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Abstract. We propose *joinwidth*, a new complexity parameter for the Constraint Satisfaction Problem (CSP). The definition of joinwidth is based on the arrangement of basic operations on relations (joins, projections, and pruning), which inherently reflects the steps required to solve the instance. We use joinwidth to obtain polynomial-time algorithms (if a corresponding decomposition is provided in the input) as well as fixed-parameter algorithms (if no such decomposition is provided) for solving the CSP.
Joinwidth is a *hybrid* parameter, as it takes both the graphical structure as well as the constraint relations that appear in the instance into account. It has, therefore, the potential to capture larger classes of tractable instances than purely *structural* parameters like hypertree width and the more general fractional hypertree width (fhtw). Indeed, we show that any class of instances of bounded fhtw also has bounded joinwidth, and that there exist classes of instances of bounded joinwidth and unbounded fhtw, so bounded joinwidth properly generalizes bounded fhtw.
We further show that bounded joinwidth also properly generalizes several other known hybrid restrictions, such as fhtw with degree constraints and functional dependencies. In this sense, bounded joinwidth can be seen as a unifying principle that explains the tractability of several seemingly unrelated classes of CSP instances.

1 Introduction

The Constraint Satisfaction Problem (CSP) is a central and generic computational problem that provides a common framework for many theoretical and practical applications in AI and other areas of Computer Science [31]. An instance of the CSP consists of a collection of variables that must be assigned values subject to constraints, where each constraint is given in terms of a relation whose tuples specify the allowed combinations of values for specified variables.

CSP is NP-complete in general. A central line of research is concerned with the identification of classes of instances for which the CSP can be solved in polynomial time. The two main approaches are to define classes either in terms of the constraint relations that may occur in the instance (*syntactic restrictions*; see, e.g., [4]), or in terms of the constraint hypergraph associated with the instance (*structural restrictions*; see, e.g., [18]).

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There are also several prominent proposals for utilizing simultaneously syntactic and structural restrictions called *hybrid restrictions* (see, e.g., [28,7,8,6]).

Grohe and Marx [20] showed that CSP is polynomial-time tractable whenever the constraint hypergraph has bounded *fractional hypertree width*, which strictly generalizes previous tractability results based on hypertree width [15] and acyclic queries [33]. Bounded fractional hypertree width is the most general known structural restriction that gives rise to polynomial-time tractability of CSP.

Our contribution: Joinwidth. We propose a new hybrid restriction for the CSP, the width parameter *joinwidth*, which is based on the arrangement of basic relational operations along a tree, and not on hypertree decompositions. Interestingly, as we will show, our notion strictly generalizes (i) bounded fractional hypertree width, (ii) recently introduced extensions of fractional hypertree width with degree constraints and functional dependencies [24], (iii) various prominent hybrid restrictions [5], as well as (iv) tractable classes based on *functionality* and *root sets* [10,9,5]. Hence, joinwidth gives rise to a common framework that captures several different tractable classes considered in the past. Moreover, none of the other hybrid parameters that we are aware of [8], such as classes based on the Broken Triangle Property or topological minors [7,6] and directional rank [28], generalize fractional hypertree width and hence all of them are either less general or orthogonal to joinwidth.

Joinwidth is based on the arrangement of the constraints on the leaves of a rooted binary tree which we call a *join decomposition*. The join decomposition indicates the order in which relational joins are formed, where one proceeds in a bottom-up fashion from the leaves to the root, labeling a node by the join of the relations at its children, and projecting away variables that do not occur in relations to be processed later. Join decompositions are related to (structural) *branch decompositions* of hypergraphs, where the hyperedges are arranged on the leaves of the tree [2,19,29]. Related notions have been considered in the context of query optimization [1,22]. However, the basic form of join decompositions using only relational joins and projections is still a weak notion that cannot be used to tackle instances of bounded fractional hypertree width efficiently. We identify a further operation that—in conjunction with relational joins and projections—gives rise to the powerful new concept of joinwidth that captures and extends the various known tractable classes mentioned. This third operation *prunes* away all the tuples from an intermediate relation that are inconsistent with a relation to be processed later.

A join decomposition of a CSP instance specifies the order in which the above three operations are applied, and its *width* is the smallest real number w such that each relation appearing within the join decomposition has at most m^w many tuples (where m is the maximum number of tuples appearing in any constraint relation of the CSP instance under consideration). The joinwidth of a CSP instance is the smallest width over all its join decompositions. Observe that joinwidth is a hybrid parameter—it depends on both the graphical structure as well as the constraint relations appearing in the instance.

