf{y}$ but $\mathbf{y}^D$ does not dominate $\mathbf{y}$, then $\mathbf{y}$ is even and $\mathbf{y}^D + \mathbf{1}$ dominates $\mathbf{y}$. The above arguments yield $\mathbf{y}_a^C + \mathbf{y}_b < \widetilde{\text{vote}}_a^{M^C}(b) + \widetilde{\text{vote}}_b^M(a) - 1$. Since $\mathbf{y}^C$ is odd and $\mathbf{y}$ is even, we have $\mathbf{y}_a^C \in \{-c, 2-c, -1, 1\}$ and $\mathbf{y}_b \in \{0, c-1\}$. It follows that $\mathbf{y}_a^C + \mathbf{y}_b \in \{-c, 2-c, -1, 1, c-2, c\}$, while $\widetilde{\text{vote}}_a^{M^C}(b) + \widetilde{\text{vote}}_b^M(a) \in \{-1-c, 1-c, c-1, 1+c\}$. This implies that even $\mathbf{y}_a^C + \mathbf{y}_b < \widetilde{\text{vote}}_a^{M^C}(b) + \widetilde{\text{vote}}_b^M(a)$ holds, proving the lemma. $\square$

Next, we show that we can restrict the set of possible nice witnesses for every connected component in such a way that the witnesses are in a weak dominance relation to each other. Note that we use here that for every connected component C of $H^{C+\mathcal{P}}$, each edge in C is contained in at least one popular matching in C .

Lemma 7 (★). *For each connected component C of $H^{C+\mathcal{P}}$, we can compute in linear time a set Y of up to four nice witnesses with the properties below (depending on whether C contains a single edge, a path, or a cycle), such that for any feasible witness $\mathbf{y}$ of a popular matching M , there exists a witness $\mathbf{y}' \in Y$ that can replace the values $\mathbf{y}_v$ for every $v \in V(C)$.*

- If C contains only a single edge e , then
 - $Y = \{\mathbf{y}^{e,1}, \mathbf{y}^{e,2}, \mathbf{y}^{e,\text{even}}\}$, where $\mathbf{y}^{e,1}$ and $\mathbf{y}^{e,2}$ are odd and $\mathbf{y}^{e,\text{even}}$ is even, and
 - $\mathbf{y}^{e,1}$ dominates $\mathbf{y}^{e,2}$ and $\mathbf{y}^{e,2}$ weakly dominates $\mathbf{y}^{e,\text{even}}$.
- If C is a path containing at least three edges, then
 - $Y = \{\mathbf{y}^{P,\text{odd}}, \mathbf{y}^{P,\text{even}}\}$ where $\mathbf{y}^{P,\text{odd}}$ is odd and $\mathbf{y}^{P,\text{even}}$ is even, and
 - $\mathbf{y}^{P,\text{odd}}$ weakly dominates $\mathbf{y}^{P,\text{even}}$.
- If C is a cycle, then
 - $Y = \{\mathbf{y}^{C,1}, \mathbf{y}^{C,2}\}$ or $Y = \{\mathbf{y}^{C,1}, \mathbf{y}^{C,2}, \mathbf{y}^{C,0}, \mathbf{y}^{C,\text{even}}\}$ where $\mathbf{y}^{C,0}$, $\mathbf{y}^{C,1}$, and $\mathbf{y}^{C,2}$ are odd and $\mathbf{y}^{C,\text{even}}$ is even, and
 - $\mathbf{y}^{C,1}$ dominates $\mathbf{y}^{C,2}$ or $\mathbf{y}^{C,1}$ dominates $\mathbf{y}^{C,0}$, witness $\mathbf{y}^{C,0}$ dominates $\mathbf{y}^{C,2}$, and $\mathbf{y}^{C,2}$ weakly dominates $\mathbf{y}^{C,\text{even}}$.

We remark that Lemma 7 requires that each agent from A has the same weight and each agent from B has the same weight (see Appendix C for an example where agents from B have different weights and the witnesses are not in a dominance relation). Considering instances with $w(a) = c' \leq 3$ for every $a \in A$ for some constant c' and $w(b) = 1$ for every $b \in B$, the reduction from Theorem 7 shows that there is no dominance relation between witnesses when $w(a) = c \leq 2$ for every $a \in A$, although the instance constructed in the reduction admits a subgraph which contains every popular edge and is a disjoint union of paths and cycles.

5.2.3 Constructing a “global” witness

Lemmas 5 and 7 lead to the following algorithm. We start by assigning to each edge component the odd witness $\mathbf{y}^{e,1}$, to each path component $\mathbf{y}^{P,\text{odd}}$, and to each cycle component $\mathbf{y}^{C,1}$. Whenever we encounter a conflicting edge $\{a, b\}$, we distinguish two cases: If the witness in the component containing a is even, then this will be the only witness remaining for the component containing a and we dismiss the witness in the component C containing b . Otherwise, we dismiss the witness in the component C containing a . In both cases, we assign a new witness (together with a matching) to C as follows: If there is an undismissed odd witness of C , then

we assign this witness to C (the only case when this witness is not unique is if C is a cycle component and neither $\mathbf{y}^{C,2}$ nor $\mathbf{y}^{C,0}$ have been dismissed; in this case we assign $\mathbf{y}^{C,0}$ to C). If no undismissed odd witness of C exists, then we assign the even witness to C . If we eventually dismissed all witnesses for one component, then we conclude that there is no popular matching. Otherwise, we eventually found a matching together with a witness of it, and we return this matching.

Theorem 9. *If all vertices in A have weight c for some $c > 3$ and every vertex in B has weight 1, then a maximum-cardinality popular matching (if one exists) can be computed in $O(n+m)$ time.*

Proof. The computation of $H^{C+\mathcal{P}}$ takes $O(n+m)$ time due to Lemma 5. There are $O(n)$ connected components of $H^{C+\mathcal{P}}$; thus at most $O(n)$ partial witnesses get dismissed. As we only need to check whether an edge is conflicting if the witness on one of the two connected component incident to it changed, we need to check for each edge only a constant number of times whether it is conflicting. From this, the running time follows.

To show the correctness of the algorithm, first observe that if a component is assigned its even witness, then it is the only remaining witness that has not been dismissed. We will now show that whenever a witness $\mathbf{y}^{\text{sub}}$ is dismissed, then there is no popular matching together with a witness $\mathbf{y}$ such that $\mathbf{y}_v^{\text{sub}} = \mathbf{y}_v$ for every $v \in V(C)$. At the beginning, the statement is clear since no witness has been dismissed. Consider the first witness $\mathbf{y}^{\text{sub}}$ of a connected component C deleted by the algorithm that is a subwitness of a witness $\mathbf{y}$ of some matching, and let $\{a, b\}$ be the conflicting edge because of which $\mathbf{y}^{\text{sub}}$ was dismissed. If $a \in V(C)$, then $\mathbf{y}^{\text{sub}}$ is odd and the current witness in the component C^b containing b weakly dominates all other non-dismissed witnesses on this component by Lemma 7. Note that $\mathbf{y}$ restricted to C^b is not dismissed by the choice of $\mathbf{y}^{\text{sub}}$. Consequently, Lemma 6 implies that $\{a, b\}$ is conflicting for $\mathbf{y}$, a contradiction.

If $b \in V(C)$, then the witness in the component containing a is even and consequently the only non-dismissed witness $\mathbf{y}^{\text{sub}}$ on this component. Therefore, by the definition of $\mathbf{y}^{\text{sub}}$, every witness $\mathbf{y}$ of some popular matching coincides with $\mathbf{y}_a^{\text{sub}}$ on a , and therefore cannot coincide with $\mathbf{y}^{\text{sub}}$ on b (as otherwise the edge $\{a, b\}$ would be conflicting for $\mathbf{y}$).

Finally, we show that there is no larger popular matching than the computed one. For every connected component that is a path, the witness corresponding to the smaller matching is even. Because we initially assign the odd witness (and thus the larger matching) to this connected component, the computed matching will contain the smaller matching only if no popular matching contains the larger matching on C . Thus, the computed matching is of maximum cardinality among all popular matchings. $\square$

6 Future directions

We discovered an unusual pattern in the complexity of POPULAR MATCHING WITH WEIGHTED VOTERS in instances where the weights on one side are fixed at 1. A solution is guaranteed to exist and easy to find if the weight c of the other side slides equals 1. Then, for $1 < c \leq 2$, the problem becomes NP-complete as Theorem 7 shows. Polynomial-time solvability then returns for $3 < c$, but no-instances occur. A straightforward open question is whether POPULAR MATCHING WITH WEIGHTED VOTERS can be solved in polynomial time when all vertices in A have weight $2 < c \leq 3$ and all vertices in B have weight 1.

Furthermore, it would be interesting whether allowing each vertex from one side an individual weight larger than three (while all vertices from the other side still have weight one) is also solvable in polynomial time.

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A Proofs from Section 4

A.1 Proof of Theorem 6

For a set of vertices $X \subseteq V$ and two matchings M and M' , we say that X *prefers* M to M' if $\sum_{v \in V^+(M, M') \cap X} w(v) - \sum_{v \in V^+(M', M) \cap X} w(v) > 0$.

Theorem 6 (★). POPULAR MATCHING WITH WEIGHTED VOTERS is NP-complete, even if all but 14 vertices have weight one, and for each vertex, its weight is one, two, or four.

Proof. By Theorem 4, POPULAR MATCHING WITH WEIGHTED VOTERS is in NP, as a popular matching can be used as a certificate. To show NP-hardness, we reduce from POPULAR MATCHING WITH TWO FORCED EDGES, which was shown to be NP-complete by Faenza et al. [15].

POPULAR MATCHING WITH TWO FORCED EDGES

Input: A popular matching instance and a set F of two edges.

Question: Does there exist a popular matching M with $F \subseteq M$?

We can assume without loss of generality that F is a matching, as otherwise no matching can contain F . Let $\mathcal{I} = (G = (A \cup B, E), (\succ_v)_{v \in A \cup B}, F)$ be an instance of this problem. **Construction.** We replace each forced edge $e = \{u, z\} \in F$, where $u \in A$ and $z \in B$, by an edge gadget. The edge gadget contains ten vertices $b_u^1, b_u^2, b_u^3, a_u^1, a_u^2$ and $a_z^1, a_z^2, a_z^3, b_z^1, b_z^2$, as depicted in Figure 2, and it is asymmetric in u and z . We set $w(v) = 1$ for all $v \in A \cup B$, $w(b_u^i) = 4 = w(a_z^i)$, $w(a_u^1) = 1$, $w(a_u^2) = 2$, and $w(b_z^1) = 1$. We call the resulting instance $\mathcal{J}$.

Note that all agents outside the two edge gadgets have weight one, while each edge gadget contains one agent of weight four and six agents of weight two. Thus, all but 14 vertices have weight one, and the remaining vertices have weight two or four.

We define a “projection” π from the matchings in $\mathcal{I}$ to the matchings in $\mathcal{J}$ via

$$\begin{aligned} \pi(M) := & (M \setminus F) \cup \{\{a_u^2, b_u^2\}, \{a_u^1, b_u^3\}, \{a_z^2, b_z^2\}, \{a_z^3, b_z^1\} : \{u, z\} \in F\} \cup \\ & \{\{u, b_u^1\}, \{a_z^1, z\} : \{u, z\} \in F \cap M\} \cup \{\{a_z^1, b_u^1\} : \{u, z\} \in F \setminus M\}. \end{aligned}$$

Intuitively, given a matching M in $\mathcal{I}$, the function π computes an “equivalent” matching $\pi(M)$ in the instance $\mathcal{J}$. Equivalent here means that M is popular if and only if $\pi(M)$ is popular, which we will prove later, in Claims 3 and 4. Furthermore, we define a “projection” ρ from the matchings in $\mathcal{J}$ to the matchings in $\mathcal{I}$ via $\rho(M') = (M' \cap E(\mathcal{I})) \cup \{\{u, z\} \in F : \{u, b_u^1\} \in M' \wedge \{a_z^1, z\} \in M'\}$. Intuitively, ρ is the inverse of π . Given a matching $M_{\mathcal{J}}$ in $\mathcal{J}$, matching $\rho(M_{\mathcal{J}})$ is popular if and only if $M_{\mathcal{J}}$ is popular in $\mathcal{J}$, as we will see later. Note that $\rho(\pi(M)) = M$ for any matching M in $\mathcal{I}$.

Before proving the correctness of the reduction, we give two helpful claims. First, we show that for every matching M in $\mathcal{I}$ containing the forced edges, in order to check $\pi(M)$ for popularity in $\mathcal{J}$, we only have to compare it with a subset of other matchings.

Claim 1. Let M be a matching in $\mathcal{I}$ containing both forced edges, and M' be any matching in $\mathcal{J}$. Let $X := \{b_u^i, a_z^i : i \in [3]\}$ and $Y := \{a_u^j, b_z^j : j \in [2]\}$ for a forced edge $\{u, z\}$. Then $X \cup Y$ does not prefer M' to $M_{\mathcal{J}} := \pi(M)$.

If M' is more popular than $M_{\mathcal{J}}$, then in $\mathcal{J}$ there exists a matching M^* that is more popular than $M_{\mathcal{J}}$ and for each forced edge $\{u, z\}$, the matching M^* contains either the edge $\{a_z^1, b_u^1\}$ or the edges $\{u, b_u^1\}$ and $\{a_z^1, z\}$.

Proof. First, we show that $X \cup Y$ does not prefer M' to $M_{\mathcal{J}}$. In order to do so, we make a case distinction on whether $\{a_z^1, b_u^1\}$ is in M' . If $\{a_z^1, b_u^1\} \in M'$, then a_z^1 prefers M' while b_u^1

prefers $M_{\mathcal{J}}$, and thus, their votes cancel out. Furthermore, if $\{a_u^1, b_u^2\} \in M'$, then a_u^1 and b_u^2 prefer M' , while b_u^3 and a_u^2 prefer $M_{\mathcal{J}}$, and otherwise none of the vertices a_u^1 , a_u^2 , b_u^2 , and b_u^3 prefers M' . Consequently, $\{a_u^1, a_u^2, b_u^2, b_u^3\}$ does not prefer M' to $M_{\mathcal{J}}$. Symmetric arguments show that $\{a_z^3, a_z^3, b_z^1, b_z^2\}$ does not prefer M' , and therefore, $X \cup Y$ does not prefer M' to $M_{\mathcal{J}}$.

If $\{a_z^1, b_u^1\} \notin M'$, then neither a_z^1 nor b_u^1 prefers M' to $M_{\mathcal{J}}$. We first look at vertices a_u^1 , a_u^2 , b_u^1 , b_u^2 , and b_u^3 . If $\{u, b_u^1\} \in M'$, then we already saw in the case of $\{a_z^1, b_u^1\} \in M'$ that $\{a_u^1, a_u^2, b_u^2, b_u^3\}$ does not prefer M' to $M_{\mathcal{J}}$. Otherwise b_u^1 prefers $M_{\mathcal{J}}$ to M' . Note that b_u^2 can prefer M' to $M_{\mathcal{J}}$ only if $\{a_u^1, b_u^2\} \in M'$, which implies that $M_{\mathcal{J}} \succ_{b_u^3} M'$. Consequently, vertices b_u^1 , b_u^2 , and b_u^3 vote for $M_{\mathcal{J}}$ by weight at least 4, while a_u^1 and a_u^2 only cast votes of weight 3. Therefore, $\{a_u^1, a_u^2, b_u^1, b_u^2, b_u^3\}$ does not prefer M' to $M_{\mathcal{J}}$. Symmetric arguments show that also $\{a_z^1, a_z^2, a_z^3, b_z^1, b_z^2\}$ does not prefer M' to $M_{\mathcal{J}}$.

