

Equilibria in Matching Markets with Soft and Hard Liquidity Constraints

P. Jean-Jacques Herings* Yu Zhou†

April 28, 2022

Abstract

We consider a matching with contracts model in the presence of liquidity constraints on the buyers side. Liquidity constraints can be either soft or hard. A convergent sequence of economies with increasingly stringent soft liquidity constraints is an economy with hard liquidity constraints at the limit. The limit of a corresponding convergent sequence of competitive equilibria may fail to be a competitive equilibrium in the limit economy. We instead establish limit results for two alternative notions of competitive equilibrium, quantity-constrained competitive equilibrium and expectational equilibrium. The limit results of stable outcomes and core outcomes follow as corollaries.

Keywords: Liquidity constraints, matching with contracts, competitive equilibrium, quantity-constrained competitive equilibrium, expectational equilibrium, limit result.

JEL Classification: C72, C78, D45, D52

1 Introduction

Financial market imperfections often cause economic agents to be subject to liquidity constraints. Such constraints have important consequences for the saving behavior of consumers, operational decisions of firms, and government auctions for procurement and sales.¹ Understanding how liquidity constraints influence market outcomes is therefore of great importance.

Liquidity constraints come in two forms. A *soft* constraint refers to the case where an agent can always get additional liquidity by paying some appropriate borrowing costs. A *hard* constraint refers to the case where an agent is subject to a strict payment limit and cannot obtain any additional liquidity.

In matching theory, notions of equilibria for competitive analysis and their properties, and monotone matching patterns among agents are qualitatively different between models with soft constraints and those with hard constraints.² We instead study the connection between economies with soft constraints and those with hard constraints. We examine the behavior of various notions of equilibrium when going from economies with increasingly stringent soft constraints to the limit economy with hard constraints. Studying the continuity of equilibria

*Department of Econometrics and Operations Research, Tilburg University, P.O. Box 90153, 5000 LE, Tilburg, the Netherlands. E-mail: P.J.J.Herings@tilburguniversity.nl

†Graduate School of Economics, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto, 606-8501, Japan. E-mail: zhouyu_0105@hotmail.com

¹See, e.g., Carroll (2001) and Blalock, Gertler, and Levine (2008), and Che and Gale (1998).

²See, e.g., Legros and Newman (2007), Talman and Yang (2015), and Herings and Zhou (2021).

with respect to a continuous change in some primitive of the economy is an important and classical theme in general equilibrium and game theory.³

We consider a one-to-one two-sided matching with contracts model in the spirit of Crawford and Knoer (1981) and Hatfield and Milgrom (2005), extended by accommodating soft or hard liquidity constraints. With soft constraints, when the payment in a contract exceeds the buyer's initial budget, the buyer can get additional liquidity on the financial markets against a particular interest rate. A higher interest rate corresponds to a more stringent soft constraint. At the limit, when the interest rate approaches plus infinity, the buyer faces a hard constraint and any monetary transfer exceeding the initial budget is not feasible.

A competitive equilibrium always exists in the case of soft constraints, but may fail to do so in the presence of hard constraints. Herings and Zhou (2021) introduce an alternative notion of competitive equilibrium, i.e., quantity-constrained competitive equilibrium (QCCE). A QCCE extends a standard competitive equilibrium by introducing endogenously determined expectations of buyers about the availability of trades. When there is a hard constraint and the price of a trade is equal to or above this constraint, a buyer is allowed to expect no supply of the trade, and therefore demands another trade.

Consider a sequence of economies with increasingly stringent soft constraints. We establish that a corresponding sequence of QCCE prices and outcomes always contains a convergent subsequence, and every convergent subsequence always leads to prices and outcomes that are compatible with a QCCE of the limit economy.

A more fundamental notion of competitive equilibrium, inspired by work on general equilibrium with price rigidities in Drèze (1975), is expectational equilibrium as introduced in Herings (2020). At an expectational equilibrium, agents have endogenously determined expectations about the tradability of contracts, those to be supplied by agents on the other side of the market. Rationing constraints are used to express that certain contracts are not expected to be supplied. At equilibrium, for each contract at least one side is not rationed. Agents choose optimal contracts subject to the rationing constraints. We establish the equivalence between QCCE outcomes and expectational equilibrium outcomes with liquidity constraints. Therefore, the existence and limit results of expectational equilibria follow as corollaries.

We further argue that the limit results related to stable outcomes and core outcomes follow as corollaries due to their equivalence with QCCEs (Herings and Zhou, 2021) while the limit results related to strongly stable and strict core outcomes fail to hold.

Notions like competitive equilibrium, stable outcome, and core outcome, and their structural properties have been intensively studied in all sorts of matching models when agents have general utility functions, see, e.g., Crawford and Knoer (1981), Quinzii (1984), Demange and Gale (1985), Legros and Newman (2007), Fleiner, Jagadeesan, Jankó, and Teytelboym (2019), and Schlegel (2021). General utility functions can be used to analyze matching models with soft constraints, but not those with hard constraints (Herings and Zhou, 2021).

In the assignment model with unit-demand agents and hard constraints, Talman and Yang (2015) propose an auction that constructively finds a core outcome. In one-to-one matching models with imperfectly transferable utility and unobserved heterogeneity in tastes, Galichon, Kominers, and Weber (2019) define the notion of aggregate equilibrium and study its empirical implications. In many-to-one matching models with general constraints, Herings (2020)

³See, e.g., Hildenbrand and Mertens (1972), Echenique (2002), and Balder and Yannelis (2006).

proposes expectational equilibrium and discusses its equivalence with competitive equilibrium and stable outcomes in a large variety of settings. In one-to-one matching models with hard liquidity constraints, Herings and Zhou (2021) propose the QCCE and investigate its structural properties. In many-to-many matching models with soft and hard constraints, Jagadeesan and Teytelboym (2022) show the existence of stable outcomes under the net substitutability condition and analyze various properties of stable outcomes. None of these results relate economies with hard constraints to those with increasingly stringent soft constraints.⁴ Our results instead throw light on novel continuity properties of these equilibria and provide a new connection between equilibria in matching models with different types of liquidity constraints.

2 The Matching Model with Liquidity Constraints

There is a finite set of buyers B , a finite set of sellers S , and a set of bilateral trades Ω . Each trade $\omega \in \Omega$ is associated with a buyer $b(\omega) \in B$ and a seller $s(\omega) \in S$. We denote the set of trades in Ω involving buyer $b \in B$ and seller $s \in S$, respectively, by $\Omega^b = \{\omega \in \Omega \mid b(\omega) = b\}$ and $\Omega^s = \{\omega \in \Omega \mid s(\omega) = s\}$.

Transactions take place by signing contracts. A contract $c = (\omega, t) \in \Omega \times \mathbb{R}$ specifies a trade $\omega \in \Omega$ and a payment $t \in \mathbb{R}$ that is transferred from buyer $b(\omega)$ to seller $s(\omega)$. Let $\omega(c)$ and $t(c)$ be the corresponding trade and payment in contract c , respectively. For a set of contracts $Y \subseteq \Omega \times \mathbb{R}$, let $\omega(Y) = \{\omega(c) \in \Omega \mid c \in Y\}$ be the set of trades related to contracts in Y .