Exploiting Joinwidth. Similarly to other width parameters, also the property that a class of CSP instances has bounded joinwidth can only be exploited for CSP solving if a decomposition (in our case a join decomposition) witnessing the bounded width is provided as part of the input. While such a join decomposition can be computed efficiently from a fractional hypertree decomposition or when the CSP instance belongs to a tractable class based on functionality or root sets mentioned earlier, we show that computing an optimal join decomposition is NP-hard in general, mirroring the corresponding NP-hardness of computing optimal fractional hypertree decompositions [13].

However, this obstacle disappears if we move from the viewpoint of polynomial-time tractability to *fixed-parameter tractability* (FPT). Under the FPT viewpoint, one considers classes of instances $\mathbf{I}$ that can be solved by a fixed-parameter algorithm—an algorithm running in time $f(k)|\mathbf{I}|^{O(1)}$, where k is the parameter (typically the number of variables or constraints), $|\mathbf{I}|$ is the size of the instance, and f is a computable function [14,16,17]. We note that it is natural to assume that k is much smaller than $|\mathbf{I}|$ in typical cases. The use of fixed-parameter tractability is well motivated in the CSP setting; see, for instance, Marx's discussion on this topic [27].

Here, we obtain two single-exponential fixed-parameter algorithms for instances of bounded joinwidth (i.e., algorithms with a running time of $2^{\mathcal{O}(k)} \cdot |\mathbf{I}|^{\mathcal{O}(1)}$): one where k is the number of variables, and the other when k is the number of constraints. In this setting, we do not require an associated join decomposition to be provided with the input.

Under the FPT viewpoint, Marx [27] previously introduced the structural parameter *submodular width* (bounded submodular width is equivalent to bounded *adaptive width* [26]), which is strictly more general than fractional hypertree width, but when bounded only gives rise to fixed-parameter tractability and not polynomial-time tractability of CSP. In fact, Marx showed that assuming the Exponential Time Hypothesis [21], bounded submodular width is the most general purely structural restriction that yields fixed-parameter tractability for CSP. However, as joinwidth is a hybrid parameter, it can (and we show that it does) remain bounded even on instances of unbounded submodular width—and the same holds also for the recently introduced extensions of submodular width based on functional dependencies and degree bounds [24].

Roadmap. After presenting the required preliminaries on (hyper-)graphs, CSP, and fractional hypertree width in Section 2, we introduce and motivate join decompositions and joinwidth in Section 3. We establish some fundamental properties of join decompositions, provide our tractability result for CSP for the case when a join decomposition is given as part of the input, and then obtain our NP-hardness result for computing join decompositions of constant width. Section 4 provides an in-depth justification for the various design choices underlying join decompositions; among others, we show that the pruning step is required if the aim is to generalize fractional hypertree width. Our algorithmic applications for joinwidth are presented in Section 5: for instance, we show that joinwidth generalizes fractional hypertree width, but also other known (and hybrid) parameters such as functionality, root sets, and Turan sets. Section 6 contains our fixed-parameter tractability results for classes of CSP instances with bounded joinwidth. Finally, in Section 7, we compare the algorithmic power of joinwidth to the power of algorithms which rely on the unrestricted use of join and projection operations.

2 Preliminaries

We will use standard graph terminology [11]. An *undirected graph* G is a pair (V, E), where V or $V(G)$ is the vertex set and E or $E(G)$ is the edge set. All our graphs are simple and loopless. For a tree T we use $L(T)$ to denote the set of its leaves. For $i \in \mathbb{N}$, we let $[i] = \{1, \ldots, i\}$.

Hypergraphs. Similarly to graphs, a *hypergraph* H is a pair (V, E) where V or $V(H)$ is its vertex set and E or $E(H) \subseteq 2^V$ is its set of hyperedges. We denote by $H[V']$ the hypergraph *induced* on the vertices in $V' \subseteq V$, i.e., the hypergraph with vertex set V' and edge set $\{\, e \cap V' : e \in E \,\}$.

The Constraint Satisfaction Problem. Let D be a set and n and n' be natural numbers. An n-ary relation on D is a subset of D^n. For a tuple $t \in D^n$, we denote by $t[i]$, the i-th entry of t, where $1 \leq i \leq n$. For two tuples $t \in D^n$ and $t' \in D^{n'}$, we denote by $t \circ t'$, the concatenation of t and t'.

