

Modular stable matching mechanisms*

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April 29, 2022

Abstract

We propose and analyze *modular stable matching mechanisms* as a candidate for a foundational framework to address issues of social welfare and equity in the stable matching model. We present two characterizations for modular stable matching mechanisms that reveal the ordinal content of optimizing a *modular* function under the constraints of *stability* and *invariance under stability* and present several examples. Then, we propose a new equity notion and characterize the class of modular stable matching mechanisms that comply with this notion. Our analysis indicates that modular matching mechanisms are both structured and rich enough to implement a wide range of objectives.

Keywords: Social welfare, equity, stable matchings lattice, rotations, modular optimization.

*We thank Battal Doğan, Serhat Doğan, Faruk Gul, Tarık Kara and participants of several seminars and conferences for valuable comments and suggestions. Kemal Yıldız’s research is supported by the BAGEP Award of the Science Academy in Turkey.

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1 Introduction

The stable matching model and its variants have been studied extensively since the pioneering work of [Gale & Shapley \(1962\)](#). In addition to its elegant theory, a particularly appealing feature of the stable matching model has been its applicability. The extremal matchings, which maximize the welfare of one side in the market, play a crucial role in the applications. However, investigation of more equitable stable matching mechanisms gathers the attention of researchers in several disciplines including computer science, operations research, mathematics, and economics. We believe that these studies have proven that the breadth of possibilities calls for a foundational framework to address issues of social welfare and equity in the stable matching model. We propose and analyze the *modular stable matching mechanisms* as a candidate to form such a framework.

To introduce the framework, consider a society consisting of equal numbers of *men* and *women*. A matching (marriage) *problem* is a preference profile in which each agent has preferences over the opposite party. A *matching* uniquely assigns each man to a woman. Our primitive objects are *matching mechanisms* that associate a nonempty set of matchings with each problem. The set of associated matchings can be thought of as a shortlist from which a final choice is to be made, the matchings that are assigned positive probability in a lottery, or the matchings used in a rotation scheme that, for example, specifies the periodical job assignments to the employees throughout their employment.

The central robustness criterion for a matching—introduced by [Gale & Shapley \(1962\)](#)—is *stability*, which requires that there is no unmatched man-woman pair who prefer each other to their assigned mates. In addition to requiring that a matching mechanism chooses only stable matchings, we impose *invariance under stability*, which requires the matchings chosen from two different problems be the same unless these problems induce different sets of stable matchings. We refer to these matching mechanisms as *stable matching mechanisms*.

Modularity of a stable matching mechanism requires the chosen matchings be the ones that optimize an explicit objective function, which is restricted to be *modular* by using the lattice structure of stable matchings for each problem. As demonstrated by several studies in the literature, formulating social objectives in the form a modular function provides analytical and computational tractability in addition to clarity.¹ Here, we provide examples demonstrating that a wide range of objectives can be implemented through modular stable matching mechanisms.

The matching mechanisms used in the literature can roughly be classified into two groups. In the first group, we can count the ones that choose the stable matchings that optimize an explicitly given objective function, which can be interpreted as measure of social welfare or fairness—such as the utilitarian objective that seeks to maximize sum of individual utilities.² As

¹For example, [Grötschel et al. \(1981\)](#) show that minimizing a (sub)modular function over a lattice family can be done in oracle-polynomial time via the ellipsoid algorithm, and [Schrijver \(2000\)](#) provided a combinatorial algorithm for the same purpose. For an introduction to submodular functions, with more examples, one can consult [Lovász \(1983\)](#). [Edmonds \(1970\)](#) and [Topkis \(1978\)](#) are two classical references on the minimization of submodular functions.

²For other examples, one can count, *minimum regret stable matchings*, *egalitarian stable matchings*, the *minimum*

discussed in Section 2.2, most of these mechanisms violate *invariance under stability*, since the objective functions in question are formulated by using agents' utilities/rankings over each other, independent of whether there is a stable matching in which these agents are matched or not. In the second group, we consider the mechanisms whose formulations are not based on optimization of an explicitly given objective function, but formulated by directly using the mathematical structure of the stable matchings,³ and thus satisfy *invariance under stability*. Modular stable matching mechanisms. Here, we provide a synthesis of these two groups of mechanisms, in that modular stable matching mechanisms have clear underlying objectives formulated in a tractable mathematical form, and satisfy *invariance under stability*.

In Theorem 1, we present two characterizations for modular stable matching mechanisms that reveal the ordinal content of optimizing a modular function under the constraints of *stability* and *invariance under stability*. Thus, we provide testable conditions to verify if the observed choice of stable matchings in a society comply with optimization of a modular function. These results guide us about normative qualitative criteria that one may consider to adopt a modular stable matching mechanism. On the technical side, we deepen and use the connection between the stable matchings lattice and the *rotations poset* (Irving & Leather 1986), which is well-known in computer science, but seems to be overlooked in economics.

Our first axiom, *convexity*, requires that for each problem and for a given pair of matchings that are chosen by the mechanism, if one can form a stable matching by assigning all the agents to one of their mates in the given pair, then this newly formed stable matching should have been chosen as well. To introduce the second axiom, suppose that we transform a given problem—the preferences of the agents—such that all the agents move their mates in the matchings chosen by the mechanism to the top of their preferences by preserving the relative rankings elsewhere. Our *independence of irrelevant rankings* requires that if a matching that is stable in the original problem remains stable in this transformed problem, then it must be one of the matchings chosen by the mechanism in the initial problem.

In the second part of the paper, we propose a new equity notion that is based on the notion of the “median attainable mate”. To introduce this notion, consider an agent i and all the agents who are *attainable* for i in the sense that for each such agent there exists a stable matching in which i is assigned to him or her in the given problem. Next, from among the agents who are attainable for i , consider the one(s) with (a) *median rank*.⁴ The *median attainable mate* for agent i is the (more preferred) attainable agent who has the lower attainable median rank. A matching mechanism is *minimally equitable* if for each matching that is chosen in a given problem, there is no other stable matching in which each agent is assigned to a mate who is either the same agent or *closer* to his/her median attainable mate than compared to his/her mate at the chosen matching.

weight stable matchings, and *sex-equal stable matchings*. For the related definitions we refer the reader to Manlove (2013, Chapter 1.3), Gusfield & Irving (1989), and the references therein.

³In Examples 1-3, we present three of such matching mechanisms introduced by Teo & Sethuraman (1998), Cheng (2010) and Cheng, McDermid & Suzuki (2016).

⁴If there is an odd number of attainable agents for i , then there is a unique attainable agent with this property, otherwise there are two such agents.

In Section 4, we present a simple problem in which there is a stable matching that assigns all the agents to their unique median attainable mates. However, several stable matching mechanisms from the literature fail to choose this matching, and thus violate *minimal equatability*. Then, in Theorem 2, we characterize the class of modular stable matching mechanisms that satisfy *minimal equatability*. Our discussion in this part of the paper demonstrates that studying modular stable matching mechanism opens new avenues through which the issues of equity can be fruitfully analyzed.

As for more general matching contexts, such as many-to-one and many-to-many matchings, it is known that the set of stable matchings forms a distributive lattice under suitable restrictions.⁵ These results together with the connection provided by Blair (1984)⁶ indicate that our results can be extended into more general matching contexts that are rich with market-design applications.

2 Modular Stable Matching Mechanisms

2.1 Preliminaries

Let M be a set of n men and W be a set of n women. Each $m \in M$ has preferences over W and each $w \in W$ has preferences over M . Let $N = M \cup W$, then preferences of each $i \in N$, denoted by $\succ_i$, is a **strict ordering**, i.e. $\succ_i$ is *complete*, *transitive*, and *asymmetric*. Let $\mathcal{P}_i$ denote the set of all possible preference relations for agent i , and $\mathcal{P} = \times_{i \in N} \mathcal{P}_i$ denote the set of all possible preference profiles. We denote a generic preference profile by $\succ$. A **matching** is a one-to-one function $\mu : M \cup W \rightarrow M \cup W$ such that for each $(m, w) \in M \times W$, we have $\mu(m) \in W$, $\mu(w) \in M$, and $\mu(m) = w$ if and only if $\mu(w) = m$.

2.2 Stable matching mechanisms

A matching μ is **stable** at a problem $\succ \in \mathcal{P}$ if there is no **blocking-pair** $(m, w) \in M \times W$ such that $m \succ_w \mu(w)$ and $w \succ_m \mu(m)$. Let $\mathcal{S}(\succ)$ denote the set of all stable matchings at a given preference profile (problem) $\succ \in \mathcal{P}$. We sometimes use $\mathcal{S}$ instead of $\mathcal{S}(\succ)$ if the problem that is referred to is clear from the context. Let $\triangleright_M$ and $\triangleright_W$ denote the **men-wise better than** and **women-wise better than** relations over $\mathcal{S}$, which are defined as follows: For each distinct $\mu, \mu' \in \mathcal{S}$, $\mu \triangleright_M \mu'$ if and only if for each $m \in M$, $\mu(m) \succ_m \mu'(m)$ or $\mu(m) = \mu'(m)$. The women-wise better than relation $\triangleright_W$ is defined similarly. A **matching mechanism** is a mapping π that associates each problem $\succ \in \mathcal{P}$ with a nonempty set of matchings.

⁵Alkan (2001) and Alkan & Gale (2003) show that in the context of many-to-one and many-to-many matchings, the distributivity of the stable matchings lattice can be guaranteed by strengthening the *substitutability* of the choice rules with *size monotonicity* (called *law of aggregate demand* by Hatfield & Milgrom (2005)).

⁶Who shows that for every distributive lattice, such as the perfect matchings of a bipartite planar graph or alternating sign matrices, there is a marriage problem with a stable matchings lattice that is order-isomorphic to the given distributive lattice.

Definition. A *stable matching mechanism* π is a matching mechanism that satisfies the following two conditions:

Stability: For each problem $\succ \in \mathcal{P}$, $\pi(\succ) \subset \mathcal{S}(\succ)$.

Invariance under stability: For each $\succ, \succ' \in \mathcal{P}$, if $\mathcal{S}(\succ) = \mathcal{S}(\succ')$ then $\pi(\succ) = \pi(\succ')$.

Stability requires the matchings chosen for each problem be stable. *Invariance under stability* requires the matchings chosen from two different problems be the same unless these problems induce different set of stable matchings. Put differently, for a stable matching mechanism the set of stable matchings associated with each problem provides the relevant information to operate. In our Examples 4-7, we will demonstrate that to formulate an objective function whose optimization leads to a stable matching mechanism that satisfies *invariance under stability*, the notion of “attainable mates” turns out to be critical. Namely, a man and a woman are *attainable* for each other in a given problem, if there exists a stable matching in which they are assigned to each other.

Several matching mechanisms in the literature are formulated as to choose the stable matchings that optimize an objective function, which can be interpreted as measure of social welfare or fairness—such as the utilitarian objective that seeks to maximize sum of individual rankings. This rich class include the ones that choose *minimum regret stable matchings*, *egalitarian stable matchings*, the *minimum weight stable matchings*, and *sex-equal stable matchings*. As a common theme in all of these mechanisms, the objective function that is to be optimized is formulated by using agents’ utilities/rankings over each other, independent of them being *attainable* or not. Therefore, they all fail to satisfy *invariance under stability*. To see this, let $\succ$ be a problem and consider the problem $\succ'$ that is obtained from $\succ$ such that agents’ *attainable mates* are moved to top of their preferences by preserving the relative rankings elsewhere. It is easy to verify that the associated set of stable matchings remains the same. However, since $\succ$ and $\succ'$ are in general different than each other, optimization of an objective function based on agents’ utilities/rankings over each other in these problems might result in different stable matchings.