We consider a one-to-one set-up. Each agent signs a contract with at most one agent on the other side. An agent $i \in B \cup S$ who does not sign any contract receives the *no-trade* option o^i .

For every buyer $b \in B$, the utility function $u^b : (\Omega^b \times \mathbb{R}) \cup \{o^b\} \rightarrow \mathbb{R}$ is such that, (b-i) for every $\omega \in \Omega^b$, $u^b(\omega, \cdot)$ is continuous and strictly decreasing in $\mathbb{R}$ and, (b-ii) for every $\omega \in \Omega^b$, the range of $u^b(\omega, \cdot)$ is all of $\mathbb{R}$. Condition (b-i) says that given a trade, a lower payment improves the buyer's utility. Condition (b-ii) says that there is no trade ω that is infinitely good or bad. For every $\omega \in \Omega^b$, we define $u^b(\omega, +\infty) = -\infty$.

For every seller $s \in S$, the utility function $u^s : (\Omega^s \times \mathbb{R}) \cup \{o^s\} \rightarrow \mathbb{R}$ is such that, (s-i) for every $\omega \in \Omega^s$, $u^s(\omega, \cdot)$ is continuous and strictly increasing in $\mathbb{R}$ and, (s-ii) for every $\omega \in \Omega^s$, the range of $u^s(\omega, \cdot)$ is all of $\mathbb{R}$. The explanation for (s-i) and (s-ii) is analogous to (b-i) and (b-ii).⁵

We denote the profile of utility functions by $u = (u^i)_{i \in B \cup S}$.

The consumption set of agent $i \in B \cup S$ is equal to $X^i = \{Y \subseteq \Omega^i \times \mathbb{R} \mid |Y| \leq 1\}$. A set of contracts $A \subseteq \Omega \times \mathbb{R}$ is an *outcome* if, for every $i \in B \cup S$, $A^i \in X^i$. Let $\mathcal{A}$ be the collection of outcomes. Notice that $\emptyset \in \mathcal{A}$, so $\emptyset$ is an outcome.

In the following, we introduce soft and hard liquidity constraints. Every buyer $b \in B$ is endowed with an amount of money $M^b \in \mathbb{R}_+$. If the payment exceeds M^b , then buyer b can borrow money on the financial markets against an interest rate $r^b \in [0, +\infty]$. In case $r^b \in [0, +\infty)$, buyer b faces a *soft liquidity constraint*. A special case of a soft liquidity constraint is the absence of a liquidity constraint when $r^b = 0$. In case $r^b = +\infty$, buyer b cannot make a payment exceeding M^b and is subject to a *hard liquidity constraint*.

Different buyers may have different amounts of collateral and/or social networks, so we allow interest rates of acquiring additional liquidity to be buyer-dependent. Let $M = (M^b)_{b \in B}$ be the

⁴Dupuy, Galichon, Jaffe, and Kominers (2020) study how changes in taxes influence the firm-worker sorting patterns and efficiency. Nevertheless, the introduction of taxes does not influence equilibrium existence.

⁵(b-ii) and (s-ii) are also assumed in Demange and Gale (1985) when agents have general utility functions.

profile of monetary endowments and $r = (r^b)_{b \in B}$ be the profile of interest rates.

For every buyer $b \in B$, for every contract $c \in \Omega^b \times \mathbb{R}$, the transfer plus borrowing cost for contract c is given by

$$t^+(c; r) = \begin{cases} t(c), & \text{if } t(c) \leq M^b, \\ t(c) + r^b(t(c) - M^b), & \text{if } t(c) > M^b. \end{cases}$$

The amount $t^+(c; r)$ is larger than or equal to $t(c)$. In case $t(c) \leq M^b$, buyer b does not need to borrow and $t^+(c; r) = t(c)$. In case $t(c) > M^b$, buyer b pays a borrowing cost $r^b(t(c) - M^b)$ and $t^+(c; r) = t(c) + r^b(t(c) - M^b)$. If buyer b faces a hard liquidity constraint, then $t(c) > M^b$ implies $t^+(c; r) = +\infty$. In such a case, such a contract is always inferior to the no-trade option.

The utility function of agent $i \in B \cup S$ over contracts induces the utility function U^i over outcomes. For every $b \in B$, every $r \in [0, +\infty]^B$, every $A \in \mathcal{A}$, in case $A^b = \emptyset$, $U^b(A; r) = u^b(o^b)$, and in case $A^b = \{c\}$, $U^b(A; r) = u^b(\omega(c), t^+(c; r))$. For every $s \in S$, every $A \in \mathcal{A}$, in case $A^s = \emptyset$, $U^s(A) = u^s(o^s)$, and in case $A^s = \{c\}$, $U^s(A) = u^s(c)$.

The primitives of the economy are summarized by $\mathcal{E} = (B, S, \Omega, u, M, r)$.

Our model unifies matching models with no liquidity constraints, soft liquidity constraints, and hard liquidity constraints. The following example illustrates this point.

Example 2.1: Take $B = \{b_1, b_2\}$, $S = \{s\}$, and $\Omega = \{\omega_1, \omega_2\}$, where buyer b_1 trades ω_1 with s , buyer b_2 trades ω_2 with s . Buyers' monetary endowments are as follows: $M^{b_1} = 2$ and $M^{b_2} = 2$. Each buyer can borrow money against a common market interest rate $\rho \in [0, +\infty]$.

Agents' utility functions are specified as follows. For seller s , $u^s(o^s) = 0$ and, for every $t \in \mathbb{R}$, $u^s(\omega_1, t) = u^s(\omega_2, t) = t$. For each buyer b , $u^b(\omega, t) = V^b(\omega) - t$ where $V^b(\omega)$ is the value that buyer b assigns to trade ω . Let $u^{b_1}(o^{b_1}) = u^{b_2}(o^{b_2}) = 0$, $V^{b_1}(\omega_{11}) = 3$, and $V^{b_2}(\omega_{21}) = 4$.

Therefore, for each buyer b , in case $A^b = \emptyset$, $U^b(A; \rho) = 0$ and in case $A^b = \{(\omega, t)\}$,

$$U^b(A; \rho) = \begin{cases} V^b(\omega) - t, & \text{if } t \leq 2, \\ V^b(\omega) - t - \rho(t - 2), & \text{if } t > 2. \end{cases}$$

3 Competitive Equilibrium

In the competitive analysis of matching models, each trade $\omega \in \Omega$ is assigned a price $p_\omega \in \mathbb{R}$, which results in a price vector $p \in \mathbb{R}^\Omega$, see, e.g., Fleiner et al. (2019) and Schlegel (2021).

The *budget set* of agent $i \in B \cup S$ is given by

$$\gamma^i(p) = \{A^i \in X^i \mid \forall c \in A^i, t(c) = p_{\omega(c)}\}, \quad p \in \mathbb{R}^\Omega.$$

The budget set always contains $\emptyset$ and is therefore non-empty.

