

Core-Stability in Assignment Markets with Financially Constrained Buyers [★]

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Abstract. We study markets where a set of indivisible items is sold to bidders with unit-demand valuations, subject to a hard budget limit. Without financial constraints and pure quasilinear bidders, this assignment model allows for a simple ascending auction format that maximizes welfare and is incentive-compatible and core-stable. Introducing budget constraints, the ascending auction requires strong additional conditions on the unit-demand preferences to maintain its properties. We show that, without these conditions, we cannot hope for an incentive-compatible and core-stable mechanism. We design an iterative algorithm that depends solely on a trivially verifiable ex-post condition and demand queries, and with appropriate decisions made by an auctioneer, always yields a welfare-maximizing and core-stable outcome. If these conditions do not hold, we cannot hope for incentive-compatibility and computing welfare-maximizing assignments and core-stable prices is hard: Even in the presence of value queries, where bidders reveal their valuations and budgets truthfully, we prove that the problem becomes NP-complete for the assignment market model. The analysis complements complexity results for markets with more complex valuations and shows that even with simple unit-demand bidders the problem becomes intractable. This raises doubts on the efficiency of simple auction designs as they are used in high-stakes markets, where budget constraints typically play a role.

1 Introduction

The idea of a market exchange automatically channeling self-interest toward welfare maximizing outcomes is a central theme in neoclassical economics. The initial conjecture of the “invisible hand” goes back to Adam Smith. Formally, the Arrow–Debreu model showed that under convex preferences and perfect competition there must be a set of Walrasian equilibrium prices [2]. In these models, market participants are price-takers, and they sell or buy divisible goods in order to maximize their total value subject to their budget or initial wealth. The more recent stream of research on competitive equilibrium theory assumes indivisible goods and quasilinear utility functions, i.e. buyers maximize value minus the price they pay (their payoff) and there are no budget constraints [4, 6, 13, 15, 18]. The underlying question is under which conditions on the preferences markets with indivisible goods can be assumed to be core-stable¹ and welfare-maximizing. This literature focuses on larger markets where bidders are assumed to be price-takers and it emphasizes core-stability over incentive-compatibility.

For two-sided matching markets where quasilinear buyers have unit-demand, referred to as assignment markets, welfare-maximization, core-stability and even incentive-compatibility can be achieved with a polynomial-time auction algorithm [22]. These auctions can be interpreted as primal-dual algorithms, where the auctioneer specifies a price vector (a demand query) in each round and the bidders respond with their demand set, i.e. the set of goods that maximize their payoff for given prices.

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¹ The core is the set of feasible outcomes that cannot be improved upon by a subset of the economy’s participants.

Unfortunately, it is well-known that we cannot hope for such positive results with more general quasilinear preferences. Incentive-compatibility and core-stability are conflicting in general markets with quasilinear utilities [3]. Even if we give up on incentive-compatibility, only very restricted types of valuations (e.g., substitutes valuations) allow for Walrasian equilibria.² As discussed in Bikhchandani and Ostroy [7], under general valuations (allowing for substitutes and complements), competitive equilibrium prices need to be non-linear and personalized and the core can be empty.

It is interesting to understand how core-stable and welfare-maximizing prices can be computed in the presence of budget constraints if we assume bidders to be price-takers. This question was recently analyzed for markets that allow for general valuations where the auctioneer has complete information about values and budgets [5]. The result is a spoiler: Computing the welfare-maximizing core-stable outcome is a Σ_2^P -hard optimization problem. Such problems are considered intractable, even for very small problem instances. The intuition behind this result is that the allocation and pricing problems cannot be treated independently anymore.

Considering that budget constraints are a reality in many markets, this casts doubts whether simple market institutions based on polynomial-time algorithms (e.g., simple ascending auctions as used in selling spectrum nowadays) can find a welfare-maximizing outcome even if bidders were price-takers. General preferences as allowed in combinatorial exchanges, might be too much to ask for. A natural question is whether we can at least hope for core-stable and welfare-maximizing outcomes in markets where bidders have unit-demand valuations, and seek to maximize payoff subject budget constraints.

1.1 Contributions

The aim of this work is to compute welfare-maximizing and core-stable outcomes in the presence of financial constraints. If we cannot achieve incentive-compatibility and core-stability with such simple valuations, markets with more complex preferences are bound not to satisfy these properties.

We first introduce and analyze an iterative process that always finds a core-stable outcome using only demand queries based on prices and no direct access to valuations. In contrast to Aggarwal et al. [1], where bidders' valuations are directly queried, a distinguishing part of our algorithm is that it relies exclusively on demand queries and provides a natural generalization of the auction by Demange et al. [10]. Moreover, we place emphasis on the question of when we can expect the outcome of the auction to not only lie in the core, but also maximize welfare among all core allocations, as welfare maximization is typically a central goal in market design. One of our main results is that, for any market instance, if the auctioneer would be able to guess the right decisions throughout the auction, we reach a welfare-maximizing core-allocation. In particular, if the auctioneer does not have to make any such decisions - which is trivial to check ex-post - our result implies that the auction always finds a welfare-maximizing core-outcome. Unfortunately, we do not know ex-ante whether the condition holds, and if it does not, welfare can be arbitrarily low.

Now, it is important to understand whether we can hope for an incentive-compatible and welfare-maximizing core-selecting mechanism without additional conditions beyond the already strong restriction to unit-demand valuations. Unfortunately, the answer to this question is negative. In a novel result, we show that no auction mechanism for the assignment market can be incentive-compatible and core-stable when buyers face budget constraints. If we give up on incentive-compatibility and assume full access to the true valuations (i.e. via value queries) and buyer budgets, we can compute a core-stable and welfare-maximizing outcome.

² A Walrasian equilibrium describes a competitive equilibrium where supply equals demand and prices are linear (i.e., there is a price for each good) and anonymous (i.e., the price is the same for all participants and there is no price differentiation) [4, 6, 17].

One might expect that the problem admits a polynomial time solution, since, without the presence of budget constraints, the problem lies in complexity class P. Unfortunately, a main finding of this paper shows that determining core-stable, welfare-maximizing outcomes with financially constrained buyers is an NP-complete optimization problem, even for the assignment market with full access to valuations and budgets. This means, the existence of budget constraints renders the problem of determining welfare-maximizing, core-stable outcomes NP-complete. These results show that, even for the simplest type of multi-object markets, those with only unit-demand bidders, we cannot expect core-stable and welfare-maximizing outcomes unless additional strong conditions are satisfied that are typically unknown ex-ante.

2 Related Literature

Two-sided matching markets describe transactions where buyers want to win at most one item (also known as the unit-demand model) and sellers sell only one item. Buyers and sellers are disjoint sets of agents and each buyer forms exclusive relationships with a seller. Such markets are central to the economic sciences. The well-known marriage model of Gale and Shapley [12] assumes ordinal preferences and non-transferable utility. Shapley and Shubik [22] analyzed such markets with quasilinear utility functions and showed that the core of this game is nonempty and encompasses all competitive equilibria. Under the quasilinear utility model, buyers maximize value minus price, while sellers maximize price minus cost. While their setting assumes access to all valuations, Demange et al. [10] showed that an ascending auction with only demand queries results in a competitive equilibrium at the lowest possible price, i.e. at the competitive equilibrium price vector that is optimal for buyers. In such an auction, the auctioneer specifies a price vector (the demand query) in each round, and buyers respond with their demand set, i.e. the set of goods that maximize payoff at the prices.

The housing market of Shapley and Scarf [21] is an example of a market without transferable utility or monetary funds. In this market, each agent is endowed with a good or house, and each agent is interested in one house only. The goal of this market is to redistribute ownership of the houses in accordance with the ordinal preferences of the agents. In such housing markets, the core set is nonempty. If no agent is indifferent between any two houses, then the economy has a unique competitive allocation, which is also the unique strict core allocation. An allocation belongs to the *strong* core, if no coalition of buyers and sellers can make all members as well off and at least one member better off by trading items among themselves. We assume an allocation belongs to the *weak* core if no coalition can lead to all members' utilities improved when redistributing items amongst themselves.

Closer to this paper is another line of research that focuses on assignment markets where buyers maximize payoff subject to a *hard* budget constraint. Aggarwal et al. [1] show that an extension of the Hungarian algorithm is incentive-compatible and bidder-optimal if the auction is in general position, a rather specific condition that is usually unknown ex-ante and hard to check. Typically, ascending auctions use only demand queries, namely the auctioneer specifies a price and the bidders respond with their demand set. The auction additionally requires value and budget queries, thereby asking for the value of a specific good to a bidder and their budget during the auction. This is quite different from the ascending auctions based on price-based demand queries only, as described in Demange et al. [10] and the subsequent literature or compared to ascending auctions used in the field. Fujishige and Tamura [11] consider two-sided markets with budget-constrained bidders whose valuation functions are more general than unit-demand. Their results imply that in the unit-demand setting, there always exists a core allocation. These prior results aim exclusively for core-stability, but do not attempt to maximize welfare as done in this work.

Finally, van der Laan and Yang [23] propose an ascending auction for the assignment market that results in an *equilibrium under allotment*, which is in general not a core-stable outcome. Core-stability and incentive-compatibility are arguably the most important axioms in market design. Whether one can design assignment markets that satisfy these axioms in the presence of hard budget constraints has not yet been answered. We show that, without strong additional assumptions, this is not possible. Importantly, even with access to all valuations and budgets, the problem is computationally intractable for large market instances.

3 Preliminaries

A two-sided matching market $M = (\mathcal{B}, \mathcal{S}, v, b, r)$ consists of two disjoint sets of agents $\mathcal{B}$ and $\mathcal{S}$, representing bidders $i \in \mathcal{B} = \{1, 2, \dots, n\}$ and goods $j \in \mathcal{S} = \{1, 2, \dots, m\} \cup \{0\}$. We identify good j with the seller owning it, i.e. each seller owns one good. The 0-item is a dummy item and does not have value to any bidder, meaning that receiving good 0 corresponds no real good. Additionally, the market is defined by each bidder i 's valuation $v_i : \mathcal{S} \rightarrow \mathbb{Z}_{\geq 0}$ with $v_i(0) = 0$ and budget $b^i \in \mathbb{Z}_{>0}$, as well as each seller j 's reserve values/ ask price $r_j \in \mathbb{Z}_{\geq 0}$.

A *price vector* is a vector $p \in \mathbb{R}^{\mathcal{S}}$ with $p(0) = 0$, assigning price $p(j)$ to every good j . Bidders have quasi-linear utilities, so if bidder i receives item j under prices p , their utility is $\pi_i(j, p) = v_i(j) - p(j)$, if $p(j) \leq b^i$, and $\pi_i(j, p) = -\infty$, otherwise. An *assignment* is represented as a map $\mu : \mathcal{B} \rightarrow \mathcal{S}$ from bidders to the items they receive, where $|\mu^{-1}(\{j\})| \leq 1$ for all $j \neq 0$, so only the dummy good may be assigned to more than one bidder. An *outcome* is a pair (μ, p) , where μ is an assignment and p is a price vector, such that no budget constraint is violated, i.e. $p(\mu(i)) \leq b^i$ for all $i \in \mathcal{B}$ and only sold items may have a positive price: $p(j) > 0$ implies that $|\mu^{-1}(\{j\})| = 1$. For our iterative auction in Section 4, for the sake of simplicity, we assume all reserve prices to be equal to 0. The results can be easily generalized by starting the auction at the reserve prices, and not at 0. The social welfare function is defined as the sum of cardinally measurable values v_i of all market participants, and an optimal allocation maximizes the social welfare function.

An important design desideratum is that of core-stability.

Definition 1 (Core outcome). *Let (μ, p) be an outcome. A bidder-seller pair $(i, j) \in \mathcal{B} \times \mathcal{S}$ is called a blocking pair, if $\pi_i(j, p) > \pi_i(\mu(i), p)$ and $p(j) < b^i$. (μ, p) is a core outcome, if there are no blocking pairs. We also say that (μ, p) is core-stable in this case.*

The idea of a blocking pair (i, j) is that both bidder i and seller j would strictly increase their utility, if i received item j instead of $\mu(i)$: if i pays $p(j) + \varepsilon$ for item j , then still $\pi_i(j, p) - \varepsilon > \pi_i(\mu(i), p)$, and at the same time, the profit of seller j is increased by ε .