We now prove the second part of the claim. Let $\{u, z\}$ be a forced edge. First, we consider the case that M' contains neither $\{u, b_u^1\}$ nor $\{a_z^1, b_u^1\}$. Then the matching M'' arising from M' by adding $\{a_z^1, b_u^1\}$ to M' (and possibly deleting edges incident to a_z^1 and b_u^1) results in a matching more popular than $M_{\mathcal{J}}$: Agent b_u^1 prefers $M_{\mathcal{J}}$ to both M' and M'' . Agent a_z^1 prefers M'' to $M_{\mathcal{J}}$ and $M_{\mathcal{J}}$ to M' while at most $M'(a_z^1)$ and $M'(b_u^1)$ prefer M' to $M_{\mathcal{J}}$ and $M_{\mathcal{J}}$ to M'' . Consequently, we have

$$\Delta_w(M'', M_{\mathcal{J}}) \leq \Delta_w(M', M_{\mathcal{J}}) - 2w(a_z^1) + 2w(M'(a_z^1)) + 2w(M'(b_u^1)) < \Delta_w(M', M_{\mathcal{J}}) < 0$$

implying that also M'' is more popular than $M_{\mathcal{J}}$.

Finally, we consider the case that M' contains $\{u, b_u^1\}$ but not $\{a_z^1, z\}$. Then the matching M'' arising from M' by adding $\{a_z^1, b_u^1\}$ to M' (and deleting $\{u, b_u^1\}$ as well as possibly an edge incident to a_z^1) is also more popular than $M_{\mathcal{J}}$: Agent b_u^1 is indifferent between $M_{\mathcal{J}}$ and M' but prefers $M_{\mathcal{J}}$ to M'' . Agent a_z^1 prefers M'' to $M_{\mathcal{J}}$ but prefers $M_{\mathcal{J}}$ to M' . At most $M'(a_z^1)$ and $M'(b_u^1) = u$ prefer $M_{\mathcal{J}}$ to M'' while preferring M' to $M_{\mathcal{J}}$. Consequently, we have

$$\Delta_w(M'', M_{\mathcal{J}}) \leq \Delta_w(M', M_{\mathcal{J}}) - 2w(a_z^1) + w(b_u^1) + 2w(M'(a_z^1)) + 2w(M'(b_u^1)) = \Delta_w(M', M_{\mathcal{J}}) < 0$$

implying that also M'' is more popular than $M_{\mathcal{J}}$. $\square$

We now show that “projecting” two matchings M and M^* from $\mathcal{I}$ to $\mathcal{J}$ via π does not influence whether M is more popular than M^* or not. This will be used to prove that for a popular matching M' in $\mathcal{J}$, matching $\rho(M')$ is popular in $\mathcal{I}$.

Claim 2. *For any two matchings M and M^* in $\mathcal{I}$, we have $\Delta(M, M^*) = \Delta_w(\pi(M), \pi(M^*))$.*

Proof. Consider a forced edge $\{u, z\}$ that appears in M , but not in M^* . Then $M \succ_{b_u^1} M^*$ and $M^* \succ_{a_z^1} M$ hold. Similarly, if the forced edge $\{u, z\}$ is contained in M^* , but not in M , then $M^* \succ_{b_u^1} M$ and $M \succ_{a_z^1} M^*$ hold.

Since b_u^1 and a_z^1 have the same weight, and all other vertices are matched identically in M and M^* , the claim follows. $\square$

Correctness. We are now ready to prove the correctness of the reduction.

Claim 3. *For each M in $\mathcal{I}$ that is a popular matching containing the set F of forced edges, there is a popular matching in $\mathcal{J}$.*

Proof. Let M be a popular matching in $\mathcal{I}$ containing the set F of forced edges. We claim that $M_{\mathcal{J}} := \pi(M)$ is popular in $\mathcal{J}$. Assume for a contradiction that $M_{\mathcal{J}}$ loses to a matching M' . By Claim 1, we may assume that for each forced edge $\{u, z\}$, matching M' contains either edges $\{u, b_u^1\}$ and $\{a_z^1, z\}$ or edge $\{a_z^1, b_u^1\}$. We define $M'_{\mathcal{I}} := \rho(M')$, and claim that $M'_{\mathcal{I}}$ wins against M , a contradiction to the popularity of M .

Since M' contains either edges $\{u, b_u^1\}$ and $\{a_z^1, z\}$ or edge $\{a_z^1, b_u^1\}$ for every forced edge $\{u, z\}$, every vertex in $A \cup B$ votes between M and $M'_{\mathcal{I}}$ the same as between $M_{\mathcal{J}}$ and M' . Furthermore,

the set S of vertices from the edge gadget do not prefer M' to $M_{\mathcal{J}}$ by Claim 1. It follows that the vertices in V prefer M' to $M_{\mathcal{J}}$, and thus, $M'_{\mathcal{I}}$ is more popular than M . $\square$

Claim 4. *For each popular matching M in $\mathcal{J}$, there is a popular matching in $\mathcal{I}$ that contains the set F of forced edges.*

Proof. Let M be a popular matching in $\mathcal{J}$. We first show by case distinction that for each forced edge $\{u, z\} \in F$, matching M contains edges $\{u, b_u^1\}$ and $\{a_z^1, z\}$. We assume for a contradiction that M does not contain these edges, and distinguish three cases. For each of the three cases, we construct a more popular matching, contradicting the popularity of M .

Case 1: Vertex b_u^1 is matched neither to u nor to a_z^1 .

In this case, we construct a more popular matching M' by replacing the edges $\{a_u^1, M(a_u^1)\}$ and $\{a_u^2, M(a_u^2)\}$ by the edges $\{a_u^1, M(a_u^2)\}$ and $\{a_u^2, b_u^j\}$, where b_u^j is unmatched in M . The matching M' is more popular than M as b_u^j and $M(a_u^2)$ prefer M' while at most $M(a_u^1)$, vertex a_u^1 , and a_u^2 prefer M to M' , and it holds that $w(b_u^j) + w(M(a_u^2)) = 8 > w(M(a_u^1)) + w(a_u^1) + w(a_u^2) = 7$.

Case 2: Vertex a_z^1 is matched neither to z nor to b_u^1 .

This case is symmetric to Case 1.

Case 3: Vertex b_u^1 is matched to a_z^1 .

In this case, we construct a more popular matching M' by matching b_u^1 to a_u^2 , and a_u^1 to $M(a_u^2)$, as $M(a_u^2)$, vertex b_u^1 , and a_u^2 prefer M' , while at most a_z^1 , vertex b_u^j for at most one $j \in \{2, 3\}$, and a_u^1 prefer M , and $w(b_u^1) = w(a_z^1) = w(M(a_u^2)) = w(b_u^j)$, and $w(a_u^2) > w(a_u^1)$.

Thus, $M_{\mathcal{I}} := \rho(M)$ contains every forced edge. It remains to show that $M_{\mathcal{I}}$ is popular. For a contradiction, assume that M' is more popular than $M_{\mathcal{I}}$. We claim that $M'_{\mathcal{J}} := \pi(M')$ wins against M , contradicting the popularity of M .

For every forced edge $\{u, z\}$, matching M has to contain the edges $\{a_u^2, b_u^2\}, \{a_u^1, b_u^3\}$, as else matching b_u^1 and b_u^2 along these edges yields a more popular matching. Symmetrically, M contains edges $\{a_z^2, b_z^2\}$ and $\{a_z^3, b_z^1\}$. Thus, we have $M = \pi(M_{\mathcal{I}})$. Since M' is more popular than M and by Claim 2, we have $0 < \Delta_w(M_{\mathcal{I}}, M') = \Delta_w(M, M'_{\mathcal{J}})$, contradicting the popularity of M . $\square\square$

A.2 Proof of Theorem 7

Theorem 7 (★). *For any constant $1 < c \leq 2$, POPULAR MATCHING WITH WEIGHTED VOTERS is NP-complete even if $w(a) = c$ for all $a \in A$ and $w(b) = 1$ for all $b \in B$.*

Proof. Membership in NP was shown in Theorem 4. Fix a constant $1 < c \leq 2$. To show NP-hardness, we reduce from 3-SAT, the restriction of SATISFIABILITY to instances where every clause contains exactly three literals. Let X be the set of variables.

Construction. We first describe the 3 types of gadgets in our reduction.

6-path gadget. The 6-path gadget consists of a path on 6 vertices a_1, a_2, a_3, b_1, b_2 , and b_3 . The preferences of the vertices are described in Figure 3 (where vertices a_x for every $x \in X$ will be defined when describing variable gadgets).

Variable gadgets. An example of a variable gadget is depicted in Figure 4. The variable gadget for a variable x contains four vertices $a_x, \bar{a}_x, b_x^t$, and b_x^f . Vertex a_x as well as $\bar{a}_x$ prefers b_x^t to b_x^f , and vertices b_x^t and b_x^f prefer a_x to $\bar{a}_x$. Intuitively, matching a_x to b_x^t (and therefore also $\bar{a}_x$ to b_x^f) corresponds to setting variable x to true, while matching $\bar{a}_x$ to b_x^t (and therefore also a_x to b_x^f) corresponds to setting variable x to false.

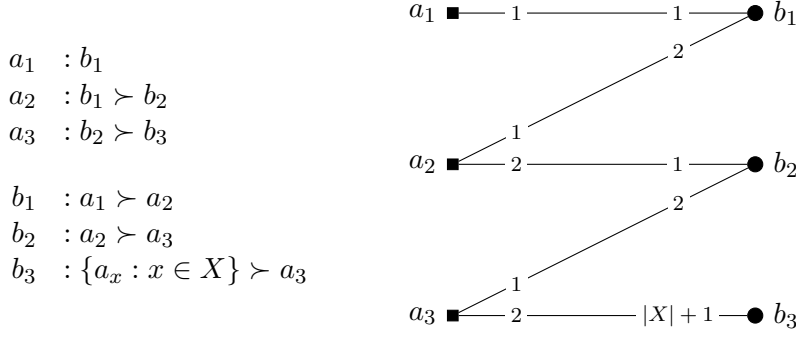


Figure 3: The 6-path gadget. Vertices in A are squared.

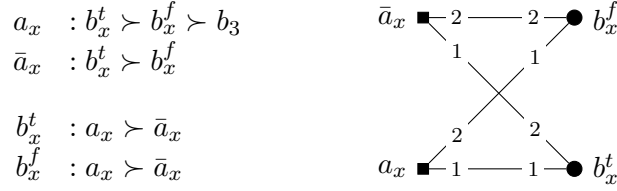


Figure 4: The variable gadget. Vertices in A are squared.

Every variable gadget is connected to the 6-path gadget via the edge $\{a_x, b_3\}$, where a_x prefers both b_x^f and b_x^t to b_3 , and b_3 prefers a_x to a_3 (the preferences of b_3 between two vertices a_x and $a_{x'}$ are arbitrary).

Clause gadgets. An example of a clause gadget is depicted in Figure 5. A clause gadget for clause C contains twelve vertices a_k^C , $\hat{a}_k^C$, b_k^C , and $\hat{b}_k^C$ for all $k \in [3]$. The preferences of these vertices over other vertices of the clause gadget are described in Figure 5. There are several popular matchings inside the depicted gadget. Our proof will use the following three popular matchings, indicating that the first, second, or third literal of the clause is satisfied.

$$\begin{aligned} M_1^C &= \left\{ \{\hat{a}_1^C, b_1^C\}, \{a_1^C, \hat{b}_1^C\}, \{\hat{a}_2^C, b_2^C\}, \{a_2^C, \hat{b}_2^C\}, \{\hat{a}_3^C, b_3^C\}, \{a_3^C, \hat{b}_3^C\} \right\} \\ M_2^C &= \left\{ \{\hat{a}_1^C, \hat{b}_1^C\}, \{a_1^C, b_1^C\}, \{\hat{a}_2^C, b_2^C\}, \{a_2^C, \hat{b}_2^C\}, \{\hat{a}_3^C, \hat{b}_3^C\}, \{a_3^C, b_3^C\} \right\} \\ M_3^C &= \left\{ \{\hat{a}_1^C, \hat{b}_1^C\}, \{a_1^C, b_1^C\}, \{\hat{a}_2^C, \hat{b}_2^C\}, \{a_2^C, b_2^C\}, \{\hat{a}_3^C, b_3^C\}, \{a_3^C, \hat{b}_3^C\} \right\} \end{aligned}$$

In Figure 5, these are marked by dotted, thick, and gray edges, in this order. Their three witnesses are shown next to the respective vertices, in green, red, and gray, in this order for M_1^C , M_2^C , and M_3^C . Intuitively, the k -th literal of clause C is selected to be satisfied if the popular matching contains edges $\{a_k^C, \hat{b}_k^C\}$ and $\{\hat{a}_k^C, b_k^C\}$, while a matching containing edges $\{a_i^C, b_i^C\}$ and $\{\hat{a}_i^C, \hat{b}_i^C\}$ poses no condition on whether the i -th literal is satisfied. The clause gadget is designed in such a way that it selects an arbitrary literal of the clause to be true.

We have that $A = \{a_1, a_2, a_3\} \cup \{a_x, \bar{a}_x : x \in X\} \cup \{a_k^C, \hat{a}_k^C : k \in [3], C \text{ clause}\}$ and $B = \{b_1, b_2, b_3\} \cup \{b_x^t, b_x^f : x \in X\} \cup \{b_k^C, \hat{b}_k^C : k \in [3], C \text{ clause}\}$ and set $w(a) := c$ for every $a \in A$ and $w(b) = 1$ for every $b \in B$.

Connection between variable and clause gadgets. We now describe the connection between variable and clause gadgets. Consider a clause C , and let y be the i -th literal in C . If $y = x$ for some variable x , then we add edge $\{a_i^C, b_x^f\}$, where a_i^C adds b_x^f to its preferences between b_i^C and $\hat{b}_i^C$, and b_x^f adds a_i^C to its preferences after $\bar{a}_x$. Otherwise we have $y = \neg x$ for some variable x . Then we add edge $\{\hat{a}_i^C, b_x^f\}$, where $\hat{a}_i^C$ adds b_x^f to the end of its preferences and b_x^f adds $\hat{a}_i^C$ to its preferences between a_x and $\bar{a}_x$.

Having described our construction, we now turn to proving the correctness of the reduction.