In a competitive equilibrium, agents take prices $p \in \mathbb{R}^\Omega$ as given and choose optimal contracts. The *demand set* of buyer $b \in B$ and *demand set* of seller $s \in S$ are given by

$$\begin{aligned} \delta^b(p; r) &= \arg \max_{A^b \in \gamma^b(p)} U^b(A^b; r), \quad p \in \mathbb{R}^\Omega, \\ \delta^s(p) &= \arg \max_{A^s \in \gamma^s(p)} U^s(A^s), \quad p \in \mathbb{R}^\Omega. \end{aligned}$$

We now define the notion of a competitive equilibrium.

Definition 3.1: An element $(p, A) \in \mathbb{R}^\Omega \times \mathcal{A}$ is a *competitive equilibrium* for the economy $\mathcal{E} = (B, S, \Omega, u, M, r)$ if:

- (i) For every $b \in B$, $A^b \in \delta^b(p; r)$.
- (ii) For every $s \in S$, $A^s \in \delta^s(p)$.

In a competitive equilibrium, agents receive their demanded contracts. The usual condition that demand equals supply is implicitly incorporated in the definition of an outcome.

We first show that there is a competitive equilibrium in the case of soft liquidity constraints.

Proposition 3.2: Let $\mathcal{E}$ be an economy such that $r \in [0, +\infty)^B$. A competitive equilibrium exists.

Proof: For every $b \in B$, we define $\tilde{u}^b : (\Omega^b \times \mathbb{R}) \cup \{o^b\} \rightarrow \mathbb{R}$ by $\tilde{u}^b(o^b) = u^b(o^b)$ and, for every $c \in \Omega^b \times \mathbb{R}$, $\tilde{u}^b(c) = u^b(\omega(c), t^+(c; r)) = U^b(\{c\}; r)$. Since $r \in [0, +\infty)^\Omega$, it is easy to see that $\tilde{u}^b$ satisfies (b-i) and (b-ii). Let $\tilde{u} = ((\tilde{u}^b)_{b \in B}, (u^s)_{s \in S})$.

Now consider the economy without liquidity constraints $\tilde{\mathcal{E}} = (B, S, \Omega, \tilde{u}, M, \tilde{r})$, where, for every $b \in B$, $\tilde{r}^b = 0$. By construction of $\tilde{u}^b$, in case $A^b = \emptyset$, $\tilde{U}^b(A^b; \tilde{r}) = \tilde{u}^b(o^b) = u^b(o^b) = U^b(A^b; r)$, and in case $A^b = \{c\}$, $\tilde{U}^b(\{c\}; \tilde{r}) = \tilde{u}^b(c) = U^b(\{c\}; r)$. Thus, for every $b \in B$, for every $p \in \mathbb{R}^\Omega$, it holds that $\tilde{\delta}^b(p; \tilde{r}) = \delta^b(p; r)$.

By construction, the competitive equilibria of $\tilde{\mathcal{E}}$ and $\mathcal{E}$ coincide. By Crawford and Knoer (1981), there is a competitive equilibrium of $\tilde{\mathcal{E}}$ and so $\mathcal{E}$ has a competitive equilibrium. **Q.E.D.**

Nevertheless, when we consider the transition from soft to hard liquidity constraints, it turns out that the buyers' utility functions over outcomes display discontinuities, which leads to the non-existence of competitive equilibria. Example 3.3 illustrates these points.

Example 3.3: Consider the economy in Example 2.1. For every $\rho \in [0, +\infty)$, $p_{\omega_1} = p_{\omega_2} = (3 + 2\rho)/(1 + \rho)$ and $A = \{(\omega_2, p_{\omega_2})\}$ constitutes a competitive equilibrium.

Now consider the case of $\rho = +\infty$. Notice that buyers' utility functions over outcomes exhibit a discontinuity at a price equal to monetary endowment. For buyer b_1 , it holds that for every $t \leq 2 = M^{b_1}$, $U^{b_1}(\{(\omega_1, t)\}; \rho) = 3 - t \geq 1$ and for every $t > 2$ we have $U^{b_1}(\{(\omega_{11}, t)\}; \rho) = -\infty$. The utility function of buyer b_2 displays similar discontinuities at price $2 = M^{b_2}$.

Moreover, in such a case, there is no competitive equilibrium. If $\max\{p_{\omega_1}, p_{\omega_2}\} \leq 2$, there is a total demand of two units, while the supply is only one unit. If $\max\{p_{\omega_1}, p_{\omega_2}\} > 2$, the trade with the highest price is supplied, but at that price, there is no demand. $\triangle$

4 Two Alternative Notions of Competitive Equilibrium

In this section, we present two alternative notions of competitive equilibrium that do not suffer from the discontinuity problems illustrated by Example 3.3.

4.1 Quantity-constrained Competitive Equilibrium

Herings and Zhou (2021) study models with hard liquidity constraints and propose the notion of a quantity-constrained competitive equilibrium by allowing buyers to form expectations about

the availability of trades. A buyer may expect that a trade is not available if the buyer faces a hard liquidity constraint and the price of the trade is equal to or above the buyer's monetary endowment. The reason is that the buyer cannot offer a higher transfer to the seller in case the seller makes another trade. For every $\omega \in \Omega$, let $q_\omega \in \{0, 1\}$ be the quantity constraint of trade ω , where $q_\omega = 1$ means that buyer $b(\omega)$ expects trade ω to be available and $q_\omega = 0$ means that buyer $b(\omega)$ expects trade ω not to be available. We denote the vector of quantity constraints by $q = (q_\omega)_{\omega \in \Omega}$.

The *constrained budget set* of buyer $b \in B$ is given by

$$\gamma^b(p, q) = \{A^b \in X^b \mid \forall c \in A^b, t(c) = p_{\omega(c)} \text{ and } q_{\omega(c)} = 1\}, \quad (p, q) \in \mathbb{R}^\Omega \times \{0, 1\}^\Omega.$$

The set $\gamma^b(p, q)$ always contains $\emptyset$ and is therefore non-empty.

The *constrained demand set* of buyer $b \in B$ is given by

$$\delta^b(p, q; r) = \arg \max_{A^b \in \gamma^b(p, q)} U^b(A^b; r), \quad (p, q) \in \mathbb{R}^\Omega \times \{0, 1\}^\Omega.$$

The constrained demand set of buyer b equals the set of contracts which maximize buyer b 's utility over the contracts in the constrained budget set.

Since a seller does not face liquidity constraints, the seller's decision problem is the same as before. A quantity-constrained competitive equilibrium is defined as follows.

Definition 4.1: An element $(p, q, A) \in \mathbb{R}^\Omega \times \{0, 1\}^\Omega \times \mathcal{A}$ is a *quantity-constrained competitive equilibrium (QCCE)* for the economy $\mathcal{E} = (B, S, \Omega, u, M, r)$ if:

- (i) For every $b \in B$, $A^b \in \delta^b(p, q; r)$.
- (ii) For every $s \in S$, $A^s \in \delta^s(p)$.
- (iii) For every $\omega \in \Omega$, if $q_\omega = 0$, then $r^{b(\omega)} = +\infty$ and $p_\omega \geq M^{b(\omega)}$.