In the literature, a core outcome is often alternatively defined in the following way: an outcome (μ, p) is in the core if there are no subsets $\mathcal{B}' \subseteq \mathcal{B}$ and $\mathcal{S}' \subseteq \mathcal{S}$ and an outcome (μ', p') on $\mathcal{B}' \times \mathcal{S}'$ such that $\pi_i(\mu'(i), p') > \pi_i(\mu(i), p)$ for all $i \in \mathcal{B}'$ and $p'(j) > p(j)$ for all $j \in \mathcal{S}'$ (see for example [25]). These definitions can easily be shown to be equivalent: first suppose that such subsets $\mathcal{B}'$ and $\mathcal{S}'$ do exist. Then it is easy to see that both sets are nonempty. In particular, let $i \in \mathcal{B}'$ and $j = \mu'(i)$. Then $p'(j) > p(j)$, so $p(j) < b^i$. Furthermore, we have $\pi_i(j, p) > \pi_i(j, p') > \pi_i(\mu(i), p)$, so (i, j) is a blocking pair. On the other hand, if (i, j) is a blocking pair, then as in the above paragraph, we can set $p'(j) = p(j) + \varepsilon$ and get $\pi_i(j, p') > \pi_i(\mu(i), p)$ and $p'(j) > p(j)$. Thus we can choose $\mathcal{B}' = \{i\}$, $\mathcal{S}' = \{j\}$, $\mu'(i) = j$ and $p'(j) = p(j) + \varepsilon$ in the alternative definition.

We focus on the problem of finding welfare-maximizing core outcomes:

$$\max \left\{ \sum_{i=1}^n v_i(\mu(i)) : (\mu, p) \text{ is a core outcome} \right\}$$

We first define the *demand set* of bidder i , which consist of the most preferred, affordable among all items at prices p :

$$D_i(p) = \{j : p(j) \leq b^i \wedge \pi_i(j, p) \geq \pi_i(k, p) \forall k \text{ with } p(k) \leq b^i\}.$$

Definition 2 (Competitive equilibrium). *An outcome (μ, p) is a competitive equilibrium, if $\mu(i) \in D_i(p)$ for all bidders i .*

If budgets are not binding, i.e., $b^i > v_i(j)$ for all bidders i and all goods j , the welfare-maximizing core outcome coincides with the competitive equilibrium. This equivalence no longer remains true if bidders have binding budgets: in general, a core outcome needs not be a competitive equilibrium, and different core outcomes might generate very different welfare.

Efficient algorithms for determining core outcomes under budget constraints have been discussed in the literature, however, desirable properties like bidder-optimality and incentive-compatibility are only guaranteed if additional assumptions on the bidders' preferences are made. Aggarwal et al. [1] introduced the notion of *general position*, a sufficient condition for ascending auctions to indeed find the welfare-maximizing core-stable outcome.

Definition 3 (Aggarwal et al. [1]). *Consider a directed bipartite graph with edges between bidders $\mathcal{B}$ and goods $\mathcal{S}$ (including dummy good 0): For $i \in \mathcal{B}$ and $j \in \mathcal{S}$, there is a*

- forward-edge from i to j with weight $-v_i(j)$
- backward-edge from j to i with weight $v_i(j)$
- maximum-price edge from i to j with weight $b^i - v_i(j)$
- terminal edge from i to the dummy good 0 with weight 0.

The auction is in general position, if for every bidder i , there are no two alternating walks, following alternating forward and backward edges and ending with a distinct maximum-price or terminal edge, having the same total weight.

Example 1. Consider an auction with two bidders 1 and 2 with $b^1 = b^2$. The number of goods and bidders' valuations may be chosen arbitrarily. Assume $j \in \mathcal{S}$ is any good. Consider the following path starting from bidder 1: $1 \rightarrow j \rightarrow 2 \rightarrow j$, where the last edge is a maximum-price edge, with total weight $-v_1(j) + v_2(j) + (b^2 - v_2(j)) = b^2 - v_1(j)$. Now consider the path $1 \rightarrow j$, where the only edge is a maximum-price edge, with weight $b^1 - v_1(j)$. Since $b^1 = b^2$, the total weight of both paths is equal, so the auction is not in general position.

As the example shows, the general position condition implies that in an ascending auction, no two bidders may reach their budget limits at the same time. Henzinger and Loitzenbauer [14] claim the general position condition is rather restrictive, as it excludes, for instance, symmetric bidders. They additionally show that no polynomial-time algorithm can determine whether a set of valuations is in general position. The general position condition is sufficient but not necessary for the existence of a unique bidder-optimal stable matching [1], which is thus also welfare-maximizing by our results below. As we will see, there are valuations not in general position, but where a welfare-maximizing core allocation can still be computed efficiently with our auction.

Let us now introduce an iterative auction that always finds a core-stable outcome in markets with budget constrained buyers and, if a simple ex-post condition is satisfied, maximizes welfare among all core outcomes.

All proofs have been deferred to the appendix.

4 An Iterative Auction

Our auction is based on the well-known auction by Demange et al. [10] (denoted as DGS auction from now on), which implements the Hungarian algorithm. In our auction, bidders only have to report their demand sets at certain prices, similar to other ascending auctions [19]. We provide conditions when reporting demand sets truthfully is incentive-compatible. Thus, we design a natural generalization of the DGS auction to markets where bidders have binding budget constraints. The simple ascending nature of our auction also naturally motivates an ex-post optimality condition for the returned allocation. Without loss of generality, we will assume $r_j = 0$ in this section.

In the auction process, we may need to “forbid” some bidder to demand a certain item. We model this by introducing subsets $R_1, \dots, R_n \subseteq \mathcal{S}$ of goods for every bidder and define the *restricted demand set* to be

$$D_i(p, R_i) = \{j \in R_i : p(j) \leq b^i \wedge \pi_i(j, p) \geq \pi_i(k, p) \forall k \in R_i \text{ with } p(k) \leq b^i\}.$$

Note that our definition of the restricted demand set coincides with the definition of demand sets by van der Laan and Yang [23]. The set consists of all affordable items that generate the highest utility among all items in R_i . We introduce the well-known notions of over- and underdemanded sets [10, 20], adjusted to our notion of restricted demand sets.

Definition 4. Let price vector p and sets $R_1, \dots, R_n \subseteq \mathcal{S}$ with $R_i \neq \emptyset \forall i$ be given. A set $T \subseteq \mathcal{S}$ is

- overdemanded, if $0 \notin T$ and $|\{i \in \mathcal{B} : D_i(p, R_i) \subseteq T\}| > |T|$, and
- underdemanded, if $p(j) > 0$ for all $j \in T$ and $|\{i \in \mathcal{B} : D_i(p, R_i) \cap T \neq \emptyset\}| < |T|$.

T is minimally over-/underdemanded, if it does not contain a proper over-/underdemanded subset.

Finally, we define the strict budget set of bidder i by $B_i(p) = \{j \in \mathcal{S} : p(j) < b^i\}$. It consists of all items with prices strictly less than the bidder’s budget.

4.1 The Auction Algorithm

Algorithm 1 describes our auction. It is based on the following observation.

Lemma 1. An outcome (μ, p) is in the core if and only if there are sets $R_1, \dots, R_n \subseteq \mathcal{S}$ such that $B_i(p) \subseteq R_i$ and $\mu(i) \in D_i(p, R_i)$ for all i .

Computing a core outcome can thus also be interpreted as computing a “competitive equilibrium” with respect to the restricted demand sets $D_i(p, R_i)$. This is quite similar to the definition of an *equilibrium under allotment* of van der Laan and Yang [23]. However, they have other requirements on the sets R_i , which, in general, cause equilibria not to lie in the core.

In view of Lemma 1, the goal of our auction procedure is to determine prices p together with sets R_i , such that there are neither over- nor underdemanded sets of items. As observed in Mishra and Talman [20], this implies existence of an assignment $\mu : \mathcal{B} \rightarrow \mathcal{S}$, such that every bidder receives an item in their demand set, and every item with positive price gets assigned to some bidder. The following result is due to the aforementioned work.

Proposition 1. Suppose that with respect to the $D_i(p, R_i)$, there is no over- or underdemanded set of items. Then there is an assignment $\mu : \mathcal{B} \rightarrow \mathcal{S}$ such that $\mu(i) \in D_i(p, R_i)$ for all i , and for all $j \in \mathcal{S}$ with $p(j) > 0$, there is some i with $\mu(i) = j$.

ALGORITHM 1: Iterative Auction

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- 1 Set $p^1 = (0, \dots, 0)$ and $R_i^1 = \mathcal{S}$ for all bidders i . Set $t = 1$, $O^0 = \emptyset$ and $I^1 = \emptyset$.
 - 2 Request $D_i(p^t, R_i^t)$ from all bidders. If $t > 1$ and the set

$$I^t = \{i \in \mathcal{B} : D_i(p^{t-1}, R_i^{t-1}) \subseteq O^{t-1} \wedge D_i(p^{t-1}, R_i^{t-1}) \setminus D_i(p^t, R_i^t) \neq \emptyset\}$$
 is nonempty, go to Step 3. Otherwise, if there is an overdemanded set, go to Step 4. Else, go to Step 5.
 - 3 Choose a bidder $i \in I^t$ and define $J_i^t = D_i(p^{t-1}, R_i^{t-1}) \setminus D_i(p^t, R_i^t)$. Set $R_i^{t+1} = R_i^t \setminus J_i^t$, $O^t = \emptyset$ and $p^{t+1} = p^{t-1}$. For all other bidders i' , the sets $R_{i'}^{t+1} = R_{i'}^t$ are unchanged. Set $t = t + 1$ and go to Step 2.
 - 4 Choose a minimally overdemanded set O^t . For all $j \in O^t$, set $p^{t+1}(j) = p^t(j) + 1$. The prices for all other goods, as well as the sets R_i^t remain unchanged. Set $t = t + 1$ and go to Step 2.
 - 5 Compute an assignment μ , such that $\mu(i) \in D_i(p^t, R_i^t)$ for all bidders i and $\mu(\mathcal{B}) \subseteq \{j \in \mathcal{S} : p^t(j) > 0\}$. Set $p = p^t$ and return (μ, p) .
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Note that they consider markets without budgets and unrestricted demand sets. However, their proof only uses combinatorial properties of the demand sets, so it can be directly adapted to our setting. Thus, a proof here is omitted.

Step 3 of the auction ensures that we do not end up with underdemanded sets of items. Moreover, sets R_i^t always contain at least all items that cost strictly less than the bidder's budget b^i . Our proof of correctness is similar to the one by van der Laan and Yang [23]: due to the budget constraints, underdemanded sets of items may appear. We show that Step 3 of the auction takes care of these sets.

Lemma 2. *Let O be minimally overdemanded and $T \subseteq O$ with $T \neq \emptyset$. Let prices p and sets R_i be given. Then*

$$|\{i : D_i(p, R_i) \subseteq O \wedge D_i(p, R_i) \cap T \neq \emptyset\}| > |T|.$$

In particular, T is not underdemanded.

Lemma 3. *For all bidders $i \in \mathcal{B}$ and all iterations t of the algorithm, we have that $B_i(p) \subseteq R_i^t$. In particular, since $p^t(0) = 0$, $R_i^t \neq \emptyset$.*

Proposition 2. *For every iteration t in the auction, it holds:*

1. *if there is a minimally underdemanded set of items T , then $T \subseteq O^{t-1}$ and Step 3 is executed*
2. *if Step 3 is executed, there is no underdemanded set of items with respect to the $D_i(p^{t+1}, R_i^{t+1})$.*

Employing the previous lemmata, we can proceed to prove correctness of our proposed auction.

Proposition 3. *The auction terminates after a finite number of iterations, and an assignment μ as is described in Step 5 exists whenever this Step is reached. The returned tuple (μ, p) constitutes a core outcome.*

The following two examples demonstrate the auction process in practice.

Example 2. Consider the following auction with three bidders 1, 2, 3 and two items A and B .