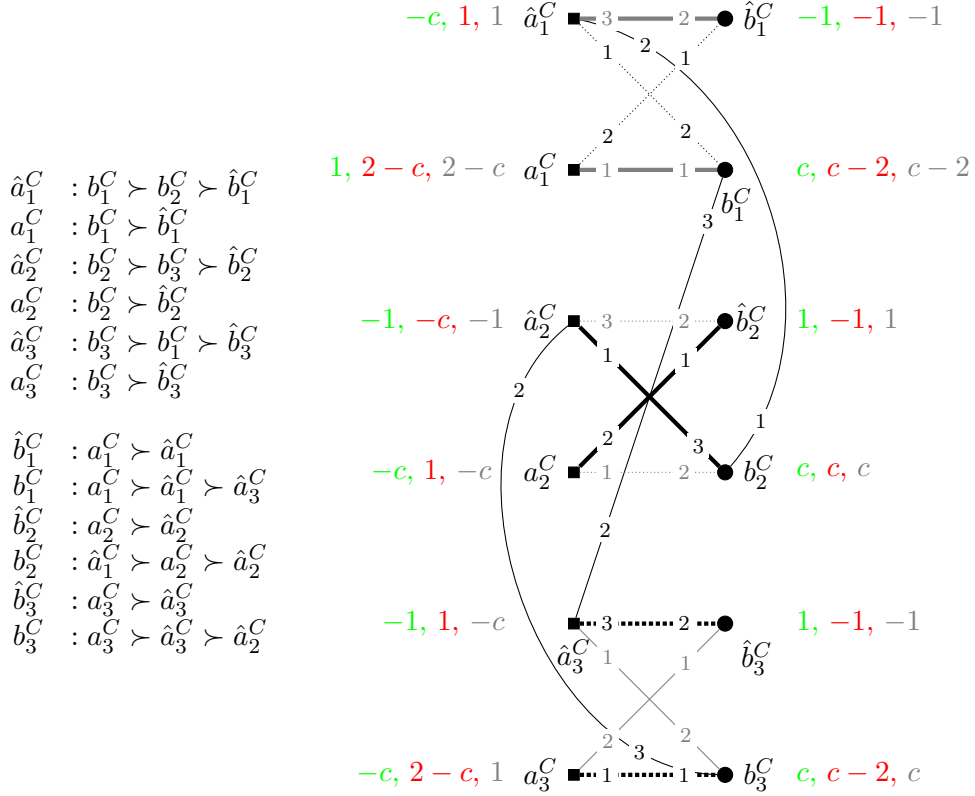


Figure 5: An example of a clause gadget for a clause C . Vertices in A are squared.

Satisfying assignment implies popular matching. Let $f : X \rightarrow \{\text{true}, \text{false}\}$ be a satisfying assignment. We construct a popular matching M as follows. First, M contains edges $\{a_i, b_i\}$ for $i \in [3]$. For every variable $x \in X$ with $f(x) = \text{true}$, we add edges $\{a_x, b_x^t\}$ and $\{\bar{a}_x, b_x^f\}$ to M , while for all other variables, we add edges $\{a_x, b_x^f\}$ and $\{\bar{a}_x, b_x^t\}$. For every clause C , there is at least one literal that is satisfied by f . We fix one such literal x and define $k_C \in [3]$ to be the number such that x is the k_C -th literal in C . We add $M_{k_C}^C$ to M , i.e., we add edges $\{a_{k_C}^C, \hat{b}_{k_C}^C\}$ and $\{\hat{a}_{k_C}^C, b_{k_C}^C\}$ as well as edges $\{a_j^C, b_j^C\}$ and $\{\hat{a}_j^C, \hat{b}_j^C\}$ for $j \in [3] \setminus \{k_C\}$ to M .

Claim 5. M is popular.

Proof. We give a witness $\mathbf{y}$ of M in Table 1. It remains to show that this is indeed a feasible witness. It is straightforward to verify that there is no conflicting edge inside any of the gadgets. For an edge $e = \{a_x, b_3\}$ between a variable and the 6-path gadget, we have $\widetilde{\text{vote}}^M(e) = 1 - c$. Since $\mathbf{y}_{a_x} \geq -1$, we have $\mathbf{y}_{a_x} + \mathbf{y}_{b_3} \geq -1 + 2 - c = 1 - c = \widetilde{\text{vote}}^M(e)$, and thus e is not conflicting. For an edge e between a variable and a clause gadget, we make a case distinction.

Case 1: $e = \{a_k^C, b_x^f\}$ for some clause C , $k \in [3]$, and $x \in X$.

In this case, the k -th literal of C is x .

Case 1.1: $k = k_C$.

Then $\widetilde{\text{vote}}^M(e) = c - 1$ and $\mathbf{y}_{a_k^C} = 1$. By the definition of k_C , we have $f(x) = \text{true}$ and thus $\mathbf{y}_{b_x^f} = 2 - c$. Thus, $\mathbf{y}_{a_k^C} + \mathbf{y}_{b_x^f} = 3 - c \geq c - 1 = \widetilde{\text{vote}}^M(e)$ using $c \leq 2$ for the inequality.

Case 1.2: $k \neq k_C$.

Then we have $M(a_k^C) = b_k^C$ and thus $\widetilde{\text{vote}}^M(e) = -c - 1$. Since $\mathbf{y}_{a_k^C} \geq -c$ and $\mathbf{y}_{b_x^f} \geq -1$, it follows that $\widetilde{\text{vote}}^M(e) \leq \mathbf{y}_{a_k^C} + \mathbf{y}_{b_x^f}$.

Case 2: $e = \{\hat{a}_k^C, b_x^f\}$ for some clause C , $k \in [3]$, and $x \in X$.

In this case, the k -th literal of C is $\neg x$.

Vertex	Value of $\mathbf{y}$	Vertex	Value of $\mathbf{y}$	Condition
a_1	$-c$	b_1	c	
a_2	-1	b_2	1	
a_3	$c-2$	b_3	$2-c$	
a_x	-1	b_x^t	1	$f(x) = \text{true}$
$\bar{a}_x$	$c-2$	b_x^f	$2-c$	
a_x	1	b_x^t	c	$f(x) = \text{false}$
$\bar{a}_x$	$-c$	b_x^f	-1	
a_1^C	1	b_1^C	c	$k_C = 1$
$\hat{a}_1^C$	$-c$	$\hat{b}_1^C$	-1	
a_2^C	$-c$	b_2^C	c	
$\hat{a}_2^C$	-1	$\hat{b}_2^C$	1	
a_3^C	$-c$	b_3^C	c	
$\hat{a}_3^C$	-1	$\hat{b}_3^C$	1	
a_1^C	$2-c$	b_1^C	$c-2$	$k_C = 2$
$\hat{a}_1^C$	1	$\hat{b}_1^C$	-1	
a_2^C	1	b_2^C	c	
$\hat{a}_2^C$	$-c$	$\hat{b}_2^C$	-1	
a_3^C	$2-c$	b_3^C	$c-2$	
$\hat{a}_3^C$	1	$\hat{b}_3^C$	-1	
a_1^C	$2-c$	b_1^C	$c-2$	$k_C = 3$
$\hat{a}_1^C$	1	$\hat{b}_1^C$	-1	
a_2^C	$-c$	b_2^C	c	
$\hat{a}_2^C$	-1	$\hat{b}_2^C$	1	
a_3^C	1	b_3^C	c	
$\hat{a}_3^C$	$-c$	$\hat{b}_3^C$	-1	

Table 1: A witness of M .

Case 2.1: $k = k_C$.

By the definition of k_C , we have $f(x) = \text{false}$ and therefore $M(b_x^f) = a_x$. Thus, we have $\widehat{\text{vote}}^M(e) = -c - 1$. Since $y_{\hat{a}_k^C} \geq -c$ and $y_{b_x^f} \geq -1$, we have $y_{\hat{a}_k^C} + y_{b_x^f} \geq \widehat{\text{vote}}^M(e)$.

Case 2.2: $k \neq k_C$.

Then we have $y_{\hat{a}_k^C} \geq -1$. If $f(x) = \text{true}$, then $M(b_x^f) = \hat{a}_x$ and $y_{b_x^f} = 2 - c$. Thus, we have $y_{\hat{a}_k^C} + y_{b_x^f} \geq -1 + 2 - c = 1 - c = \widehat{\text{vote}}^M(e)$. If $f(x) = \text{false}$, then $\widehat{\text{vote}}^M(e) = -c - 1 \leq y_{\hat{a}_k^C} + y_{b_x^f}$. $\square$

Popular matching implies satisfying assignment. Let M be a popular matching. We start with a technical observation we will frequently rely on in our proof.

Observation 1. *In instances where $w(a_1) = w(a_2)$ for each $a_1, a_2 \in A$ and $w(b) > 0$ for each $b \in B$, each popular matching M covers every vertex $b \in B$ that is the first choice of some $a \in A$.*

Proof. If M leaves b unmatched, then $(M \setminus \{\{a, M(a)\}\}) \cup \{\{a, b\}\}$ is more popular than M , a contradiction to the popularity of M . Notice that in our construction, the vertex weights are identical and positive on each side, and thus, each first-choice vertex must be matched in all popular matchings. $\square$

Claim 6. *Matching M contains edges $\{a_k, b_k\}$ for every $k \in [3]$.*

Proof. Observation 1 implies that a_1, a_2, b_1 , and b_2 are matched in M . Thus, M contains $\{a_1, b_1\}$ and $\{a_2, b_2\}$. Assume for a contradiction that M does not contain $\{a_3, b_3\}$. Then a_3 is unmatched. If b_3 was also unmatched, then $M \cup \{\{a_3, b_3\}\}$ would be more popular than M , so we must have $\{a_x, b_3\} \in M$ for some variable $x \in X$. Then $(M \setminus \{\{a_x, b_3\}, \{\bar{a}_x, b_x^t\}\}) \cup \{\{a_3, b_3\}, \{a_x, b_x^t\}\}$ (note that b_x^t is either unmatched or matched to $\bar{a}_x$) is more popular than M , a contradiction. Thus, M contains $\{a_3, b_3\}$. $\square$

Claim 7. *M contains edges $\{a_x, b_x^t\}$ and $\{\bar{a}_x, b_x^f\}$ or edges $\{a_x, b_x^f\}$ and $\{\bar{a}_x, b_x^t\}$ for every variable $x \in X$.*

Proof. Assume for a contradiction that the claim does not hold for a variable $x \in X$. Due to the maximality of M and to Claim 6, M contains edge $\{a_k^C, b_x^f\}$ or $\{\hat{a}_k^C, b_x^f\}$ for some clause C and $k \in [3]$. Claim 6 together with Observation 1 implies that a_x is matched to b_x^t and therefore $\bar{a}_x$ is unmatched. We will now show that there exists an alternating path starting in $\bar{a}_x$ such that augmenting M with this path results in a matching more popular than M . If M contains $\{a_k^C, b_x^f\}$, then $(M \setminus \{\{a_k^C, b_x^f\}\}) \cup \{\{\bar{a}_x, b_x^f\}\}$ is more popular than M , contradicting the popularity of M . Thus, M contains $\{\hat{a}_k^C, b_x^f\}$. If b_k^C or $\hat{b}_k^C$ was unmatched (call this unmatched vertex b), then $(M \setminus \{\{\hat{a}_k^C, b_x^f\}\}) \cup \{\{a_k^C, b\}\}$ would be more popular than M , contradicting the popularity of M . Thus, M contains edges $\{a_k^C, \hat{b}_k^C\}$ and $\{\hat{a}_{k-1}^C, b_k^C\}$ (interpreting $k - 1$ modulo 3). By Observation 1, b_{k-1}^C is not unmatched. If $\{a_{k-1}^C, b_{k-1}^C\} \in M$, then $(M \setminus \{\{\hat{a}_k^C, b_x^f\}, \{\hat{a}_{k-1}^C, b_k^C\}, \{a_{k-1}^C, b_{k-1}^C\}\}) \cup \{\{\bar{a}_x, b_x^f\}, \{\hat{a}_k^C, b_k^C\}, \{\hat{a}_{k-1}^C, b_{k-1}^C\}, \{a_{k-1}^C, \hat{b}_{k-1}^C\}\}$ is more popular than M , contradicting the popularity of M . Thus, we have $\{\hat{a}_{k-2}^C, b_{k-1}^C\} \in M$ (also interpreting $k - 2$ modulo 3). Then one of b_{k-2}^C and $\hat{b}_{k-2}^C$ is unmatched and we call the unmatched vertex b . Then $(M \setminus \{\{\hat{a}_k^C, b_x^f\}, \{\hat{a}_{k-1}^C, b_k^C\}, \{\hat{a}_{k-2}^C, b_{k-1}^C\}\}) \cup \{\{\bar{a}_x, b_x^f\}, \{\hat{a}_k^C, b_k^C\}, \{\hat{a}_{k-1}^C, b_{k-1}^C\}, \{\hat{a}_{k-2}^C, b\}\}$ is more popular than M , contradicting the popularity of M . $\square$

Claim 8. *M does not contain edge $\{\hat{a}_k^C, b_{k+1}^C\}$ for any clause C and any $k \in [3]$ (where $k + 1$ is taken modulo k).*

Proof. Assume for a contradiction that M contains $\{\hat{a}_k^C, b_{k+1}^C\}$ for some $k \in [3]$. If M contains $\{\hat{a}_i^C, b_{i+1}^C\}$ for every $i \in [3]$ (considering $i+1$ modulo 3), then $(M \setminus \{\{\hat{a}_k^C, b_{k+1}^C\}, \{\hat{a}_{k-1}^C, b_k^C\}, \{\hat{a}_{k+1}^C, b_{k-1}^C\}\}) \cup \{\{\hat{a}_i^C, b_i^C\} : i \in [3]\}$ is more popular than M , a contradiction. Thus, we may assume that M does not contain $\{\hat{a}_{k-1}^C, b_k^C\}$. By Observation 1, vertex b_k^C is not unmatched. Thus, we have $\{a_k^C, b_k^C\} \in M$ and $\hat{b}_k^C$ is unmatched. If a_{k+1}^C or $\hat{a}_{k+1}^C$ was unmatched (call this unmatched vertex a), then $(M \setminus \{\{\hat{a}_k^C, b_{k+1}^C\}, \{a_k^C, b_k^C\}\}) \cup \{\{a, b_{k+1}^C\}, \{\hat{a}_k^C, b_k^C\}, \{a_k^C, \hat{b}_k^C\}\}$ would be more popular than M , contradicting the popularity of M . Thus, using Claim 7, M contains edges $\{a_{k+1}^C, \hat{b}_{k+1}^C\}$ and $\{\hat{a}_{k+1}^C, b_{k-1}^C\}$. Using Claim 7, one of a_{k-1}^C or $\hat{a}_{k-1}^C$ is unmatched (call this unmatched vertex a^*). Then $(M \setminus \{\{\hat{a}_k^C, b_{k+1}^C\}, \{a_k^C, b_k^C\}, \{\hat{a}_{k+1}^C, b_{k-1}^C\}\}) \cup \{\{a^*, b_{k-1}^C\}, \{\hat{a}_{k+1}^C, b_{k+1}^C\}, \{\hat{a}_k^C, b_k^C\}, \{a_k^C, \hat{b}_k^C\}\}$ is more popular than M , a contradiction to the popularity of M . $\square$

Claim 9. *For every clause C , there exists some $k \in [3]$ such that matching M contains edges $\{a_k^C, \hat{b}_k^C\}$ and $\{\hat{a}_k^C, b_k^C\}$.*

Proof. From the fact that every popular matching is maximal and from Claims 7 and 8 follows that M contains either edges $\{a_i^C, b_i^C\}$ and $\{\hat{a}_i^C, \hat{b}_i^C\}$ or edges $\{a_i^C, \hat{b}_i^C\}$ and $\{\hat{a}_i^C, b_i^C\}$ for every $i \in [3]$. So assume for a contradiction that M contains $\{a_i^C, b_i^C\}$ and $\{\hat{a}_i^C, \hat{b}_i^C\}$ for every $i \in [3]$. Then

$$(M \setminus \{\{a_i^C, b_i^C\}, \{\hat{a}_i^C, \hat{b}_i^C\}\}) \cup \{\{\hat{a}_1^C, b_2^C\}, \{a_1^C, \hat{b}_1^C\}, \{\hat{a}_2^C, b_3^C\}, \{a_2^C, \hat{b}_2^C\}, \{\hat{a}_3^C, b_1^C\}, \{a_3^C, \hat{b}_3^C\}\}$$

is more popular than M , a contradiction to the popularity of M . $\square$

Claim 10. *Assume that M contains edges $\{a_k^C, \hat{b}_k^C\}$ and $\{\hat{a}_k^C, b_k^C\}$ for some clause C and some $k \in [3]$. If the k -th literal of C is x for a variable $x \in X$, then M contains $\{a_x, b_x^t\}$ and $\{\bar{a}_x, b_x^f\}$. If the k -th literal of C is $\neg x$ for a variable $x \in X$, then M contains $\{a_x, b_x^f\}$ and $\{\bar{a}_x, b_x^t\}$.*

Proof. First assume that the k -th literal of C is x but M does not contain edges $\{a_x, b_x^t\}$ and $\{\bar{a}_x, b_x^f\}$. By Claim 7, M contains edges $\{a_x, b_x^f\}$ and $\{\bar{a}_x, b_x^t\}$. Then $(M \setminus \{\{\bar{a}_x, b_x^t\}, \{a_x, b_x^f\}, \{a_k^C, \hat{b}_k^C\}\}) \cup \{\{a_x, b_x^t\}, \{a_k^C, b_x^f\}\}$ is more popular than M , a contradiction to the popularity of M .