The first two conditions of Definition 4.1 correspond to optimization by buyers that take p and q as given and optimization by sellers that take p as given. These two conditions also imply equality of demand and supply at a QCCE. The third condition reflects that when buyer $b(\omega)$ expects trade ω not to be supplied, then buyer $b(\omega)$ faces a hard liquidity constraint, i.e., $r^{b(\omega)} = +\infty$, and the price is larger than or equal to the buyer's monetary endowment, i.e., $p_\omega \geq M^{b(\omega)}$. Since this condition only applies in case of a hard liquidity constraint, a QCCE coincides with a competitive equilibrium for models with soft liquidity constraints. Both p and q are endogenously determined in a QCCE. Example 4.2 illustrates a QCCE.

Example 4.2: Consider the economy in Example 2.1 when $\rho = +\infty$. As argued in Example 3.2, it has no competitive equilibrium. We illustrate that there is a QCCE.

Let $(\bar{p}, \bar{q}, \bar{A}) \in \mathbb{R}^\Omega \times \{0, 1\}^\Omega \times \mathcal{A}$ be defined by $\bar{p} = (\bar{p}_{\omega_1}, \bar{p}_{\omega_2}) = (2, 2)$, $\bar{q} = (\bar{q}_{\omega_1}, \bar{q}_{\omega_2}) = (1, 0)$, and $\bar{A} = \{(\omega_1, 2)\}$. We show that $(\bar{p}, \bar{q}, \bar{A})$ is a QCCE by verifying that it satisfies Conditions (i), (ii), and (iii) of Definition 4.1.

First, $(\bar{p}, \bar{q}, \bar{A})$ satisfies Condition (i). For buyer b_1 , it holds that $\gamma^{b_1}(\bar{p}, \bar{q}) = \{\emptyset, \{(\omega_1, 2)\}\}$ and $\bar{A}^{b_1} = \bar{A} \in \delta^{b_1}(\bar{p}, \bar{q}; \rho) = \{\{(\omega_1, 2)\}\}$. We can similarly show $\bar{A}^{b_2} = \emptyset \in \delta^{b_2}(\bar{p}, \bar{q}; \rho) = \{\emptyset\}$.

Second, $(\bar{p}, \bar{q}, \bar{A})$ satisfies Condition (ii). For seller s , $\bar{A}^s = \bar{A} \in \delta^{s_1}(\bar{p}) = \{\{(\omega_1, 2)\}, \{(\omega_2, 2)\}\}$.

It is straightforward that Condition (iii) holds. Therefore, $(\bar{p}, \bar{q}, \bar{A})$ is a QCCE. $\triangle$

The following existence result of QCCEs is shown by Herings and Zhou (2021).

Proposition 4.3: Let $\mathcal{E}$ be an economy such that $r \in [0, +\infty]^B$. A QCCE exists.

We turn next to the analysis of an economy with hard liquidity constraints as the limit of an economy with increasingly stringent soft liquidity constraints.

The sequence of economies $(\mathcal{E}^n)_{n \in \mathbb{N}} = (B, S, \Omega, u, M, r^n)_{n \in \mathbb{N}}$ is said to converge to an economy $\bar{\mathcal{E}} = (B, S, \Omega, u, M, \bar{r})$ if $\bar{r} = \lim_{n \rightarrow \infty} r^n$. A sequence of outcomes $(A^n)_{n \in \mathbb{N}}$ in $\mathcal{A}$ is said to converge to an outcome $A \in \mathcal{A}$ if there is $n' \in \mathbb{N}$ such that, for every $n \geq n'$, $\omega(A^n) = \omega(A)$ and, for every $(\omega, t) \in A$, for every $c^n \in A^n$ with $\omega(c^n) = \omega$, $\lim_{n \rightarrow \infty} t(c^n) = t$. A sequence of equilibria, e.g., competitive equilibria or QCCEs, is convergent if the sequence of corresponding equilibrium prices and outcomes is convergent.

We establish the following general limit result of QCCEs.

Theorem 4.4: Let $(\mathcal{E}^n)_{n \in \mathbb{N}}$ be a sequence of economies that converges to an economy $\bar{\mathcal{E}}$. For every $n \in \mathbb{N}$, let (p^n, q^n, A^n) be a QCCE of the economy $\mathcal{E}^n$.

- (i) The sequence $(p^n, A^n)_{n \in \mathbb{N}}$ has a convergent subsequence.
- (ii) Let $(\bar{p}, \bar{A})$ be the limit of a convergent subsequence of $(p^n, A^n)_{n \in \mathbb{N}}$. Then there is $\bar{q} \in \{0, 1\}^\Omega$ such that $(\bar{p}, \bar{q}, \bar{A})$ is a QCCE of $\bar{\mathcal{E}}$.

Proof: By (b-ii), for every $b \in B$, every $\omega \in \Omega^b$, there is $\bar{t}_\omega^b \in \mathbb{R}$ such that $u^b(\omega, \bar{t}_\omega^b) = u^b(o^b)$. By (s-ii), for every $s \in S$, every $\omega \in \Omega^s$, there is $\underline{t}_\omega^s \in \mathbb{R}$ such that $u^s(\omega, \underline{t}_\omega^s) = u^s(o^s)$.

Part (i): Since the set of trades is finite, there is a subsequence $(A^{n_m})_{m \in \mathbb{N}}$ of $(A^n)_{n \in \mathbb{N}}$ such that the set of trades $\omega(A^{n_m})$ does not depend on n_m . Define $\bar{\Omega} = \omega(A^{n_m})$.

Step 1: For every $\omega \in \bar{\Omega}$, the sequence $(p_\omega^{n_m})_{m \in \mathbb{N}}$ is bounded.

Fix some $m \in \mathbb{N}$. Let $\omega \in \bar{\Omega}$, $b = b(\omega)$, and $s = s(\omega)$. Since $(p^{n_m}, q^{n_m}, A^{n_m})$ is a QCCE of $\mathcal{E}^{n_m}$, it holds that $U^s(A^{n_m}) = u^s(\omega, p_\omega^{n_m}) \geq u^s(o^s) = u^s(\omega, \underline{t}_\omega^s)$. It follows that $\underline{t}_\omega^s \leq p_\omega^{n_m}$.

There is $t_\omega^{n_m} \geq p_\omega^{n_m}$ such that $u^b(\omega, t_\omega^{n_m}) = U^b(\{\omega, p_\omega^{n_m}\}; r^{n_m})$. Since $(p^{n_m}, q^{n_m}, A^{n_m})$ is a QCCE of $\mathcal{E}^{n_m}$, it holds that $u^b(\omega, t_\omega^{n_m}) = U^b(\{\omega, p_\omega^{n_m}\}; r^{n_m}) \geq u^b(o^b) = u^b(\omega, \bar{t}_\omega^b)$. It follows that $p_\omega^{n_m} \leq t_\omega^{n_m} \leq \bar{t}_\omega^b$. We conclude that, for every $m \in \mathbb{N}$, $\underline{t}_\omega^s \leq p_\omega^{n_m} \leq \bar{t}_\omega^b$.

Step 2: For every $\omega \in \Omega \setminus \bar{\Omega}$, the sequence $(p_\omega^{n_m})_{m \in \mathbb{N}}$ is bounded.

Fix some $m \in \mathbb{N}$. Let $\omega \in \Omega \setminus \bar{\Omega}$, $b = b(\omega)$, and $s = s(\omega)$.