There is a another class of matching mechanisms in the literature whose formulation are not explicitly based on optimization of an objective function, but are built on the exquisite geometric structure of stable matchings. That is, for each pair of stable matchings μ and μ' , consider $\mu \vee \mu'$ ($\mu \wedge \mu'$) that maps each man to his best (worst) mate among the women he is matched at μ or μ' ; it turns out that both $\mu \vee \mu'$ and $\mu \wedge \mu'$ are stable matchings as well, which implies that the pair $\langle \mathcal{S}, \triangleright_M \rangle$ forms a (distributive) **lattice**.⁷ Next, we present three of such matching mechanisms introduced by Teo & Sethuraman (1998), Cheng (2010), and Cheng, McDermid & Suzuki (2016). Since formulations of these mechanisms are entirely based on the structure of the stable matching lattice, they satisfy *invariance under stability*, and thus form stable matching mechanisms.

Example 1 (Median stable matchings). Let $\succ$ be a problem such that there are K stable matchings. For each man m , collect his mates from all these K stable matchings and arrange them from his most preferred to his least preferred woman. Let $w^k(m)$ denote the k -th woman in this sorted

⁷Knuth (1976), pp. 92-93, attributes the discovery of this lattice structure to J. H. Conway.

list in which the same woman can be counted multiple times. For each $k \in \{1, \dots, K\}$, define $\mu^k : M \cup W \rightarrow M \cup W$ such that $\mu^k(m) = w^k(m)$ for each $m \in M$. Teo & Sethuraman (1998) show that μ^k is a stable matching. Then, they define the *median stable matching(s)* as $\mu^{(K+1)/2}$ when K is odd, and $\mu^{K/2}$ and $\mu^{(K/2)+1}$ when K is even.⁸

Example 2 (Median of the stable matchings lattice). Consider the undirected graph associated with the stable matching lattice $\langle \mathcal{S}, \triangleright_M \rangle$, in which each $\mu \in \mathcal{S}$ is a vertex and for each $\mu, \mu' \in \mathcal{S}$, μ and μ' are *adjacent* if there is no $\mu'' \in \mathcal{S}$ with $\mu \triangleright_M \mu'' \triangleright_M \mu'$.⁹ For each $\mu, \mu' \in \mathcal{S}$, the *distance* between μ and μ' is the length of (the number of edges on) the shortest path (a.k.a. *geodesic*) between μ and μ' in this graph. Cheng (2010) analyzes stable matchings that are *median(s) of the stable matchings lattice* whose total or average distance from all other stable matchings is the least. She shows that when n is odd, the median stable matching is the unique median of the stable matchings lattice and when n is even, a stable matching μ is a median of the stable matchings lattice if and only if μ is between the median stable matchings according to $\triangleright_M$, i.e. $\mu^{(K/2)+1} \triangleright_M \mu \triangleright_M \mu^{K/2}$.

Example 3 (Center of the stable matchings lattice). Cheng, McDermid & Suzuki (2016) formulate *center stable matching* as the one(s) whose maximum distance (as described in Example 3) from any other stable matchings is the least. They provide a characterization of all center stable matchings, and show that a (specific) center stable matching can be computed in polynomial time.

2.3 Modularity

The additional restriction of being *modular*, which we would like to impose on a stable matching mechanism, requires optimization of an explicit objective function that is restricted in a specific way by using the lattice structure of stable matchings. That is, for a given problem $\succ$, let $F : \mathcal{S}(\succ) \rightarrow \mathbb{R}$ be a mapping that attaches a value $F(\mu)$ to each stable matching $\mu \in \mathcal{S}(\succ)$. Then, F is **modular** if for each $\mu, \mu' \in \mathcal{S}(\succ)$,

$$F(\mu) + F(\mu') = F(\mu \vee \mu') + F(\mu \wedge \mu') \quad (1)$$

Definition. Let π be a stable matching mechanism. Then, π is **modular** if for each problem $\succ$, there exists a modular $F : \mathcal{S}(\succ) \rightarrow \mathbb{R}$ such that $\pi(\succ)$ is the set of matchings that minimizes F , that is $\pi(\succ) = \operatorname{argmin}_{\mu \in \mathcal{S}(\succ)} F(\mu)$.

We believe that formulating social objectives in the form a modular function provides clarity and—as demonstrated by several studies in the literature—analytical and computational tractability. Additionally, we will provide several examples demonstrating that a wide range of objectives

⁸The existence of (generalized) median stable matchings has been studied in other settings including: one-to-one matching with wages (Schwarz & Yenmez 2011), the college admissions model with responsive preferences (Klaus & Klijn 2006a, Sethuraman et al. 2006), the roommates problem (Klaus & Klijn 2010), many-to-many matching markets with contracts (Chen et al. 2016).

⁹This graph is called the (undirected) Hasse diagram of $\langle \mathcal{S}, \triangleright_M \rangle$.

can be implemented through modular stable matching mechanisms. The mechanisms in these examples are build on the aforementioned notion of the attainable mates. Formally, let $\succ$ be a given problem, then a man m and a woman w are **attainable** for each other if $\mu(m) = w$ for some $\mu \in \mathcal{S}(\succ)$. For each $i \in N$, let A_i denote the set of attainable agents for i . For each m and w who are attainable for each other, let $\text{Rank}_m^A(w)$ be the rank of w in $\succ_m \upharpoonright_{A_m}$, which is obtained by restricting $\succ_m$ to the women who are attainable for m . Similarly, $\text{Rank}_w^A(m)$ is the rank of m in $\succ_w \upharpoonright_{A_w}$, which is obtained by restricting $\succ_w$ to the men who are attainable for w .

Example 4 (Maximizing total attainable ranks). In the vein of utilitarian welfare measures, it may be reasonable to evaluate each stable matching according to the sum of agents' attainable ranks in that matching. That is, for each problem $\succ$, let $\pi(\succ)$ be the set of stable matchings that maximize $\sum_{mw \in \mu} (\text{Rank}_m^A(w) + \text{Rank}_w^A(m))$. As proven in Lemma 1, it turns out that this sum is constant among all stable matchings, and therefore does not differentiate any stable matching from the others.

In Example 5, we present a general class of modular stable matching mechanisms in which only attainable mates are taken into consideration in assessing the value of a matching. In Examples 6 and 7, we present two special mechanisms out of this general class, so their modularity immediately follows.

Example 5. For each problem $\succ \in \mathcal{P}$, let $\pi(\succ)$ be the set of matchings that minimizes $\sum_{i \in N} F_i(\mu(i))$, where $F_i : A_i \rightarrow \mathbb{R}$. To see that π is modular, let $\succ \in \mathcal{P}$ and $i \in N$. Since for each $\mu, \mu' \in \mathcal{S}(\succ)$, we have $\{(\mu \vee \mu')(i), (\mu \wedge \mu')(i)\} = \{\mu(i), \mu'(i)\}$, we obtain that $\{F_i((\mu \vee \mu')(i)), F_i((\mu \wedge \mu')(i))\} = \{F_i(\mu(i)), F_i(\mu'(i))\}$. It directly follows that F_i is modular for each agent i , and thus π is modular.

Example 6 (Minimizing total spread from the ideals). For each agent $i \in N$, from among the agents who are attainable for i , let $I(i)$ be the *ideal* partner for i in the sense that assigning i to $I(i)$ makes agent i reach the welfare level that is determined for him. We assume that two different agents can have the same ideal partner. For a given stable matching μ , we can measure the **spread from the ideal** for agent i by $|\text{Rank}_i^A(\mu(i)) - \text{Rank}_i^A(I(i))|$. Then, consider the stable matching mechanism π that chooses the set of stable matchings that minimize the *total spread from the ideals*. That is, for each problem $\succ$, $\pi(\succ)$ is the set of matchings that minimizes $\sum_{i \in N} |\text{Rank}_i^A(\mu(i)) - \text{Rank}_i^A(I(i))|$.

Example 7 (Minimizing total spread from attainable medians). We may add further structure into the previous example as follows. For each agent i , from among the agents who are attainable for i , consider the one(s) with (a) *median rank*. Note that if there is an odd number of attainable agents for i , then there is a unique attainable agent with this property, otherwise there are two such agents. Let med_i^A be the (more preferred) attainable agent with the lowest attainable median rank, i.e. $\text{Rank}_i^A(\text{med}_i^A) = \lfloor |A_i|/2 \rfloor$. Suppose that med_i^A is viewed as the ideal partner for i . Then, consider the stable matching mechanism π that chooses the set of stable matchings that minimize the total spread from the median. That is, for each problem $\succ$, let $\pi(\succ)$ be the set of matchings that minimizes $\sum_{i \in N} |\text{Rank}_i^A(\mu(i)) - \text{Rank}_i^A(\text{med}_i^A)|$.

3 Characterizations

We present two characterizations for modular stable matching mechanisms revealing the ordinal content of optimizing a modular function under the constraints of *stability* and *invariance under stability*. Thus, we provide testable conditions to verify if the observed choices of stable matchings in a society comply with optimization of a modular function. These results guide us about normative qualitative criteria that one may consider to adopt a modular stable matching mechanism.

Our first axiom is *convexity*, which requires that for each problem and for a given pair of matchings that are chosen by the mechanism, if one can form a stable matching by assigning each agent to one of their mates in the given pair, then this newly formed matching should be chosen as well.

Convexity: For each problem $\succ \in \mathcal{P}$, if $\mu', \mu'' \in \pi(\succ)$ and $\mu(m) \in \{\mu'(m), \mu''(m)\}$ for each $m \in M$, then $\mu \in \pi(\succ)$.

Note that convexity is weaker than requiring that all the stable matchings that are between the chosen matchings—according to the men-wise better than relation—must also be chosen as well, which is also referred to as convexity in lattice theory literature. To see this, consider a stable matching mechanism π such that for each problem $\succ$, π chooses only the extremal matchings whenever there is no other $\mu \in \mathcal{S}(\succ)$ such that $\mu(m) \in \{\mu^M(m), \mu^W(m)\}$ for each $m \in M$ (such as the problem in Example 8); and chooses all of the stable matchings, otherwise. Although π satisfies *convexity*, it clearly violates the latter requirement.

To introduce our second axiom, we need to introduce the notion of a π -transformed problem. For each problem $\succ \in \mathcal{P}$ and each agent $i \in N$, let $\pi_i(\succ)$ be the set of agents that i is assigned to at any matching $\mu \in \pi(\succ)$, i.e. $\pi_i(\succ) = \{\mu(i) \in N \mid \mu \in \pi(\succ)\}$. Then, the π -transformed problem $\succ^\pi$ is the problem obtained from $\succ$ such that for each agent $i \in N$, each member of $\pi_i(\succ)$ is moved to the top of agent i 's preferences by preserving the relative rankings elsewhere.

Now, let us first make a simple observation. For each $\mu \in \mathcal{S}(\succ)$, if $\mu \in \pi(\succ)$, then in transforming $\succ$ into $\succ^\pi$, for each $i \in N$, the set of agents that i prefers to $\mu(i)$ remains the same or shrinks, i.e. $\{j \mid j \succ_i^\pi \mu(i)\} \subset \{j \mid j \succ_i \mu(i)\}$. Therefore, for each $\mu \in \mathcal{S}(\succ)$, if $\mu \in \pi(\succ)$ then $\mu \in \mathcal{S}(\succ^\pi)$. Our second axiom requires the converse, that is, if a stable matching remains stable after the transformation, then it must be one of the matchings chosen by the mechanism in the initial problem. In Theorem 1, we show that *convexity* and *IIR* independently characterize modular stable matching mechanisms.