An instance of a *constraint satisfaction problem* (CSP) $\mathbf{I}$ is a triple $\langle V, D, C \rangle$, where V is a finite set of variables over a finite set (domain) D, and C is a set of constraints. A *constraint* $c \in C$ consists of a *scope*, denoted by $S(c)$, which is a completely ordered subset of V, and a relation, denoted by $R(c)$, which is a $|S(c)|$-ary relation on D. We let $|c|$ denote the number of tuples in $R(c)$ and $|\mathbf{I}| = |V| + |D| + \sum_{c \in C} |c|$. Without loss of generality, we assume that each variable occurs in the scope of at least one constraint.

A *solution* for $\mathbf{I}$ is an assignment $\theta : V \to D$ of the variables in V to domain values (from D) such that for every constraint $c \in C$ with scope $S(c) = (v_1, \ldots, v_{|S(c)|})$, the relation R contains the tuple $\theta(S(c)) = (\theta(v_1), \ldots, \theta(v_{|S(c)|}))$. We denote by SOL($\mathbf{I}$) the constraint containing all solutions of $\mathbf{I}$, i.e., the constraint with scope $V = \{v_1, \ldots, v_n\}$, whose relation contains one tuple $(\theta(v_1), \ldots, \theta(v_n))$ for every solution θ of $\mathbf{I}$. The task in CSP is to decide whether the instance $\mathbf{I}$ has at least one solution or in other words whether SOL($\mathbf{I}$) $\neq \emptyset$. Here and in the following we will for convenience (and with a slight abuse of notation) sometimes treat constraints like sets of tuples.

For a variable $v \in S(c)$ and a tuple $t \in R(c)$, we denote by $t[v]$, the i-th entry of t, where i is the position of v in $S(c)$. Let V' be a subset of V and let V'' be all the variables that appear in V' and $S(c)$. With a slight abuse of notation, we denote by $S(c) \cap V'$, the sequence $S(c)$ restricted to the variables in V' and we denote by $t[V']$ the tuple $(t[v_1], \ldots, t[v_{|V''|}])$, where $S(c) \cap V' = (v_1, \ldots, v_{|V''|})$.

Let c and c' be two constraints of $\mathbf{I}$. We denote by $S(c) \cup S(c')$, the ordered set (i.e., tuple) $S(c) \circ (S(c') \setminus S(c))$. The *(natural) join* between c and c', denoted by $c \bowtie c'$, is the constraint with scope $S(c) \cup S(c')$ containing all tuples $t \circ t'[S(c') \setminus S(c)]$ such that $t \in R(c)$, $t' \in R(c')$, and $t[S(c) \cap S(c')] = t'[S(c) \cap S(c')]$. The *projection* of c to V', denoted by $\pi_{V'}(c)$, is the constraint with scope $S(c) \cap V'$, whose relation contains all tuples $t[V']$ with $t \in R(c)$. We note that if c contains at least one tuple, then projecting it onto a set V' with $V' \cap S(c) = \emptyset$ results in the constraint with an empty scope and a relation containing the empty tuple (i.e., a tautological constraint). On the other hand, if $R(c)$ is the relation containing the empty tuple, then every projection of c will also result in a relation containing the empty tuple.

For a CSP instance $\mathbf{I} = \langle V, D, C \rangle$ we sometimes denote by $V(\mathbf{I})$, $D(\mathbf{I})$, $C(\mathbf{I})$, and $\sharp_{tup}(\mathbf{I})$ its set of variables V, its domain D, its set of constraints C, and the maximum number of tuples in any constraint relation of $\mathbf{I}$, respectively. For a subset $V' \subseteq V$, we will also use $\mathbf{I}[V']$ to denote the sub-instance of $\mathbf{I}$ induced by the variables in $V' \subseteq V$, i.e., $\mathbf{I}[V'] = \langle V', D, \{\, \pi_{V'}(c) : c \in C \,\}\rangle$. The *hypergraph* $H(\mathbf{I})$ of a CSP instance $\mathbf{I} = \langle V, D, C\rangle$ is the hypergraph with vertex set V and edge set $\{\, S(c) : c \in C \,\}$.