Assume $(A^{n_m})^b = \emptyset$. Consider the case where $p_\omega^{n_m} < M^b$, so $q_\omega^{n_m} = 1$. Since $\min\{0, \bar{t}_\omega^b\} \leq M^b$ and $(p^{n_m}, q^{n_m}, A^{n_m})$ is a QCCE of $\mathcal{E}^{n_m}$, it holds that

$$\begin{aligned} U^b(\{\omega, \min\{0, \bar{t}_\omega^b\}\}; r^{n_m}) &= u^b(\omega, \min\{0, \bar{t}_\omega^b\}) \geq u^b(\omega, \bar{t}_\omega^b) = u^b(o^b) = U^b(A^{n_m}; r^{n_m}) \\ &\geq U^b(\{\omega, p_\omega^{n_m}\}; r^{n_m}), \end{aligned}$$

and so $p_\omega^{n_m} \geq \min\{0, \bar{t}_\omega^b\}$. In case $p_\omega^{n_m} \geq M^b \geq 0$, it also holds that $p_\omega^{n_m} \geq \min\{0, \bar{t}_\omega^b\}$.

Assume $(A^{n_m})^b \neq \emptyset$. Consider the case where $p_\omega^{n_m} < M^b$, so $q_\omega^{n_m} = 1$. Let $(A^{n_m})^b = \{(\omega', p_{\omega'}^{n_m})\}$ and $s(\omega') = s'$. Since $\omega' \in \bar{\Omega}$, by Step 1, it holds that $\underline{t}_{\omega'}^{s'} \leq p_{\omega'}^{n_m} \leq \bar{t}_{\omega'}^b$. Let $t_\omega^b \in \mathbb{R}$ be such that $u^b(\omega', t_{\omega'}^{s'}) = u^b(\omega, t_\omega^b)$. Since $t_{\omega'}^{s'} \leq p_{\omega'}^{n_m}$, $\min\{0, t_\omega^b\} \leq M^b$, and $(p^{n_m}, q^{n_m}, A^{n_m})$ is a QCCE of $\mathcal{E}^{n_m}$, it holds that

$$\begin{aligned} U^b(\{\omega, \min\{0, t_\omega^b\}\}; r^{n_m}) &= u^b(\omega, \min\{0, t_\omega^b\}) \geq u^b(\omega', t_{\omega'}^{s'}) \geq u^b(\omega', p_{\omega'}^{n_m}) \\ &\geq U^b(A^{n_m}; r^{n_m}) \geq U^b(\{\omega, p_\omega^{n_m}\}; r^{n_m}), \end{aligned}$$

and so $p_\omega^{n_m} \geq \min\{0, t_\omega^b\}$. In case $p_\omega^{n_m} \geq M^b \geq 0$, it also holds that $p_\omega^{n_m} \geq \min\{0, t_\omega^b\}$.

Assume $(A^{n_m})^s = \emptyset$. Since $(p^{n_m}, q^{n_m}, A^{n_m})$ is a QCCE of $\mathcal{E}^{n_m}$, it holds that $U^s(A^{n_m}) = u^s(o^s) = u^s(\omega, t_\omega^s) \geq u^s(\omega, p_\omega^{n_m})$. It follows that $p_\omega^{n_m} \leq t_\omega^s$.

Assume $(A^{n_m})^s \neq \emptyset$. Let $(A^{n_m})^s = \{(\omega', p_\omega^{n_m})\}$ and $b(\omega') = b'$. Since $\omega' \in \bar{\Omega}$, by Step 1, it holds that $t_{\omega'}^s \leq p_\omega^{n_m} \leq \bar{t}_{\omega'}^{b'}$. By (s-ii), there is $t_\omega^s \in \mathbb{R}$ such that $u^s(\omega', \bar{t}_{\omega'}^{b'}) = u^s(\omega, t_\omega^s)$. Since $(p^{n_m}, q^{n_m}, A^{n_m})$ is a QCCE of $\mathcal{E}^{n_m}$, it holds that

$$U^s(\{\omega, t_\omega^s\}) = U^s(\{\omega', \bar{t}_{\omega'}^{b'}\}) \geq U^s(\{\omega', p_\omega^{n_m}\}) = U^s(A^{n_m}) \geq U^s(\{\omega, p_\omega^{n_m}\}),$$

and so $p_\omega^{n_m} \leq t_\omega^s$. Consequently, Step 2 holds.

By Steps 1 and 2, it follows that the sequence $(p^n, A^n)_{n \in \mathbb{N}}$ has a convergent subsequence.

Part (ii): Without loss of generality, assume the sequence $(p^n, A^n)_{n \in \mathbb{N}}$ is convergent and has limit $(\bar{p}, \bar{A})$. Define $\bar{\Omega} = \omega(\bar{A})$. Let $\bar{q} \in \{0, 1\}^\Omega$ be such that (i) for every $\omega \in \bar{\Omega}$, $\bar{q}_\omega = 1$, and (ii), for every $\omega \in \Omega \setminus \bar{\Omega}$, if $r^{b(\omega)} = +\infty$ and $\bar{p}_\omega \geq M^{b(\omega)}$, $\bar{q}_\omega = 0$, and if $r^{b(\omega)} < +\infty$ or $\bar{p}_\omega < M^{b(\omega)}$, $\bar{q}_\omega = 1$. Since for every $n \in \mathbb{N}$, A^n is a QCCE outcome and the sequence $(A^n)_{n \in \mathbb{N}}$ converges to $\bar{A}$, it is easily seen that $\bar{A}$ is an outcome. We show next that $(\bar{p}, \bar{q}, \bar{A})$ is a QCCE of $\bar{\mathcal{E}}$.

Step 1: Condition (i) of Definition 4.1 holds.

By contradiction, suppose that there is a buyer $b \in B$ such that $\bar{A}^b \notin \delta^b(\bar{p}, \bar{q}; \bar{r})$.

Case 1: $\bar{A}^b = \emptyset$. Since $\bar{A}^b = \emptyset \notin \delta^b(\bar{p}, \bar{q}; \bar{r})$, there is $\omega' \in \Omega^b$ such that $\{(\omega', \bar{p}_{\omega'})\} \in \delta^b(\bar{p}, \bar{q}; \bar{r})$, $U^b(\emptyset; \bar{r}) < U^b(\{(\omega', \bar{p}_{\omega'})\}; \bar{r})$, and $\bar{q}_{\omega'} = 1$. If $\bar{r}^b = +\infty$, it holds by the construction of $\bar{q}$ that $\bar{p}_{\omega'} < M^b$. For sufficiently large n , we have that $p_{\omega'}^n < M^b$, $q_{\omega'}^n = 1$, and $U^b(\emptyset; r^n) = u^b(o^b) < u^b(\omega', p_{\omega'}^n) = U^b(\{(\omega', p_{\omega'}^n)\}; r^n)$, contradicting that (p^n, q^n, A^n) is a QCCE of $\mathcal{E}^n$. If $\bar{r}^b < +\infty$, we have for sufficiently large n that $U^b(\emptyset; r^n) = u^b(o^b) < U^b(\{(\omega', p_{\omega'}^n)\}; r^n)$, contradicting that (p^n, q^n, A^n) is a QCCE of $\mathcal{E}^n$.