Now assume that the k -th literal of C is $\neg x$ but M does not contain edges $\{a_x, b_x^f\}$ and $\{\bar{a}_x, b_x^t\}$. By Claim 7, M contains edges $\{a_x, b_x^t\}$ and $\{\bar{a}_x, b_x^f\}$. Then $(M \setminus \{\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}, \{a_x, b_x^t\}, \{\bar{a}_x, b_x^f\}, \{a_k^C, \hat{b}_k^C\}, \{\bar{a}_k^C, b_k^C\}\}) \cup \{\{a_2, b_1\}, \{a_3, b_2\}, \{a_x, b_3\}, \{\bar{a}_x, b_x^t\}, \{\hat{a}_k^C, b_x^f\}, \{a_k^C, b_k^C\}\}$ is more popular than M , a contradiction to the popularity of M . $\square$

Claim 11. *Any popular matching implies a satisfying assignment.*

Proof. Let M be a popular matching. By Claim 7, for each variable x , matching M contains either $\{a_x, b_x^t\}$ and $\{\bar{a}_x, b_x^f\}$, or the edges $\{\bar{a}_x, b_x^t\}$ and $\{a_x, b_x^f\}$. In the former case, we set the variable to **true**, while we set the variable to **false** in the latter case. It remains to show that every clause is satisfied by one variable. By Claim 9, for every clause C , there is some $k \in [3]$ such that $\{a_k^C, b_k^C\}$ and $\{\hat{a}_k^C, \hat{b}_k^C\}$ are in M . By Claim 10, it follows that the k -th variable of C is set to **true**, while all other variables appearing in this clause are set to **false**. $\square$

With this we have finished the proof of correctness for our reduction. $\square$

Note that except for Case 1.1 in the proof of Claim 5, the whole proof works for all $c \leq 3$. The condition $c \leq 3$ is used in the 6-path gadget: if $c > 3$, then the 6-path gadget does not admit a popular matching and thus the whole constructed instance does not have a popular matching (the given witness $\mathbf{y}$ would not be feasible because $\mathbf{y}_{b_3} = 2 - c < -1$ for $c > 3$).

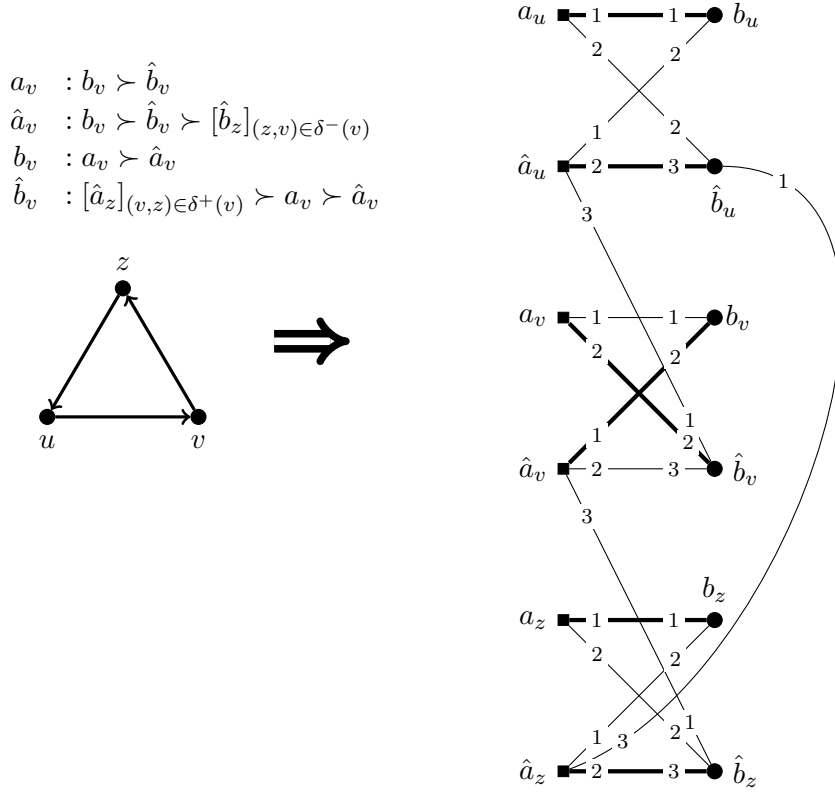


Figure 6: An example instance constructed from the triangle on the left in the proof of Theorem 8. Squared brackets denote an arbitrary ordering of the corresponding vertices. The set of edges leaving v is denoted by $\delta^-(v)$, while the set of edges entering v is denoted by $\delta^+(v)$. Thick edges mark the popular matching corresponding to the independent vertex set $\{v\}$.

Remark 1. Reducing from a restricted version of 3-SAT where every variable appears at most three times and making one copy the 6-path gadget for every variable gadget, the hardness also extends to graphs of maximum degree five.

A.3 Proof of Theorem 8

Theorem 8 (★). Unless $P=NP$, no $\varepsilon > 0$ exists such that there is a $O(n^{1-\varepsilon})$ -approximation for OPTIMAL POPULAR MATCHING WITH WEIGHTED VOTERS even if $w(a) = c > 3$ for all $a \in A$, $w(b) = 1$ for all $b \in B$, and every edge $e \in E$ has $\omega(e) \in \{0, 1\}$.

Proof. Fix a constant $c > 3$. We reduce from INDEPENDENT SET. The input (G, ℓ) consists of a graph G and an integer ℓ , while the goal is to decide whether G has an independent vertex set of size ℓ . Zuckerman [42] showed that this problem cannot be approximated within $O(|V(G)|^{1-\varepsilon})$ for any $\varepsilon > 0$, unless $P=NP$. To simplify notation in our reduction, we fix an arbitrary orientation for each edge in G .

Construction. For each $v \in V(G)$, we add a 4-cycle containing vertices $a_v, \hat{a}_v, b_v$, and $\hat{b}_v$ —not unlike in the variable gadget in Theorem 7. For each edge $e = \{u, z\} \in E(G)$, we fix an orientation (u, z) and add the edge $\{\hat{a}_z, \hat{b}_u\}$. The preferences are depicted in Figure 6. We have $A = \{a_v, \hat{a}_v : v \in V(G)\}$ and $B = \{b_v, \hat{b}_v : v \in V(G)\}$. We set $w(a) = c$ for every $a \in A$ and $w(b) = 1$ for every $b \in B$. For every $v \in V(G)$, we set $\omega(\{a_v, \hat{b}_v\}) := 1$, while for all other edges e we set $\omega(e) = 0$.

Correctness. We now prove the correctness of the above reduction.

Claim 12. If G has an independent set of size ℓ , then there exists a popular matching M with $\omega(M) = \sum_{e \in M} \omega(e) \geq \ell$.

Proof. Let X be an independent set of size ℓ . Consider matching $M := \{\{a_v, b_v\}, \{\hat{a}_v, \hat{b}_v\} : v \in V(G) \setminus X\} \cup \{\{a_v, \hat{b}_v\}, \{\hat{a}_v, b_v\} : v \in X\}$. Clearly, $\omega(M) = \ell$, so it remains to show that M is popular. We define a witness $\mathbf{y}$ of M in Table 2.

Vertex	Value of $\mathbf{y}$	Vertex	Value of $\mathbf{y}$	Condition
a_v	$-c$	b_v	c	$v \in V(G) \setminus X$
$\hat{a}_v$	-1	$\hat{b}_v$	1	
a_v	1	b_v	c	$v \in X$
$\hat{a}_v$	$-c$	$\hat{b}_v$	-1	

Table 2: A witness of M .

Consider an edge $e = \{\hat{a}_v, \hat{b}_z\}$. Then $\widetilde{\text{vote}}^M(e) = 1 - c$. If $v \notin X$, then $\mathbf{y}_{\hat{a}_v} + \mathbf{y}_{\hat{b}_z} \geq -1 - 1 = -2 \geq \widetilde{\text{vote}}^M(e)$ as $c > 3$. Otherwise we have $z \notin X$ (as X is an independent set) and consequently, $\mathbf{y}_{a_v} + \mathbf{y}_{b_z} = -c + 1 = \widetilde{\text{vote}}^M(e)$. Thus, e is not conflicting. It is straightforward to verify that for every $v \in V(G)$, none of the edges $\{a_v, b_v\}$, $\{a_v, \hat{b}_v\}$, $\{\hat{a}_v, b_v\}$, and $\{\hat{a}_v, \hat{b}_v\}$ is conflicting. Thus, M is a popular matching. $\square$

Claim 13. *No popular matching contains an edge $\{\hat{a}_v, \hat{b}_z\}$ for $z \neq v$.*

Proof. For every $v \in V(G)$, we have that $f(\hat{a}_v) = b_v$ and $s(\hat{a}_v) = \hat{b}_v$. Lemma 4 and theorem 1 imply that for any vertex $v \in V$, any popular matching contains $\{\hat{a}_v, b_v\}$ or $\{\hat{a}_v, \hat{b}_v\}$. Consequently, no popular matching can contain edge $\{\hat{a}_v, \hat{b}_z\}$ for $z \neq v$. $\square$

Claim 14. *If there exists a popular matching M with $\omega(M) = \sum_{e \in M} \omega(e) \geq \ell$, then G admits an independent set of size ℓ .*

Proof. Let $X := \{v \in V(G) : \{a_v, \hat{b}_v\} \in M\}$. Clearly $|X| \geq \ell$, so it is enough to show that X is an independent set. Assume for a contradiction that there exists an edge $e = (v, z) \in E(G)$ with $v \in X$ and $z \in X$. By Claim 13 and the fact that every popular matching is maximal, matching M contains edges $\{\hat{a}_v, b_v\}$ and $\{\hat{a}_z, b_z\}$. Then $M' := (M \setminus \{\{\hat{a}_v, b_v\}, \{a_v, \hat{b}_v\}, \{a_z, \hat{b}_z\}, \{\hat{a}_z, b_z\}\}) \cup \{\{a_v, b_v\}, \{\hat{a}_z, \hat{b}_v\}, \{a_z, b_z\}\}$ is more popular than M (because a_v, a_z, b_v, b_z , and $\hat{b}_v$ prefer M' while only $\hat{a}_v, \hat{a}_z$, and $\hat{b}_z$ prefer M), a contradiction to the popularity of M . $\square$

The correctness of the reduction follows from Claims 12 and 14 and the corresponding hardness result for INDEPENDENT SET [42]. $\square$

B Proofs from Section 5

B.1 Proof of Lemma 1

Lemma 1 (★). *Let e be an edge which is contained in a popular matching M^* . For any popular matching M with witness $\mathbf{y}$, we have that $\mathbf{y}$ is tight on e .*

Proof. Let P and D be the primal LP (1) and dual LP (2) from theorem 5 for M . Because both M and M^* are popular, it holds that $\Delta_w(M, M^*) = 0$. Thus, we have that M^* is an optimal solution for P . The statement now follows from complementary slackness (complementary slackness states that for every optimal solutions $\mathbf{x}$ to a primal LP, in any optimal solution to the dual LP the inequality corresponding to any non-zero variable in $\mathbf{x}$ must be tight, i.e. fulfilled with equality; see e.g. [39, Equation 5.13]). $\square$

B.2 Proof of Lemma 2

Lemma 2 (★). *Let M be a popular matching. Then there exists a nice witness of M .*

Proof. For a vertex v , let R^v be the set of possible values of a nice witness at v , i.e. $R^v = \{-c, 1-c, 2-c, -1, 0, 1\}$ if $v \in A$ and $R^v = \{-1, 0, 1, c-2, c-1, c\}$ if $v \in B$. Let $\mathbf{y}$ be a witness of M , and let $S(\mathbf{y})$ be the set of vertices v such that $\mathbf{y}_v \notin R^v$. We assume that $\mathbf{y}$ is a witness minimizing the cardinality of $S(\mathbf{y})$. Assume for a contradiction that $S(\mathbf{y}) \neq \emptyset$. Let $v \in S(\mathbf{y})$ and $r \in R^v$ such that $|\mathbf{y}_v - r|$ is minimized. Set $x := r - \mathbf{y}_v$.