It is well known that for every instance $\mathbf{I}$ and every instance $\mathbf{I}'$ obtained by either (1) replacing two constraints in $C(\mathbf{I})$ by their natural join or (2) adding a projection of a constraint in $C(\mathbf{I})$, it holds that $\mathrm{SOL}(\mathbf{I}) = \mathrm{SOL}(\mathbf{I}')$. As a consequence, $\mathrm{SOL}(\mathbf{I})$ can be computed by performing, e.g., a sequence of joins over all the constraints in C.

Fractional Hypertree Width. Let H be a hypergraph. A *fractional edge cover* for H is a mapping $\gamma : E(H) \to \mathbb{R}$ such that $\sum_{e \in E(H) \wedge v \in e} \gamma(e) \geq 1$ for every $v \in V(H)$. The *weight* of γ, denoted by $w(\gamma)$, is the number $\sum_{e \in E(H)} \gamma(e)$. The *fractional edge cover number* of H, denoted by $\mathsf{fec}(H)$, is the smallest weight of any fractional edge cover of H.

A *fractional hypertree decomposition* $\mathcal{T}$ of H is a triple $\mathcal{T} = (T, (B_t)_{t \in V(T)}, (\gamma_t)_{t \in V(T)})$, where $(T, (B_t)_{t \in V(T)})$ is a tree decomposition [30,12] of H and $(\gamma_t)_{t \in V(T)}$ is a family of mappings from $E(H)$ to $\mathbb{R}$ such that for every $t \in V(T)$, it holds that γ_t is a fractional edge cover for $H[B_t]$. We call the sets B_t the *bags* and the mappings γ_t the *fractional guards* of the decomposition. The *width* of $\mathcal{T}$ is the maximum $w(\gamma_t)$ over all $t \in V(T)$. The *fractional hypertree width* of H, denoted by $\mathsf{fhtw}(H)$, is the minimum width of any fractional hypertree decomposition of H. Finally, the *fractional hypertree width* of a CSP instance $\mathbf{I}$, denoted by $\mathsf{fhtw}(\mathbf{I})$, is equal to $\mathsf{fhtw}(H(\mathbf{I}))$.

Proposition 1 *Let $\mathbf{I}$ be a CSP instance with hypergraph H and let $\mathcal{T} = (T, (B_t)_{t \in V(T)}, (\gamma_t)_{t \in V(T)})$ be a fractional hypertree decomposition of H of width at most ω. For every node $t \in V(T)$ and every subset $B \subseteq B_t$, it holds that $|\mathrm{SOL}(\mathbf{I}[B])| \leq (\sharp_{\mathsf{tup}}(\mathbf{I}))^{\omega}$.*

3 Join Decompositions and Joinwidth

This section introduces two notions that are central to our contribution: *join decompositions* and *joinwidth*. In the following, let us consider an arbitrary CSP instance $\mathbf{I} = \langle V, D, C\rangle$.

Definition 2 *A* join decomposition *for* $\mathbf{I}$ *is a pair (J, ϱ), where J is a rooted binary tree and ϱ is a bijection between the leaves $L(J)$ of J and C.*

Let j be a node of J. We denote by J_j the subtree of J rooted at j and we denote by $X(j)$, $V(j)$, $\overline{V}(j)$, and $S(j)$ the (unordered) sets $\{\, \varrho(\ell) : \ell \in L(J_j) \,\}$, $\bigcup_{c \in X(j)} S(c)$, $\bigcup_{c \notin X(j)} S(c)$, and $V(j) \cap \overline{V}(j)$, respectively; infuitively, $X(j)$ is the set of constraints that occur in the subtree rooted at j, $V(j)$ is the set of variables that occur in the scope of constraints in $X(j)$, $\overline{V}(j)$ is the set of variables that occur in the scope of constraints not in $X(j)$, and $S(j)$ is the set of variables that occur in $V(j)$ and $\overline{V}(j)$. In some cases, we will also consider *linear join decompositions*, which are join decompositions where every inner node is adjacent to at least one leaf.

Semantics of Join Decompositions. Intuitively, every internal node of a join decomposition represents a join operation that is carried out over the constraints obtained for the two children; in this way, a join decomposition can be seen as a procedure for performing joins, with the aim of determining whether $\text{SOL}(\mathbf{I})$ is non-empty (i.e., solving the CSP instance $\mathbf{I}$). Crucially, the running time of such a procedure depends on the size of the constraints obtained and stored by the algorithm which performs such joins. The aim of this subsection is to formally define and substantiate an algorithmic procedure which uses join decompositions to solve CSP.