Independence of Irrelevant Rankings (IIR): For each problem $\succ \in \mathcal{P}$ and $\mu \in \mathcal{S}(\succ)$, if $\mu \in \mathcal{S}(\succ^\pi)$, then $\mu \in \pi(\succ)$.

Theorem 1. Let π be a stable matching mechanism. Then, the following are equivalent.

- i. π satisfies convexity.
- ii. π satisfies independence of irrelevant rankings.
- iii. π is modular.

Proof. Please see Section 7. □

To see one side of the connection between *convexity* and *IIR*, here, we will show that if a stable matching mechanism π satisfies *IIR*, then π satisfies *convexity*. To see this, for a given problem $\succ$, let $\mu', \mu'' \in \pi(\succ)$ and $\mu \in \mathcal{S}(\succ)$ such that for each $m \in M$, $\mu(m) \in \{\mu'(m), \mu''(m)\}$. Since π satisfies *IIR*, we have $\pi(\succ) = \mathcal{S}(\succ) \cap \mathcal{S}(\succ^\pi)$. Therefore, to conclude that $\mu \in \pi(\succ)$ it is sufficient to show that $\mu \in \mathcal{S}(\succ^\pi)$. By contradiction, suppose that there is a blocking pair (m, w) . Therefore, $w \succ_m^\pi \mu(m)$ and $m \succ_w^\pi \mu(w)$. Now, since $\mu(m) \in \{\mu'(m), \mu''(m)\}$ and $\mu', \mu'' \in \pi(\succ)$, it follows from the definition of $\succ^\pi$ that $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$, contradicting that $\mu \in \mathcal{S}(\succ)$.

We conclude this section by revisiting the three stable matching mechanisms from the literature that we discussed in Section 2.2. By using our Theorem 1, we can easily check if these mechanisms are modular or not. It turns out that the matching mechanism that chooses the median(s) of the stable matching lattice (Example 2) for each problem is modular since it is *convex*. To see this, note that this mechanism either chooses the unique median stable matching or all the stable matchings that are between the median stable matchings according to the men-wise better than relation. However, the following simple example demonstrates that the same is not true for the mechanisms presented in Examples 1 and 3.

Example 8. Consider the problem with four men and women whose preferences are represented by the table in Figure 1 such that each entry ij is associated with man m_i and women w_j , the $\succ_{m_i}$ -rank of w_j is written in the bottom corner and the rank of m_i in $\succ_{w_j}$ is written in the top corner.

In Figure 1, we also present the stable matching lattice associated with this problem such that each stable matching is represented as an array $[r_1, \dots, r_4]$, where each r_i is the $\succ_{m_i}$ -rank of the women who is matched with m_i . Then, it is easy to see that the median stable matchings are

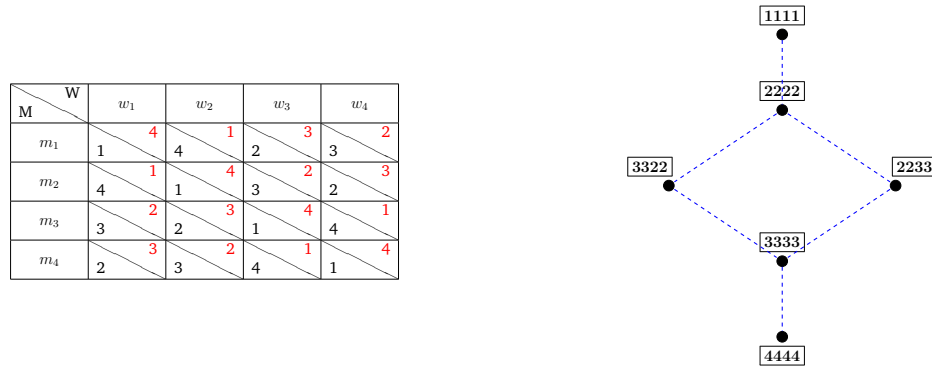


Figure 1: The problem and the associated stable matchings lattice.

[2222] and [3333]. The center of the stable matching lattice are [3322] and [2233]. Therefore, both mechanisms fail to satisfy *convexity*. Note that the union of the two sets gives the set of median attainable matchings.

4 A new equity notion

We believe that studying modular stable matching mechanism opens new avenues through which the issues of equity can be fruitfully analyzed. To demonstrate this, we consider a problem with six men and women whose preferences are represented by the table in Figure 2, where each entry ij is associated with man m_i and women w_j , and the $\succ_{m_i}$ -rank of w_j (the rank of m_i in $\succ_{w_j}$) is written in the bottom (top) corner.

M \ W	w_1	w_2	w_3	w_4	w_5	w_6
m_1	1 / 6	2 / 5	3 / 4	4 / 3	5 / 1	6 / 2
m_2	2 / 5	1 / 6	6 / 2	3 / 4	4 / 3	5 / 1
m_3	5 / 1	6 / 2	1 / 6	2 / 5	3 / 4	4 / 3
m_4	4 / 3	5 / 1	2 / 5	1 / 6	6 / 2	3 / 4
m_5	3 / 4	4 / 3	5 / 1	6 / 2	1 / 6	2 / 5
m_6	6 / 2	3 / 4	4 / 3	5 / 1	2 / 5	1 / 6

Figure 2: The problem.

In Figure 3, we present the stable matching lattice associated with the problem in Figure 2 such that each stable matching is represented as an array $[r_1, \dots, r_6]$, where each r_i is the $\succ_{m_i}$ -rank of the women who is matched with m_i .

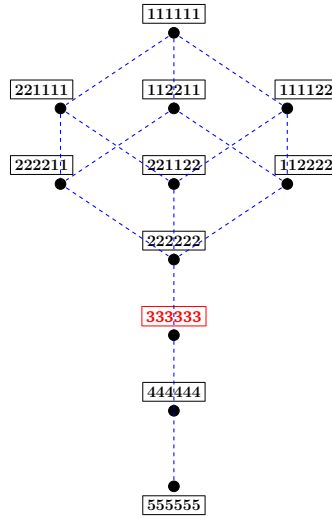


Figure 3: The associated stable matching lattice.

It is easy to see that the median stable matching, the unique median and the unique center of the stable matching lattice (as defined in Section 2.2), is the one in which each man is matched to his second-ranked woman and each woman is matched to her fifth-ranked man. However, one

can argue that the matching in which each agent–man or woman–is matched to his third-ranked attainable mate is more equitable, since each agent is matched to his/her median attainable mate.

To formalize this intuition as a principle, let $i \in N$ and $j, j' \in A_i$. Then, j is **closer to** med_i^A than j' if $|Rank_i^A(j) - Rank_i^A(med_i^A)| < |Rank_i^A(j') - Rank_i^A(med_i^A)|$. Our next axiom requires that if a matching mechanism chooses a stable matching μ , then there should be no other stable matching in which each agent is assigned to a mate who is closer to his/her median attainable mate than compared to his/her mate at μ .

Minimal equatability: For each problem $\succ \in \mathcal{P}$, if $\mu \in \pi(\succ)$, then there is no $\mu' \in \mathcal{S}(\succ)$ such that for each $i \in N$ with $\mu(i) \neq \mu'(i)$ we have $\mu'(i)$ is closer to med_i^A than $\mu(i)$.

In our previous example, this principle uniquely pins down the matching in which each agent is matched to his/her third-ranked mate. Next, we characterize the class of modular stable matching mechanisms that satisfy *minimal equatability*.

Let i be an agent, a mapping $F_i : A_i \rightarrow \mathbb{R}$ is **single peaked** with respect to the ordering $\succ_i$ with the ideal point med_i^A : If F_i attains its maximum at med_i^A , and for each $j, j' \in A_i \setminus \{med_i^A\}$, we have $F_i(j) > F_i(j')$ if and only if j' is further away from med_i^A compared to j according to $\succ_i$, i.e. $j' \succ_i j \succ_i med_i^A$ or $med_i^A \succ_i j \succ_i j'$.

Theorem 2. A modular stable matching mechanism π satisfies minimal equatability if and only if for each problem $\succ \in \mathcal{P}$, $\pi(\succ)$ is the set of matchings that minimizes $-\sum_{i \in N} F_i(\mu(i))$, where $F_i : A_i \rightarrow \mathbb{R}$ is single peaked with respect to $\succ_i$ with the ideal point med_i^A for each $i \in N$.

Proof. Please see Section 8. □

A notable aspect of the objective function $-\sum_{i \in N} F_i(\mu(i))$ is that the social value of a stable matching is obtained by adding $-F_i(\mu(i))$ for each agent i , which can be interpreted as the social value of matching agent i with $\mu(i)$. This social value might be determined by the productivity of agent i when he/she is matched with $\mu(i)$. The objective function of total spread from attainable medians formulated in Example 7, can be expressed as a member of this additive single peaked family, where for each agent i and stable matching μ , $F_i(\mu(i)) = -|Rank_i^A(\mu(i)) - Rank_i^A(med_i^A)|$. The notion of *minimal equatability* and Theorem 2 can be generalized by replacing median attainable mates by ideal mates as defined in Example 6.

5 Related literature and final comments

We proposed a general foundational framework to address the issues of social welfare and equity in the stable matching model and presented axiomatic characterizations for modular stable matching mechanisms. To best of our knowledge, this is the first such general analysis of non-extremal matching mechanisms. Existing studies have focused only on mechanisms that singles

out an extremal matching, including [Balinski & Sönmez \(1999\)](#), [Ehlers & Klaus \(2006\)](#) and [Kojima & Manea \(2010\)](#) who offer different characterizations of the Gale-Shapley student-optimal stable matching mechanism. A general axiomatic approach to the problem of fair algorithms was presented by [Masarani & Gokturk \(1989\)](#). However, their approach concentrates on the algorithm, not on the resulting matchings, and concludes with an impossibility result. Relatedly, [Klaus & Klijn \(2006b\)](#) introduce the *procedural fairness* notion that labels a probabilistic stable matching mechanisms as fair if each agent has the same probability to move at a certain point in the procedure that determines the final probability distribution. Their approach does not provide any criterion for fairness or equatability of a stable matching.

Modularity properties of agents' preferences was investigated by [Kreps \(1979\)](#) in decision-theory literature and by [Milgrom & Shannon \(1994\)](#) in the monotone comparative statics literature. [Chambers & Echenique \(2008\)](#) clarifies the connection between these two different approaches. The closest papers to ours are [Kreps \(1979\)](#) and [Chambers & Echenique \(2009\)](#) who provide representations for modular preferences over lattices under the additional assumption of *monotonicity*. However, violation of monotonicity is the departure point of our question, since a stable matching mechanism that would satisfy *monotonicity* would choose one of the extremal matchings. As demonstrated in our proofs, here, the neat mathematical structure of the stable matchings paves the way for our results in the absence of *monotonicity*. As another key modelling difference, our primitives are not agents' preferences but stable matching mechanisms that can be thought of as choice rules over the set of stable matchings.

In the second part of the paper, we propose and analyze a novel equity notion to demonstrate that the class of modular stable matching mechanisms are rich enough to formulate and analyze similar notions of equity. This new framework leads to a wide variety of open problems to be solved. To name a few, a major concern for stable matching mechanisms is to provide the agents with the right incentives to report their preferences truthfully. Recall that the set of matchings chosen by a mechanism can be interpreted as the ones that are assigned positive probability in a lottery. Therefore, it seems reasonable to consider probabilistic assignment rules, as formulated by [Bogomolnaia & Moulin \(2001\)](#), associated with a modular stable matching mechanism and investigate general conditions that guarantee its strategy-proofness in terms of stochastic-dominance. Another direction that we find worth to explore is developing efficient algorithms to implement specific modular stable matching mechanisms that can be useful in inequality-oriented market design.