Case 2: $\bar{A}^b \neq \emptyset$. Let ω be the unique element of $\omega(\bar{A}^b)$. First, we show that if $\bar{r}^b = +\infty$, then $\bar{p}_\omega \leq M^b$. By contradiction, suppose that $\bar{p}_\omega > M^b$. There is $\varepsilon > 0$ such that $\bar{p}_\omega > M^b + \varepsilon$. For sufficiently large n , we have that $p_\omega^n \geq M^b + \varepsilon$ and $r^n \varepsilon > \bar{t}_\omega^b$. This implies that

$$U^b(A^n; r^n) = u^b(\omega, r^n(p_\omega^n - M^b) + p_\omega^n) \leq u^b(\omega, r^n \varepsilon) < u^b(\omega, \bar{t}_\omega^b) = u^b(o^b) = U^b(\emptyset; r^n),$$

contradicting that (p^n, q^n, A^n) is a QCCE of $\mathcal{E}^n$. Consequently, we have that $\bar{p}_\omega \leq M^b$.

Since $\bar{A}^b \notin \delta^b(\bar{p}, \bar{q}; \bar{r})$, there is $\tilde{A}^b \in \delta^b(\bar{p}, \bar{q}; \bar{r})$ such that $U^b(\bar{A}^b; \bar{r}) < U^b(\tilde{A}^b; \bar{r})$.

Case 2-1: $\bar{r}^b = +\infty$. In such a case, it holds that $\bar{p}_\omega \leq M^b$. For every $n \in \mathbb{N}$, we have that

$$U^b(\{(\omega, p_\omega^n)\}; r^n) = u^b(\omega, p_\omega^n + r^{n,b} \max\{0, p_\omega^n - M^b\}) \leq u^b(\omega, p_\omega^n).$$

Assume $\tilde{A}^b = \emptyset$. Since $\lim_{n \rightarrow \infty} u^b(\omega, p_\omega^n) = u^b(\omega, \bar{p}_\omega) = U^b(\bar{A}^b; \bar{r})$, for sufficiently large n , we have that $U^b(\{(\omega, p_\omega^n)\}; r^n) < U^b(\emptyset; r^n)$, contradicting that (p^n, q^n, A^n) is a QCCE of $\mathcal{E}^n$.

Assume $\tilde{A}^b \neq \emptyset$. Let $\tilde{\omega}$ be the unique element of $\omega(\tilde{A}^b)$. It holds that $\bar{q}_{\tilde{\omega}} = 1$. Since $\bar{r}^b = +\infty$, by the construction of $\bar{q}$, it holds that $\bar{p}_{\tilde{\omega}} < M^b$. Since $\lim_{n \rightarrow \infty} u^b(\omega, p_\omega^n) = u^b(\omega, \bar{p}_\omega) = U^b(\bar{A}^b; \bar{r})$, it follows that $U^b(\{(\omega, p_\omega^n)\}; r^n) < U^b(\{(\tilde{\omega}, p_{\tilde{\omega}}^n)\}; r^n)$ for sufficiently large n , contradicting that (p^n, q^n, A^n) is a QCCE of $\mathcal{E}^n$.

Case 2-2: $\bar{r}^b < +\infty$. If $\tilde{A}^b = \emptyset$, we have for n sufficiently large that $U^b(\{(\omega, p_\omega^n)\}; r^n) < U^b(\emptyset; r^n)$, contradicting that (p^n, q^n, A^n) is a QCCE of $\mathcal{E}^n$. If $\tilde{A}^b \neq \emptyset$, let $\tilde{\omega}$ be the unique element of $\omega(\tilde{A}^b)$. Since $\bar{r}^b < +\infty$, it holds that $\bar{q}_{\tilde{\omega}} = 1$. We have for n sufficiently large that $U^b(\{(\omega, p_\omega^n)\}; r^n) < U^b(\{(\tilde{\omega}, p_{\tilde{\omega}}^n)\}; r^n)$, contradicting that (p^n, q^n, A^n) is a QCCE of $\mathcal{E}^n$.

Thus Step 1 holds.

Step 2: Condition (ii) of Definition 4.1 holds.

By contradiction, suppose that there is $s \in S$ such that $\bar{A}^s \notin \delta^s(\bar{p})$. In case $\bar{A}^s = \emptyset$, there is $\omega' \in \Omega^s$ such that $U^s(\bar{A}^s) < U^s(\{(\omega', \bar{p}_{\omega'})\})$. We have for n sufficiently large that $U^s(\emptyset) = U^s(A^n) < U^s(\{(\omega', p_{\omega'}^n)\})$, contradicting that (p^n, q^n, A^n) is a QCCE of $\mathcal{E}^n$.

In case $\bar{A}^s \neq \emptyset$, let ω be the unique element of $\omega(\bar{A}^s)$. Then there is $\tilde{A}^s \in \delta^s(\bar{p})$ such that $U^s(\bar{A}^s) < U^s(\tilde{A}^s)$. If $\tilde{A}^s = \emptyset$, we have for n sufficiently large that $U^s(\{\omega, p_{\omega}^n\}) = U^s(A^n) < U^s(\emptyset)$, contradicting that (p^n, q^n, A^n) is a QCCE of $\mathcal{E}^n$. If $\tilde{A}^s \neq \emptyset$, let $\tilde{\omega}$ be the unique element of $\omega(\tilde{A}^s)$. For n sufficiently large, we have that $U^s(\{\omega, p_{\omega}^n\}) = U^s(A^n) < U^s(\{\tilde{\omega}, p_{\tilde{\omega}}^n\})$, contradicting that (p^n, q^n, A^n) is a QCCE of $\mathcal{E}^n$.

Condition (iii) of Definition 4.1 follows immediately from the construction of $\bar{q}$. Consequently, $(\bar{p}, \bar{q}, \bar{A})$ is a QCCE of $\bar{\mathcal{E}}$. **Q.E.D.**

Theorem 4.4 considers sequences $(p^n, A^n)_{n \in \mathbb{N}}$ of prices and outcomes and *leaves out* the quantity constraints q^n . Indeed, the limit of any convergent sequence of QCCEs $(p^n, q^n, A^n)_{n \in \mathbb{N}}$ of $\mathcal{E}^n$ may not be a QCCE of the limit economy.⁶

In addition, not every set of QCCE trades at the limit economy corresponds to the limit of competitive equilibrium trades in economies with increasingly stringent liquidity constraints.⁷

A particularly interesting special case of Theorem 4.4 connects competitive equilibria in economies with increasingly stringent soft liquidity constraints to a QCCE at the limit economy with hard liquidity constraints. The formal statement is as follows.

Corollary 4.5: Let $(\mathcal{E}^n)_{n \in \mathbb{N}}$ be a sequence of economies with soft liquidity constraints that converges to a limit economy $\bar{\mathcal{E}}$. For every $n \in \mathbb{N}$, let $(p^n, A^n)_{n \in \mathbb{N}}$ be a competitive equilibrium of the economy $\mathcal{E}^n$. Then $(p^n, A^n)_{n \in \mathbb{N}}$ contains a convergent subsequence whose limit $(\bar{p}, \bar{A})$ correspond to QCCE prices and outcome in the limit economy $\bar{\mathcal{E}}$.