A naive way of implementing the above idea would be to simply compute and store the natural join at each node of the join decomposition and proceed up to the root; see for instance the work of [3]. Formally, we can recursively define a constraint $C_{\mathit{naive}}(j)$ for every node $j \in V(J)$ as follows. If j is a leaf, then $C_{\mathit{naive}}(j) = \varrho(j)$. Otherwise $C_{\mathit{naive}}(j)$ is equal to $C_{\mathit{naive}}(j_1) \bowtie C_{\mathit{naive}}(j_2)$, where j_1 and j_2 are the two children of j in J. It is easy to see that this approach can create large constraints even for very simple instances of CSP: for example, at the root r of T it holds that $\text{SOL}(\mathbf{I}) = C_{\mathit{naive}}(r)$, and hence $C_{\mathit{naive}}(r)$ would have superpolynomial size for every instance of CSP with a superpolynomial number of solutions. In particular, an algorithm which computes and stores $C_{\mathit{naive}}(j)$ would never run in polynomial time for CSP instances with a superpolynomial number of solutions.

An efficient way of joining constraints along a join decomposition is to only store projections of constraints onto those variables that are still relevant for constraints which have yet to appear; this idea has been used, e.g., in algorithms which exploit hypertree width [15]. To formalize this, let $C_{\mathit{proj}}(j)$ be recursively defined for every node $j \in V(T)$ as follows. If j is a leaf, then $C_{\mathit{proj}}(j) = \pi_{\overline{V}(j)}(\varrho(j))$. Otherwise $C_{\mathit{proj}}(j)$ is equal to $\left(\pi_{\overline{V}(j)}(C_{\mathit{proj}}(j_1) \bowtie C_{\mathit{proj}}(j_2))\right)$, where j_1 and j_2 are the two children of j in J. In this case, $\mathbf{I}$ is a YES-instance if and only if $C_{\mathit{proj}}(r)$ does not contain the empty relation. Clearly, for every node j of J it holds that $C_{\mathit{proj}}(j)$ has at most as many tuples as $C_{\mathit{naive}}(j)$, but can have arbitrarily fewer tuples; in particular, an algorithm which uses join decompositions to compute C_{proj} in a bottom-up fashion can solve CSP instances in polynomial time even if they have a superpolynomial number of solutions (see also Observation 8).

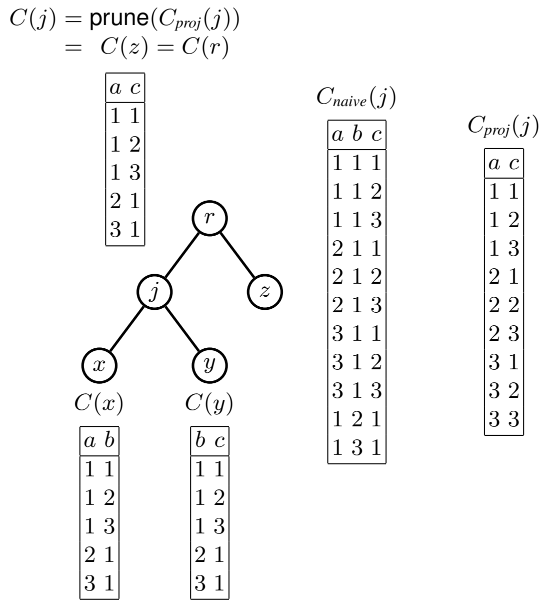
However, the above approach still does not capture the algorithmic power offered by dynamically computing joins along a join decomposition. In particular, similarly as has been done in the evaluation algorithm for fractional edge cover [20, Theorem 3.5], we can further reduce the size of each constraint $C_{\mathit{proj}}(j)$ computed in the above procedure by *pruning* all tuples that would immediately violate a constraint c in $\mathbf{I}$ (and, in particular, in $C \setminus C(j)$). To formalize this operation, we let $\mathsf{prune}(c)$ denote the *pruned constraint* w.r.t. $\mathbf{I}$, i.e., $\mathsf{prune}(c)$ is obtained from c by removing all tuples $t \in R(c)$ such that there is a constraint $c' \in C$ with $t[S(c')] \notin \pi_{S(c)}(c')$. This leads us to our final notion of dynamically computed constraints: for a node j, we let $C(j) = \mathsf{prune}(C_{\mathit{proj}}(j))$. We note that this, perhaps inconspicuous, notion of pruning is in fact critical—without it, one cannot use join decompositions to efficiently solve instances of small fractional hypertree width or even small fractional edge cover. A more in-depth discussion on this topic is provided in Section 4.