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6 Appendix A: Stepping stones

6.1 Rotations and some preliminary observations

For a given fixed problem $\succ \in \mathcal{P}$, *rotations* are the incremental changes that need to be made so that a stable matching μ can be transformed into another stable matching μ' such that $\mu \triangleright_M \mu'$ and there is no other stable matching μ'' such that $\mu \triangleright_M \mu'' \triangleright_M \mu'$.

Let μ^M and μ^W denote the men-optimal and women-optimal stable matchings. Let μ be a stable matching such that $\mu \neq \mu^W$. Then, $\mu(m) \neq \mu^W(m)$ for some man m . For each such man m , define his **successor woman** at μ , denoted by $s_\mu(m)$, as the first attainable woman for m on his preference list who is less preferred than $\mu(m)$ (and prefers him over her current partner in μ) i.e. $s_\mu(m)$ is the $\succ_m$ -best attainable woman w such that $\mu(m) \succ_m w$ (and $m \succ_w \mu(m)$). A **rotation ρ exposed in μ** is an (ordered) cyclic sequence of distinct man-woman pairs $\rho = [(m_1, w_1), (m_2, w_2), \dots, (m_k, w_k)]$ such that $m_i w_i \in \mu$ and $s_\mu(m_i) = w_{i+1}$ for each $i \in \{1, \dots, k\}$, where the addition in the subscripts is modulo k . To **eliminate** a rotation ρ exposed in a stable matching μ , each man m_i in ρ is matched to w_{i+1} while the rest of the pairs not in ρ are kept the same. The result is another stable matching, denoted by $\mu \circ \rho$, such that $\mu \triangleright_M \mu \circ \rho$ and there is no other stable matching μ' with $\mu \triangleright_M \mu' \triangleright_M \mu \circ \rho$.

$\rho^{11} = [(m_1, w_1), (m_2, w_2)]$
 $\rho^{12} = [(m_3, w_3), (m_4, w_4)]$
 $\rho^{13} = [(m_5, w_5), (m_6, w_6)]$
 $\rho^2 = [(m_1, w_2), (m_4, w_3), (m_5, w_6), (m_2, w_1), (m_3, w_4), (m_6, w_5)]$
 $\rho^3 = [(m_1, w_3), (m_2, w_4), (m_3, w_5), (m_4, w_6), (m_5, w_1), (m_6, w_2)]$
 $\rho^4 = [(m_1, w_4), (m_2, w_5), (m_3, w_6), (m_4, w_1), (m_5, w_2), (m_6, w_3)]$

	W	w_1	w_2	w_3	w_4	w_5	w_6
M		$\begin{smallmatrix} 1 \\ 6 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 5 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ 4 \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ 3 \end{smallmatrix}$	$\begin{smallmatrix} 5 \\ 1 \end{smallmatrix}$	$\begin{smallmatrix} 6 \\ 2 \end{smallmatrix}$
m_1		1	2	3	4	5	6
m_2		2	1	6	3	4	5
m_3		5	6	1	2	3	4
m_4		4	5	2	1	6	3
m_5		3	4	5	6	1	2
m_6		6	3	4	5	2	1

Figure 4: The problem in Section 4 and the associated rotations.

Let $\mathcal{R}$ denote the set of all rotations. A rotation ρ **precedes** another rotation ρ' , denoted by $\rho \rightarrow \rho'$, if in order to obtain a stable matching in which ρ' is exposed, ρ must be eliminated first. We assume that a rotation precedes itself. A rotation ρ **immediately precedes** another rotation ρ' if $\rho \rightarrow \rho'$ and there is no other rotation ρ'' such that $\rho \rightarrow \rho'' \rightarrow \rho'$. A distinct pair of rotations ρ and ρ' are **independent** if none of them precedes the other. The pair $\langle \mathcal{R}, \rightarrow \rangle$ is called the **rotation poset**.

A subset of $\mathcal{R}$, generically denoted by R , is **closed** if whenever a rotation is in the subset, all the rotations that precede it are also in the subset. Let $Cl(\mathcal{R})$ denote all closed subsets of $\mathcal{R}$. We suppress the reference to the specific problem $\succ \in \mathcal{P}$, whenever it is clear from the context.

We will use the following basic properties of rotations. First, a man-woman pair (m, w) belongs to a rotation if and only if it appears in some stable matching (i.e. if they are attainable) and w is not the worst mate of m in all stable matchings. Since a man-woman pair can be part of at most one rotation, each rotation has at least two man-woman pairs. Note that a pair of

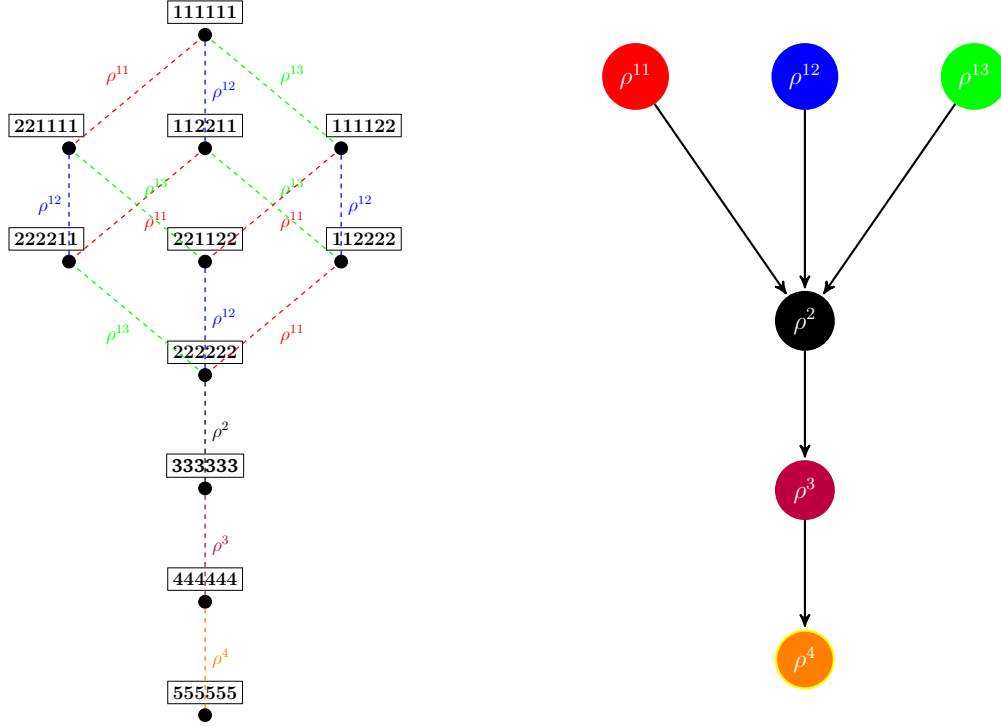


Figure 5: The stable matchings lattice and the rotation poset for the problem in Section 4 .

rotations ρ and ρ' are independent if and only if there is no man and no woman who appears both in ρ and ρ' .

In their main result, [Irving & Leather \(1986\)](#) show that the closed subsets of $\mathcal{R}$ endowed with the set containment relation $\langle Cl(\mathcal{R}), \subset \rangle$ is a lattice that is *order isomorphic*¹⁰ to $\langle S, \triangleright_M \rangle$.¹¹ Next, we make some simple observations using this result. Let $\succ \in \mathcal{P}$ be a given problem with the associated set of stable matching $\mathcal{S}$ and set of rotations $\mathcal{R}$. For each $\mu \in \mathcal{S}$, let R_μ be the associated closed subset of the rotation poset $\langle \mathcal{R}, \rightarrow \rangle$.

Lemma 1. *For each $\mu \in \mathcal{S}$, $\sum_{mw \in \mu} (Rank_m^A(w) + Rank_w^A(m))$ is the same.*

Proof. Let $\mathcal{R}$ be the set of rotations associated with $\mathcal{S}$. We first show that for each $\mu, \mu' \in \mathcal{S}$, if $\mu' = \mu \circ \rho$ for some $\rho \in \mathcal{R}$, then for each $(m, w) \in \rho$, we have

$$Rank_m^A(\mu'(m)) = Rank_m^A(w) + 1 \text{ and } Rank_w^A(\mu'(w)) = Rank_w^A(m) - 1 \quad (2)$$

To see this, note that since $\mu' = \mu \circ \rho$, for each $(m, w) \in \rho$, we have $w \succ_m \mu'(m)$ and $\mu'(w) \succ_w m$. Since there is also no other stable matching μ'' such that $\mu \triangleright_M \mu'' \triangleright_M \mu'$, it directly follows that (2) holds.

¹⁰Given two posets $(S, \leq_S)$ and $(T, \leq_T)$, an *order isomorphism* from $(S, \leq_S)$ to $(T, \leq_T)$ is a bijective function f from S to T that is an *order embedding*, i.e. for each $x, y \in S$, $x \leq_S y$ if and only if $f(x) \leq_T f(y)$.

¹¹[Gusfield \(1987\)](#) showed that a representation for the rotation poset $\langle \mathcal{R}, \rightarrow \rangle$ can be constructed in $O(n^2)$ time.

Now, for each $\mu, \mu' \in \mathcal{S}$ such that $\mu' = \mu \circ \rho$, by the above observation, we have $\sum_{mw \in \mu'} \text{Rank}_m^A(w) = \sum_{mw \in \mu} \text{Rank}_m^A(w) + |\rho|$ and $\sum_{mw \in \mu'} \text{Rank}_w^A(m) = \sum_{mw \in \mu} \text{Rank}_w^A(m) - |\rho|$. Since each $\mu \in \mathcal{S}$ can be obtained from μ^M by sequentially eliminating the rotations in $R_\mu \setminus R_{\mu^M}$, we reach the conclusion. $\square$

Lemma 2. *A function $F : \mathcal{S} \rightarrow \mathbb{Z}$ is modular if and only if there exists an additive set function $G : Cl(\mathcal{R}) \rightarrow \mathbb{Z}$ such that for each $\mu \in \mathcal{S}$, $F(\mu) = G(R_\mu)$.*

Proof. Let $R \in Cl(\mathcal{R})$. Then, it follows from Irving & Leather (1986) that there exists $\mu \in \mathcal{S}$ such that $R = R_\mu$. We define $G : Cl(\mathcal{R}) \rightarrow \mathbb{Z}$ such that for each $\mu \in \mathcal{S}$, $G(R_\mu) = F(\mu)$. To see that G is additive, let $\mu, \mu' \in \mathcal{S}$. Since $\langle Cl(\mathcal{R}), \subset \rangle$ is order isomorphic to $\langle \mathcal{S}, \triangleright_M \rangle$, we have $R_{\mu \vee \mu'} = R_\mu \cap R_{\mu'}$ and $R_{\mu \wedge \mu'} = R_\mu \cup R_{\mu'}$. Then, it follows from the modularity of F that $G(R_\mu \cup R_{\mu'}) = G(R_\mu) + G(R_{\mu'}) - G(R_\mu \cap R_{\mu'})$. The proof for the other direction is similar. $\square$

Lemma 3. *Let $G : Cl(\mathcal{R}) \rightarrow \mathbb{Z}$ be an additive set function. Then, there exists $g : \mathcal{R} \rightarrow \mathbb{Z}$ such that for each $R \in Cl(\mathcal{R})$, $G(R) = G(\emptyset) + \sum_{\rho \in R} g(\rho)$.*