First assume that $v \in A$. We construct a new witness $\mathbf{y}'$ as follows: For each vertex $a \in A \cap S(\mathbf{y})$, we set $\mathbf{y}'_a := \mathbf{y}_a + x$, and for each vertex $b \in B \cap S(\mathbf{y})$, we set $\mathbf{y}'_b := \mathbf{y}_b - x$. Since $-c \in R^a$ for every $a \in A$ and $-1 \in R^b$ for every $b \in B$, the definition of x implies that $\mathbf{y}'_v \geq w(v)$ for every $v \in V$. Note that for each unmatched vertex u , we have $\mathbf{y}_u = 0 \in R^u$. Thus, a vertex u fulfills $\mathbf{y}_u \in R^u$ if and only if $\mathbf{y}_{M(u)} \in R^{M(u)}$, and therefore, $|A \cap S(\mathbf{y})| = |B \cap S(\mathbf{y})|$. Thus, we have $\sum_{v \in A \cup B} \mathbf{y}'_v = \sum_{v \in A \cup B} \mathbf{y}_v + x \cdot |A \cap S(\mathbf{y})| - x \cdot |B \cap S(\mathbf{y})| = 0$. For any edge $\{a, b\}$ with either $a, b \in S(\mathbf{y})$ or $a, b \notin S(\mathbf{y})$, we have $\mathbf{y}'_a + \mathbf{y}'_b = \mathbf{y}_a + \mathbf{y}_b$, and thus, $\mathbf{y}'$ also satisfies $\mathbf{y}'_a + \mathbf{y}'_b \geq \widetilde{\text{vote}}^M(\{a, b\})$. So assume that only one end vertex of edge $\{a, b\}$ is in $S(\mathbf{y})$. This implies $\{a, b\} \notin M$ by Lemma 1. Thus, we have $\widetilde{\text{vote}}^M(\{a, b\}) \in \{-c-1, 1-c, c-1, 1+c\}$. If $\mathbf{y}_a \in R^a$, then (since $\widetilde{\text{vote}}^M(\{a, b\}) - \mathbf{y}_a \leq \mathbf{y}_b \leq c$) it follows that $\widetilde{\text{vote}}^M(\{a, b\}) - \mathbf{y}_a \in [-c-2, -1] \cup \{0, 1, c-2, c-1, c\}$. If $\mathbf{y}_b \in R^b$, then (since $\widetilde{\text{vote}}^M(\{a, b\}) - \mathbf{y}_b \leq \mathbf{y}_a \leq 1$) it follows that $\widetilde{\text{vote}}^M(\{a, b\}) - \mathbf{y}_b \in [-2c-1, -c] \cup \{1-c, 2-c, -1, 0, 1\}$. Thus, we have by the definition of x that $\mathbf{y}_a + \mathbf{y}_b \geq \widetilde{\text{vote}}^M(\{a, b\}) + |x|$. Since $|\mathbf{y}_a + \mathbf{y}_b - (\mathbf{y}'_a + \mathbf{y}'_b)| = |x|$, we have that $\mathbf{y}'$ is a witness of M . Clearly, $S(\mathbf{y}') \subseteq S(\mathbf{y})$, and since $v \notin S(\mathbf{y}')$, it follows that $|S(\mathbf{y}')| < |S(\mathbf{y})|$, a contradiction.

The case $v \in B$ is symmetric. $\square$

B.3 Proof of Lemma 3

Lemma 3 (★). *Let M_1 and M_2 be popular matchings and C a connected component of $M_1 \triangle M_2$. Then every nice witness $\mathbf{y}$ of M_1 or M_2 on C is either even or odd.*

Proof. Since the witness is nice, it can only take values $-c, 1-c, 2-c, -1, 0, 1, c-2, c-1, c$. For any edge $\{a, b\} \in E(C)$ with $\{a, b\} \notin M_i$, we have that $\widetilde{\text{vote}}^{M_i}(\{a, b\}) \in \{-c-1, 1-c, c-1, 1+c\}$. Fix a vertex $v \in V(C)$. If $\mathbf{y}_v \in \{-c, 2-c, -1, 1, c-2, c\}$, then Lemma 1 implies that $\mathbf{y}_u \in \{-c, 2-c, -1, 1, c-2, c\}$ for all neighbors $u \in N_C(v)$. By iterating this argument, it follows that $\mathbf{y}$ is odd. Otherwise we have $\mathbf{y}_v \in \{1-c, 0, c-1\}$. Then Lemma 1 implies that $\mathbf{y}_u \in \{1-c, 0, c-1\}$ for all neighbors u of v , implying that $\mathbf{y}$ is even. $\square$

B.4 Proof of Lemma 5

Lemma 5 (★). *We can compute in $O(n+m)$ time a subgraph $H^{C+\mathcal{P}}$ of G such that $H^{C+\mathcal{P}}$ contains every popular edge, $H^{C+\mathcal{P}}$ is a disjoint union of paths and cycles, and every edge in $H^{C+\mathcal{P}}$ is contained in a popular matching in $H^{C+\mathcal{P}}$.*

We start by reducing $H^{\deg(A) \leq 2}$ to a subgraph $H^{C+\mathcal{T}}$ that contains every popular edge and whose connected components are trees or cycles. The basic idea here is that any edge incident to a cycle can be deleted as such an edge cannot be contained in a popular matching.

Claim 15. *Let H be the graph with edges $\{a, f(a)\}$ and $\{a, s(a)\}$ for all $a \in A$. For every cycle C in H , each popular matching M matches all vertices in C along edges of C , i.e. for every $v \in V(C)$ with $\{v, v_1\}$ and $\{v, v_2\} \in E(C)$, we have $M(v) = v_1$ or $M(v) = v_2$.*

Proof. If the statement is not true, then there exists a cycle C containing a vertex v that is unmatched or is matched to a partner $M(v) \notin C$. Theorem 1 and Lemma 4 imply that $M(a) \in V(C)$ for every $a \in A \cap V(C)$. Since $|A \cap V(C)| = |B \cap V(C)|$, it follows that every vertex $b \in B \cap V(C)$ is matched to a vertex $a \in A \cap V(C)$. Since no edge $\{a, b\} \in E(G) \setminus E(C)$ with $a \in A \cap V(C)$ and $b \in B \cap V(C)$ can be part of a stable matching by Theorem 1 and Lemma 4, the claim follows. $\square$

Thus, we define $H^{C+\mathcal{T}}$ to be the graph arising from $H^{\deg(A) \leq 2}$ by exhaustively deleting for every cycle $C \in H^{\deg(A) \leq 2}$ all edges incident to a vertex in $V(C)$ if they are not in $E(C)$. Note that $H^{C+\mathcal{T}}$ consists of a disjoint union of cycles and trees. As our goal is to reach a graph that has maximum degree two, we now take care of high-degree vertices in $H^{C+\mathcal{T}}$.

Claim 16. *For any vertex $v \in V(H^{C+\mathcal{T}})$, there are at most two edges incident to v that can appear in a popular matching in $H^{C+\mathcal{T}}$.*

Proof. As every vertex in A has degree at most two, the lemma holds for every vertex in A .

So consider some $b \in B$. Assume for a contradiction that there are three popular matchings M_1, M_2 , and M_3 such that b prefers $a_1 := M_1(b)$ to $a_2 := M_2(b)$ to $a_3 := M_3(b)$. Let $\mathbf{y}^i$ be a nice witness of M_i for $i \in \{1, 2, 3\}$. We distinguish two cases:

Case 1: $b = f(a)$ for some $a \in A$.

Theorem 1 and Lemma 4 imply that $f(a_1) = b$. Thus, we have $\widetilde{\text{vote}}^{M_i}(\{a_1, b\}) = c + 1$ for every $i \in \{2, 3\}$ and therefore, $\mathbf{y}_b^2 = c = \mathbf{y}_b^3$. By Lemma 3, both $\mathbf{y}^2$ and $\mathbf{y}^3$ are odd on the connected component C of $M_2 \triangle M_3$ that contains b . However, since $M_2 \triangle M_3$ is not a cycle (otherwise $\{a_1, b\}$ would have been deleted from $H^{C+\mathcal{T}}$), it follows that there is a vertex $u \in V(C)$ that is unmatched in M_i for some $i \in \{2, 3\}$. This implies that $\mathbf{y}_u^i = 0$. By Lemma 3, it follows that $\mathbf{y}^i$ is even on C , a contradiction to $\mathbf{y}^i$ being odd on C .

Case 2: $b = s(a)$ for some $a \in A$.

By Theorem 1 and Lemma 4, it follows that $M_1(a_3) = f(a_3) = M_2(a_3)$. Thus, we have $\widetilde{\text{vote}}^{M_i}(\{a_3, b\}) = -c - 1$ for every $i \in \{2, 3\}$. By Lemma 1, we have $\mathbf{y}_b^1 = -1 = \mathbf{y}_b^2$. By Lemma 3, both $\mathbf{y}^1$ and $\mathbf{y}^2$ are odd on the connected component C of $M_1 \triangle M_2$ containing b . However, since $M_1 \triangle M_2$ is not a cycle (otherwise $\{a_3, b\}$ would have been deleted from H), it follows that there is a vertex u in C that is unmatched in M_i for some $i \in \{1, 2\}$. This implies that $\mathbf{y}_u^i = 0$. By Lemma 3, it follows that $\mathbf{y}^i$ is even on C , a contradiction to $\mathbf{y}^i$ being odd on C . $\square$

Let now $H^{C+\mathcal{P}}$ be the graph whose edge set is the disjoint union of popular matchings in each component of $H^{C+\mathcal{T}}$. We now show that $H^{C+\mathcal{P}}$ can be computed in polynomial time, starting with the connected components consisting of cycles.

Observation 2. *Let C be a cycle in $H^{C+\mathcal{T}}$. Then we can compute the set of popular matchings in C in linear time.*

Proof. Theorem 1 and Lemma 4 imply that every popular matching must match every vertex in $V(C)$. Thus, there are only two different candidates for a popular matching, and these can be checked for popularity in polynomial time by Theorem 4. Note that on cycles, a simple dynamic program allows to find a maximum-weight matching in linear time. $\square$

The next lemma shows how to compute the set of edges in a popular matching for a tree, implying that $H^{C+\mathcal{P}}$ can be computed in polynomial time.

Claim 17. *If T is a tree, then the set of edges that appear in at least one popular matching in T can be computed in linear time.*

Proof. We compute the set of edges contained in a popular matching via dynamic programming. Fix an arbitrary vertex $r \in V(T)$ to be the root of the tree. For a vertex $v \in V(T)$, we denote by T_v the subtree of T rooted at v . Let $X := \{-c, 1-c, 2-c, -1, 0, 1, c-2, c-1, c\}$ be the set of values a nice witness can attain. The dynamic program performs bottom-up induction on T . Note that to decide whether a matching on T_v together with a witness $\mathbf{y}$ can be extended to a popular matching on T , it is enough to know to which vertex (if any) v is matched and the value of $\mathbf{y}_v$. Thus, for each vertex $v \in V(T)$, every $e \in \delta_T(v) \cup \{\perp\}$ (representing the edge with which v is matched, where $\perp$ indicates that v remains unmatched) and every $x \in X$, the dynamic program contains one entry $F_v^{e,x}$. If $e = \{v, z\} \in \delta_T(v)$, then define $G_v^{e,x}$ to be the union over all popular matchings M on $T_v \cup \{z\}$ with $e \in M$ and having a witness $\mathbf{y}$ with $\mathbf{y}_v = x$ assuming that at least one such popular matching exists; if no such matching exists, then we set $G_v^{e,x} = \square$. If $e = \perp$, then we define $G_v^{e,x}$ to be the union over all popular matchings M on T_v with v being unmatched and having a witness $\mathbf{y}$ with $\mathbf{y}_v = x$ assuming that at least one such popular matching exists; if no such matching exists, then we set $G_v^{e,x} = \square$. We will now give a dynamic program computing values $F_v^{e,x}$ for which we will later show that $F_v^{e,x} = G_v^{e,x}$.

The dynamic program computes the sets $F_v^{e,x}$ via bottom-up induction (i.e. processing the vertices of the starting with the leaves and then "going up" the tree towards the root). For a leaf v , this set can clearly be computed in constant time: If $e = \perp$, then $F_v^{e,x} = \emptyset$ if $x = 0$ and $F_v^{e,x} = \square$ otherwise. If $e = \{v, z\}$, then $F_v^{e,x} = \{\{v, z\}\}$ if $x \geq -w(v)$ and $-x \geq -w(z)$. Otherwise, we have $F_v^{e,x} = \square$.

For the root r , the union of $F_r^{e,x}$ over every $e \in \delta_T(r) \cup \{\perp\}$ and every $x \in X$ is the set of popular edges.

It remains to show how to compute $F_v^{e,x}$ for a vertex v from the sets $F_d^{e_d, x_d}$ for every child d of v , every $e_d \in \delta_T(d) \cup \{\perp\}$, and $x_d \in X$. Denote the set of children of v by $C(v)$. For every child d of v with $d \notin e$, we define $S_d^{e,x}$ to be the set containing every pair (e_d, x_d) with $F_d^{e_d, x_d} \neq \square$ such that $\{d, v\}$ does not violate the condition of a witness. More precisely, $S_d^{e,x}$ contains every pair (e_d, x_d) with $e_d \in \delta_T(d) \cup \{\perp\}$ such that $F_d^{e_d, x_d} \neq \square$ and $\widetilde{\text{vote}}_d^{e_d}(v) + \text{vote}_v^e(d) \leq x + x_d$. For the child d^* with $e = \{v, d^*\}$, we set $S_{d^*}^{e,x} := \{(e, -x)\}$. If $S_d^{e,x} = \emptyset$ for some child d of v , then we set $F_v^{e,x} := \square$. Otherwise we set $F_v^{e,x} := (\{e\} \setminus \{\square\}) \cup \bigcup_{d \in C(v)} \bigcup_{(e_d, x_d) \in S_d^{e,x}} F_d^{e_d, x_d}$.

We now prove by bottom-up induction that $F_v^{e,x}$ is computed correctly. For the leaves, correctness is obvious. So fix a non-leaf node v , an element $e \in \delta_T(v) \cup \{\perp\}$, and some $x \in X$. First, we show that $G_v^{e,x} \subseteq F_v^{e,x}$. Consider an edge $f \in G_v^{e,x}$. By the definition of $G_v^{e,x}$, there is a popular matching M on $T_v \cup e$ with $e \in M$ if $e \in \delta(v)$ and on T_v with v being unmatched in M if $e = \perp$ such that there exists a witness $\mathbf{y}$ of M with $\mathbf{y}_v = x$. This implies that for every child d of v , we have that $G_{\{d, M(d)\}, \mathbf{y}_d} \neq \square$. By induction hypothesis, we have $F_{\{d, M(d)\}, \mathbf{y}_d} = G_{\{d, M(d)\}, \mathbf{y}_d} \neq \square$. Thus, we have that $F_v^{e,x} \neq \square$. If $f = \{v, p\} = e$ where p is the parent of v , then clearly $f \in F_v^{e,x}$. Otherwise $f = \{v, d\}$ or f is contained in $E(T_d)$ for some child d of v . By induction, we have that $f \in F_d^{\{d, M(d)\}, \mathbf{y}_d}$, implying that $f \in F_v^{e,x}$.

Next, we show that $F_v^{e,x} \subseteq G_v^{e,x}$. Consider an edge $f \in F_v^{e,x}$. We will show that there exists a popular matching M containing f on $T_v \cup e$ with $e \in M$ if $e \in \delta_T(v)$ and on T_v with v being unmatched in M if $e = \perp$ such that there exists a witness $\mathbf{y}$ of M with $\mathbf{y}_v = x$. Since $f \in F_v^{e,x}$ there exist $(e_d, x_d) \in S_d^{e,x}$ for every $d \in C$ such that $f \in (\{e\} \setminus \{\square\}) \cup \bigcup_{d \in C(v)} F_d^{e_d, x_d}$ and $F_d^{e_d, x_d} \neq \square$. By induction, we have $F_d^{e_d, x_d} = G_d^{e_d, x_d}$ for every child d . Consequently, for every child d , there exists a popular matching M_d with witness $\mathbf{y}^d$ such that $\mathbf{y}_d^d = x_d$ and $M_d(d) \neq v$, and if $f \in E(T_v)$, then we can pick M_d such that it contains f . Then $(\{e\} \setminus \{\perp\}) \cup \bigcup_{d \in C(v)} M_d$ is a popular matching containing f , of which a witness can be constructed by setting $\mathbf{y}_u := \mathbf{y}_u^d$ for every $u \in T_d$ and every child d of v and $\mathbf{y}_v := x$. No conflicting edge can be contained in T_d for some d because $\mathbf{y}^d$ is a feasible witness, and $\{v, d\}$ is not conflicting for every child d because $\widetilde{\text{vote}}_d^{e_d}(v) + \text{vote}_v^e(d) \leq x + x_d$.