We can now proceed to formally define the considered width measures.

Definition 3 *Let $\mathcal{J} = (J, \varrho)$ be a join decomposition for $\mathbf{I}$ and let $j \in V(J)$. The* join-width *of j, denoted* $\mathsf{jw}(j)$*, is the smallest real number ω such that $|C(j)| \leq (\sharp_{\text{tup}}(\mathbf{I}))^{\omega}$, i.e., $\omega = \log_{\sharp_{\text{tup}}(\mathbf{I})} |C(j)|$. The joinwidth of $\mathcal{J}$ (denoted* $\mathsf{jw}(\mathcal{J})$*) is then the maximum* $\mathsf{jw}(j)$ *over all $j \in V(J)$. Finally, the joinwidth of $\mathbf{I}$ (denoted* $\mathsf{jw}(\mathbf{I})$*) is the minimum* $\mathsf{jw}(\mathcal{J})$ *over all join decompositions $\mathcal{J}$ for* $\mathbf{I}$.

In general terms, an instance $\mathbf{I}$ has joinwidth ω if it admits a join decomposition where the number of tuples of the produced constraints never increases beyond the ω-th power of the size of the largest relation in $\mathbf{I}$. Analogously as above, we denote by $\mathsf{ljw}(\mathbf{I})$ the minimum joinwidth of any linear join decomposition of a CSP instance $\mathbf{I}$.

Example 4 *Let $N \in \mathbb{N}$ and consider the CSP instance $\mathbf{I}$ having three variables a, b, and c and three constraints x, y, and z with scopes (a, b), (b, c), and (a, c), respectively. Assume furthermore that the relations of all three constraints are identical and contain all tuples $(1, i)$ and $(i, 1)$ for every $i \in [N]$. Refer also to Figure 1 for an illustration of the example. Then $|x| = |y| = |z| = \sharp_{\text{tup}}(\mathbf{I}) = 2N - 1$ and due to the symmetry of $\mathbf{I}$ any join-tree $\mathcal{J}$ of $\mathbf{I}$ has the same joinwidth, which (as we will show) is equal to 1. To see this consider for instance the join-tree $\mathcal{J}$ that has one inner node j joining x and y and a root node r joining $C(j)$ and z. Then* $\mathsf{jw}(\ell) = 1$ *for any leaf node ℓ of $\mathcal{J}$. Moreover $|C(j)| = |\mathit{prune}(C_{\text{proj}}(j))| = |z| = \sharp_{\text{tup}}(\mathbf{I})$ since the pruning step removes all tuples from $C_{\text{proj}}(j)$ that are not in z and consequently $C(r) = z$ and* $\mathsf{jw}(\mathcal{J}) = 1$*. Note that in this example* $\mathsf{jw}(\mathbf{I}) = 1 < \mathsf{fhtw}(\mathbf{I}) = 3/2$.



$C(j)$:

a	c
1	1
1	2
1	3
2	1
3	1

$C_{naive}(j)$:

a	b	c
1	1	1
1	1	2
1	1	3
2	1	1
2	1	2
2	1	3
3	1	1
3	1	2
3	1	3
1	2	1
1	3	1

$C_{proj}(j)$:

a	c
1	1
1	2
1	3
2	1
2	2
2	3
3	1
3	2
3	3

$C(x)$:

a	b
1	1
1	2
1	3
2	1
3	1

$C(y)$:

b	c
1	1
1	2
1	3
2	1
3	1

Fig. 1: The join decomposition given in Example 4 for $N = 3$ together with the intermediate constraints obtained for the node j.

Finally, we remark that one could in principle also define joinwidth in terms of a (rather tedious and technically involved) variant of hypertree decompositions. However, the inherent algorithmic nature of join-trees makes them much better suited for the definition of joinwidth.

Properties of Join Decompositions. Our first task is to formalize the intuition behind the constraints $C(j)$ computed when proceeding through the join tree.