	$v_i(A)$	$v_i(B)$	b^i
Bidder $i = 1$	10	0	1
Bidder $i = 2$	0	10	2
Bidder $i = 3$	10	10	10

The auction proceeds as follows.

	p^t	$D_1(p^t, R_1^t)$	$D_2(p^t, R_2^t)$	$D_3(p^t, R_3^t)$	R_1^t	R_2^t	R_3^t	O^t	I^t
$t = 1$	(0, 0)	$\{A\}$	$\{B\}$	$\{A, B\}$	$\mathcal{S}$	$\mathcal{S}$	$\mathcal{S}$	$\{A, B\}$	$\emptyset$
$t = 2$	(1, 1)	$\{A\}$	$\{B\}$	$\{A, B\}$	$\mathcal{S}$	$\mathcal{S}$	$\mathcal{S}$	$\{A, B\}$	$\emptyset$
$t = 3$	(2, 2)	$\{0\}$	$\{B\}$	$\{A, B\}$	$\mathcal{S}$	$\mathcal{S}$	$\mathcal{S}$	$\emptyset$	$\{1\}$
$t = 4$	(1, 1)	$\{0\}$	$\{B\}$	$\{A, B\}$	$\{0, B\}$	$\mathcal{S}$	$\mathcal{S}$	$\emptyset$	$\emptyset$

In iterations $t = 1, 2$, there is a unique minimally overdemanded set $O^t = \{A, B\}$, and I^t is empty. Thus, Step 4 of the auction is executed and the prices for A and B are raised. In iteration $t = 3$, the set $I^t = \{1\}$ is nonempty which indicates that bidder 1's budget was tight for A at prices (1, 1). Thus, we forbid 1 to receive item A and reset the prices to (1, 1). Now, in iteration $t = 4$, there is no overdemanded set and I^t is empty. Thus, there exists an assignment μ with $\mu(i) \in D_i(p^4, R_i^4)$ for all $i \in \mathcal{B}$, namely $\mu(1) = 0$, $\mu(2) = B$ and $\mu(3) = A$. It is easily checked that (μ, p) is indeed a core outcome.

Example 3. Let us now consider an example of an auction where a non-trivial decision has to be made in Step 3.

	$v_i(A)$	$v_i(B)$	b^i
Bidder $i = 1$	10	0	3
Bidder $i = 2$	0	11	1
Bidder $i = 3$	5	3	10

It is easy to see that after 3 iterations through Step 4 of our auction, we reach prices $p^4 = (3, 1)$, where bidder 1 demands $\{A\}$, bidder 2 demands $\{B\}$ and bidder 3 demands $\{A, B\}$. Since $\{A, B\}$ is minimally overdemanded, we execute Step 4 once again to reach $p^5 = (4, 2)$, where due to the budget constraints, we have $D_1(p^5, R_1^5) = \{0\}$ and $D_2(p^5, R_2^5) = \{0\}$, while bidder 3 still demands $\{A, B\}$. Thus, Step 3 of the auction is executed with $I^5 = \{1, 2\}$, so both bidders 1 or 2 would be valid bidders to choose in Step 3. For the choice $i = 1$, we have $J_i^5 = \{A\}$, while for the choice $i = 2$, we have $J_i^5 = \{B\}$. We could thus either remove A from R_1^5 , or B from R_2^5 . Depending on our choice, we get two different core outcomes, both supported by the prices $p = (3, 1)$: one where bidder 1 receives nothing, bidder 2 receives B and bidder 3 receives A , and one where bidder 1 receives A , bidder 2 receives nothing and bidder 3 receives B . The total welfare of the former allocation is 16, while the one of the latter is 13.

4.2 Economic Properties

The output produced by our iterative auction is not uniquely defined - it may depend on which bidder $i \in I^t$ is chosen whenever Step 3 is executed. Indeed, we prove the following result.

Proposition 4. *Let (ν, q) be an arbitrary core outcome. Then bidders $i \in I^t$ in Step 3 can be chosen in such a way, that for the resulting outcome (μ, p) we have that $p \leq q$ coefficient-wise, and $\pi_i(\mu(i), p) \geq \pi_i(\nu(i), q)$ for all bidders i .*

We say that a core outcome is (μ, p) *Pareto optimal for the bidders*, if for every core outcome (ν, q) with $\pi_i(\nu(i), q) > \pi_i(\mu(i), p)$ for some bidder i , there is a bidder i' with $\pi_{i'}(\mu(i'), p) > \pi_{i'}(\nu(i'), q)$. Proposition 4 directly implies that for every core outcome which is Pareto optimal for the bidders, there is an outcome (μ, p) reachable by the auction with $\pi_i(\mu(i), p) = \pi_i(\nu(i), q)$ for all $i \in \mathcal{B}$.

Aggarwal et al. [1] prove that their algorithm for computing a core-stable outcome always finds the bidder-optimal core outcome (μ, p) , whenever the auction is in general position. Here, *bidder-optimal* means that for every other core outcome (ν, q) we have that $\pi_i(\mu(i), p) \geq \pi_i(\nu(i), q)$ for all $i \in \mathcal{B}$. Bidder-optimality thus implies Pareto optimality. We show a similar result for our auction: if the bidder to choose in Step 3 of our auction is always unique, our auction also finds a bidder-optimal core outcome.

Corollary 1. *Suppose that whenever Step 3 is executed, $|I^t| = 1$, i.e., there is a unique bidder to choose, and let (μ, p) be the uniquely determined outcome of the auction. Then for any core outcome (ν, q) we have that $p \leq q$ and $\pi_i(\mu(i), p) \geq \pi_i(\nu(i), q)$ for all bidders i , i.e., (μ, p) is bidder-optimal.*

In particular, if the general position condition is satisfied, it can be shown that I^t never contains more than one bidder. Thus, our auction always finds a bidder-optimal outcome.

Proposition 5. *Suppose the auction is in general position. Then in every iteration through Step 3 of our iterative auction, we have that $|I^t| = 1$, and for the unique $i \in I^t$, we have $|J_i^t| = 1$.*

Note that in general our ex-post condition $|I^t| = 1$ whenever Step 3 is reached is less demanding than the general position condition, since we do not require $|J_i^t| = 1$, and it is easy to construct examples where $|J_i^t| > 1$, but the ex-post condition is fulfilled. While our condition is only ex-post, it is straight-forward for the auctioneer to check when the auction is actually performed.

Let us now consider welfare-maximization properties of our auction. We first observe that a welfare maximizing core outcome can always be found among the ones which are Pareto optimal for the bidders.

Proposition 6. *Let (μ, p) and (ν, q) be core outcomes. If $\pi_i(\mu(i), p) \geq \pi_i(\nu(i), q)$ for all bidders i , then*

$$\sum_{i \in \mathcal{B}} v_i(\mu(i)) \geq \sum_{i \in \mathcal{B}} v_i(\nu(i)).$$

As we described above, by Proposition 4, we can reach any core outcome which is Pareto optimal for the bidders with our auction, and Proposition 6 says that one of them must be welfare-maximizing. Now if we always have $|I^t| = 1$, the outcome of our auction is unique which proves our first main result.

Theorem 1. *Bidders in I^t in Step 3 of the auction can be chosen such that the outcome of the auction is a welfare-maximizing core outcome.*

In particular, if $|I^t| = 1$ whenever Step 3 is reached, the unique outcome of the auction is a welfare-maximizing core outcome.

Knowledge of the bidders' demand sets does not suffice in order to always choose the "correct" bidders in Step 3 to reach a welfare-maximizing outcome. Our hardness result in Section 5 implies that even with perfect knowledge of the bidders' preferences, choosing the correct bidders in Step 3 is NP-hard. However, our simple ex-post condition $|I^t| = 1$ at least gives the auctioneer a simple certificate of optimality.

If the auction is in general position, then the auction is ex-post incentive-compatible, which follows from the original work by Demange et al. [10] and the paper by Aggarwal et al. [1]. The question is if this algorithm or any other algorithm where the bidders' preferences are no further restricted can be incentive-compatible. Unfortunately, the answer is no because incentive-compatibility goes against envy-freeness and therefore the core definition as we show next.

Theorem 2. *In assignment markets with payoff-maximizing but budget constrained bidders there is no incentive-compatible mechanism terminating in a core-stable solution for every input.*

Theorem 2 does not preclude an incentive-compatible and welfare-maximizing auction (that is not core-stable). Overall, these iterative auctions require bidders to reveal that they are indifferent to not winning the good once the price equals the valuation of a bidder. Only this allows auctioneers to differentiate between a bidder dropping out due to reaching his valuation or his budget. In practice, bidders might not always bid the null set when price reaches value even in an incentive-compatible auction, which can lead to inefficiencies in such iterative auctions.

5 A Sealed-Bid Auction

If in some iteration of the above algorithm Step 3 is reached with $|I^t| > 1$, the ascending auction with only demand queries does not necessarily find the welfare-maximizing core-stable outcome. In such case, a combination of value and demand queries is required in order to obtain the desired assignment. To this end, for the remaining part of this paper, we assume the auctioneer has unlimited access to all item valuations and bidder budgets. Unfortunately, the outcome and pricing problem do not only need a different oracle, but it also becomes NP-complete as we will show next. Besides, as mentioned above, the property of strategyproofness does not hold.

5.1 A MILP Formulation

First, we show that the problem belongs to complexity class NP by modeling it as a mixed integer linear program (MILP). Bi-linear terms present in the quadratic formulation (q-BC), namely products of continuous prices $p(j)$ and binary variables, can easily be linearized to obtain the resulting MILP.

$$\begin{aligned}
& \text{maximize} && \sum_{i \in \mathcal{B}} \pi_i + \sum_{j \in \mathcal{S}} \pi_j \\
& \text{subject to} && \pi_i = \sum_{j \in \mathcal{S}} (v_i(j) - p(j)) m_i(j) && \forall i \in \mathcal{B} && (1) \\
& && \pi_j = \sum_{i \in \mathcal{B}} (p(j) - r_j) m_i(j) && \forall j \in \mathcal{S} && (2) \\
& && \sum_{j \in \mathcal{S}} m_i(j) \leq 1 && \forall i \in \mathcal{B} && (3) \\
& && \sum_{i \in \mathcal{B}} m_i(j) \leq 1 && \forall j \in \mathcal{S} && (4) \\
& && \pi_i \geq (v_i(j) - p(j)) \alpha_i(j) && \forall i \in \mathcal{B}, j \in \mathcal{S} && (5) \\
& && \pi_j \geq \min(v_i(j), b^i) (1 - y_i(j)) && \forall i \in \mathcal{B}, j \in \mathcal{S} && (6) \\
& && b^i \geq p_j (1 - \beta_i(j)) && \forall i \in \mathcal{B}, j \in \mathcal{S} && (7) \\
& && p_j \geq b^i \beta_i(j) && \forall i \in \mathcal{B}, j \in \mathcal{S} && (8) \\
& && (1 - \alpha_i(j)) + (1 - \beta_i(j)) - 2 \leq 2(1 - y_i(j)) + \epsilon y_i(j) && \forall i \in \mathcal{B}, j \in \mathcal{S} && (9) \\
& && r_j m_i(j) \leq p(j) \leq \min(v_i(j), b^i) m_i(j) + M(1 - m_i(j)) && \forall i \in \mathcal{B}, j \in \mathcal{S} && (10) \\
& && m_i(j), y_i(j), \alpha_i(j), \beta_i(j) \in \{0, 1\} && \forall i \in \mathcal{B}, j \in \mathcal{S} && (11) \\
& && p(j) \geq 0 && \forall j \in \mathcal{S} && (12)
\end{aligned} \tag{q-BC}$$

An assignment of buyer i to seller j is denoted as a binary variable $m_i(j)$, where the equivalence to the previous definitions is $m_i(j) = 1 \Leftrightarrow \mu(i) = j$. In order to check for the existence of deviating coalitions, two additional binary variables are introduced. Setting $\alpha_i(j) = 0$ represents the case where bidder i has a benefit from deviating by trading with seller j , and $\alpha_i(j) = 1$ means that the

bidder is best satisfied under the current assignment. The second helper binary variable is set to $\beta_i(j) = 0$ if i possesses a sufficient amount of money to purchase j , and set to 1 otherwise. Variable $y_i(j) = 0$ reflects the case where bidder i prefers to trade with seller j and has sufficient budget, and the variable is set to 1 if one of the two necessary conditions does not hold.