It remains to analyze the running time. Consider a vertex $v \in V(G)$ and fix some $x \in X$. We process the edges from $\delta(v)$ in order according to the preferences of v . Thus, when turning

from one edge $e = \{v, c\}$ to the next edge $e' = \{v, c'\}$, sets $S_d^{e,x}$ and $S_d^{e',x}$ can only change for $d \in \{c, c'\}$. Consequently, the computation of $F_v^{e,x}$ can be done in amortized constant time (when not storing all edges contained in a popular matching, but only the corresponding backtracking information instead). It follows that we need $O(\deg(v))$ time for vertex v , and consequently $O(n)$ time in total. $\square$

Proof of Lemma 5. By Lemma 4 and Theorem 1 we can compute a graph $H^{\deg(A) \leq 2}$ containing every popular edge and fulfilling $\deg(a) \leq 2$ for every $a \in A$. Applying Claim 15 results in a graph $H^{C+\mathcal{T}}$ such that $H^{C+\mathcal{T}}$ contains every popular edge and every connected component of $H^{C+\mathcal{T}}$ is a cycle or a tree. Applying Claims 16 and 17, we compute from $H^{C+\mathcal{T}}$ a subgraph $H^{\leq 2}$ such that $H^{\leq 2}$ contains every popular edge and has maximum degree two. For every connected component C of $H^{\leq 2}$, we apply Observation 2 or Claim 17 to exhaustively delete every edge not contained in a popular matching in C . The resulting graph $H^{C+\mathcal{P}}$ fulfills the requirements of the lemma.

The running time follows from the observation that $H^{\deg(A) \leq 2}$ can be computed in $O(n+m)$ time. $\square$

B.5 Proof of Lemma 7

Lemma 7 (★). *For each connected component C of $H^{C+\mathcal{P}}$, we can compute in linear time a set Y of up to four nice witnesses with the properties below (depending on whether C contains a single edge, a path, or a cycle), such that for any feasible witness $\mathbf{y}$ of a popular matching M , there exists a witness $\mathbf{y}' \in Y$ that can replace the values $\mathbf{y}_v$ for every $v \in V(C)$.*

- If C contains only a single edge e , then
 - $Y = \{\mathbf{y}^{e,1}, \mathbf{y}^{e,2}, \mathbf{y}^{e,\text{even}}\}$, where $\mathbf{y}^{e,1}$ and $\mathbf{y}^{e,2}$ are odd and $\mathbf{y}^{e,\text{even}}$ is even, and
 - $\mathbf{y}^{e,1}$ dominates $\mathbf{y}^{e,2}$ and $\mathbf{y}^{e,2}$ weakly dominates $\mathbf{y}^{e,\text{even}}$.
- If C is a path containing at least three edges, then
 - $Y = \{\mathbf{y}^{P,\text{odd}}, \mathbf{y}^{P,\text{even}}\}$ where $\mathbf{y}^{P,\text{odd}}$ is odd and $\mathbf{y}^{P,\text{even}}$ is even, and
 - $\mathbf{y}^{P,\text{odd}}$ weakly dominates $\mathbf{y}^{P,\text{even}}$.
- If C is a cycle, then
 - $Y = \{\mathbf{y}^{C,1}, \mathbf{y}^{C,2}\}$ or $Y = \{\mathbf{y}^{C,1}, \mathbf{y}^{C,2}, \mathbf{y}^{C,0}, \mathbf{y}^{C,\text{even}}\}$ where $\mathbf{y}^{C,0}$, $\mathbf{y}^{C,1}$, and $\mathbf{y}^{C,2}$ are odd and $\mathbf{y}^{C,\text{even}}$ is even, and
 - $\mathbf{y}^{C,1}$ dominates $\mathbf{y}^{C,2}$ or $\mathbf{y}^{C,1}$ dominates $\mathbf{y}^{C,0}$, witness $\mathbf{y}^{C,0}$ dominates $\mathbf{y}^{C,2}$, and $\mathbf{y}^{C,2}$ weakly dominates $\mathbf{y}^{C,\text{even}}$.

The proof of Lemma 7 is split into three parts (Claims 18, 20 and 22), where Claim 18 shows the statement for components consisting of a single edge, Claim 20 shows the statement for paths with at least three vertices, and Claim 22 shows the statement for cycles.

We first consider components consisting of a single edge.

Claim 18. *Let P be a connected component of $H^{C+\mathcal{P}}$ consisting of a single edge $e = \{a, b\}$, and assume that G admits at least one popular matching.*

Then there exist three witnesses $\mathbf{y}^{e,1}$, $\mathbf{y}^{e,2}$, and $\mathbf{y}^{e,\text{even}}$ such that $\mathbf{y}^{e,1}$ dominates $\mathbf{y}^{e,2}$, witness $\mathbf{y}^{e,2}$ weakly dominates $\mathbf{y}^{e,\text{even}}$, and for every popular matching M in G with nice witness $\mathbf{y}$, there exists some $\mathbf{y}^e \in \{\mathbf{y}^{e,1}, \mathbf{y}^{e,2}, \mathbf{y}^{e,\text{even}}\}$ such that $\mathbf{y}'$ defined by $\mathbf{y}'_a := \mathbf{y}_a^e$, $\mathbf{y}'_b := \mathbf{y}_b^e$, and $\mathbf{y}'_v := \mathbf{y}_v$ is a witness of M .

Proof. Let M be a popular matching. By Lemma 2, there exists a nice witness $\mathbf{y}$. Furthermore, M contains e since $H^{C+\mathcal{P}}$ contains every popular edge and therefore, we have $\mathbf{y}_a = -\mathbf{y}_b$.

We first consider the case that $b = f(a)$. We define

- $\mathbf{y}_a^{e,1} := -c$ and $\mathbf{y}_b^{e,1} := c$,
- $\mathbf{y}_a^{e,2} := 2 - c$ and $\mathbf{y}_b^{e,2} := c - 2$, and
- $\mathbf{y}_a^{e,\text{even}} := 1 - c$ and $\mathbf{y}_b^{e,\text{even}} := c - 1$.

Clearly, $\mathbf{y}^{e,1}$ dominates $\mathbf{y}^{e,2}$ and $\mathbf{y}^{e,2}$ weakly dominates $\mathbf{y}^{e,\text{even}}$. It remains to show that, given a popular matching M with nice witness $\mathbf{y}$, there exists some $\mathbf{y}^e \in \{\mathbf{y}^{e,1}, \mathbf{y}^{e,2}, \mathbf{y}^{e,\text{even}}\}$ such that $\mathbf{y}'$ defined by $\mathbf{y}'_a := \mathbf{y}_a^e$, $\mathbf{y}'_b := \mathbf{y}_b^e$, and $\mathbf{y}'_v := \mathbf{y}_v$ is a witness of M . If $\mathbf{y}_a \leq 2 - c$, then $\mathbf{y}_a = \mathbf{y}_a^e$ and $\mathbf{y}_b = \mathbf{y}_b^e$ for some $\mathbf{y}^e \in \{\mathbf{y}^{e,1}, \mathbf{y}^{e,2}, \mathbf{y}^{e,\text{even}}\}$, and this $\mathbf{y}^e$ fulfills the claim. Thus, we assume $\mathbf{y}_a \geq -1$. We claim that we can choose $\mathbf{y}^e := \mathbf{y}^{e,2}$ in this case. We assume for a contradiction that there exists a conflicting edge $\{a', b'\}$. Note that $\mathbf{y}'_v \geq \mathbf{y}_v$ for all $v \neq a$, and thus we have $a = a'$. Since $b = f(a)$, we have $\widetilde{\text{vote}}_a^M(\{a, b'\}) = -c$ and consequently $\widetilde{\text{vote}}^M(\{a, b'\}) \leq 1 - c$. Because $\mathbf{y}'_{b'} \geq -1$, it follows that $\{a', b'\}$ is not conflicting, a contradiction.

We now handle the case $b = s(a)$. We define

- $\mathbf{y}_a^{e,1} := -1$ and $\mathbf{y}_b^{e,1} := 1$,
- $\mathbf{y}_a^{e,2} := 1$ and $\mathbf{y}_b^{e,2} := -1$, and
- $\mathbf{y}_a^{e,\text{even}} := 0$ and $\mathbf{y}_b^{e,\text{even}} := 0$.

Again, $\mathbf{y}^{e,1}$ clearly dominates $\mathbf{y}^{e,2}$ and $\mathbf{y}^{e,2}$ weakly dominates $\mathbf{y}^{e,\text{even}}$. It remains to show that, given a popular matching M with nice witness $\mathbf{y}$, there exists some $\mathbf{y}^e \in \{\mathbf{y}^{e,1}, \mathbf{y}^{e,2}, \mathbf{y}^{e,\text{even}}\}$ such that $\mathbf{y}'$ defined by $\mathbf{y}'_a := \mathbf{y}_a^e$, $\mathbf{y}'_b := \mathbf{y}_b^e$, and $\mathbf{y}'_v := \mathbf{y}_v$ is a witness of M . If $\mathbf{y}_a \geq -1$, then $\mathbf{y}_a = \mathbf{y}_a^e$ and $\mathbf{y}_b = \mathbf{y}_b^e$ for some $\mathbf{y}^e \in \{\mathbf{y}^{e,1}, \mathbf{y}^{e,2}, \mathbf{y}^{e,\text{even}}\}$, and this $\mathbf{y}^e$ fulfills the claim. Thus, we assume that $\mathbf{y}_a \leq 2 - c$. Then $\widetilde{\text{vote}}^M(\{a, f(a)\}) \geq c - 1 > 2 = 2 - c + c \geq \mathbf{y}_a + \mathbf{y}_{f(a)}$, a contradiction to $\mathbf{y}$ being a witness of M . $\square$

We now turn to paths with at least three vertices. We start with a helpful claim.

Claim 19. *Let M_1 and M_2 be two popular matchings. Then every path P in $M_1 \triangle M_2$ has one end vertex in A and the other one in B .*

Proof. Assume for a contradiction that P has both end vertices in A , and let a_1 and a_2 be these end vertices. Without loss of generality a_1 is matched in M_1 and a_2 is matched in M_2 . Let $\mathbf{y}^1$ and $\mathbf{y}^2$ be nice witnesses of M_1 and M_2 . Since a_1 is unmatched in M_2 , it follows that $\mathbf{y}_{a_1}^2 = 0$ and thus, $\mathbf{y}^2$ is even on P . Since a_2 is unmatched in M_1 , it follows that $\mathbf{y}_{a_2}^1 = 0$ and thus, $\mathbf{y}^1$ is even on P . Let $b := M_2(a_2)$, $a := M_1(b)$, and $b' := M_2(a)$. We have $\widetilde{\text{vote}}_{a_2}^{M_1}(b) = 1$ and $\mathbf{y}_{a_2}^1 + \mathbf{y}_b^1 \leq c$, it follows by Lemma 1 that $\widetilde{\text{vote}}^{M_1}(\{a_2, b\}) = \mathbf{y}_b^1 = 1 - c = -\mathbf{y}_a^1$. Thus, $a \succ_b a_2$. Again Lemma 1 implies that $\widetilde{\text{vote}}^{M_1}(\{a, b'\}) = 1 - c$. Consequently, $b \succ_a b'$ holds. Thus, we have $\widetilde{\text{vote}}^{M_2}(\{a, b\}) = c + 1$, and consequently, $\mathbf{y}_a^2 = 1$ and $\mathbf{y}_b^2 = c$. However, this implies that $\mathbf{y}^2$ is odd, a contradiction to $\mathbf{y}^2$ being even.

The case that both end vertices of P are in B leads to a contradiction by symmetric arguments. $\square$

We now describe the witnesses for path components.

Claim 20. *Let P be a connected component of $H^{C+\mathcal{P}}$ that is a path with at least three vertices and let M_1 and M_2 be the two popular matchings on P such that M_2 leaves at least one vertex in P unmatched.*

Then there is a unique nice witness $\mathbf{y}^{P,\text{odd}}$ for M_1 and a unique nice witness $\mathbf{y}^{P,\text{even}}$ for M_2 , and $\mathbf{y}^{P,\text{odd}}$ weakly dominates $\mathbf{y}^{P,\text{even}}$.

Proof. Let M_1^P and M_2^P be the two popular matchings in P . By Claim 19, one end vertex a_1 of P is in A and the other end vertex is contained in B . Let $P = \langle a_1, b_1, a_2, b_2, a_3, b_3, \dots, a_k, b_k \rangle$. We assume without loss of generality that a_1 (and b_k) are unmatched in M_2^P . First, we prove by induction on i that for every $i \leq \lceil \frac{k}{2} \rceil$,

- $f(a_{2i}) = b_{2i-1} = f(a_{2i-1})$ and $s(a_{2i}) = b_{2i} = s(a_{2i+1})$ (in particular, a_{2i} and b_{2i} exist),
- $a_{2i} \succ_{b_{2i-1}} a_{2i-1}$ and if $2i < k$, then $a_{2i} \succ_{b_{2i}} a_{2i+1}$,
- every witness $\mathbf{y}^{P,\text{odd}}$ for M_1^P fulfills $\mathbf{y}_{a_{2i-1}}^{P,\text{odd}} = -c$, $\mathbf{y}_{a_{2i}}^{P,\text{odd}} = 1$, $\mathbf{y}_{b_{2i-1}}^{P,\text{odd}} = c$, and $\mathbf{y}_{b_{2i}}^{P,\text{odd}} = -1$, and
- every witness $\mathbf{y}^{P,\text{even}}$ for M_2^P fulfills $\mathbf{y}_{a_{2i-1}}^{P,\text{even}} = 0$, $\mathbf{y}_{a_{2i}}^{P,\text{even}} = 1 - c$, $\mathbf{y}_{b_{2i-1}}^{P,\text{even}} = c - 1$, and $\mathbf{y}_{b_{2i}}^{P,\text{even}} = 0$.

From this, one easily observes that $\mathbf{y}^{P,\text{odd}}$ weakly dominates $\mathbf{y}^{P,\text{even}}$ and the lemma follows.