4.2 Expectational Equilibrium

Herings (2020) proposes the notion of expectational equilibrium. An expectational equilibrium does not explicitly depend on prices but rather on expectations related to tradable contracts. The concept unifies all the existing approaches of competitive equilibrium and, in particular, can be applied to both settings with and settings without monetary transfers.

Let $\bar{Y} = \Omega \times \mathbb{R}$ be the set of all possible contracts and $2^{\bar{Y}}$ be the power set of $\bar{Y}$. Let $Q \subseteq \bar{Y}$ denote a set of rationing constraints on the buyers side. For every $b \in B$, Q^b represents the set of contracts for which buyer b expects no supply from the sellers side. Similarly, Let $R \subseteq \bar{Y}$ denote a set of rationing constraints on the sellers side. For every $s \in S$, R^s is the set of contracts for which seller s expects no demand from the buyers side. An expectational equilibrium corresponds to an endogenously determined profile of rationing constraints (Q, R) together with an outcome A .

Given a set of rationing constraints Q , the *rationed budget set* of buyer $b \in B$ is given by

$$\gamma^b(Q) = \{A^b \in X^b \mid A^b \cap Q^b = \emptyset\}$$

⁶In Example 3.3, a sequence of competitive equilibria (p^n, A^n) is a sequence of QCCEs (p^n, q^n, A^n) with $q^n = (1, \dots, 1)$ when $\rho^n \in [0, +\infty)$. However, at the limit economy shown in Example 4.2, there is no competitive equilibrium so there is no QCCE that is compatible with quantity constraints $\bar{q} = (1, \dots, 1)$.

⁷In Example 4.2, $(\bar{p}, \bar{q}, \bar{A})$ is a QCCE with set of equilibrium trades equal to $\{\omega_1, \omega_2\}$. However, $\{\omega_1, \omega_2\}$ is not compatible with any competitive equilibrium in an economy with soft liquidity constraints in Example 3.3.

and the *rationed demand set* of buyer $b \in B$ is given by

$$\delta^b(Q; r) = \arg \max_{A^b \in \gamma^b(Q)} U^b(A^b; r).$$

The rationed budget set of buyer b consists of all contracts for which the buyer does not expect rationing together with the no-trade option. The rationed demand set of buyer b collects all choices that maximize buyer b 's utility over the rationed budget set.

Similarly, given a set of rationing constraints R , the *rationed budget set* and *rationed demand set* of seller $s \in S$ are respectively given by

$$\gamma^s(R) = \{A^s \in X^s \mid A^s \cap R^s = \emptyset\},$$

$$\delta^s(R) = \arg \max_{A^s \in \gamma^s(R)} U^s(A^s).$$

The definition of an expectational equilibrium is as follows.

Definition 4.6: An element $(A, Q, R) \in \mathbb{R}^\Omega \times 2^{\bar{Y}} \times 2^{\bar{Y}}$ is an *expectational equilibrium* for the economy $\mathcal{E} = (B, S, \Omega, u, M, r)$ if:

- (i) For every $b \in B$, $A^b \in \delta^b(Q; r)$.
- (ii) For every $s \in S$, $A^s \in \delta^s(R)$.
- (iii) $Q \cap R = \emptyset$.

The first two conditions correspond to optimization by buyers and sellers given the profile of rationing constraints. The third condition expresses that markets are transparent. For a given contract, it cannot be the case that both sides expect to be rationed at the same time.

Example 4.7 illustrates an expectational equilibrium.

Example 4.7: Consider the economy in Example 2.1 when $\rho = +\infty$.

Let $(A, Q, R) \in \mathbb{R}^\Omega \times 2^{\bar{Y}} \times 2^{\bar{Y}}$ be defined by $A = \{(\omega_1, 2)\}$, $Q^{b_1} = \{(\omega_1, t) \in \bar{Y} \mid t < 2\}$, $Q^{b_2} = \{(\omega_2, t) \in \bar{Y} \mid t \leq 2\}$, and $R = \bar{Y} \setminus Q$.

We show that (A, Q, R) satisfies Conditions (i), (ii), and (iii) of Definition 4.6.

First, (A, Q, R) satisfies Condition (i). For buyer b_1 , it holds that $\gamma^{b_1}(Q) = \{(\omega_1, t) \in \bar{Y} \mid t \geq 2\} \cup \{\emptyset\}$ so $A^{b_1} = A \in \delta^{b_1}(Q; \rho) = \{\{(\omega_1, 2)\}\}$. We can similarly show $A^{b_2} = \emptyset \in \delta^{b_2}(Q; \rho) = \{\emptyset\}$.

Second, (A, Q, R) satisfies Condition (ii). For seller s , $\gamma(R) = \{(\omega_1, t) \in \bar{Y} \mid t \leq 2\} \cup \{(\omega_2, t) \in \bar{Y} \mid t < 2\} \cup \{\emptyset\}$ so $A^s = A \in \delta^{s_1}(\bar{p}) = \{\{(\omega_1, 2)\}\}$.

It is straightforward that Condition (iii) holds. Therefore, (A, Q, R) is an expectational equilibrium. $\triangle$

In what comes next, we show the equivalence between QCCE and expectational equilibrium outcomes. Then the existence and convergence results of expectational equilibrium follow directly from Proposition 4.3 and Theorem 4.4.

Theorem 4.8: Let $\mathcal{E} = (B, S, \Omega, u, M, r)$ be an economy such that $r \in [0, +\infty]^B$. The sets of QCCE outcomes and expectational equilibrium outcomes coincide.

Proof: The proof consists of the following two steps.

Step 1: If (p, q, A) is a QCCE of $\mathcal{E}$, there is $(Q, R) \in 2^{\bar{Y}} \times 2^{\bar{Y}}$ such that (A, Q, R) is an expectational equilibrium.

For every $b \in B$, we define

$$Q^b = \{(\omega, t) \in \Omega^b \times \mathbb{R} \mid \omega \in \omega(A) \wedge t < p_\omega\} \cup \{(\omega, t) \in \Omega^b \times \mathbb{R} \mid \omega \in \Omega \setminus \omega(A) \wedge t \leq p_\omega\}.$$

For every $s \in S$, we define $R^s = \{(\omega, t) \in \Omega^s \times \mathbb{R} \mid t > p_\omega\}$.

By construction, we have that $Q \cap R = \emptyset$ and so Condition (iii) of Definition 4.6 holds for (A, Q, R) . We show below that Conditions (i) and (ii) of Definition 4.6 hold for (A, Q, R) .