Lemma 5 *Let (J, ϱ) be a join decomposition for $\mathbf{I} = \langle V, D, C\rangle$ and let $j \in V(J)$. Then $C(j) = \pi_{\overline{V}(j)}(\text{SOL}(\mathbf{I}'))$, where $\mathbf{I}' = \mathbf{I}[V(j)]$.*

Proof. We prove the lemma by leaf-to-root induction along J. If j is a leaf such that $\varrho(j) = c$, then $C(j)$ is the constraint obtained from c by projecting onto $\overline{V}(j)$ and then applying pruning with respect to $\mathbf{I}$. Crucially, pruning c w.r.t. $\mathbf{I}'$ produces the same result as pruning c w.r.t. $\mathbf{I}$. Since pruning cannot remove tuples which occur in SOL$(\mathbf{I}')$, each tuple in SOL$(\mathbf{I}')$ must also occur in $C(j)$ (as a projection onto $\overline{V}(j)$). On the other hand, consider a tuple α in $C(j)$ and assume for a contradiction that α is not present in $\pi_{\overline{V}(j)}(\text{SOL}(\mathbf{I}'))$. Since variables outside of $\overline{V}(j)$ do not occur in the scopes of constraints other than c, this means that there would exist a constraint c' in $\mathbf{I}'$ which is not satisfied by an assignment corresponding to α—but in that case α would be removed from $C(j)$ via pruning. Hence $C(j) = \pi_{\overline{V}(j)}(\text{SOL}(\mathbf{I}'))$ holds for every leaf in T.

For the induction step, consider a node j with children j_1 and j_2 (with their corresponding instances being $\mathbf{I}'_1$ and $\mathbf{I}'_2$, respectively), and recall that $C(j)$ is obtained from $C(j_1) \bowtie C(j_2)$ by projecting onto $\overline{V}(j)$ and then pruning (w.r.t. $\mathbf{I}$ or, equivalently, w.r.t. $\mathbf{I}'$). We will also implicitly use the fact that $\overline{V}(j) \subseteq \overline{V}(j_1) \cup \overline{V}(j_2)$ and $V(j) = V(j_1) \cup V(j_2)$. First, consider for a contradiction that there exists a tuple β in SOL$(\mathbf{I}'[\overline{V}(j)])$ which does not occur in $C(j)$. Clearly, β could not have been removed by pruning, and hence this would mean that there exists no tuple in $C(j_1) \bowtie C(j_2)$ which results in β after projection onto $\overline{V}(j)$; in particular, w.l.o.g. we may assume that every tuple in $C(j_1)$ differs from β in (the assignment of) at least one variable. However, since β occurs in SOL$(\mathbf{I}'[\overline{V}(j)])$, there must exist at least one tuple, say β', which occurs in SOL$(\mathbf{I}')$, and consequently there exists a tuple in SOL$(\mathbf{I}'_1)$ which matches β in (the assignment of) all variables. At this point, we have reached a contradiction with the inductive assumption that SOL$(\mathbf{I}'_1[\overline{V}(j_1)]) = C(j_1)$.

For the final case, consider a tuple γ in $C(j)$ and assume for a contradiction that γ is not present in $\pi_{\overline{V}(j)}(\text{SOL}(\mathbf{I}'))$. This means that there exists at least one constraint, say c', in $\pi_{\overline{V}(j)}(\text{SOL}(\mathbf{I}'))$ which would be invalidated by (an assignment corresponding to) γ. Let us assume that c' occurs in the subtree rooted in j_2, and let γ_1 be an arbitrary "projection" of γ onto $\overline{V}(j_1)$. Since $\overline{V}(j_1) \supseteq S(c')$, this means that γ_1 would have been removed from $C(j_1)$ by pruning; in particular, we see that there exists no tuple γ_1 in $C(j_1)$ which could produce γ in a join, contradicting our assumptions about γ. By putting everything together, we conclude that indeed $C(j) = \pi_{\overline{V}(j)}(\text{SOL}(\mathbf{I}'))$. □

Next, we show how join decompositions can be used to solve CSP.