Variable r_j describes the reserve value or ask price of seller j . Constraints (1) and (2) represent the utilities of buyers and sellers, respectively. Constraints (5) and (6) guarantee core-stability. Constraint (5) examines whether the corresponding payoff π_i a buyer i receives in the selected assignment m is higher or equal to the alternative assignment (i, j) in question, in which case $\alpha_i(j) = 0$. Constraint (6) tests whether a seller j 's payoff π_j on the optimal matching m is higher or equal to the minimum value between any buyer i 's budget constraint b^i and i 's valuation for the item $v_i(j)$, which represents the maximum possible payment seller j could receive from any buyer. One or both of these conditions need to be true. Constraints (7) and (8) examine whether bidder i has a sufficient budget to obtain item j under price p_j . Constraint (9) is responsible for handling the value of $y_i(j)$ in an appropriate manner, to reflect whether a deviating coalition of (i, j) is indeed profitable and budget-feasible, for any positive value $\epsilon < 1$. The value of $y_i(j)$ depends on binary values $\alpha_i(j), \beta_i(j)$, and we verify our claim by examining the inequalities formed by the different value combinations:

$\alpha_i(j)$	$\beta_i(j)$	$y_i(j)$	Inequality
0	0	0	$1 + 1 - 2 \leq 2 \cdot 1 - \epsilon \cdot 0$ (1*)
0	0	1	$1 + 1 - 2 \leq 2 \cdot 0 - \epsilon \cdot 1$ (2*)
0	1	1	$1 + 0 - 2 \leq 2 \cdot 0 - \epsilon \cdot 1$ (3*)
1	0	1	$0 + 1 - 2 \leq 2 \cdot 0 - \epsilon \cdot 1$ (4*)
1	1	1	$0 + 0 - 2 \leq 2 \cdot 0 - \epsilon \cdot 1$ (5*)

In cases (1*) and (2*), agent i has a sufficient budget and can profit from deviating. However, inequality (2*) is infeasible, therefore the value of $y_i(j)$ cannot be set to 1 for this combination of $\alpha_i(j), \beta_i(j)$ and is forced to 0. For the remaining cases, either i does not have sufficient budget ($\beta_i(j) = 1$), or has no profit from trading with j ($\alpha_i(j) = 1$), or both conditions hold. In all the aforementioned cases, $y_i(j) = 1$, and thus reflects the case where no deviation is preferable from the buyer's side.

Constraint (10) then makes sure that if an item j is assigned to buyer i , then the price is less than the minimum of the budget of this buyer or his value, and it is higher than the reserve price of the seller. We can conclude that the above formulation always results in a the welfare-maximizing core-stable outcome for assignment markets with budget constraints.

5.2 Complexity Analysis

We now proceed to the proof of NP-completeness. The existence of endogenous pricing variables renders the proof non-trivial: there is no standard method of encoding both continuous and discrete variables, for prices and assignment, in a problem instance. Therefore, the proof requires special assumptions regarding the handling of item prices. We define the decision version of the problem.

Maximum Welfare Budget Constrained Stable Bipartite Matching (MBSBM)

Input: Two disjoint sets $\mathcal{S}$ (sellers) and $\mathcal{B}$ (buyers) of n agents each, a budget b^i for each $i \in \mathcal{B}$ and a reserve value r_j for each $j \in \mathcal{S}$, a value $v_i(j)$ for each pair of agents i, j , and a non-negative integer k .

Output: Boolean value

Question: Does there exist a stable outcome μ with total value $\sum_{i \in \mathcal{B}} (v_i(\mu(i)) - r_{\mu(i)}) \geq k$?

The problem belongs to class NP, since MBSBM can be modeled as a MILP where all the reserve values r_j of sellers $j \in \mathcal{S}$ are set to 0 [9]. For the case when all budgets are too high and therefore non-binding, $b^i \geq v_i(j)$ for all $i \in \mathcal{B}$ and $j \in \mathcal{S}$, the problem is equivalent to the maximum-weight bipartite matching, which admits a polynomial time solution via the Hungarian algorithm [16]. Core-stable prices can be derived from the duals of the corresponding linear program [22]. Therefore, the case of interest that can result in increased computational cost is when budgets are binding for participating buyers. In our problem, we reduce from the Maximum Independent Set (MIS) problem, known to be APX-hard, thus implying NP-hardness. Chen et al. [8] follow a similar approach for fractional matching without transferable utility. However, an environment with partially transferable utility as in our case, requires special attention and leads to significant differences.

Maximum Independent Set (MIS)

Input: A graph $G = (V, E)$, with vertices V and edges E , and a non-negative integer k .

Output: Boolean value

Question: Does there exist an Independent Set (IS) of size at least k , where as IS we define a set of vertices no two of which are adjacent?

The construction and proofs are discussed in detail in the Appendix. The following hardness results concludes our computational complexity analysis of the problem.

Theorem 3. MBSBM is NP-complete.

6 Conclusions

In this work, we showed that there is no incentive-compatible mechanism that selects an outcome in the core for bidders with unit-demand valuations. However, it is possible to expand the auction algorithm by Demange et al. [10], based on demand queries, which terminates with a core allocation if bidders truthfully reveal their demand set in each round. With an additional condition on the unit-demand valuations, this mechanism is incentive-compatible and maximizes welfare. It is typically unknown ex-ante whether this condition holds.

Overall, the analysis of markets with unit-demand bidders is important as it aids our understanding about how restrictive the assumptions need to be for a market to be efficient. In his seminal paper, Vickrey [24] showed that markets can be designed such that it is a dominant strategy for participants to reveal their preferences truthfully. He allows for general valuations (including substitutes and complements) and only poses a seemingly innocuous assumption of payoff-maximizing bidders. Budget constraints are wide-spread and violate the assumptions of the VCG mechanism. We prove that, even in the simplest markets where bidders only have unit-demand valuations, incentive-compatibility and core-stability are conflicting. In fact, even if all unit-demand valuations and budgets were known to the auctioneer, computing a core-stable and efficient outcome is NP-complete. As a result, claims about the efficiency of simple (polynomial-time) market designs need to be considered with care in the many markets where financial constraints of bidders play a role.

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A Appendix

A.1 Proof of Lemma 1

Proof. Suppose first that (μ, p) is a core outcome. Set $R_i = \{j \in \mathcal{S} : p(j) < b^i\} \cup \{\mu(i)\}$. Then $\mu(i) \in D_i(p, R_i)$, since otherwise there would exist an item j with $p(j) < b^i$ generating a higher utility than $\mu(i)$ - this would constitute a blocking pair.

Now let's assume that there are sets R_i as described with $\mu(i) \in D_i(p, R_i)$ for all i . Suppose there is a blocking pair (i, j) . Then j costs strictly less than b^i , so $j \in R_i$, and j generates a higher utility than $\mu(i)$. This would contradict $\mu(i) \in D_i(p, R_i)$. Thus, (μ, p) is a core outcome.

A.2 Proof of Lemma 2

Proof. Since $O \setminus T \subseteq O$ and O is minimally overdemanded, $O \setminus T$ is not overdemanded, so

$$|O| - |T| = |O \setminus T| \geq |\{i : D_i(p, R_i) \subseteq O \setminus T\}|.$$

Now

$$\{i : D_i(p, R_i) \subseteq O \setminus T\} = \{i : D_i(p, R_i) \subseteq O\} \setminus \{i : D_i(p, R_i) \subseteq O \wedge D_i(p, R_i) \cap T \neq \emptyset\}$$

so

$$|O| - |T| \geq |\{i : D_i(p, R_i) \subseteq O\}| - |\{i : D_i(p, R_i) \subseteq O \wedge D_i(p, R_i) \cap T \neq \emptyset\}|.$$

By rearranging terms we get

$$|\{i : D_i(p, R_i) \subseteq O \wedge D_i(p, R_i) \cap T \neq \emptyset\}| \geq |\{i : D_i(p, R_i) \subseteq O\}| - |O| + |T|$$

and since O is overdemanded, this implies

$$|\{i : D_i(p, R_i) \subseteq O \wedge D_i(p, R_i) \cap T \neq \emptyset\}| > |T|.$$

A.3 Proof of Lemma 3

Proof. Assume to the contrary that there is a minimal iteration $t + 1$, such that a bidder i^* and a good j^* exist with $p^{t+1}(j^*) < b^{i^*}$, but $j^* \notin R_{i^*}^{t+1}$. Then in iteration t , Step 3 was executed, since otherwise $p^t \leq p^{t+1}$ and $R_{i^*}^t = R_{i^*}^{t+1}$, so $t + 1$ would not be minimal. Hence, in iteration t , we have $j^* \in J_{i^*}^t$ and in particular $j^* \in D_{i^*}(p^{t-1}, R_{i^*}^{t-1})$. Because Step 3 is executed, we have $O^{t-1} \neq \emptyset$, so in iteration $t - 1$ Step 4 was executed and $p^t(j^*) = p^{t-1}(j^*) + 1 = p^{t+1}(j^*) + 1 \leq b^{i^*}$. Thus, since from iteration $t - 1$ to t , all prices for all preferred goods of bidder i^* were raised and i^* can still afford j^* at prices p^t , $j^* \in D_{i^*}(p^t, R_{i^*}^t)$, so $j^* \notin J_{i^*}^t$. This is a contradiction.

A.4 Proof of Proposition 2

Proof. We prove this by induction on t . For $t = 1$, there clearly is no underdemanded set of items, and Step 3 is not executed.

Suppose now that $t > 1$ and that the statement is true for all $1 \leq s < t$.

First suppose that there exists an underdemanded set of items T . Therefore, by induction, in iteration $t - 1$, Step 4 must have been executed - otherwise, there would not exist an underdemanded set. But then, using the same inductive reasoning, there was no underdemanded set in iteration $t - 1$. It is thus easy to see that, since in iteration $t - 1$ only prices for items in O^{t-1} were raised,

only the demand for those items could decrease, so T must be a subset of O^{t-1} . By Lemma 2, we have

$$|\{i \in \mathcal{B} : D_i(p^{t-1}, R_i^{t-1}) \subseteq O^{t-1} \cap D_i(p^{t-1}, R_i^{t-1}) \cap T \neq \emptyset\}| > |T|.$$

Thus, since $|\{i \in \mathcal{B} : D_i(p^t, R_i^t) \cap T \neq \emptyset\}| < |T|$, there must be a bidder i^* with $D_{i^*}(p^{t-1}, R_{i^*}^{t-1}) \subseteq O^{t-1}$ and $D_{i^*}(p^{t-1}, R_{i^*}^{t-1}) \cap T \neq \emptyset$, but $D_{i^*}(p^t, R_{i^*}^t) \cap T = \emptyset$. This implies that $i^* \in I^t$, so Step 3 is executed in iteration t .

Now suppose that Step 3 is executed in iteration t . Then again, in iteration $t-1$, Step 4 was executed, since otherwise we would have $O^{t-1} = \emptyset$, which implies $I^t = \emptyset$. By induction, there was no underdemanded set of items in iteration $t-1$. Note that $p^{t-1} = p^{t+1}$, so only the demand of a single bidder $i^* \in I^t$ chosen in Step 3 does change. Since $D_{i^*}(p^{t-1}, R_{i^*}^{t-1}) \subseteq O^{t-1}$, so $J_{i^*}^t \subseteq O^{t-1}$, only the demand for items in O^{t-1} can decrease. However, for $T \subseteq O^{t-1}$ we have again by Lemma 2 that

$$|\{i \in \mathcal{B} : D_i(p^{t-1}, R_i^{t-1}) \subseteq O^{t-1} \cap D_i(p^{t-1}, R_i^{t-1}) \cap T \neq \emptyset\}| > |T|,$$

and, since we only changed $R_{i^*}^{t-1}$, the demand for items in T can at most decrease by 1. Thus, T is not underdemanded in iteration $t+1$.

A.5 Proof of Proposition 3

Proof. Whenever Step 3 is executed, at least one item is removed from the set R_i^t of one bidder. Hence, Step 3 can only be called a finite number of times. Also, prices can only be increased a finite number of times in Step 4 - if prices of goods go to infinity, they are clearly not overdemanded at some point anymore. Thus, in some iteration t^* , Step 5 is executed. By Lemma 2, there is no underdemanded set in iteration t^* , because otherwise Step 3 would have been executed. Similarly, there is no overdemanded set. Finally, because of Lemma 3, no set $R_i^{t^*}$ is empty, so by Proposition 1, an assignment μ as required exists. By Lemma 1, (μ, p) is a core outcome.

In the following we need a simple auxiliary lemma.

Lemma 4. *Let (ν, q) be an arbitrary outcome, and let $O \subseteq \mathcal{S}$ be a minimally overdemanded set with respect to the demand sets $D_i(q, R_i)$. Let $\emptyset \neq J \subsetneq O$. Then there exists a bidder i^* with $D_{i^*}(q, R_{i^*}) \subset O$, $D_{i^*}(q, R_{i^*}) \cap J \neq \emptyset$ and $\nu(i^*) \notin J$.*

A.6 Proof of Proposition 4

Proof. We prove the following statements by induction on the iteration t , which imply Proposition 4.