Let $b \in B$. By the construction of Q^b , we have

$$\gamma^b(Q) = \{\{(\omega, t)\} \in X^b \mid \omega \in \omega(A) \wedge t \geq p_\omega\} \cup \{\{(\omega, t)\} \in X^b \mid \omega \in \Omega \setminus \omega(A) \wedge t > p_\omega\} \cup \{\emptyset\}.$$

By Condition (i) of Definition 4.1, we have that $U^b(A^b; r) \geq U^b(\emptyset)$. Let $\{c\} \in \gamma^b(Q)$ be such that $\omega(c) \in \Omega \setminus \omega(A)$. In case $q_{\omega(c)} = 1$, by Condition (i) of Definition 4.1, it holds that $U^b(A^b; r) \geq U^b(\{(\omega, p_\omega)\}; r) \geq U^b(\{c\}; r)$. In case $q_{\omega(c)} = 0$, by Condition (iii) of Definition 4.1, it holds that $r^b = +\infty$ and $t(c) > p_{\omega(c)} \geq M^b$. Thus $U^b(A^b; r) > U^b(\{c\}; r)$. If $A^b = \emptyset$, we are done. If $A^b = \{(\omega', p_{\omega'})\}$, it holds that $U^b(A; r) \geq U^b(\{(\omega', t')\}; r)$ for every $t' \geq p_{\omega'}$. Consequently, Condition (i) of Definition 4.6 holds.

Let $s \in S$. By the construction of R^s , $\gamma^s(R) = \{\{(\omega, t)\} \in X^s \mid t \leq p_\omega\} \cup \{\emptyset\}$. Since (p, q, A) is a QCCE, by Condition (ii) of Definition 4.1, we have that $U^s(A) \geq U^s(\emptyset)$, and for every $\{(\omega, t)\} \in \gamma^s(R)$, it holds that $U^s(A) \geq u^s(\omega, p_\omega) \geq u^s(\omega, t)$. Thus Condition (ii) of Definition 4.6 holds.

Step 2: If (A, Q, R) is an expectational equilibrium of $\mathcal{E}$, there is $(p, q) \in \mathbb{R}^\Omega \times \{0, 1\}^\Omega$ such that (p, q, A) is a QCCE of $\mathcal{E}$.

For every $s \in S$, for every $\omega \in \Omega^s$, there is $p_\omega \in \mathbb{R}$ such that $u^s(A) = u^s(\omega, p_\omega)$. Let $p_\omega \in \mathbb{R}$ be the price of trade ω . For every $\omega \in \Omega$, if $r^{b(\omega)} = +\infty$ and $p_\omega \geq M^{b(\omega)}$, $q_\omega = 0$, and if $r^{b(\omega)} < +\infty$ or $p_\omega < M^{b(\omega)}$, $q_\omega = 1$. By the construction of $(p, q) \in \mathbb{R}^\Omega \times \{0, 1\}^\Omega$, it is easily seen that Conditions (ii) and (iii) of Definition 4.1 hold for (p, q, A) . We show next that Condition (i) of Definition 4.1 holds for (p, q, A) .

By contradiction, suppose Condition (i) of Definition 4.1 fails to hold. Then there is $b \in B$ such that $A^b \notin \delta^b(p, q; r)$. Since (A, Q, R) is an expectational equilibrium, by Condition (i) of Definition 4.6, $U^b(A^b; r) \geq U^b(\emptyset)$ and so there is $\{(\omega, p_\omega)\} \in \delta^b(p, q; r)$ such that $U^b(A^b; r) < U^b(\{(\omega, p_\omega)\}; r)$ and $q_\omega = 1$. By the construction of q , either $r^{b(\omega)} < +\infty$ or $p_\omega < M^{b(\omega)}$ holds. In either case, there is $t > p_\omega$ such that $U^b(A^b; r) < U^b(\{(\omega, t)\}; r)$ and $U^{s(\omega)}(A) = U^{s(\omega)}(\{(\omega, p_\omega)\}) < U^{s(\omega)}(\{(\omega, t)\})$. Since (A, Q, R) is an expectational equilibrium, by Conditions (i) and (ii) of Definition 4.6, we have that $\{(\omega, t)\} \in Q^b$ and $\{(\omega, t)\} \in R^{s(\omega)}$, contradicting Condition (iii) of Definition 4.6. **Q.E.D.**

Proposition 4.3 and Theorem 4.8 together implies the following existence result.

Corollary 4.9: Let $\mathcal{E}$ be an economy such that $r \in [0, +\infty]^B$. An expectational equilibrium exists.⁸

By Theorems 4.4 and 4.8, the following limit result holds for expectational equilibrium.

⁸Since both the model with soft and the model with hard liquidity constraints satisfy the assumptions of Theorem 5.5 of Herings (2020), the existence of expectational equilibrium follows as well from that result.

Corollary 4.10: Let $(\mathcal{E}^n)_{n \in \mathbb{N}}$ be a sequence of economies that converges to an economy $\bar{\mathcal{E}}$. For every $n \in \mathbb{N}$, let (A^n, Q^n, R^n) be an expectational equilibrium of the economy $\mathcal{E}^n$.

- (i) The sequence $(A^n)_{n \in \mathbb{N}}$ has a convergent subsequence.
- (ii) Let $\bar{A}$ be the limit of a convergent subsequence of $(A^n)_{n \in \mathbb{N}}$. Then there is $(\bar{Q}, \bar{R}) \in 2^{\bar{Y}} \times 2^{\bar{Y}}$ such that $(\bar{A}, \bar{Q}, \bar{R})$ is an expectational equilibrium of $\bar{\mathcal{E}}$.

Corollary 4.10 implies that expectational equilibrium does not suffer from discontinuity problems in case a sequence of economies with increasingly stringent soft liquidity constraints converges to a limit economy with hard liquidity constraints.

5 Discussion

In this section, we study the continuity of stable outcomes as well as core outcomes.

A *stable outcome* is a set of contracts that contains at most one contract for each agent such that every agent involved in a contract prefers this contract weakly to the no-trade option and there are no two agents who can sign a contract that makes both of them strictly better off. An outcome is a *core outcome* if no coalition of agents can propose a better outcome satisfying the following two conditions: Agents outside the coalition obtain the no-trade option and each agent in the coalition is strictly better off.⁹

Herings and Zhou (2021) show the equivalence between QCCE outcomes, stable outcomes, and core outcomes for different types of liquidity constraints. As corollaries of Theorem 4.4, we conclude that stable outcomes and core outcomes do not suffer from discontinuity problems when going from economies with increasingly stringent soft constraints to the limit economy with hard constraints.

The sets of strongly stable outcomes and strict core outcomes¹⁰ coincide in an economy with soft liquidity constraints and are both non-empty. However, in an economy with hard liquidity constraints, the sets of strongly stable outcomes and strict core outcomes can be empty (Herings and Zhou, 2021). Thus, a convergent sequence of strongly stable outcomes, or strict core outcomes in economies with increasingly stringent soft liquidity constraints may not be a strongly stable outcome or strict core outcome at the limit economy with hard liquidity constraints, respectively.

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⁹For formal definitions of stable outcomes and core outcomes, we refer to Crawford and Knoer (1981) and Herings and Zhou (2021).

¹⁰An outcome is strongly stable if (i) no agent involved in a contract prefers the no-trade option and (ii) there are no two agents who can sign a contract making both agents weakly better off and at least one strictly better off. An outcome is a strict core outcome if there is no outcome and a coalition of agents such that (i) agents outside the coalition obtain $\emptyset$, and (ii) each agent in the coalition is weakly better off, with at least one being strictly better off. For formal definitions, we refer to Herings and Zhou (2021).

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