Proof. Let $\rho \in \mathcal{R}$ and define its closure $C(\rho) = \{\rho' \in \mathcal{R} \mid \rho' \rightarrow \rho\}$. Since the precedence relation is transitive, $C(\rho) \setminus \{\rho\}$ is closed. Therefore, for each $\rho \in \mathcal{R}$, we can define $g(\rho) = G(C(\rho)) - G(C(\rho) \setminus \{\rho\})$. Then, for each $\rho \in \mathcal{R}$, we have $G(C(\rho)) = G(\emptyset) + \sum_{\rho' \in C(\rho)} g(\rho')$. Now, let $R \in Cl(\mathcal{R})$. Then, since $R = \bigcup_{\rho \in R} C(\rho)$, by additivity of G , we conclude that $G(R) = G(\emptyset) + \sum_{\rho \in R} g(\rho)$. $\square$

6.2 An order isomorphism result

Let π be a stable matching mechanism and $\succ \in \mathcal{P}$ be a problem with the associated rotation poset $\langle \mathcal{R}, \rightarrow \rangle$ such that $\pi(\succ)$ is a sublattice of $\langle \mathcal{S}(\succ), \triangleright_M \rangle$. First, let $\bar{\mu}(\underline{\mu})$ be the $\triangleright_M$ -best(worst) matching in $\pi(\succ)$. Then, define $\mathcal{R}^\delta = R_\mu \setminus R_{\bar{\mu}}$ and

$$Cl^0(\mathcal{R}^\delta) = \{R_\mu \setminus R_{\bar{\mu}} \mid \mu \in \pi(\succ)\}.$$

Next, we recursively define two set collections $\{X^i\}_{i=1}^K$ and $\{\Lambda^i\}_{i=1}^K$. Then, we prove related structural result, Proposition 1, which will be an important stepping stone to show that if (i) holds, then (iii) holds in proving Theorem 1.

For $k = 1$: Consider $\min(Cl^0(\mathcal{R}^\delta), \subset)$ that consists of $R \in Cl^0(\mathcal{R}^\delta)$ such that there is no $R' \in Cl^0(\mathcal{R}^\delta) \setminus \emptyset$ with $R' \subsetneq R$. Then, let $X^1 = Cl^0(\mathcal{R}^\delta)$ and $\Lambda^1 = \min(X^1, \subset)$.

For $k \geq 2$: Define $X^2 = \{R \setminus \bigcup_{R' \in \Lambda^1} R' \mid R \in X^1\}$ and $\Lambda^2 = \min(X^2, \subset)$ (the elements of X^2 and Λ^2 are sets of rotations that are not closed). Similarly, for each $k \geq 1$, define $X^{k+1} = \{R \setminus \bigcup_{R' \in \Lambda^k} R' \mid R \in X^k\}$ and $\Lambda^{k+1} = \min(X^{k+1}, \subset)$. Let $K \geq 1$ be the smallest number such that $X^{K+1} = \emptyset$. Then, $\{X^k\}_{k=1}^K$ and $\{\Lambda^k\}_{k=1}^K$ are the ordered collection of the disjoint nonempty sets that are constructed. Define $\Lambda = \bigcup_{k=1}^K \Lambda^k$. We will generically denote a member of Λ by λ . Figure 6 presents a simple demonstration of Λ is formed.

Lemma 4. For each $k \in \{1, \dots, K\}$, $\langle X^k, \subset \rangle$ is a lattice.

Proof. By induction, first, consider the case that $k = 1$, where $X^1 = Cl^0(\mathcal{R}^\delta)$. Since $\pi(\succ)$ is a sublattice of $\langle \mathcal{S}(\succ), \triangleright_M \rangle$, $Cl^0(\mathcal{R}^\delta)$ is a sublattice of $\langle Cl(\mathcal{R}^\delta), \subset \rangle$. Therefore, $\langle X^1, \subset \rangle$ is a lattice. Next, for each $k \in \{1, \dots, K-1\}$ assume that $\langle X^k, \subset \rangle$ is a lattice, and let $Q, Q' \in X^{k+1}$. By construction of X^{k+1} , there exist $R \in X^k$ and $R' \in X^k$ such that $R = Q \cup \bigcup_{\lambda \in \Lambda^k} \lambda$ and $R' = Q' \cup \bigcup_{\lambda \in \Lambda^k} \lambda$. Since $\langle X^k, \subset \rangle$ is a lattice, $R \cap R' \in X^k$ and $R \cup R' \in X^k$. Therefore, $Q \cap Q' \in X^{k+1}$ and $Q \cup Q' \in X^{k+1}$. Thus, we conclude that $\langle X^{k+1}, \subset \rangle$ is a lattice. $\square$

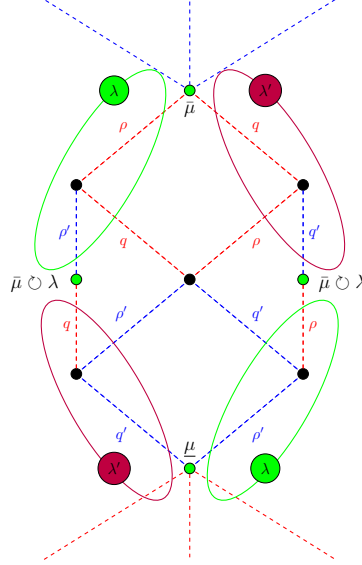


Figure 6: A demonstration of our construction for Λ , where $\pi(\succ)$ are the green labelled nodes.

Lemma 5. Let $\rho \in \mathcal{R}^\delta$. Then, there exists unique $\lambda_\rho \in \Lambda$ such that $\rho \in \lambda_\rho$.

Proof. First, note that by construction, $\{\Lambda^k\}_{k=1}^K$ is collection of disjoint sets such that for each $\rho \in \mathcal{R}^\delta$, there exists $k \in \{1, \dots, K\}$ and $\lambda \in \Lambda^k$ such that $\rho \in \lambda$. Next, let $k \in \{1, \dots, K\}$ and $\lambda, \lambda' \in \Lambda^k$ be distinct. Since, by Lemma 4, $\langle X^k, \subset \rangle$ is a lattice, $\lambda \cap \lambda' \in X^k$. But, $\Lambda^k = \min(X^k, \subset)$ implies that $\lambda \cap \lambda' = \emptyset$. $\square$

Lemma 6. Let $R \in Cl^0(\mathcal{R}^\delta)$. Then, $\{\lambda_\rho\}_{\rho \in R}$ partitions R .

Proof. For each $k \in \{1, \dots, K\}$, define $\Lambda_R^k = \{\lambda \in \Lambda^k \mid \lambda \subset R\}$. Recall that $X^1 = Cl^0(\mathcal{R}^\delta)$ and since $X^{K+1} = \emptyset$, we have $X^K = \Lambda^K$. Then, since $R \in Cl^0(\mathcal{R}^\delta)$, by construction of $\{X^k\}_{k=1}^K$ and $\{\Lambda^k\}_{k=1}^K$, we have $R = \bigcup_{k=1}^K \bigcup_{\lambda \in \Lambda_R^k} \lambda$. Since for each $\rho \in R$, by Lemma 5, λ_ρ is the unique member of Λ with $\rho \in \lambda_\rho$, it follows that $R = \bigcup_{\rho \in R} \lambda_\rho$ and for each distinct $\rho, \rho' \in R$, $\lambda_\rho \cap \lambda_{\rho'} = \emptyset$. $\square$

Now, we are ready to prove Proposition 1, which we will use to prove Theorem 1. To introduce this result, first, recall that $\Lambda = \bigcup_{k=1}^K \Lambda^k$. Then, for each distinct $\lambda, \lambda' \in \Lambda$, we define λ **precedes** λ' , denoted by $\lambda \Rightarrow \lambda'$, if for each $R \in Cl^0(\mathcal{R}^\delta)$, $\lambda' \subset R$ implies that $\lambda \subset R$. Put differently, $\lambda \Rightarrow \lambda'$ if

$$\lambda \subset \bigcap_{\{R \in Cl^0(\mathcal{R}^\delta) | \lambda' \subset R\}} R.$$

Note that $\langle \Lambda, \Rightarrow \rangle$ is a finite poset. Let $Cl(\Lambda)$ be the closed subsets of Λ with respect to the precedence relation $\Rightarrow$. In Proposition 1, we show that $\langle Cl^0(\mathcal{R}^\delta), \subset \rangle$ is *order isomorphic* to $\langle Cl(\Lambda), \subset \rangle$. To show this, define $\Lambda : Cl^0(\mathcal{R}^\delta) \rightarrow Cl(\Lambda)$ such that for each $R \in Cl^0(\mathcal{R}^\delta)$, $\Lambda(R) = \{\lambda_\rho\}_{\rho \in R}$.

Proposition 1. *The Λ mapping induces an order isomorphism between $\langle Cl^0(\mathcal{R}^\delta), \subset \rangle$ and $\langle Cl(\Lambda), \subset \rangle$.*

Proof. Let $R \in Cl^0(\mathcal{R}^\delta)$. First, we verify that $\Lambda(R) \in Cl(\Lambda)$. To see this, let $\lambda \in \Lambda(R)$ and $\lambda' \in \Lambda$ such that $\lambda' \Rightarrow \lambda$. Then, since $R \in Cl^0(\mathcal{R}^\delta)$ with $\lambda \subset R$, it follows from $\lambda' \Rightarrow \lambda$ that $\lambda' \subset R$. Therefore, $\lambda' \in \Lambda(R)$. To see that Λ is one-to-one, note that for each $R, R' \in Cl^0(\mathcal{R}^\delta)$, if $R \neq R'$ then $\Lambda(R) \neq \Lambda(R')$. To see that Λ is an order embedding, by Lemma 6, for each $R, R' \in Cl^0(\mathcal{R}^\delta)$, $\{\lambda_\rho\}_{\rho \in R}$ partitions R and $\{\lambda_{\rho'}\}_{\rho' \in R'}$ partitions R' . It follows that $R \subset R'$ if and only if $\{\lambda_\rho\}_{\rho \in R} \subset \{\lambda_{\rho'}\}_{\rho' \in R'}$.

Finally, we show that Λ is onto. To see this, let $Q \in Cl(\Lambda)$ and define $R_Q = \bigcup_{\lambda \in Q} \lambda$. We show that $R_Q \in Cl^0(\mathcal{R}^\delta)$ and $\Lambda(R_Q) = Q$. To see this, let $\lambda \in \Lambda$ and recall that, by construction, there exists $R \in Cl^0(\mathcal{R}^\delta)$ such that $\lambda \subset R$. Therefore, we can define $R_\lambda = \bigcap_{\{R \in Cl^0(\mathcal{R}^\delta) | \lambda \subset R\}} R$. Then, since $Cl^0(\mathcal{R}^\delta)$ is a lattice, we have $R_\lambda \in Cl^0(\mathcal{R}^\delta)$ and $\bigcup_{\lambda \in Q} R_\lambda \in Cl^0(\mathcal{R}^\delta)$. Next, to conclude, we show that $R_Q = \bigcup_{\lambda \in Q} R_\lambda$.