- If in iteration t Step 4 or 5 is executed, then $p^t \leq q$.
- If in iteration t Step 3 is executed, then $p^{t+1} \leq q$ and we can choose bidder $i \in I^t$ such that $\nu(i) \notin J_i^t$, and consequently, by choosing this bidder, we have $\nu(i) \in R_i^{t+1}$ for all i .

For iteration $t = 1$ this is obviously true, since only Step 4 or Step 5 can be executed, and $p^1 = (0, \dots, 0) \leq q$.

Now suppose that $t > 1$. First assume that in iteration t Step 4 or 5 is executed. If in Step $t-1$ Step 3 was executed, we have that $p^t = p^{t-2}$, and since Step 3 cannot be executed twice in a row, we have by induction $p^t = p^{t-2} \leq q$. Now assume that Step 4 was executed in iteration $t-1$. Then by induction we have $p^{t-1} \leq q$. Towards a contradiction, assume that the set $J = \{j \in O^{t-1} : p^t(j) > q(j)\}$ is not empty. Note that for all $j \in J$ we must have $q(j) = p^{t-1}(j)$. By Lemma 4 there

exists a bidder i^* with $D_{i^*}(p^{t-1}, R_{i^*}^{t-1}) \subset O^{t-1}$, $D_{i^*}(p^{t-1}, R_{i^*}^{t-1}) \cap J \neq \emptyset$ and $\nu(i^*) \notin J$. Thus, we have $\nu(i^*) \notin O^{t-1}$ or $p^t(j) \leq q(j)$. Since $\nu(i^*) \in R_{i^*}^{t-1}$, we have for every $j \in D_{i^*}(p^{t-1}, R_{i^*}^{t-1}) \cap J$ that $\pi_{i^*}(\nu(i^*), q) < \pi_{i^*}(j, q)$. Consequently, since (ν, q) is a core outcome, so (i^*, j) is no blocking pair, we must have that $q(j) = b^{i^*}$. But since $q(j) = p^{t-1}(j)$ and $j \in O^{t-1}$, it would follow that $j \notin D_{i^*}(p^t, R_{i^*}^t)$, so $I^t \neq \emptyset$ and Step 3 is executed in iteration t . This contradicts our assumption that Step 4 is executed.

Now consider the case where Step 3 is executed in iteration t . Since Step 3 cannot be executed twice in a row, we have by induction that $p^{t+1} = p^{t-1} \leq q$. It remains to show that there is some bidder $i \in I^t$ with $\nu(i) \notin J^t$. Again, towards a contradiction, assume that for all $i \in I^t$ we have $\nu(i) \in J^t$. Hence, $p^{t-1}(\nu(i)) = b^i$ for all $i \in I^t$, and since $q \geq p^{t-1}$ by induction, we have $q(\nu(i)) = b^i$ for all $i \in I^t$. Our argument is now very similar to the one above: Consider the set $J = \{j \in O^{t-1} : q(j) = p^{t-1}(j)\}$. Then $\nu(I^t) \subseteq J$, and in particular J^t is nonempty. Hence there is a bidder i^* with $D_{i^*}(q, R_{i^*}) \subset O$, $D_{i^*}(q, R_{i^*}) \cap J \neq \emptyset$ and $\nu(i^*) \notin J$. Again, we have $\nu(i^*) \notin O^{t-1}$, or $q(\nu(i^*)) > p^{t-1}(\nu(i^*))$. In both cases, i^* would prefer any good j in the intersection $D_{i^*}(q, R_{i^*}) \cap J \neq \emptyset$ to $\nu(i^*)$ at prices q . But since $i^* \notin I^t$, $b^{i^*} > q(j)$, so (i^*, j) would form a blocking pair.

$$I = \{i \in \mathcal{B} : D_i(p^{t-1}, O^{t-1}) \subseteq O^{t-1} \wedge \nu(i) \in O^{t-1} \wedge p^{t-1}(\nu(i)) = q^{t-1}(\nu(i))\}.$$

Then $I^t \subseteq I$, so I is not empty. By Lemma 4, there is a bidder $i^* \notin I$ with $D_{i^*}(p^{t-1}, R_{i^*}^{t-1}) \subseteq O^{t-1}$ and $\nu(I) \cap D_{i^*}(p^{t-1}, R_{i^*}^{t-1}) \neq \emptyset$. The $\nu(i^*) \notin O^{t-1}$, or $q(\nu(i^*)) > p^{t-1}(\nu(i^*))$. Since by induction we have $\nu(i^*) \in R_{i^*}^{t-1}$, we have that $\pi_{i^*}(\nu(i^*), q) < \pi_{i^*}(j, q)$ for all $j \in \nu(I) \cap D_{i^*}(p^{t-1}, R_{i^*}^{t-1})$. But since (ν, q) is a core outcome, for no such j , (i^*, j) can be a blocking pair - implying that $b^{i^*} = q(j) = p^{t-1}(j)$. This is a contradiction, since it would follow that $i^* \in I$.

A.7 Proof of Corollary 1

Proof. Since $|I^t| = 1$ in every iteration through Step 3, the outcome (μ, p) of the auction is unique. Proposition 4 now directly implies that for every core outcome (ν, q) , we have $\pi_i(\mu(i), p) \geq \pi_i(\nu(i), q)$ and $p \leq q$.

A.8 Proof of Proposition 5

Proof. Consider the undirected bipartite graph G with vertex set $I \dot{\cup} O^{t-1}$, where $I = \{i \in \mathcal{B} : D_i(p^{t-1}, R_{i^*}^{t-1}) \subseteq O^{t-1}\}$, containing the edge $\{i, j\}$ if and only if $j \in D_i(p^{t-1}, R_{i^*}^{t-1})$. Since O^{t-1} is minimally overdemanded, this graph is connected by Lemma 4. For the sake of clarity, we denote the bidders in I by $1, \dots, r$, and the goods in O^{t-1} by $1', \dots, s'$, where $r = |I|$ and $s = |O^{t-1}|$.

Let $J = \{j' \in O^{t-1} : \exists i \in I : j' \in D_i(p^{t-1}, R_{i^*}^{t-1}) \wedge p(j') = b^i\}$ be the set of all goods j' demanded by some bidder in I with tight budget for j' . We show that $|J| = 1$. To that goal, assume that this is not the case. Then, after a possible re-indexing, there exists a path

$$j_1 \rightarrow i_1 \rightarrow j_2 \rightarrow \dots \rightarrow i_{m-1} \rightarrow j_m$$

alternating between goods and bids in G , where j_1 and j_m are distinct goods contained in J . Then by following the corresponding forward- and backward edges according to Definition 3, the total weight of this walk is

$$\sum_{k=1}^{m-1} v_{i_k}(j_k) - v_{i_k}(j_{k+1}).$$

For all $k \in \{1, \dots, m-1\}$, we have that $\pi_{i_k}(j_k, p^{t-1}) = \pi_{i_k}(j_{k+1}, p^{t-1})$, which is equivalent to $v_{i_k}(j_k) - v_{i_k}(j_{k+1}) = p^{t-1}(j_k) - p^{t-1}(j_{k+1})$. Thus, the weight of the above walk is equal to $p^{t-1}(j_1) - p^{t-1}(j_m)$. Now let $i_0, i_m \in I$ be bidders demanding j_1 , respectively j_m with $b^{i_0} = p(j_1)$ and $b^{i_m} = p(j_m)$. Let us consider two walks starting from i_0 : the first one consists of a single maximum-price edge $i_0 \rightarrow j_1$ and has weight $b^{i_0} - v_{i_0}(j_1)$. The second one is

$$i_0 \rightarrow j_1 \rightarrow i_1 \rightarrow j_2 \rightarrow \dots \rightarrow i_{m-1} \rightarrow j_m \rightarrow i_m \rightarrow j_m$$

where the last edge is a maximum-price edge. This walk has weight

$$-v_{i_0}(j_1) + (p^{t-1}(j_1) - p^{t-1}(j_m)) + v_{i_m}(j_m) + (b^{i_m} - v_{i_m}(j_m)).$$

Since $b^{i_0} = p(j_1)$ and $b^{i_m} = p(j_m)$, this is equal to $b^{i_0} - v_{i_0}(j_1)$. Thus, we found two distinct paths with the same total weight, so the auction is not in general position. It follows that $|J| = 1$.

It remains to show that $|I^t| = 1$. To that goal, assume that $|I^t| > 1$ and let $i_1, i_2 \in I^t$ be distinct. Since $|J| = 1$, there is a good $j \in O^{t-1}$, demanded by both i_1 and i_2 with $p(j) = b^{i_1} = b^{i_2}$. The walk $i_1 \rightarrow j$ consisting of a single maximum-price edge has weight $b^{i_1} - v_{i_1}(j)$, and the walk $i_1 \rightarrow j \rightarrow i_2 \rightarrow j$ ending with a maximum-price edge, also has weight $-v_{i_1}(j) + v_{i_2}(j) + (b^{i_2} - v_{i_2}(j)) = b^{i_1} - v_{i_1}(j)$. Thus, the auction is not in general position, a contradiction. Hence, $|I^t| = 1$, and since $J = |1|$, the good $j \in D_i(p^{t-1}, R_i^{t-1})$ with $p(j) = b^i$ for the bidder $i \in I^t$ is unique.

A.9 Proof of Proposition 6

Proof. We have that

$$\sum_{i \in \mathcal{B}} v_i(\mu(i)) = \sum_{i \in \mathcal{B}} \pi_i(\mu(i), p) + \sum_{j \in \mathcal{S}} p(j) \geq \sum_{i \in \mathcal{B}} \pi_i(\mu(i), p) + p(\nu(i))$$

and

$$\sum_{i \in \mathcal{B}} v_i(\nu(i)) = \sum_{i \in \mathcal{B}} \pi_i(\nu(i), q) + \sum_{j \in \mathcal{S}} q(j) = \sum_{i \in \mathcal{B}} \pi_i(\nu(i), q) + q(\nu(i))$$

since each good j with $q(j) > 0$ is assigned to some bidder by ν . Thus, it suffices to show that

$$\sum_{i \in \mathcal{B}} \pi_i(\mu(i), p) - \pi_i(\nu(i), q) + p(\nu(i)) - q(\nu(i)) \geq 0.$$

We show that each summand is non-negative by distinguishing two cases:

Case 1: $p(\nu(i)) = b^i$. Then, since bidder i receives $\nu(i)$ in the outcome (ν, q) , $q(\nu(i)) \leq b^i = p(\nu(i))$. Since by assumption $\pi_i(\mu(i), p) \geq \pi_i(\nu(i), q)$, it follows that $\pi_i(\mu(i), p) - \pi_i(\nu(i), q) + p(\nu(i)) - q(\nu(i)) \geq 0$.

Case 2: $p(\nu(i)) < b^i$. Then $\pi_i(\mu(i), p) \geq \pi_i(\nu(i), p)$ - otherwise, $(i, \nu(i))$ would be a blocking pair with respect to prices p . Since $\pi_i(\nu(i), p) = \pi_i(\nu(i), q) + q(\nu(i)) - p(\nu(i))$, we again get that $\pi_i(\mu(i), p) - \pi_i(\nu(i), q) + p(\nu(i)) - q(\nu(i)) \geq 0$.

A.10 Proof of Theorem 2

Proof. By the direct revelation principle, we may assume that bidders report their exact valuations, as well as their budgets to the auctioneer. Consider a market with three bidders 1, 2, 3 and two items

A, B . Let $\mathcal{M}((v_1, b^1), \dots, (v_3, b^3)) = (\mu, p)$ denote a mechanism, mapping the bidders' reported valuations and budgets to a core-stable outcome with respect to their reports.

We consider instances of the above described market, where all bidders have the same values for both items: $v_i(A) = v_i(B) = 10$ for $i = 1, 2, 3$. Let us consider two instances, where the bidders vary their reported budget.