For $i = 0$, due to the popularity of M_2^P , it follows that $a_2 \succ_{b_1} a_1$ (otherwise $M_2' := (M_2^P \setminus \{\{a_2, b_1\}\}) \cup \{\{a_1, b_1\}\}$ is more popular). As a_1 is unmatched in M_2 , we have $\mathbf{y}_{a_1}^{P,\text{even}} = 0$. By Lemma 1, it follows that $\mathbf{y}_{b_1}^{P,\text{even}} = c - 1$ and thus $\mathbf{y}_{a_2}^{P,\text{even}} = 1 - c$. Therefore, b_1 prefers a_2 to a_1 . Applying again Lemma 1 to the edge $\{a_2, b_2\}$, we conclude that $\mathbf{y}_{b_2}^{P,\text{even}} = 0$, and if a_3 exists, then b_2 prefers a_2 to a_3 . Since $\widetilde{\text{vote}}^{M_1}(\{a_2, b_1\}) = c + 1$, it follows that $\mathbf{y}_{b_1}^{P,\text{odd}} = c$ and $\mathbf{y}_{a_2}^{P,\text{odd}} = 1$, implying $\mathbf{y}_{b_2}^{P,\text{odd}} = -1$ and $\mathbf{y}_{a_1}^{P,\text{odd}} = -c$.

The argument for the induction step is basically identical: Since $\mathbf{y}_{b_{2i}}^{P,\text{even}} = 0$, we have $\mathbf{y}_{a_{2i+1}}^{P,\text{even}} = 0$. By Lemma 1, it follows that $\mathbf{y}_{b_{2i+1}}^{P,\text{even}} = c - 1$ and thus $\mathbf{y}_{a_{2i+2}}^{P,\text{even}} = 1 - c$. Therefore, b_{2i+1} prefers a_{2i+2} to a_{2i+1} . Applying again Lemma 1 to the edge $\{a_{2i+2}, b_{2i+2}\}$, we conclude that $\mathbf{y}_{b_{2i+2}}^{P,\text{even}} = 0$, and if a_{2i+3} exists, then b_{2i+2} prefers a_{2i+2} to a_{2i+3} . Since $\widetilde{\text{vote}}^{M_1}(\{a_{2i+2}, b_{2i+1}\}) = c + 1$, it follows that $\mathbf{y}_{b_{2i+1}}^{P,\text{odd}} = c$ and $\mathbf{y}_{a_{2i+2}}^{P,\text{odd}} = 1$, implying $\mathbf{y}_{b_{2i+2}}^{P,\text{odd}} = -1$ and $\mathbf{y}_{a_{2i+1}}^{P,\text{odd}} = -c$. $\square$

We now turn to the cycle components of H^{C+P} . We say that a witness $\mathbf{y}^1$ of M_1 *dominates* a witness $\mathbf{y}^2$ of M_2 at a vertex $b \in B$ if $\mathbf{y}_b^1 \geq \mathbf{y}_b^2 + 2$ or $\mathbf{y}_b^1 \geq \mathbf{y}_b^2$ and b prefers M_1 to M_2 . First, we show that if $\mathbf{y}^1$ dominates $\mathbf{y}^2$ at one vertex $b \in B$, then this already implies that $\mathbf{y}^1$ dominates $\mathbf{y}^2$.

Claim 21. *Let C be a cycle component of H^{C+P} . Let M_1 and M_2 be two popular matchings in C , and $\mathbf{y}^1$ and $\mathbf{y}^2$ be odd witnesses of M_1 and M_2 such that $\mathbf{y}^1$ dominates $\mathbf{y}^2$ at some $b^* \in V(B) \cap V(C)$. Then $\mathbf{y}^1$ dominates $\mathbf{y}^2$ at every $b \in V(B) \cap V(C)$.*

Proof. Let $\mathbf{y}^1$ be a witness of M_1 , and $\mathbf{y}^2$ be a witness of M_2 . If $M_1 = M_2$, then the claim follows from Lemma 1. Thus, we assume $M_1 \neq M_2$. Let $M_1 = \{\{a_i, b_i\} : i \in [k]\}$ for some $k \in \mathbb{N}$, and $M_2 = \{\{a_{i+1}, b_i\} : i \in [k-1]\} \cup \{\{a_1, b_k\}\}$.

Let $b^* = b_j$ such that $\mathbf{y}^1$ dominates $\mathbf{y}^2$ at b^* . We will show that $\mathbf{y}^1$ dominates $\mathbf{y}^2$ also at b_{j+1} , implying the claim. To simplify notation, we assume without loss of generality that $j = 1$.

Case 1: $b_1 = f(a_1)$.

Theorem 1 and Lemma 4 imply that $b_{2i-1} = f(a_{2i-1}) = f(a_{2i})$ for every i , and $b_{2i} = s(a_{2i-1}) = s(a_{2i})$. Because a_2 prefers b_1 to $M_1(a_2) = b_2$, we have $\mathbf{y}_{b_1}^1 \geq c - 2$. Similarly, because a_1 prefers b_1 to $M_2(a_1) = b_k$, we have $\mathbf{y}_{b_2}^2 \geq c - 2$.

Case 1.1: $\mathbf{y}_{b_1}^1 = c - 2$.

We will show that this case leads to a contradiction and therefore cannot occur. Since $\mathbf{y}^1$ dominates $\mathbf{y}^2$ at b_1 , it follows that $\mathbf{y}_{b_1}^2 = c - 2$, and b_1 prefers a_1 to a_2 . However, $\widetilde{\text{vote}}^{M_2}(\{a_1, b_1\}) = c + 1$. Since $\mathbf{y}_{a_1}^2 \leq 1$, this contradicts $\mathbf{y}^2$ being a witness of M_2 .

Case 1.2: $\mathbf{y}_{b_1}^1 = c$.

Case 1.2.1: $\mathbf{y}_{b_1}^2 = c - 2$.

Then b_1 prefers a_2 to a_1 , since otherwise $\widetilde{\text{vote}}^{M_2}(\{a_1, b_1\}) = c+1 > \mathbf{y}_{a_1}^2 + \mathbf{y}_{b_1}^2$, contradicting $\mathbf{y}^2$ being a witness of M_2 . It follows that $\mathbf{y}_{a_2}^1 = 1$, which implies $\mathbf{y}_{b_2}^1 = -1$. Because $\mathbf{y}_{a_2}^2 = 2 - c$, Lemma 1 implies $\mathbf{y}_{b_2}^2 = -1$, from which then follows $\mathbf{y}_{a_3}^2 = 1$. There are now two possibilities: either $a_2 \succ_{b_2} a_3$, or $a_3 \succ_{b_2} a_2$. In the first case, it follows that $\mathbf{y}^1$ dominates $\mathbf{y}^2$ at b_2 , so we only consider the second case in the following, and derive a contradiction from it. By Lemma 1, it follows that $\mathbf{y}_{a_3}^1 = 2 - c$ and $\mathbf{y}_{b_3}^1 = c - 2$, and that $a_3 \succ_{b_3} a_4$ (because otherwise $\widetilde{\text{vote}}^{M_1}(\{a_4, b_3\}) = c + 1 > 1 + c - 2 \geq \mathbf{y}_{a_4}^1 + \mathbf{y}_{b_3}^1$). From this, it follows that $\mathbf{y}_{a_4}^1 = 1$, $\mathbf{y}_{b_4}^1 = -1$, $\mathbf{y}_{b_3}^2 = c$, and $\mathbf{y}_{a_4}^2 = -c$.

There are now two possibilities: either $a_4 \succ_{b_4} a_5$, or $a_5 \succ_{b_4} a_4$. In the first case, it follows that $\mathbf{y}_{a_5}^1 = -c$, $\mathbf{y}_{b_5}^1 = c$, $\mathbf{y}_{b_4}^2 = 1$, $\mathbf{y}_{a_5}^2 = -1$, $a_6 \succ_{b_5} a_5$, $\mathbf{y}_{b_5}^2 = c$, $\mathbf{y}_{a_6}^2 = -c$, $\mathbf{y}_{a_6}^1 = 1$ and $\mathbf{y}_{b_6}^1 = -1$. In the second case, it follows that $\mathbf{y}_{a_5}^1 = 2 - c$, $\mathbf{y}_{b_5}^1 = c - 2$, $a_5 \succ_{b_5} b_6$, $\mathbf{y}_{a_6}^1 = 1$, $\mathbf{y}_{b_6}^1 = -1$, $\mathbf{y}_{b_4}^2 = -1$, $\mathbf{y}_{a_5}^2 = 1$, $\mathbf{y}_{b_5}^2 = c$, and $\mathbf{y}_{a_6}^2 = -c$. In both cases, we have $\mathbf{y}_{a_6}^2 = -c$, $\mathbf{y}_{a_6}^1 = 1$ and $\mathbf{y}_{b_6}^1 = -1$. By iterating the above arguments, it follows that $\mathbf{y}_{a_{2i}}^2 = -c$, $\mathbf{y}_{a_{2i}}^1 = 1$ and $\mathbf{y}_{b_{2i}}^1 = -1$ for every $i \geq 2$. However, because $\widetilde{\text{vote}}_{a_1}^{M_2}(b_1) = c$, $\widetilde{\text{vote}}_{b_1}^{M_2}(a_1) = -1$ and $\mathbf{y}_{b_1}^2 = c - 2$, we have $\mathbf{y}_{a_1}^2 = 1$, implying $\mathbf{y}_{b_k}^2 = -1$. Since $\mathbf{y}_{a_k}^2 = -c$, it follows that $a_1 \succ_{b_k} a_k$. This implies that $\widetilde{\text{vote}}^{M_1}(\{a_1, b_k\}) = -c + 1 > \mathbf{y}_{a_1}^1 + \mathbf{y}_{b_k}^1 = -c - 1$, a contradiction to $\mathbf{y}^1$ being a witness.

Case 1.2.2: $\mathbf{y}_{b_1}^2 = c$.

Then we have $\mathbf{y}_{a_2}^2 = -c$. Since $\mathbf{y}^1$ dominates $\mathbf{y}^2$ at b_1 , we have $a_1 \succ_{b_1} a_2$. It follows that $\mathbf{y}_{a_2}^1 = -1$ and $\mathbf{y}_{b_2}^1 = 1$. If $\mathbf{y}_{b_2}^2 = -1$, then $\mathbf{y}^1$ dominates $\mathbf{y}^2$ at b_2 . Otherwise we have $\mathbf{y}_{b_2}^2 = 1$ and Lemma 1 implies $a_2 \succ_{b_2} a_3$, and therefore, $\mathbf{y}^1$ dominates $\mathbf{y}^2$ at b_2 .

Case 2: $b_1 = s(a_1)$.

Theorem 1 and Lemma 4 imply that $b_{2i-1} = s(a_{2i-1}) = s(a_{2i})$ for every i , and $b_{2i} = f(a_{2i-1}) = f(a_{2i})$. Lemma 1 implies that $\mathbf{y}_{b_1}^1 \leq 1$ and $\mathbf{y}_{b_2}^2 \leq 1$.

Case 2.1: $\mathbf{y}_{b_1}^1 = -1$.

Since $\mathbf{y}^1$ dominates $\mathbf{y}^2$ at b_1 , we have $\mathbf{y}_{b_1}^2 = -1$, and $a_1 \succ_{b_1} a_2$. Thus, we have $\mathbf{y}_{a_2}^1 = -c$ and $\mathbf{y}_{b_2}^1 = c$, and $\mathbf{y}_{a_2}^2 = 1$. If $\mathbf{y}_{b_2}^2 = c - 2$, then $\mathbf{y}^1$ dominates $\mathbf{y}^2$ at b_2 . Otherwise, we have $\mathbf{y}_{b_2}^2 = c$. By Lemma 1, it follows that $a_2 \succ_{b_2} a_3$, and thus $\mathbf{y}^1$ dominates $\mathbf{y}^2$ at b_2 .

Case 2.2: $\mathbf{y}_{b_1}^1 = 1$.

If $a_1 \succ_{b_1} a_2$, then $\widetilde{\text{vote}}^{M_1}(\{a_2, b_1\}) = -1 - c < -c + 1 \leq \mathbf{y}_{a_2}^1 + \mathbf{y}_{b_1}^1$, a contradiction to Lemma 1. Thus, we have $a_2 \succ_{b_1} a_1$. We also have $\mathbf{y}_{a_2}^1 = -c$ and thus $\mathbf{y}_{b_2}^1 = c$. As $\mathbf{y}^1$ dominates $\mathbf{y}^2$ at b_1 , it follows that $\mathbf{y}_{b_1}^2 = -1$ and thus $\mathbf{y}_{a_2}^2 = 1$. If $\mathbf{y}_{b_2}^2 = c - 2$, then $\mathbf{y}^1$ dominates $\mathbf{y}^2$ at b_2 . Otherwise $\mathbf{y}_{b_2}^2 = c$, and by Lemma 1, it follows that $a_2 \succ_{b_2} a_3$, and thus $\mathbf{y}^1$ dominates $\mathbf{y}^2$ at b_2 . $\square$

We can now show that the witnesses in a cycle component (weakly) dominate each other.

Claim 22. *Let C be a connected component of $H^{C+\mathcal{P}}$ that is a cycle. Then C has two popular matchings M_0^C and M_1^C such that M_0^C has exactly one (odd) witness $\mathbf{y}^{C,0}$, while M_1^C has either one witness $\mathbf{y}^{C,1}$ or three nice witnesses $\mathbf{y}^{C,1}$, $\mathbf{y}^{C,2}$, and $\mathbf{y}^{C,\text{even}}$.*

If M_1^C has one witness, then $\mathbf{y}^{C,1}$ dominates $\mathbf{y}^{C,0}$.

If M_1^C has three nice witnesses, then $\mathbf{y}^{C,1}$ dominates $\mathbf{y}^{C,0}$, witness $\mathbf{y}^{C,0}$ dominates $\mathbf{y}^{C,2}$ and $\mathbf{y}^{C,2}$ weakly dominates $\mathbf{y}^{C,\text{even}}$.

Proof. Since every vertex $a \in A \cap V(C)$ can have at most two edges to $f(a)$ and $s(a)$, it follows that $f(a) \in V(C)$ for all $a \in V(C) \cap A$. Let $b \in B \cap H^{C+\mathcal{P}}$ be the f -post of at least one vertex in A . Then b prefers one of its neighbors in C to the other, say, $a_1 \succ_b a_2$. Assuming that M_0^C contains $\{a_2, b\}$, it follows that $\widetilde{\text{vote}}^{M_0^C}(\{a_1, b\}) = c + 1$. Thus, we have $\mathbf{y}_{a_1} = 1$ and $\mathbf{y}_b = c$ for every witness of M_0^C . By Lemma 1, it follows that the witness of M_0^C is unique.