Since for each $\lambda \in Q$, $\lambda \subset R_\lambda$ and $R_Q = \bigcup_{\lambda \in Q} \lambda$, we have $R_Q \subset \bigcup_{\lambda \in Q} R_\lambda$. To see the converse, let $\lambda \in Q$ and $\rho \in R_\lambda$. We show that $R_\lambda \subset R_Q$. Since $R_\lambda \in Cl^0(\mathcal{R}^\delta)$, by Lemma 6, $\{\lambda_{\rho'}\}_{\rho' \in R_\lambda}$ partitions R_λ . Therefore, $\lambda_\rho \subset R_\lambda$. Then, it follows from the construction of R_λ that for each $R \in Cl^0(\mathcal{R}^\delta)$, if $\lambda \subset R$ then $\lambda_\rho \subset R$, that is $\lambda_\rho \Rightarrow \lambda$. Now, since $Q \in Cl(\Lambda)$ and $\lambda \in Q$, $\lambda_\rho \Rightarrow \lambda$ implies that $\lambda_\rho \in Q$. Since $R_Q = \bigcup_{\lambda \in Q} \lambda$, we have $\lambda_\rho \subset R_Q$ indicating that $\rho \in R_Q$. Thus, we conclude that $R_Q = \bigcup_{\lambda \in Q} R_\lambda$. Then, it directly follows from the formulation of Λ that $\Lambda(R_Q) = Q$. $\square$

7 Appendix B: Proof of Theorem 1

7.1 If (iii) then (i)

Since (iii) holds, $\pi(\succ)$ is a sublattice of $\langle \mathcal{S}(\succ), \triangleright_M \rangle$. Moreover, by Lemma 2, there exists an additive set function $G : Cl(\mathcal{R}) \rightarrow \mathbb{Z}$ such that $\pi(\succ) = \arg_{\mu \in \mathcal{S}(\succ)} \min G(R_\mu)$ and by Lemma 3 there exists $g : \mathcal{R} \rightarrow \mathbb{Z}$ such that for each $R \in Cl(\mathcal{R})$, $G(R) = G(\emptyset) + \sum_{\rho \in R} g(\rho)$.

To see that (i) holds, let $\mu^*, \mu^{**} \in \pi(\succ)$ and $\mu(m) \in \{\mu^*(m), \mu^{**}(m)\}$ for each $m \in M$. We show that $\mu \in \pi(\succ)$. First, let $\mu' = \mu^* \vee \mu^{**}$ and $\mu'' = \mu^* \wedge \mu^{**}$. Note that, we have $\mu(m) \in \{\mu'(m), \mu''(m)\}$ for each $m \in M$, and $\mu', \mu'' \in \pi(\succ)$ since $\pi(\succ)$ is a lattice. Therefore, if $\mu = \mu'$ or $\mu = \mu''$ then we conclude that $\mu \in \pi(\succ)$; if not then $\mu' \triangleright_M \mu \triangleright_M \mu''$, which implies that $R_{\mu'} \subsetneq R_\mu \subsetneq R_{\mu''}$. Let $P = R_\mu \setminus R_{\mu'}$ and $P' = R_{\mu''} \setminus R_\mu$. Next, we show that each $\rho \in P$ and $\rho' \in P'$ are independent. To see this, let N_ρ denote the set of agents in a rotation ρ . Let $\rho \in P$ and $i \in N_\rho$, then in moving from μ' to μ it must be that agent i 's mate is changed and $\mu(i) = \mu''(i)$. Therefore, in moving from μ to μ'' , i 's mate can not change, that is there is no $\rho' \in P'$ such that $i \in N_{\rho'}$. Thus, we conclude that for each $\rho \in P$ and $\rho' \in P'$, $N_\rho \cap N_{\rho'} = \emptyset$, which implies that ρ and ρ' are independent.

Now, since $\mu', \mu'' \in \pi(\succ)$, we have $G(R_{\mu'}) = G(R_{\mu''})$. Since G is additive and $\{P, P'\}$ partitions $R_{\mu''} \setminus R_{\mu'}$, it follows that $\sum_{\rho \in P} g(\rho) + \sum_{\rho' \in P'} g(\rho') = 0$. Now, if $\mu \notin \pi(\succ)$, then we must have $\sum_{\rho \in P} g(\rho) > 0$, which implies that $\sum_{\rho' \in P'} g(\rho') < 0$. Since each $\rho \in P$ and $\rho' \in P'$ are independent, $R_{\mu'} \cup P' \in Cl(\mathcal{R})$. Let $\mu''' \in \mathcal{S}(\succ)$ be such that $R_{\mu'''} = R_{\mu'} \cup P'$. Then, we have $G(R_{\mu'''}) < G(R_{\mu'})$, contradicting that μ' minimizes G . Thus, we conclude that $\mu \in \pi(\succ)$.

7.2 (i) if and only if (ii)

The *if part* was proven in the main text after Theorem 1. To see the *only if part*, let $\succ \in \mathcal{P}$. Since π satisfies *convexity*, $\pi(\succ)$ is a sublattice of $\langle \mathcal{S}(\succ), \triangleright_M \rangle$. To see that π satisfies *IIR*, first, let $\bar{\mu}(\mu)$ be the $\triangleright_M$ -best(worst) matching in $\pi(\succ)$. Then, recall that we obtain the men(women)-optimal stable matching for the problem $\succ^\pi$ by running the men(women)-proposing Gale-Shapley algorithm, in which each man(woman) proposes to women(men) by following his(her) transformed preference list. Then, it directly follows that the men(women)-optimal stable matching for the problem $\succ^\pi$ is $\bar{\mu}(\mu)$. Therefore, for each $\mu \in \mathcal{S}(\succ)$, if $\mu \triangleright_M \bar{\mu}$ or $\bar{\mu} \triangleright_M \mu$, then $\mu \notin \mathcal{S}(\succ^\pi)$. Now, let $\mu \in \mathcal{S}(\succ)$ be such that $\bar{\mu} \triangleright_M \mu \triangleright_M \underline{\mu}$. Next, we show that if $\mu \notin \pi(\succ)$, then $\mu \notin \mathcal{S}(\succ^\pi)$.

First, we show that there exists $m \in M$ such that $\mu(m) \notin \pi_m(\succ)$. To see this, let $\mu', \mu'' \in \pi(\succ)$ be the $\triangleright_M$ -worst and $\triangleright_M$ -best matchings such that $\mu' \triangleright_M \mu \triangleright_M \mu''$. Now, since π satisfies *convexity* and $\mu \notin \pi(\succ)$, there exists $m \in M$ such that $\mu(m) \notin \{\mu'(m), \mu''(m)\}$. Next, we show that $\mu(m) \notin \pi_m(\succ)$. By contradiction, suppose that there exists $\tilde{\mu} \in \pi(\succ)$ with $\tilde{\mu}(m) = \mu(m)$. Then, let $\mu^* = (\mu' \wedge \tilde{\mu}) \vee \mu''$. Since $\pi(\succ)$ is a lattice, $\mu^* \in \pi(\succ)$. Moreover, $\mu' \triangleright_M \mu^* \triangleright_M \mu''$, since $\mu^*(m) = \mu(m)$. Let ρ be the rotation with $(m, \mu(m)) \in \rho$ and $\lambda = \{\rho' \in \mathcal{R} \mid \rho' \rightarrow \rho\}$. Then, we have $R_{\mu'} \cup \lambda \in Cl(\mathcal{R})$ and $R_{\mu'} \cup \lambda \subset R_{\mu^*}$. Let $\mu^{**} \in \mathcal{S}(\succ)$ be such that $R_{\mu^{**}} = R_{\mu'} \cup \lambda$. Now, since $\mu^{**}(m) \in \{\mu'(m), \mu^*(m)\}$ for each $m \in M$, by *convexity*, $\mu^{**} \in \pi(\succ)$. However, since $R_{\mu'} \cup \lambda \subset R_\mu$, we have $\mu' \triangleright_M \mu^{**} \triangleright_M \mu$. This contradicts our choice of μ' .

Finally, we conclude that $\mu \notin \mathcal{S}(\succ^\pi)$. To see this, let $w = \underline{\mu}(m)$. Then, by the construction of $\succ^\pi$, m is the $\succ_w^\pi$ -best man for w . Since, by the above observation, $\mu(m) \notin \pi_m(\succ)$, we also have $w \succ_m^\pi \mu(m)$. Therefore, m and w form a blocking pair.

7.3 If (i) then (iii)

Let π be a stable matching mechanism that satisfies *convexity*. Let $\succ \in \mathcal{P}$ be a problem with the associated rotation poset $\langle \mathcal{R}, \rightarrow \rangle$. To show that there exists a modular fairness measure $F : \mathcal{S}(\succ) \rightarrow \mathbb{Z}$ such that $\pi(\succ) = \arg_{\mu \in \mathcal{S}(\succ)} \min F(\mu)$, by Lemma 2, it is sufficient to show that there exists an additive set function $G : Cl(\mathcal{R}) \rightarrow \mathbb{Z}$ such that $\pi(\succ) = \arg_{\mu \in \mathcal{S}(\succ)} \min G(R_\mu)$.

Since π satisfies *convexity*, $\pi(\succ)$ is a sublattice of $\langle \mathcal{S}(\succ), \triangleright_M \rangle$. Therefore, Proposition 1 holds for π . In what follows, we assume that the formal objects, such as $Cl^0(\mathcal{R}^\delta)$ and Λ , defined in Section 6.2 are associated with π . Note that for each $\mu \in \mathcal{S}(\succ)$, $\mu \in \pi(\succ)$ if and only if $R_\mu \setminus R_{\bar{\mu}} \in Cl^0(\mathcal{R}^\delta)$. Therefore, to prove the result, we show that there exists an additive set function $G : Cl(\mathcal{R}) \rightarrow \mathbb{Z}$ such that for each $R \in Cl(\mathcal{R})$, R minimizes G if and only if $R \setminus R_{\bar{\mu}} \in Cl^0(\mathcal{R}^\delta)$. Next, by using Proposition 1 and *convexity*, we prove the following structural result that paves the way for constructing the desired additive set function.

Lemma 7 (Partition lemma). *Let $\lambda \in \Lambda$ that contains at least two rotations and $\{P, P'\}$ be a partition of λ . Then, there exist $\rho \in P$ and $\rho' \in P'$ such that $\rho \rightarrow \rho'$ or $\rho' \rightarrow \rho$.*

Proof. By contradiction, suppose each $\rho \in P$ and $\rho' \in P'$ are independent. Suppose that $\lambda \in \Lambda^j$ for some $j \in \{1, \dots, K\}$. Recall that $\Lambda^j = \min(X^j, \subset)$, and consider the set $A = \bigcup_{k=1}^{j-1} \Lambda^k$, in the case that $j = 1$, assume that $\Lambda^0 = \emptyset$. Then, we have $A \in Cl(\Lambda)$. By construction of $\langle \Lambda, \Rightarrow \rangle$, for each $\lambda' \in \Lambda$, if $\lambda' \Rightarrow \lambda$, then $\lambda' \in \Lambda^i$ for some $i < j$. Therefore, we have $A \cup \{\lambda\} \in Cl(\Lambda)$. Then, by Proposition 1, there exist $R, R' \in Cl^0(\mathcal{R}^\delta)$ such that $\Lambda(R) = A$ and $\Lambda(R') = A \cup \{\lambda\}$. Let μ and μ' be the stable matchings in $\mathcal{S}(\succ)$ associated with R and R' , i.e. $R = R_\mu \setminus R_{\bar{\mu}}$ and $R' = R_{\mu'} \setminus R_{\bar{\mu}}$.

First, we show that there is no $\mu'' \in \pi(\succ)$ with $\mu \triangleright_M \mu'' \triangleright_M \mu'$. Otherwise, let $R'' = R_{\mu''} \setminus R_{\bar{\mu}}$. Since $\mu'' \in \pi(\succ)$, we have $R'' \in Cl^0(\mathcal{R}^\delta)$ and since $\mu \triangleright_M \mu'' \triangleright_M \mu'$, we have $R \subsetneq R'' \subsetneq R'$. Then, by Proposition 1, $\Lambda(R'') \in Cl(\Lambda)$ and $\Lambda(R) \subsetneq \Lambda(R'') \subsetneq \Lambda(R')$. But, since $\Lambda(R) = A$ and $\Lambda(R') = A \cup \{\lambda\}$, there can not exist such $\Lambda(R'')$, a contradiction.