1. If all bidders report $b^i = 1$ for $i = 1, 2, 3$, then obviously, since there are only two items, one bidder does not receive one: for $(\mu, p) = \mathcal{M}((v_1, 1), (v_2, 1), (v_3, 1))$, there is an i with $\mu(i) = 0$. Without loss of generality, we assume that $i = 3$. It is easy to see that for core-stability to hold, bidders 1 and 2 both receive an item, and that $p(A) = p(B) = 1$. Bidders 1 and 2 have utility 9, while bidder 3 has utility 0.
2. If bidder 3 reports $b^3 = 2$, and the other bidders report $b^1 = b^2 = 1$, then clearly bidder 3 receives an item in any core-stable outcome, and without loss of generality $\mu(3) = B$. Also the other item A must necessarily be assigned to some bidder. Again, without loss of generality, we assume that $\mu(1) = A$ and $\mu(2) = 0$. It is easy to see that $p(A)$ must be equal to 1 in a core outcome. Additionally, we must have $p(A) = p(B)$, since otherwise bidder 3 would strictly prefer item A to item B , which would not be envy-free. Thus, $p(A) = p(B) = 1$, and bidder 3 has a utility of 9.

This already shows that $\mathcal{M}$ is not incentive-compatible: If all bidders' true budgets are equal to 1 and they report truthfully, bidder 3 has a utility of 0. However, if bidder 3 misreports $b^3 = 2$, they would receive an item and have a utility of 9. Note, that $p(B) = p(A) = 1$ in this case, so bidder 3 can still afford the received item.

A.11 Construction

Assume an instance of MIS defined on a cubic graph $G = ((V, E), k)$. A cubic graph is a graph in which all vertices have degree three. We define the transformed instance as a bipartite graph $G' = (V', E')$, with $V' = \mathcal{B} \cup \mathcal{S}$, and functions $\pi_i : \mathcal{S} \rightarrow \mathbb{R}_{\geq 0}$ for each buyer $i \in \mathcal{B}$, and $\pi_j : \mathcal{B} \rightarrow \mathbb{R}_{\geq 0}$ for each seller $j \in \mathcal{S}$ that represents agents' payoffs.

- V' represents the total set of agents
- $\mathcal{B}$ and $\mathcal{S}$ denote the sets of buyers and sellers respectively
- E' represents the potential transactions between buyers and sellers
- π_i specifies the difference between the true valuation of buyer $i \in \mathcal{B}$ for the item of seller $j \in \mathcal{S}$ and her total budget b^i , namely $v_i(j) - b^i$
- π_j specifies difference between the budget of buyer $i \in \mathcal{B}$ and the reserve value of seller $s \in \mathcal{S}$, namely $b^i - r_j$

Since we assume that all assigned buyers pay prices for items equal to their budgets, we omit prices from the payoff formulae π_i, π_j . In Lemma 7 we show that this restriction of the prices to be at the budgets is without loss of generality for the construction. Observe that in our construction multiple buyers share the same budget constraint, which violates the general position condition from Definition 3.

An *assignment* $\mu : \mathcal{B} \rightarrow \mathcal{S}$ in G' assigns each edge $e \in E'$ according to the condition $\sum_{j \in \mathcal{S}} |\mu(i) = j| \leq 1$ for any agent $i \in \mathcal{B}$. The *utility* of agent $i \in \mathcal{B}$ under assignment μ of (G', π_i, π_j) is defined as $\pi_i(\mu) := \pi_i(\mu(i))$ and similarly for agent $j \in \mathcal{S}$ it holds that $\pi_j(\mu) := \pi_j(\mu^{-1}(j))$. Given an assignment μ , an edge $(i, j) \in E'$ is a *blocking pair/ edge* if $\pi_i(\mu(i)) < \pi_i(j)$ and $\pi_j(\mu^{-1}(j)) < \pi_j(i)$.

An assignment μ is *stable* if it does not contain any blocking pair of agents.

Edge Gadget. Starting from an edge $e \in E$ of the original graph G , we construct the edge gadget $G'_e = \mathcal{B}_e \cup \mathcal{S}_e$ as a bipartite graph, with preferences for the subgraph defined as mentioned above and noted as $\pi_i : \mathcal{S}_e \rightarrow \mathbb{R}_{\geq 0}$ and $\pi_j : \mathcal{B}_e \rightarrow \mathbb{R}_{\geq 0}$ for each of the two sets respectively. The vertices represent participating *agents*.

For each edge $e = (u, u') \in E$, we proceed to the following construction:

- Add to $\mathcal{B}_e$ three agents $\beta_e, \gamma_e, \delta_e$
- Add to $\mathcal{S}_e$ three agents $\eta_e, \alpha_e^u, \alpha_e^{u'}$
- Add to $\mathcal{B}_e$ two extra agents ϵ_u and $\epsilon_{u'}$, if not present already. These represent the connection from edge gadget to vertex gadget.

Vertices α_e^u and $\alpha_e^{u'}$ represent the gates to the vertex gadgets, which are connected to the vertices $\epsilon_u, \epsilon_{u'}$ of the vertex gadgets of u, u' of the original graph.

For each edge $e \in E$ of the original graph G , the corresponding edge gadget consists of subgraph that agents $\{\beta_e, \gamma_e, \delta_e\} \cup \{\eta_e, \alpha_e^u, \alpha_e^{u'}\}$ induce. The detailed construction is depicted in Figure 1. Any edge not present in the figure is assigned a value of zero.

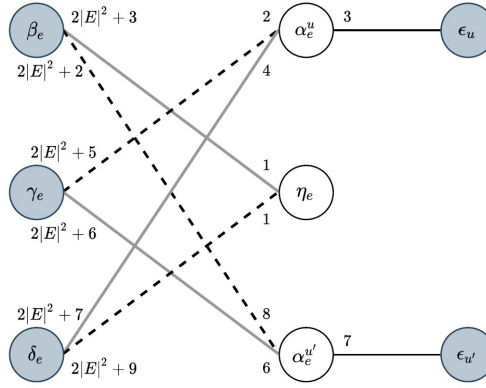


Fig. 1: The edge gadget, with corresponding buyer and seller preferences π_i, π_j . If u belongs to the independent set $IS(G)$, then we integrally match the pairs corresponding to the solid edges, otherwise we match the dashed edges.

	α_e^u	η_e	$\alpha_e^{u'}$	b^i
β_e	0	$2 E ^2 + 12$	$2 E ^2 + 11$	9
γ_e	$2 E ^2 + 12$	0	$2 E ^2 + 13$	7
δ_e	$2 E ^2 + 16$	$2 E ^2 + 18$	0	9
ϵ_u	3	0	0	8
$\epsilon_{u'}$	0	0	7	8
r_j	5	8	1	

Table 1: Valuation table representing valuations $v_i(j)$ where $i \in \{\beta_e, \gamma_e, \delta_e, \epsilon_u, \epsilon_{u'}\}$ and $j \in \{\alpha_e^u, \eta_e, \alpha_e^{u'}\}$. b^i corresponds to the budget of each buyer and r_j to the reserve value of each seller.

An important observation is, that any edge gadget G'_e should contain two feasible stable assignments, with an equal total social welfare. In every stable assignment μ , at least one of $\{\alpha_e^u, \alpha_e^{u'}\}$ is

unsatisfied under μ , namely agents α_e^u or $\alpha_e^{u'}$ have a preference towards ϵ_u or $\epsilon_{u'}$ respectively.

Vertex Gadget. In a similar manner, we construct the vertex gadget corresponding to vertex $u \in V$.

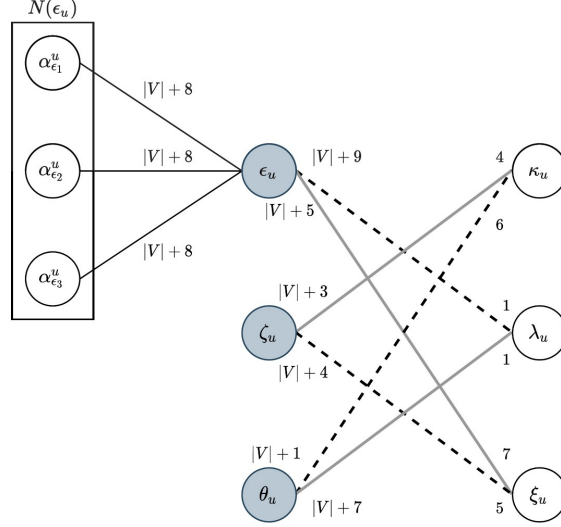


Fig. 2: The vertex gadget, with corresponding buyer and seller preferences. If u belongs to the independent set $IS(G)$, then we integrally match the pairs corresponding to the solid edges, otherwise, we match the dashed edges.

	κ_u	λ_u	ξ_u	$N(\epsilon_u)$	b^i
ϵ_u	0	$ V + 17$	$ V + 13$	$n + 16$	8
ζ_u	$ V + 9$	0	$ V + 10$	0	6
θ_u	$ V + 9$	$ V + 15$	0	0	8
r_j	2	7	1	5/1	

Table 2: Valuation table representing valuations $v_i(j)$ where $i \in \{\epsilon_u, \zeta_u, \theta_u\}$ and $j \in \{\kappa_u, \lambda_u, \xi_u, N(\epsilon_u)\}$. $N(\epsilon_u)$ consists of the neighboring vertices of ϵ_u in the edge gadgets of $\alpha_{\epsilon_1}^u, \alpha_{\epsilon_2}^u, \alpha_{\epsilon_3}^u$, since each vertex u has a degree of 3. b^i corresponds to the budget of each buyer and r_j to the reserve value of each seller in the vertex gadget.

Figure 2 illustrates the vertex gadget G'_u corresponding to a vertex $u \in V$, along with the preferences of buyers and sellers. Agents colored in gray belong to set $\mathcal{B}_u$, and agents in white belong to $\mathcal{S}_u$, defined in an analogous manner to $\mathcal{B}_e, \mathcal{S}_e$. Two feasible stable outcomes exist in G'_u , resulting in welfare that differs by 1: in case an assignment indicated by the solid edges is chosen, induced welfare equals $|V| + 27$, while for an assignment of the dashed edges, welfare has a value of $|V| + 26$. The set $N(\epsilon_u)$ represents the neighborhood of ϵ , and consists of three vertices from three distinct edge gadgets.

The following lemma provides a useful bound for the achieved social welfare of an optimal assignment in an edge gadget.

Lemma 5. *Let μ be a stable assignment for edge gadget G'_e . The total welfare SW achieved by μ is at most $|E| \cdot (6|E|^2 + 27)$. If there is an edge $(u, u') \in E$, where for the gadgets it holds that $\pi_{\epsilon_u}(\mu(\epsilon_u)) < \pi_{\epsilon_u}(\alpha_e^u)$ and $\pi_{\epsilon_{u'}}(\mu(\epsilon_{u'})) < \pi_{\epsilon_{u'}}(\alpha_e^{u'})$, then $SW < |E| \cdot (6|E|^2 + 27) - |V|$.*

Proof. We examine the edge gadget G'_e of edge $e = (u, u') \in E$, and define $\mathcal{B}_e := \{\beta_e, \gamma_e, \delta_e\}$ as the set of buyers and $\mathcal{S}_e := \{\alpha_e^u, \eta_e, \alpha_e^{u'}\}$ as the set of sellers. The total social welfare that can be achieved by agents in G'_e under assignment μ is $SW_e = \sum_{i \in \mathcal{B}_e} \pi_i(\mu(i)) + \sum_{j \in \mathcal{S}_e} \pi_j(\mu^{-1}(j))$.

For each individual edge gadget G'_e , we prove that the maximum total achieved welfare is bound by

$$SW_e \leq 3 \cdot 2|E|^2 + 27$$

We prove the claim as follows:

$$\begin{aligned} SW_e &= \sum_{\substack{(i,j) \in \mathcal{B}_e \times \mathcal{S}_e \\ (i,j) \in E(G_e)}} \left(|\mu(i) = j| \cdot (\pi_i(j) + \pi_j(i)) \right) + \left(|\mu(\epsilon_u) = \alpha_e^u| \cdot \pi_{\alpha_e^u}(\epsilon_u) \right) + \left(|\mu(\epsilon_{u'}) = \alpha_e^{u'}| \cdot \pi_{\alpha_e^{u'}}(\epsilon_{u'}) \right) \\ &= \sum_{i \in \{\gamma_e, \delta_e\}} \left(|\mu(i) = \alpha_e^u| \cdot (\pi_i(\alpha_e^u) + \pi_{\alpha_e^u}(i)) \right) + \sum_{i \in \{\beta_e, \gamma_e\}} \left(|\mu(i) = \alpha_e^{u'}| \cdot (\pi_i(\alpha_e^{u'}) + \pi_{\alpha_e^{u'}}(i)) \right) \\ &\quad + \sum_{i \in \{\beta_e, \delta_e\}} \left(|\mu(i) = \eta_e| \cdot (\pi_i(\eta_e) + \pi_{\eta_e}(i)) \right) + \left(|\mu(\epsilon_u) = \alpha_e^u| \cdot \pi_{\alpha_e^u}(\epsilon_u) \right) + \left(|\mu(\epsilon_{u'}) = \alpha_e^{u'}| \cdot \pi_{\alpha_e^{u'}}(\epsilon_{u'}) \right) \\ &\leq 3 \cdot 2|E|^2 + 27 \end{aligned}$$

We observe that each seller η_e is connected to two buyer nodes inside the edge gadget G'_e , therefore, under assignment μ , strictly one of conditions $|\mu(\beta_e) = \eta_e|$ and $|\mu(\delta_e) = \eta_e|$ returns a value of 1, while the other is set to 0. Sellers α_e^u and $\alpha_e^{u'}$ can be potentially matched to 3 different buyers: two of them within G'_e , and one belonging to the vertex gadget, u or u' respectively. We compute the upper bound on the total sum of valuations of sellers and buyers, on any possible assignment μ , maintaining that every time, one pair is matched, and the remaining conditions output 0 for all other edges connected to the pair. The resulting social welfare SW_e is equal to $3 \cdot 2|E|^2 + 27$ only for the case where assignment μ does not satisfy neither condition $|\mu(\epsilon_u) = \alpha_e^u|$ nor $|\mu(\epsilon_{u'}) = \alpha_e^{u'}|$.