Considering M_1^C , note that we have $\widetilde{\text{vote}}^{M_1^C}(\{a_2, b\}) = c - 1$, and thus, we have that $\mathbf{y}_b \in \{c - 2, c - 1, c\}$ for every nice witness $\mathbf{y}$ of M_1^C . If $\mathbf{y}_b = c - 1$, then also $\mathbf{y} + 1$ and $\mathbf{y} - 1$ are feasible witnesses. If there exists a feasible witness $\mathbf{y}_b^{C,1} = c$ and one with $\mathbf{y}_b^3 = c - 2$, then also

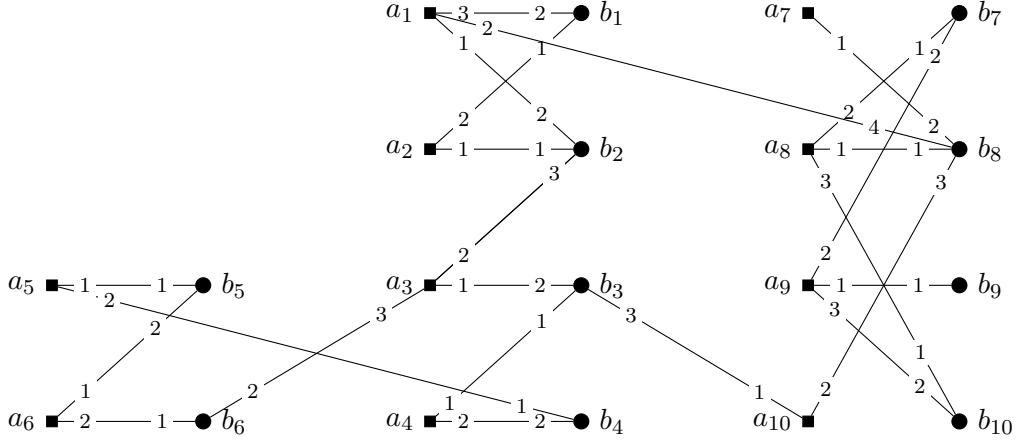


Figure 7: The input instance for our example execution of the algorithm.

$\mathbf{y}_b^{C,1} - \mathbf{1}$ is a feasible witness. Consequently, M_1^C admits either one witness or three witnesses. The claim now follows by Lemma 1.

Assume that M_1^C admits exactly one witness $\mathbf{y}^{C,1}$. Then we have $\mathbf{y}_b^{C,1} \in \{c, c-2\}$. Thus, either $\mathbf{y}^{C,1}$ dominates $\mathbf{y}^{C,0}$ at b or $\mathbf{y}^{C,0}$ dominates $\mathbf{y}^{C,1}$ at b . Claim 21 implies that either $\mathbf{y}^{C,1}$ dominates $\mathbf{y}^{C,0}$ or $\mathbf{y}^{C,0}$ dominates $\mathbf{y}^{C,1}$. The claim follows (possibly by switching the names of M_0^C and M_1^C).

Assume that M_1^C admits three witnesses $\mathbf{y}^{C,1}$, $\mathbf{y}^{C,\text{even}}$, and $\mathbf{y}^{C,2}$ with $\mathbf{y}_b^{C,1} = c$, $\mathbf{y}_b^{C,\text{even}} = c-1$, and $\mathbf{y}_b^{C,2} = c-2$. Because b prefers M_1^C to M_0^C , it follows that $\mathbf{y}^{C,1}$ dominates $\mathbf{y}^{C,0}$ at b . Since $\mathbf{y}_b^{C,0} = c = \mathbf{y}_b^{C,2} + 2$, it follows that $\mathbf{y}^{C,0}$ dominates $\mathbf{y}^{C,2}$ at b . Claim 21 implies that $\mathbf{y}^{C,1}$ dominates $\mathbf{y}^{C,0}$ and $\mathbf{y}^{C,0}$ dominates $\mathbf{y}^{C,2}$. Since $\mathbf{y}^{C,2} + \mathbf{1} = \mathbf{y}^{C,\text{even}}$, it follows that $\mathbf{y}^{C,2}$ weakly dominates $\mathbf{y}^{C,\text{even}}$. $\square$

B.6 Example execution of the algorithm

We now present an example execution for our algorithm from Theorem 9. The input instance is depicted in Figure 7.

B.6.1 Phase 1: Pruning edges

In the first phase of the algorithm, see Section 5.2.1, we prune non-popular edges in three steps, each of which results in the graph $H^{\deg(A) \leq 2}$, $H^{C+\mathcal{T}}$, and $H^{C+\mathcal{P}}$, respectively. Figures 8 to 10 depict these three graphs.

1. In the first step, we compute $f(v)$ and $s(v)$ for every vertex $v \in V(G)$. This results in graph $H^{\deg(A) \leq 2}$, depicted in Figure 8. For example, for vertex a_1 , $f(a_1) = b_1$, because b_1 is the first choice of a_1 . The second choice of a_1 is b_8 , but since $b_8 = f(a_8)$, $s(a_1) \neq b_8$. Instead, $s(a_1) = b_1$.
2. In the second step, we delete every edge incident to a cycle, resulting in graph $H^{C+\mathcal{T}}$ (see Figure 9). There are two cycles in the graph $H^{\deg(A) \leq 2}$ in Figure 8, but only one of them is incident to an edge. This edge is $\{a_{10}, b_3\}$.
3. In the third step, we consider each connected component, which is either a tree or a cycle, and exhaustively delete every edge not contained in a popular matching in this component. This results in the graph $H^{C+\mathcal{P}}$, depicted in Figure 10. In our example, only the component on vertices $\{a_7, a_8, a_9, b_7, b_8, b_9\}$ has such an edge, all other components have popular matchings that cover all edges.

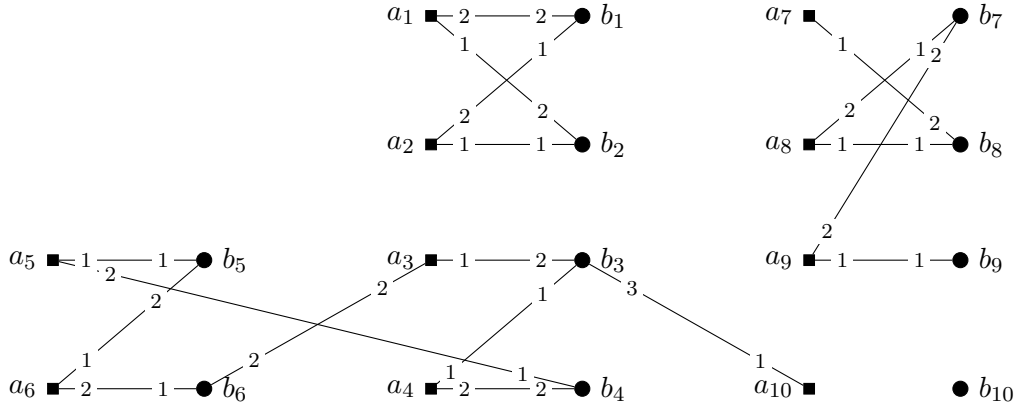


Figure 8: The graph $H^{\deg(A) \leq 2}$.

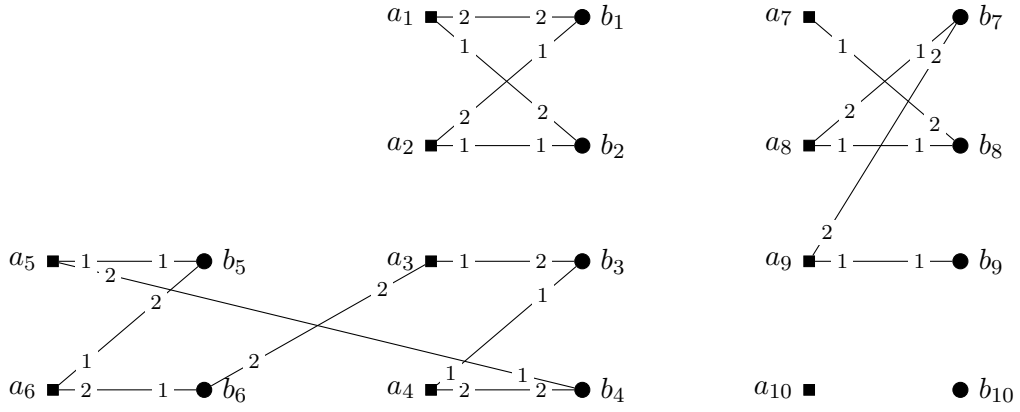


Figure 9: The graph H^{C+T} .

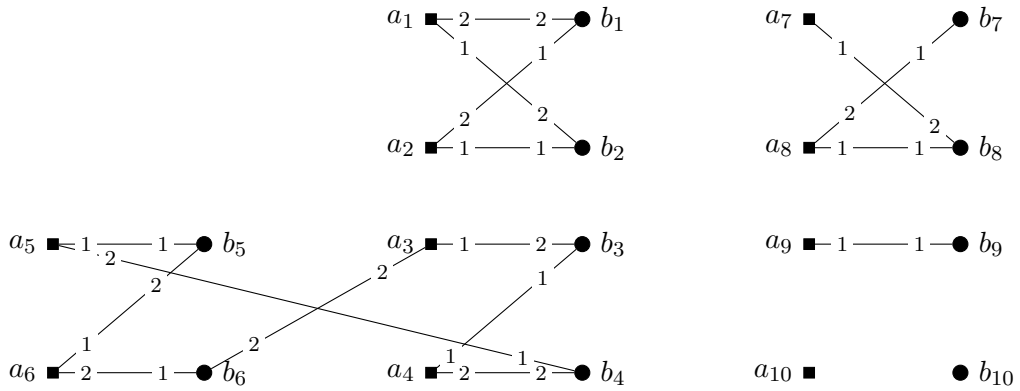


Figure 10: The graph H^{C+P} .

Witness	Matching	Vertex	Value of $\mathbf{y}$	Vertex	Value of $\mathbf{y}$
$\mathbf{y}^{C_{1,1}}$	$\{\{a_2, b_2\}, \{a_1, b_1\}\}$	a_2	$-c$	b_2	c
		a_1	-1	b_1	1
$\mathbf{y}^{C_{1,0}}$	$\{\{a_2, b_1\}, \{a_1, b_2\}\}$	a_2	1	b_2	c
		a_1	$-c$	b_1	-1
$\mathbf{y}^{C_{1,2}}$	$\{\{a_2, b_2\}, \{a_1, b_1\}\}$	a_2	$2 - c$	b_2	$c - 2$
		a_1	1	b_1	-1
$\mathbf{y}^{C_{1,\text{even}}}$	$\{\{a_2, b_2\}, \{a_1, b_1\}\}$	a_2	$1 - c$	b_2	$c - 1$
		a_1	0	b_1	0
$\mathbf{y}^{C_{2,1}}$	$\{\{a_4, b_3\}, \{a_3, b_6\}, \{a_6, b_5\}, \{a_5, b_4\}\}$	a_4	$-c$	b_4	-1
		a_3	-1	b_3	c
		a_6	$-c$	b_6	1
		a_5	1	b_5	c
$\mathbf{y}^{C_{2,2}}$	$\{\{a_4, b_4\}, \{a_3, b_3\}, \{a_6, b_6\}, \{a_5, b_5\}\}$	a_4	1	b_4	-1
		a_3	$-c$	b_3	c
		a_6	1	b_6	-1
		a_5	$2 - c$	b_5	$c - 2$
$\mathbf{y}^{P,\text{odd}}$	$\{\{a_8, b_7\}, \{a_7, b_8\}\}$	a_8	1	b_8	c
		a_7	$-c$	b_7	-1
$\mathbf{y}^{P,\text{even}}$	$\{\{a_8, b_8\}\}$	a_8	$1 - c$	b_8	$c - 1$
		a_7	0	b_7	0
$\mathbf{y}^{e,1}$	$\{\{a_9, b_9\}\}$	a_9	$-c$	b_9	c
$\mathbf{y}^{e,2}$	$\{\{a_9, b_9\}\}$	a_9	$2 - c$	b_9	$c - 2$
$\mathbf{y}^{e,\text{even}}$	$\{\{a_9, b_9\}\}$	a_9	$1 - c$	b_9	$c - 1$
	$\emptyset$	a_{10}	0		
	$\emptyset$			b_{10}	0

Table 3: The set of witnesses for the connected components of $H^{C+\mathcal{P}}$. The components are separated by a double horizontal line in the table.

B.6.2 Phase 2: Computing “local” witnesses

In the second phase, we compute witnesses for each connected component of $H^{C+\mathcal{P}}$, as described in Section 5.2.2. These “local” witnesses are depicted in Table 3.

B.6.3 Phase 3: Constructing a “global” witness

We now turn to the final step of the algorithm, which prunes “local” witnesses step-by-step until we found a “global” witness. Throughout this phase, each connected component of $H^{C+\mathcal{P}}$ is assigned a “local” witness. Initially, witnesses $\mathbf{y}^{C_{1,1}}$, $\mathbf{y}^{C_{2,1}}$, $\mathbf{y}^{P,\text{odd}}$, and $\mathbf{y}^{e,1}$ are assigned to the four non-trivial components of $H^{C+\mathcal{P}}$. We now choose an arbitrary order of conflicting edges. We start with edge $\{a_9, b_{10}\}$. Because $\mathbf{y}^{e,1}$ is odd, we dismiss $\mathbf{y}^{e,1}$ and assign $\mathbf{y}^{e,2}$. Next, we choose edge $\{a_{10}, b_8\}$. Because the witness $\mathbf{0}$ for a_{10} is even, we dismiss $\mathbf{y}^{P,\text{odd}}$ and assign $\mathbf{y}^{P,\text{even}}$. Now edge $\{a_1, b_8\}$ becomes conflicting and we choose it. Because $\mathbf{y}^{P,\text{even}}$ is even, we dismiss $\mathbf{y}^{C_{1,1}}$ and assign $\mathbf{y}^{C_{1,0}}$. Now edge $\{a_3, b_2\}$ becomes conflicting and we choose it. Because $\mathbf{y}^{C_{2,1}}$ is odd, we dismiss witness $\mathbf{y}^{C_{2,1}}$ and assign $\mathbf{y}^{C_{2,2}}$. Afterwards, there are no conflicting edges, and thus, we found the popular matching $M = \{\{a_2, b_1\}, \{a_1, b_2\}, \{a_4, b_3\}, \{a_3, b_6\}, \{a_6, b_5\}, \{a_5, b_3\}, \{a_8, b_8\}, \{a_9, b_9\}\}$.

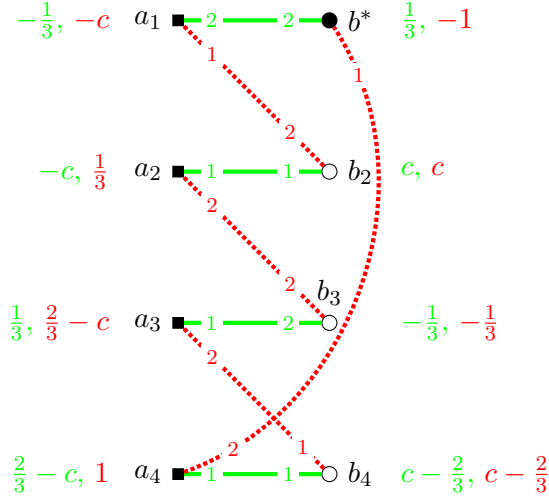


Figure 11: An example for an instance where each agent from A has weight $c > 3$ and each agent from B has weight at most 1 such that the witnesses are not in a dominance relation. Each squared vertex has weight $c > 3$, each black circular vertex has weight 1, and each white (circular) vertex has weight $\frac{1}{3}$.

C Example for non-dominating witnesses for non-unit weights for B

If we drop the assumption that vertices from B have weight 1 and allow them instead to have arbitrary weights not larger than 1, then Lemma 7 does not hold any more: As an example, consider Figure 11. There are two popular matchings (one containing the green edges, the other containing the red dotted edges), each having a unique witness (indicated by the green and red numbers). At vertex b^* , neither witness dominates the other.