Now, let μ'' be the matching obtained from μ by eliminating all the rotations in P . Since each $\rho \in P$ and $\rho' \in P'$ are independent, $N_\rho \cap N_{\rho'} = \emptyset$. It follows that for each $m \in M$, $\mu''(m) \in \{\mu(m), \mu'(m)\}$. Therefore, by *convexity*, $\mu'' \in \pi(\succ)$. But, this contradicts that there is no $\mu'' \in \pi(\succ)$ with $\mu \triangleright_M \mu'' \triangleright_M \mu'$. $\square$

We are now ready to construct an additive function $G : Cl(\mathcal{R}) \rightarrow \mathbb{Z}$ such that for each $R \in Cl(\mathcal{R})$, R minimizes G if and only if $R \setminus R_{\bar{\mu}} \in Cl^0(\mathcal{R}^\delta)$. First, we construct an additive function $H : Cl(\mathcal{R}^\delta) \rightarrow \mathbb{Z}$ such that for each $R \in Cl(\mathcal{R}^\delta)$, $H(R) = 0$ if and only if $R \in Cl^0(\mathcal{R}^\delta)$.

Let $\lambda \in \Lambda$ and define $\lambda^\downarrow$ as the set of rotations in λ that has no successor in λ , i.e. $\lambda^\downarrow = \{q \in \lambda \mid \text{there is no } \rho' \in \lambda \text{ with } q \rightarrow \rho'\}$ (we denote a generic element of $\lambda^\downarrow$ by q). Similarly define $\lambda^\uparrow$ as the set of rotations in λ that has no predecessor in λ , i.e. $\lambda^\uparrow = \{\rho \in \lambda \mid \text{there is no } \rho' \in \lambda \text{ with } \rho' \rightarrow \rho\}$. Since precedence is a transitive relation and λ is a finite set, $\lambda^\downarrow \neq \emptyset$ and $\lambda^\uparrow \neq \emptyset$. If λ is not singleton, then $\lambda^\uparrow \cap \lambda^\downarrow = \emptyset$. To see this, supposing that there exists $\rho \in \lambda^\uparrow \cap \lambda^\downarrow$, consider $\{p\}$ and $\lambda \setminus \{p\}$, which partitions λ . Then, there is no $\rho' \in \lambda$ such that $\rho \rightarrow \rho'$ (since $\rho \in \lambda^\downarrow$) or

$\rho' \rightarrow \rho$ (since $\rho \in \lambda^\uparrow$). But, this contradicts to Lemma 7. Thus, we conclude that $\lambda^\uparrow \cap \lambda^\downarrow = \emptyset$ whenever λ is not singleton. Now, let $\lambda^\uparrow$ be any set of rotations in λ that contains $\lambda^\uparrow$, and $\lambda^\downarrow$ be any set of rotations in λ that contains $\lambda^\downarrow$ such that $\lambda^\uparrow \cap \lambda^\downarrow = \emptyset$.

Remark 1. For the current proof, we can assume that $\lambda^\uparrow = \lambda^\uparrow$ and $\lambda^\downarrow = \lambda^\downarrow$. We present this general construction foreseeing that it will helpful to prove Theorem 2 and similar results.

Let $\lambda \in \Lambda$ that is not singleton and $\rho \in \lambda$. Then, define $\lambda^\downarrow(\rho)$ as the set of rotations in $\lambda^\downarrow$ that are preceded by ρ , i.e. $\lambda^\downarrow(\rho) = \{q \in \lambda^\downarrow \mid \rho \rightarrow q\}$. Similarly, define $\lambda^\uparrow(\rho)$ as the set of rotations in $\lambda^\uparrow$ that precede ρ , i.e. $\lambda^\uparrow(\rho) = \{\rho' \in \lambda^\uparrow \mid \rho' \rightarrow \rho\}$.

Next, recall that, by Lemma 5, for each $\rho \in \mathcal{R}^\delta$, there exists unique $\lambda_\rho \in \Lambda$ such that $\rho \in \lambda_\rho$. Therefore, we can define $h : \mathcal{R}^\delta \rightarrow \mathbb{Z}$ such that for each $\rho \in \mathcal{R}^\delta$, if $\lambda_\rho = \{\rho\}$, then $h(\rho) = 0$; if not, then let $\lambda = \lambda_\rho$ and define

$$h(\rho) = \begin{cases} -1 & \text{if } \rho \in \lambda^\downarrow, \\ \sum_{q \in \lambda^\downarrow(\rho)} \frac{1}{|\lambda^\uparrow(q)|} & \text{if } \rho \in \lambda^\uparrow, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

For each nonempty $Q \subset \mathcal{R}^\delta$, define $H(Q) = \sum_{\rho \in Q} h(\rho)$ and $H(\emptyset) = 0$. Note that by construction of h , for each $\lambda \in \Lambda$, $H(\lambda) = 0$. Moreover, the construction of h together with Lemma 7 guarantees that the following assertion holds.

Lemma 8. Let $\lambda \in \Lambda$ and $Q \subset \lambda^\downarrow$ be a nonempty set of rotations such that $\lambda^\downarrow \setminus Q \neq \emptyset$. Then, we have $H(\bigcup_{q \in Q} \lambda^\uparrow(q)) > |Q|$.

Proof. To see this, first, let $\lambda^\uparrow(Q) = \bigcup_{q \in Q} \lambda^\uparrow(q)$ and note that, by construction of h ,

$H(\lambda^\uparrow(Q)) \geq |Q|$. In what follows, we show that there exists $\rho^* \in \lambda^\uparrow(Q) \cap \lambda^\uparrow$ such that $\lambda^\downarrow(\rho^*)$ contains some $q' \in \lambda^\downarrow \setminus Q$. Thus, we will conclude that $h(\rho^*) > \sum_{q \in Q \cap \lambda^\downarrow(\rho^*)} \frac{1}{|\lambda^\uparrow(q)|}$ and $H(\lambda^\uparrow(Q)) > |Q|$. To see this, let P be the set of rotations in λ that precede an element of Q . Note that $P \neq \lambda$, since

$\lambda^\downarrow \setminus Q \neq \emptyset$. Then, consider P and $\lambda \setminus P$, which partitions λ . It follows from Lemma 7 that there exists $\rho \in P$ and $\rho' \in \lambda \setminus P$ such that $\rho \rightarrow \rho'$ or $\rho' \rightarrow \rho$. By our choice of P , the latter is not possible, so we have $\rho \rightarrow \rho'$. Since $\rho' \in \lambda \setminus P$, there exists $q' \in \lambda^\downarrow \setminus Q$ such that $\rho' \rightarrow q'$, and thus $\rho \rightarrow q'$. Since $\rho \in P$, there exists $\rho^* \in \lambda^\uparrow(Q) \cap \lambda^\uparrow$ such that $\rho^* \rightarrow \rho$. Thus, we conclude that $\rho^* \rightarrow q'$ as desired. $\square$

Next, we show that by restricting the domain of H to $Cl(\mathcal{R}^\delta)$, we will obtain the desired additive function. Figure 7 presents a rough demonstration of our construction.

Lemma 9. For each $R \in Cl(\mathcal{R}^\delta)$, $H(R) = 0$ if and only if $R \in Cl^0(\mathcal{R}^\delta)$.

Proof. (If part) As we noted before, by the construction of h , for each $\lambda \in \Lambda$, we have $H(\lambda) = 0$. Since, by Lemma 6, for each $R \in Cl^0(\mathcal{R}^\delta)$, $\{\lambda_\rho\}_{\rho \in R}$ partitions R , it follows that $H(R) = 0$.

(Only if part) Let $R \in Cl(\mathcal{R}^\delta)$ such that $R \notin Cl^0(\mathcal{R}^\delta)$. We show that $H(R) > 0$. First, recall that, by Lemma 5, for each $\rho \in \mathcal{R}^\delta$, there exists unique $\lambda \in \Lambda$ such that $\rho \in \lambda$. Therefore, $\{R \cap \lambda\}_{\lambda \in \Lambda}$ partitions R . Since H is additive, it follows that $H(R) = \sum_{\lambda \in \Lambda} H(R \cap \lambda)$.

Since $R \in Cl(\mathcal{R}^\delta)$ but $R \notin Cl^0(\mathcal{R}^\delta)$, by Proposition 1, there exists $\lambda \in \Lambda$, such that $R \cap \lambda \neq \emptyset$ and $\lambda \setminus R \neq \emptyset$. Therefore, λ is not singleton, and by the construction of h , for each $\rho \in \lambda$, $h(\rho) < 0$ ($h(\rho) > 0$) if and only if $\rho \in \lambda^\downarrow$ ($\rho \in \lambda^\uparrow$). Thus, we have $H(R \cap \lambda) = H(R \cap \lambda^\uparrow) + H(R \cap \lambda^\downarrow)$.

Since R is closed and for each $\rho \in \lambda$, either $\rho \in \lambda^\uparrow$ or ρ is preceded by a rotation in $\lambda^\uparrow \subset \lambda^\uparrow$, it follows from $R \cap \lambda \neq \emptyset$ that $R \cap \lambda^\uparrow \neq \emptyset$. Now, first, suppose that $R \cap \lambda^\downarrow = \emptyset$. Then, $H(R \cap \lambda^\downarrow) = 0$, and thus $H(R \cap \lambda) > 0$. Next, suppose that $R \cap \lambda^\downarrow \neq \emptyset$ and let $Q = R \cap \lambda^\downarrow$. Note that, since R is closed, we have $\bigcup_{q \in Q} \lambda^\uparrow(q) = R \cap \lambda^\uparrow$. Therefore, $H(R \cap \lambda) = H(\bigcup_{q \in Q} \lambda^\uparrow(q)) - |Q|$.

Finally, by Lemma 8, we will conclude that $H(R \cap \lambda) > 0$. To apply the lemma, we need to show that $\lambda^\downarrow \setminus Q \neq \emptyset$. But, if we would have $\lambda^\downarrow \subset Q$, then since R is closed and for each $\rho \in \lambda$, either $\rho \in \lambda^\downarrow$ or ρ precedes a rotation in $\lambda^\downarrow$, we must have $\lambda \subset R$, contradicting that $\lambda \setminus R \neq \emptyset$. $\square$

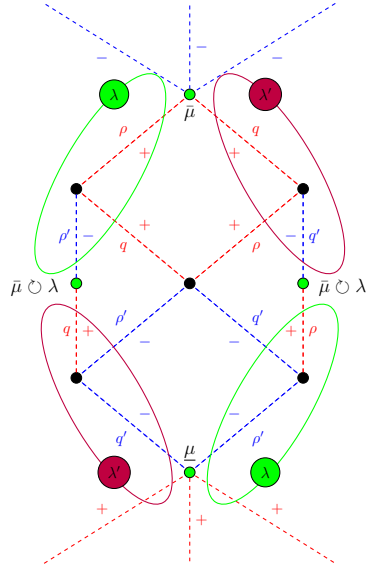


Figure 7: A demonstration of our construction of h and g .