We now proceed to bound the total welfare achieved under the condition that $\pi_{\epsilon_u}(\mu(\epsilon_u)) < \pi_{\epsilon_u}(\alpha_e^u)$ and $\pi_{\epsilon_{u'}}(\mu(\epsilon_{u'})) < \pi_{\epsilon_{u'}}(\alpha_e^{u'})$ hold simultaneously. Since μ is a stable outcome, from the first condition, we derive that

$$\pi_{\alpha_e^u}(\mu^{-1}(\alpha_e^u)) \geq \pi_{\alpha_e^u}(\epsilon_u) = 3 \quad (1)$$

Similarly, from the second condition, we get that

$$\pi_{\alpha_e^{u'}}(\mu^{-1}(\alpha_e^{u'})) \geq \pi_{\alpha_e^{u'}}(\epsilon_{u'}) = 7 \quad (2)$$

In order to ensure that Equation (3) holds, assignment μ needs to force node α_e^u to be matched with δ_e , as $\pi_{\alpha_e^u}(\delta_e) = 4 > 3$ in this case. If the pair $\{\gamma_e, \alpha_e^u\}$ was matched instead, it would hold that $\pi_{\alpha_e^u}(\gamma_e) = 2 < 3$, thus contradicting condition (3).

In a similar manner, for Equation (4), μ matches pair $\{\beta_e, \alpha_e^{u'}\}$, achieving utility $\pi_{\alpha_e^{u'}}(\beta_e) = 8 > 7$.

The total welfare of assignment μ is therefore:

$$\begin{aligned}
SW_e &= \sum_{\substack{(i,j) \in \mathcal{B}_e \times \mathcal{S}_e \\ (i,j) \in E(G_e)}} \left(|\mu(i) = j| \cdot (\pi_i(j) + \pi_j(i)) \right) + \left(|\mu(\epsilon_u) = \alpha_e^u| \cdot \pi_{\alpha_e^u}(\epsilon_u) \right) + \left(|\mu(\epsilon_{u'}) = \alpha_e^{u'}| \cdot \pi_{\alpha_e^{u'}}(\epsilon_{u'}) \right) \\
&= |\mu(\delta_e) = \alpha_e^u| \cdot (\pi_{\delta_e}(\alpha_e^u) + \pi_{\alpha_e^u}(\delta_e)) + |\mu(\beta_e) = \alpha_e^{u'}| \cdot (\pi_{\beta_e}(\alpha_e^{u'}) + \pi_{\alpha_e^{u'}}(\beta_e)) \\
&\leq 2|E|^2 + 11 + 2|E|^2 + 10 \\
&\leq 4|E|^2 + 22 \\
&< 6|E|^2 + 27 - |V|
\end{aligned}$$

since, without loss of generality, we can assume that $2|E|^2 - |V| + 5 > 0$, as we are referring to cubic graphs, where $3|V| = 2|E|$ and as a result the claim trivially holds.

This lemma implies that, in any welfare-maximizing assignment, edges between $\{\alpha_e^u, \epsilon_u\}$ and $\{\alpha_e^{u'}, \epsilon_{u'}\}$ are never chosen. Furthermore, at most one of the conditions $\pi_{\epsilon_u}(\mu(\epsilon_u)) < \pi_{\epsilon_u}(\alpha_e^u)$ and $\pi_{\epsilon_{u'}}(\mu(\epsilon_{u'})) < \pi_{\epsilon_{u'}}(\alpha_e^{u'})$ must hold, otherwise assignment μ is not maximizing welfare.

In order to prove NP-hardness for MBSBM, we need to provide an intermediate result concerning the existence of blocking pairs under certain assignments.

Lemma 6. *Consider edge $e = (u, u')$ of G , and let μ be an assignment for edge gadget G'_e . The following two statements hold:*

1. *If assignment μ satisfies $\mu(\beta_e) = \eta_e$, $\mu(\gamma_e) = \alpha_e^{u'}$ and $\mu(\delta_e) = \alpha_e^u$ (solid edges in Figure 1), then no blocking pair of μ involves any agent from $\{\beta_e, \gamma_e, \delta_e, \alpha_e^u, \eta_e\}$.*
2. *If assignment μ satisfies $\mu(\beta_e) = \alpha_e^{u'}$, $\mu(\gamma_e) = \alpha_e^u$ and $\mu(\delta_e) = \eta_e$ (dashed edges in Figure 1), then no blocking pair of μ involves any agent from $\{\beta_e, \gamma_e, \delta_e, \alpha_e^{u'}, \eta_e\}$.*

Proof. For statement (1), we assume outcome μ sets values as suggested. Computing the utilities under μ for each agent, we get:

$$\pi_{\beta_e}(\eta_e) = 2|E|^2 + 3, \quad \pi_{\gamma_e}(\alpha_e^{u'}) = 2|E|^2 + 6, \quad \pi_{\delta_e}(\alpha_e^u) = 2|E|^2 + 7$$

$$\pi_{\alpha_e^u}(\delta_e) = 4, \quad \pi_{\eta_e}(\beta_e) = 1$$

One can easily verify that agents $\beta_e, \gamma_e, \alpha_e^u$ are maximizing their utility under μ , and since agent η_e is indifferent between agents δ_e and β_e , they have no incentive to deviate by forming a blocking pair with agent δ_e . Therefore, there does not exist a blocking pair that includes agents $\{\beta_e, \gamma_e, \delta_e, \alpha_e^u, \eta_e\}$. For statement (2), we assume outcome μ sets values as suggested. Computing the utilities under μ for each agent, we get:

$$\pi_{\beta_e}(\alpha_e^{u'}) = 2|E|^2 + 2, \quad \pi_{\gamma_e}(\alpha_e^u) = 2|E|^2 + 5, \quad \pi_{\delta_e}(\eta_e) = 2|E|^2 + 9$$

$$\pi_{\alpha_e^{u'}}(\beta_e) = 8, \quad \pi_{\eta_e}(\delta_e) = 1$$

One can easily verify that agents $\delta_e, \alpha_e^{u'}$ are maximizing their utility under μ , and since agent η_e is indifferent between agents δ_e and β_e , they have no incentive to deviate by forming a blocking pair with agent β_e . Given that agent $\alpha_e^{u'}$ is assigned to agent β_e , and is maximizing their utility, agent γ_e is assigned to agent α_e^u , as this constitutes their best available choice. Therefore, there does not exist a blocking pair that includes agents $\{\beta_e, \gamma_e, \delta_e, \alpha_e^{u'}, \eta_e\}$.

In our construction, we fix the price for each matching to the budget constraint of the buyer to whom a seller is matched. Next, we show that this restriction is without loss of generality. We show in each bipartite graph, regardless of whether it belongs to the vertex or edge gadget, the current scheme yields all possible welfare-maximizing, core-stable outcomes.

Lemma 7. *Assuming that all buyers pay a price equal to their budget for assigned items does not impact generality, namely there does not exist a welfare-maximizing, core-stable outcome that is not reachable through this pricing scheme.*

Proof. We examine the edge and vertex gadgets separately, and argue that, in both cases, setting prices equal to the winning bidders' budgets yields all feasible welfare-maximizing, core-stable outcomes. Formally, the set of all welfare-maximizing, core-stable assignments coincides with the set of optimal assignments when prices are set at the budget limit.

In a two-sided matching the welfare is defined as the gains from trade, the value of the buyers minus that of the sellers. This means, for each match between buyer i and seller j , the corresponding welfare is computed as the sum of buyer $\pi_i(j) = v_i(j) - p_j$ and seller payoff $\pi_j(i) = p_j - r_j$, therefore prices are not included in the final sum, which is the result of the difference between assigned items' valuations and seller reserve values.

We begin by analyzing the edge gadget, as seen in Figure 1. The values depicted in Table 1 represent the true valuation of each buyer among $\{\beta_e, \gamma_e, \delta_e\}$ for the item of each seller among $\{\alpha_e^u, \eta_e, \alpha_e^{u'}\}$. One can trivially observe that there exist 6 feasible assignments between buyers and sellers in the bipartite graph. As stated above, prices do not participate in the welfare computation, and thus we can calculate the welfare-maximizing assignment based on the buyer valuation table. Since valuations $v_{\beta_e}(\alpha_e^u) = v_{\gamma_e}(\eta_e) = v_{\delta_e}(\alpha_e^{u'}) = 0$, only 2 among 6 assignments are stable, and simultaneously welfare-maximizing. The two assignments μ_1, μ_2 are $\mu_1(\beta_e) = \eta_e, \mu_1(\gamma_e) = \alpha_e^{u'}, \mu_1(\delta_e) = \alpha_e^u$ and $\mu_2(\beta_e) = \alpha_e^{u'}, \mu_2(\gamma_e) = \alpha_e^u, \mu_2(\delta_e) = \eta_e$. The total welfare admitted by assignments μ_1, μ_2 is equal to $6|E|^2 + 27$, while the remaining feasible assignments yield a strictly lower welfare. Additionally, according to Lemma 6, assignments μ_1, μ_2 (grey, dotted) with seller payoffs set as $\pi_{\alpha_e^u}(\delta_e) = b_{\delta_e} - r_{\alpha_e^u} = 9 - 5 = 4$, $\pi_{\eta_e}(\beta_e) = b_{\beta_e} - r_{\eta_e} = 9 - 8 = 1$, $\pi_{\alpha_e^{u'}}(\gamma_e) = b_{\gamma_e} - r_{\alpha_e^{u'}} = 7 - 1 = 6$ under assignment μ_1 , and $\pi_{\alpha_e^u}(\gamma_e) = b_{\gamma_e} - r_{\alpha_e^u} = 7 - 5 = 2$, $\pi_{\eta_e}(\delta_e) = b_{\delta_e} - r_{\eta_e} = 9 - 8 = 1$, $\pi_{\alpha_e^{u'}}(\beta_e) = b_{\beta_e} - r_{\alpha_e^{u'}} = 9 - 1 = 8$ under assignment μ_2 , do not generate any blocking pairs. This argument leads to the conclusion that, for the edge gadget and corresponding valuation table, setting prices for winning buyers equal to their budgets produces every feasible core-stable, welfare-maximizing assignment, namely setting prices to lower values cannot yield different stable assignments of higher or equal welfare.

In a similar analysis, we observe that, for the vertex gadget of Figure 2, the welfare-maximizing assignment μ_3 , based on valuations described on Table 2, is $\mu_3(\epsilon_u) = N(\epsilon_u), \mu_3(\zeta_u) = \xi_u, \mu_3(\theta_u) = \lambda_u$. However, as suggested in Lemma 5, assignment $\mu_0(\epsilon_u)$ results in sub-optimal welfare for the overall subgraph including vertex and edge gadget. Therefore, the welfare-maximizing assignment μ_4 is a result of an assignment between the vertices within the gadget. The aforementioned assignment is $\mu_4(\epsilon_u) = \xi_u, \mu_3(\zeta_u) = \kappa_u, \mu_3(\theta_u) = \lambda_u$ (solid assignment in Figure 2), admitting welfare equal to $3|V| + 27$. In Theorem 3, we have shown that assignment μ_4 , for winning buyer prices equal to their budgets is core-stable. Thus, the initial claim is true for each vertex gadget, concluding the proof of the lemma.

A.12 Proof of Theorem 3

Finally, we build the main complexity result for MBSBM. The proof uses no assumption on specific properties (such as general position) to hold.

Proof. Let $G = (V, E)$ be a cubic graph, with sizes of vertex and edge sets defined as $|V|$ and $|E|$ respectively. Since G is assumed to be cubic, it does not possess any isolated vertices. An instance of MIS is defined by G and an integer k .

A key property of cubic graphs is that all nodes must have a degree of 3. According to the *Handshaking lemma*, it holds that $\sum_{u \in V} \deg(u) = 2|E|$, namely the sum of degrees of all vertices of the graph is twice as large as than the size of the edge set.

We construct an instance $\langle G', \pi_i, \pi_j, SW \rangle$ of MBSBM, where $G' = (V', E')$ is a bipartite graph, with $SW = |V| \cdot (3|V| + 26) + k + |E| \cdot (6|E|^2 + 27)$. For each edge $e \in E$, buyer and seller sets of the corresponding edge gadget are defined as $\mathcal{B}_e, \mathcal{S}_e$. For each vertex $u \in V$, the respective sets for the corresponding vertex gadget are defined as $\mathcal{B}_u, \mathcal{S}_u$. Unifying for all edge gadgets in G' , we define $\mathcal{B}_E = \bigcup_{e \in E} \mathcal{B}_e, \mathcal{S}_E = \bigcup_{e \in E} \mathcal{S}_e$, and for all vertex gadgets, sets $\mathcal{S}_V = \bigcup_{u \in V} \mathcal{S}_u$ and $\mathcal{B}_V = \bigcup_{u \in V} \mathcal{B}_u$. Vertex set V' consists of the union over all vertex and edge gadgets, and therefore $\mathcal{B} = \mathcal{B}_E \cup \mathcal{B}_V$ and $\mathcal{S} = \mathcal{S}_E \cup \mathcal{S}_V$ represent the total number of buyers and sellers respectively. Each edge gadget corresponding to edge $e \in E$ consists of vertices $\mathcal{B}_e = \{\beta_e, \gamma_e, \delta_e\}$ and $\mathcal{S}_e = \{\alpha_e^u, \eta_e, \alpha_e^{u'}\}$. Similarly, the vertex gadget corresponding to vertex $u \in V$ consists of two disjoint sets of vertices defined as $\mathcal{B}_u = \{\epsilon_u, \zeta_u, \theta_u\}$ and $\mathcal{S}_u = \{\kappa_u, \lambda_u, \xi_u\}$. The vertex and edge gadgets are defined for each vertex $u \in V$ and each edge $e \in E$ of the original graph G . The total size of each set is $|\mathcal{B}| = |\mathcal{S}| = 6|E| + 6|V|$.

Formally, the following claim should be proven: G has an independent set $IS(G)$ of size at least k **if and only if** G' admits a stable outcome with welfare at least SW .

The transformation from an instance of MIS to an instance of MBSBM is performed in polynomial time. An important aspect of our construction is the assumption that, both in edge and vertex gadgets, there exist pairs of buyers with equal budgets. From Tables 1 and 2, one can observe that pairs of agents $\{\beta_e, \delta_e\}$ and $\{\epsilon_u, \theta_u\}$ verify the claim. Thus, the instance is not in general position, and therefore the hardness proof holds for cases where the property is violated.

We now proceed to the forward part of the proof, namely prove that, given an independent set $IS(G)$ of size at least k , we construct an assignment μ as follows:

- For each $u \in V$, if $u \in IS(G)$, set $\mu(\epsilon_u) = \xi_u, \mu(\zeta_u) = \kappa_u, \mu(\theta_u) = \lambda_u$.
- For each $u \in |V|$, if $u \notin IS(G)$, set $\mu(\epsilon_u) = \lambda_u, \mu(\zeta_u) = \xi_u, \mu(\theta_u) = \kappa_u$.
- For each edge $e = \{u, u'\} \in E$ do:
 - If $u \in IS(G)$, set $\mu(\beta_e) = \eta_e, \mu(\gamma_e) = \alpha_e^{u'}, \mu(\delta_e) = \alpha_e^u$, which matches condition (1) of Lemma 6.
 - If $u \notin IS(G)$, set $\mu(\beta_e) = \alpha_e^{u'}, \mu(\gamma_e) = \alpha_e^u, \mu(\delta_e) = \eta_e$, which matches condition (2) of Lemma 6.
- For any pair of agents $(i, j) \in G'$ not assigned above, set $\mu(i) = 0, \mu^{-1}(j) = 0$.

Since the size of the independent set is at least k , we can trivially verify that, according to the aforementioned rules, the welfare attained by the agents of all vertex gadgets is equal to $|V| \cdot (3|V| + 26) + k$ and of all edge gadgets is $|E| \cdot (6|E|^2 + 27)$, thus achieving a total welfare of SW . In what follows, we prove that outcome μ is indeed stable.

Edge Gadget. Examining the edge gadget of edge $e = \{u, u'\} \in E$, we need to prove that there does not exist any blocking pair of agents. There are two distinct cases, as mentioned previously.

1. If $u \in IS(G)$, then μ corresponds to condition (1) of Lemma 6. Therefore, no blocking pair involving agents $\{\beta_e, \gamma_e, \delta_e, \alpha_e^u, \eta_e\}$ exists. In this case, assigning $\alpha_e^{u'}$ to $\epsilon_{u'}$ would yield a higher utility for agent $\alpha_e^{u'}$. However, since $u' \notin IS(G)$, by definition $\mu(\epsilon_{u'}) = \lambda_u$ and $\epsilon_{u'}$ is assigned

to her most preferred agent, with no incentive to deviate. We conclude that no blocking pair involving agents $\{\alpha_e^{u'}, \epsilon_{u'}\}$, guaranteeing stability for μ .

2. If $u \notin IS(G)$, then μ corresponds to condition (2) of Lemma 6. Therefore, no blocking pair involving agents $\{\beta_e, \gamma_e, \delta_e, \alpha_e^{u'}, \eta_e\}$ exists. In this case, assigning α_e^u to ϵ_u would yield a higher utility for agent α_e^u . However, since $u \notin IS(G)$, by definition $\mu(\epsilon_u) = \lambda_u$ and ϵ_u is integrally matched to her most preferred agent, with no incentive to deviate. We conclude that no blocking pair involving agents $\{\alpha_e^u, \epsilon_u\}$, guaranteeing stability for μ .

We prove a similar result for the vertex gadgets of G' .

Vertex Gadget. Examining the vertex gadget of vertex $u \in V$, we again detect two cases.

1. If $u \in IS(G)$, then no blocking pair of μ involves agents $\{\xi_u, \theta_u, \lambda_u\}$, as μ assigns them to their most preferred agents. Therefore, no blocking pair involves agents $\{\zeta_u, \kappa_u\}$, since no possible combination can yield improved payoff for all participants. As previously argued, no agent from the edge gadget participates in a blocking pair, and thus ϵ_u is also not involved in a blocking pair.
2. If $u \notin IS(G)$, then no blocking pair of μ involves agents $\{\kappa_u, \zeta_u, \lambda_u\}$, as μ assigns them to their most preferred agents. The same holds for agent ϵ_u . Therefore, no blocking pair involves agents $\{\theta_u, \xi_u\}$, since no possible combination can yield improved payoff for all participants.

We conclude that neither the edge nor vertex gadget contain agents involved in blocking pairs under μ . Thus, μ is stable and achieves a welfare of SW .

For the reverse direction, assume μ is a stable outcome for $\langle G', \pi_i, \pi_j \rangle$ with welfare $welfare(\mu) \geq SW$. Defining a subset of the vertices as $IS(G) = \{u \in V \mid \mu(\epsilon_u) = \xi_u\}$, we prove that $IS(G)$ is an independent set of G of size at least k .

Firstly, we prove the desired lower bound on $|IS(G)|$. We define $\mu(\epsilon_u) = N(\epsilon_u)$ as the assignment between vertex ϵ_u and any of the neighboring vertices from the edge gadgets. Summing up for all $|V|$ vertices of G , we get:

$$\begin{aligned}
& \sum_{u \in V} \left(\sum_{i \in \{\epsilon_u, \zeta_u, \theta_u\}} \pi_i(\mu(i)) + \sum_{j \in \{\kappa_u, \lambda_u, \xi_u\}} \pi_j(\mu^{-1}(j)) \right) \\
&= \sum_{u \in V} (|V| + 10) \cdot |\mu(\epsilon_u) = \lambda_u| + (|V| + 12) \cdot |\mu(\epsilon_u) = \xi_u| + (|V| + 8) \cdot |\mu(\epsilon_u) = N(\epsilon_u)| + \\
& (|V| + 7) \cdot |\mu(\zeta_u) = \kappa_u| + (|V| + 9) \cdot |\mu(\zeta_u) = \xi_u| + (|V| + 7) \cdot |\mu(\theta_u) = \kappa_u| + (|V| + 8) \cdot |\mu(\theta_u) = \lambda_u| \\
&\leq \sum_{u \in V} (|V| + 10) |\mu(\epsilon_u) = \lambda_u| + |\mu(\epsilon_u) = \xi_u| + |\mu(\epsilon_u) = N(\epsilon_u)| + (|V| + 8) \cdot |\mu(\zeta_u) = \kappa_u| + |\mu(\zeta_u) = \xi_u| + \\
& (|V| + 8) \cdot |\mu(\theta_u) = \kappa_u| + |\mu(\theta_u) = \lambda_u| + \sum_{u \in V} |\mu(\epsilon_u) = \lambda_u| \\
&\leq |V| \cdot (3|V| + 26) + \sum_{u \in V} |\mu(\epsilon_u) = \lambda_u|
\end{aligned}$$

Since μ corresponds to a binary assignment, only one condition holds true for each of $\epsilon_u, \zeta_u, \theta_u$, and examining all possible outcomes, we use the mean of values to provide an upper bound on utilities. The sum of values for each vertex $\kappa_u, \lambda_u, \xi_u$ is at most 1, which is also taken into account when computing the upper bound. We can trivially verify that, the total welfare of μ is maximized if and only if pair ϵ_u, ξ_u is integrally matched.

We now need to show that $|IS(G)| \geq k$. Set $IS(G)$ has been defined as the set of vertices ϵ_u for $u \in V$, that are integrally matched to ξ_u . This can be expressed as $|IS(G)| = \sum_{u \in V} |\mu(\epsilon_u) = \xi_u|$. It then suffices to prove inequality $\sum_{u \in V} |\mu(\epsilon_u) = \xi_u| \geq k$, to prove the lower bound on the size of the independent set. Since the induced welfare from all $|E|$ edge gadgets is at most $welfare(\mu) \leq |E| \cdot (6|E|^2 + 27)$. At least $|V| \cdot (3|V| + 26) + \sum_{u \in V} |\mu(\epsilon_u) = \xi_u|$ must be derived from the vertex gadgets. Using the upper bound provided above, we conclude that $\sum_{u \in V} |\mu(\epsilon_u) = \xi_u| \geq k$, as required, thus proving the lower bound on the size of $IS(G)$.

Finally, we need to prove that $IS(G)$ is in fact an independent set of G . We prove the claim by contradiction. Suppose there is an edge $e = \{u, u'\} \in E$ and $\{u, u'\} \subseteq IS(G)$. Then, for both nodes u, u' under assignment μ , it must hold that $\pi_{\epsilon_u}(\mu(\epsilon_u)) < |V| + 8 = \pi_{\epsilon_u}(\alpha_e^u)$ and $\pi_{\epsilon_{u'}}(\mu(\epsilon_{u'})) < |V| + 8 = \pi_{\epsilon_{u'}}(\alpha_e^{u'})$, as $IS(G)$ is defined as the set of nodes u that are integrally matched to ξ_u . From Lemma 5, the induced welfare from agents of the edge gadgets is at most $6|E|^2 + 27 - |V|$, and using the previously shown bound, the welfare induced from agents of the vertex gadgets is at most $3|V| + 26 + |V|$. Therefore, the total welfare of G' under μ is lower than the original assumption, which is a contradiction. Note that the restriction of our prices to be at the budget constraint b^i of the buyers $i \in \mathcal{B}$ in the edge and vertex gadgets is without loss of generality as shown in Lemma 7. Concluding, we have proven that indeed $IS(G)$ is an independent set of G .