Next, we extend the domain of H from $Cl(\mathcal{R}^\delta)$ to $Cl(\mathcal{R})$. For this, first define $g : \mathcal{R} \rightarrow \mathbb{Z}$ such that for each $\rho \in \mathcal{R}(\succ)$,

$$g(\rho) = \begin{cases} -1 & \text{if } \rho \in R_{\bar{\mu}}, \\ 1 & \text{if } \rho \notin R_{\underline{\mu}}, \\ h(\rho) & \text{otherwise, i.e. } \rho \in \mathcal{R}^\delta. \end{cases}$$

Then, define the additive function $G : Cl(\mathcal{R}) \rightarrow \mathbb{Z}$ such that $G(\emptyset) = 0$ and for each nonempty $R \in Cl(\mathcal{R})$, $G(R) = \sum_{\rho \in R} g(\rho)$.

It follows from the construction of g that for each $R \in Cl(\mathcal{R})$, if R minimizes G , then $R_{\bar{\mu}} \subset R \subset R_{\underline{\mu}}$. Since $R \in Cl(\mathcal{R})$, this means that $R \setminus R_{\bar{\mu}} \in Cl(\mathcal{R}^\delta)$. Then, since for each $\rho \in \mathcal{R}^\delta$, $g(\rho) = \bar{h}(\rho)$, it directly follows from Lemma 9 that for each $R \in Cl(\mathcal{R})$, R minimizes G if and only if $R \setminus R_{\bar{\mu}} \in Cl^0(\mathcal{R}^\delta)$. Thus, we complete the proof.

8 Appendix C: Proof of Theorem 2

(If part) To see that *minimal equatability* is satisfied, let $\succ \in \mathcal{P}$ and let $\mu \in \mathcal{S}(\succ)$ that minimizes $-\sum_{i \in N} F_i(\mu(i))$. Since for each $i \in N$, F_i is single peaked with respect to $\succ_i$ with the ideal point med_i^A , it follows that there is no $\mu' \in \mathcal{S}(\succ)$ such that for each $i \in N$, $\mu'(i)$ is closer to med_i^A than $\mu(i)$.

(Only if part) Let $\succ \in \mathcal{P}$ be a problem, $\mathcal{S}$ and $\mathcal{R}$ be the associated set of stable matchings and rotations.

Step 1. Let $\rho = [(m_1, w_1), (m_2, w_2), \dots, (m_k, w_k)]$ be a rotation. For each $i \in \{1, \dots, k\}$, define $\rho(m_i) = w_i$ and $s_\rho(m_i) = w_{i+1}$; $\rho(w_i) = m_i$ and $s_\rho(w_i) = m_{i-1}$, where the addition and subtraction in the subscripts is modulo k . Now, for each $i \in N$ and $\rho \in \mathcal{R}$, define

$$\phi_i(\rho) = \begin{cases} 0 & \text{if there is no pair with } i \text{ in } \rho, \\ -1 & \text{if } s_\rho(i) \text{ is closer to } med_i^A \text{ than } \rho(i), \\ 1 & \text{if } \rho(i) \text{ is closer to } med_i^A \text{ than } s_\rho(i). \end{cases} \quad (3)$$

Then, since each $\mu \in \mathcal{S}$ is obtainable from μ^M by eliminating the rotations in R_μ , for each $i \in N$, we have

$$|Rank_i^A(\mu(i)) - Rank_i^A(med_i^A)| = K + \sum_{\rho \in R_\mu} \phi_i(\rho) \quad (4)$$

where $K = |Rank_i^A(\mu^M(i)) - Rank_i^A(med_i^A)| = \lfloor |A_i|/2 \rfloor - 1$.

For each $i \in N$ and $j \in A_i$, let ρ_{ij} be the unique rotation elimination of which makes i matched to j . By construction of ϕ_i , for each $\rho \in R_\mu$, if there is no pair with i , then $\phi_i(\rho) = 0$ and if there is a pair with i , then $\rho \rightarrow \rho_{i\mu(i)}$. Therefore, we have

$$|Rank_i^A(\mu(i)) - Rank_i^A(med_i^A)| = K + \sum_{\{\rho | \rho \rightarrow \rho_{i\mu(i)}\}} \phi_i(\rho). \quad (5)$$

Step 2. By using the generality of our results in Section 7.3, we construct an additive function $G : Cl(\mathcal{R}) \rightarrow \mathbb{R}$ for π such that $G(\emptyset) = 0$ and for each nonempty $R \in Cl(\mathcal{R})$, $G(R) = \sum_{\rho \in R} g(\rho)$. To see this, let $\rho \in \mathcal{R}$ and N_ρ^+ (N_ρ^-) be the set of agents such that $\phi_i(\rho) > 0$ ($\phi_i(\rho) < 0$). First,

note that for each $\rho \in \mathcal{R}$, if $N_\rho^+ = \emptyset$ ($N_\rho^- = \emptyset$), then we must have $g(\rho) < 0$ ($g(\rho) > 0$), otherwise some agent $i \in N_\rho^-$ ($i \in N_\rho^+$) must receive a positive (negative) $g_i(\rho)$ value, contradicting that the resulting F_i will be single peaked. To guarantee that this is not the case, for each $\lambda \in \Lambda$ that is not singleton choose $\lambda^\downarrow$ as the union $\lambda^\downarrow$ and all $\rho \in \lambda$ such that $N_\rho^+ = \emptyset$, and $\lambda^\uparrow$ as the union $\lambda^\uparrow$ and all $\rho \in \lambda$ such that $N_\rho^- = \emptyset$, i.e. $\lambda^\downarrow = \lambda^\downarrow \cup \{\rho' \in \lambda \mid N_{\rho'}^+ = \emptyset\}$ and $\lambda^\uparrow = \lambda^\uparrow \cup \{\rho' \in \lambda \mid N_{\rho'}^- = \emptyset\}$. Note that, we have $\lambda^\uparrow \cap \lambda^\downarrow = \emptyset$, as we know that $\lambda^\uparrow \cap \lambda^\downarrow = \emptyset$. Now, by Lemma 9, for the associated mapping H , for each $\mu \in \mathcal{S}$ with $R_{\bar{\mu}} \subset R_\mu \subset R_{\underline{\mu}}$, we have $H(R_\mu) = 0$ if and only if $\mu \in \pi(\succ)$.

In moving to G from H , there is a minor problem of directly using the construction presented at the end of Section 7.3. To fix this, we need to modify our construction of g by using the additional structure brought by *minimal equatability* of π to guarantee that each F_i can be constructed as to be single peaked. To show this, first, we make two simple observations. Let $\mu \in \pi(\succ)$ and $\rho \in \mathcal{R}$. (i) If ρ is exposed in μ , then $N_\rho^+ \neq \emptyset$; otherwise let $\mu' = \mu \circ \rho$, then for each agent $i \in N$ with $\mu(i) \neq \mu'(i)$ we have $\mu'(i)$ is closer to med_i^A than $\mu(i)$, which contradicts that π satisfies *minimal equatability*. (ii) If μ is obtained from another matching μ' by eliminating ρ , i.e. $\mu = \mu' \circ \rho$, then $N_\rho^- \neq \emptyset$; otherwise for each agent $i \in N$ with $\mu(i) \neq \mu'(i)$ we have $\mu'(i)$ is closer to med_i^A than $\mu(i)$, which contradicts that π satisfies *minimal equatability*.

Now, let $\rho \notin R_{\bar{\mu}}$ ($\rho \in R_{\bar{\mu}}$). Then, as it was defined in Section 7.3, $g(\rho) = 1$ ($g(\rho) = -1$) even if $N_\rho^+ = \emptyset$ ($N_\rho^- = \emptyset$). To fix this, first, update g such that if $N_\rho^+ = \emptyset$ ($N_\rho^- = \emptyset$), then $g(\rho) = -1$ ($g(\rho) = 1$). However, for $\bar{\mu} \neq \mu^M$, it should still be the case that if $R \in Cl(\mathcal{R})$ minimizes G , then $R_{\bar{\mu}} \subset R \subset R_{\underline{\mu}}$. To guarantee this, let $\rho \in \mathcal{R}$ be such that $\bar{\mu} = \mu' \circ \rho$ for some $\mu \in \mathcal{S}$. By (ii), we have $N_\rho^- \neq \emptyset$. Therefore, we can pick $g(\rho)$ small enough as to guarantee that for each $R \in Cl(\mathcal{R})$, if $R \subsetneq R_{\bar{\mu}}$, then $G(R_{\bar{\mu}}) < G(R)$. Similarly, let $\rho \in \mathcal{R}$ such that ρ is exposed in $\underline{\mu}$. By (i), we have $N_\rho^+ \neq \emptyset$. Therefore, we can pick $g(\rho)$ big enough as to guarantee that for each $R \in Cl(\mathcal{R})$, if $R_{\underline{\mu}} \subsetneq R$, then $G(R_{\underline{\mu}}) < G(R)$.

Step 3. First, for the mapping $g : \mathcal{R} \rightarrow \mathbb{Z}$ that is constructed in the previous step, we show that for each $\rho \in \mathcal{R}$, there exists $\{g_i(\rho)\}_{i \in N}$ that satisfies

$$\sum_{i \in N} g_i(\rho) = g(\rho), \text{ and} \quad (6)$$

$$\text{for each } i \in N, g_i(\rho) = \alpha \phi_i(\rho) \text{ for some } \alpha > 0. \quad (7)$$

To see this, recall that g is constructed such that for each $\rho \in \mathcal{R}$, if $N_\rho^+ = \emptyset$ ($N_\rho^- = \emptyset$), then $g(\rho) < 0$ ($g(\rho) > 0$). Now, for each $i \in N$ and $\rho \in \mathcal{R}$, define $g_i(\rho)$ such that if $g(\rho) < 0$, then

$$g_i(\rho) = \begin{cases} 0 & \text{if } \phi_i(\rho) = 0, \\ \frac{(g(\rho) - |N_\rho^+|)}{|N_\rho^-|} & \text{if } \phi_i(\rho) < 0, \\ 1 & \text{if } \phi_i(\rho) > 0. \end{cases} \quad (8)$$

If $g(\rho) \geq 0$, then

$$g_i(\rho) = \begin{cases} 0 & \text{if } \phi_i(\rho) = 0, \\ -1 & \text{if } \phi_i(\rho) < 0, \\ \frac{(g(\rho) + |N_\rho^-|)}{|N_\rho^+|} & \text{if } \phi_i(\rho) > 0. \end{cases} \quad (9)$$

Note that, by this construction, $\{g_i(\rho)\}_{i \in N}$ satisfies (6) and (7). Next, recall that for each $i \in N$ and $j \in A_i$, ρ_{ij} was the unique rotation elimination of which makes i matched to j . Then, we define

$$F_i(j) = \sum_{\{\rho | \rho \rightarrow \rho_{ij}\}} g_i(\rho). \quad (10)$$

It directly follows from (5) and (7) that for each $i \in N$, F_i is single peaked with respect to $\succ_i$ with the ideal point med_i^A . Finally, let $\mu \in \mathcal{S}$ and define $F(\mu) = \sum_{i \in N} F_i(\mu(i))$. To see that $F(\mu) = G(R_\mu)$, first, note that since $R_\mu \in Cl(\mathcal{R})$, we have $R_\mu = \bigcup_{i \in N} \{\rho | \rho \rightarrow \rho_{i\mu(i)}\}$. By (7), for each $i \in N$ and $\rho \in \mathcal{R}$, if $\phi_i(\rho) = 0$, then $g_i(\rho) = 0$. Therefore, it follows from (6) that

$$G(R_\mu) = \sum_{\rho \in R_\mu} g(\rho) = \sum_{i \in N} \sum_{\{\rho | \rho \rightarrow \rho_{i\mu(i)}\}} g_i(\rho). \quad (11)$$

By substituting (10) into (11), we conclude that $F(\mu) = G(R_\mu)$.