Theorem 6 CSP *can be solved in time $\mathcal{O}(|\mathbf{I}|^{2\omega+4})$ provided that a join decomposition of width at most ω is given in the input.*

Proof. Let $\mathcal{J} = (J, \varrho)$ be the provided join decomposition of width ω for $\mathbf{I}$. As noted before, the algorithm for solving $\mathbf{I}$ computes $C(j)$ for every $j \in V(J)$ in a bottom-up manner. Since J has exactly $2|C| - 1$ nodes, it remains to analyse the maximum time required to compute $C(j)$ for any node of J. If j is a leaf, then $C(j) = \mathsf{prune}(\pi_{S(j)}(\varrho(j)))$ and since the time required to compute the projection $P = \pi_{S(j)}(\varrho(j))$ from $\varrho(j)$ is at most $\mathcal{O}(\sharp_{tup}(\mathbf{I})|S(j)|)$ and the time required to compute the pruned constraint $\mathsf{prune}(P)$ from P is at most $\mathcal{O}(|C|(\sharp_{tup}(\mathbf{I}))^2|S(j)|)$, we obtain that $C(j)$ can be computed in time $\mathcal{O}(|C|(\sharp_{tup}(\mathbf{I}))^2|S(j)|) \in \mathcal{O}(|\mathbf{I}|^4)$. Moreover, if j is an inner node with children j_1 and j_2, then $C(j) = \mathsf{prune}(\pi_{S(j)}(C(j_1) \bowtie C(j_2)))$ and since we require at most $\mathcal{O}((\sharp_{tup}(\mathbf{I}))^{2\omega}|S(j_1) \cup S(j_2)|)$ time to compute the join $Q = C(j_1) \bowtie C(j_2)$ from $C(j_1)$ and $C(j_2)$, at most $\mathcal{O}((\sharp_{tup}(\mathbf{I}))^{2\omega}|S(j)|)$ time to compute the projection $P = \pi_{S(j)}(Q)$ from Q, and at most $\mathcal{O}(|C|(\sharp_{tup}(\mathbf{I}))^{2\omega+1} \cdot |S(j)|)$ time to compute the pruned constraint $\mathsf{prune}(P)$ from P, we obtain $\mathcal{O}(|C|(\sharp_{tup}(\mathbf{I}))^{2\omega+1} \cdot |S(j_1) \cup S(j_2)|) = \mathcal{O}(|\mathbf{I}|^{2\omega+3})$ as the total time required to compute $C(j)$. Multiplying the time required to compute $C(j)$ for an inner node $j \in V(J)$ with the number of nodes of T yields the running time stated in the lemma. □

Computing Join Decompositions. Next, let us address the problem of computing join decompositions of bounded joinwidth, formalized as follows.

ω-JOIN DECOMPOSITION
Input: A CSP instance $\mathbf{I}$.
Question: Compute a join decomposition for $\mathbf{I}$ of width at most ω, or correctly determine that $\mathsf{jw}(\mathbf{I}) > \omega$.

We show that ω-JOIN DECOMPOSITION is NP-hard even for width $\omega = 1$. This is similar to fractional hypertree width, where it was only very recently shown that deciding whether $\mathsf{fhtw}(\mathbf{I}) \leq 2$ is NP-hard [13], settling a question which had been open for about a decade. Our proof is, however, entirely different from the corresponding hardness proof for fractional hypertree width and uses a reduction from the NP-complete BRANCHWIDTH problem [32].

Theorem 7 1-JOIN DECOMPOSITION *is* NP-*hard, even on Boolean CSP instances.*

4 Justifying Joinwidth

Below, we substantiate the use of both pruning and projections in our definition of join decomposition. In particular, we show that using pruning and projections allows the joinwidth to be significantly lower than if we were to consider joins carried out via C_{naive} or C_{proj}. More importantly, we show that join decompositions without pruning do not cover CSP instances with bounded fractional edge cover (and by extension bounded fractional hypertree width). To formalize this, let $\mathsf{jw}_{naive}(\mathbf{I})$ and $\mathsf{jw}_{proj}(\mathbf{I})$ be defined analogously as $\mathsf{jw}(\mathbf{I})$, with the distinction being that these measure the width in terms of C_{naive} and C_{proj} instead of C.

We also justify the use of trees for join decompositions by showing that there is an arbitrary difference between linear join decompositions (which precisely correspond to simple sequences of joins) and join decompositions.

Observation 8 *For every integer ω there exists a CSP instance $\mathbf{I}_\omega$ such that* $\mathsf{jw}_{\mathrm{proj}}(\mathbf{I}_\omega) \leq 1$*, but* $\mathsf{jw}_{\mathrm{naive}}(\mathbf{I}_\omega) \geq \omega$*.*

Proof. Consider the CSP instance $\mathbf{I}_\omega$ with variables $x, v_1, \dots, v_\omega$ and for each $i \in [\omega]$ a constraint c_i with scope $\{x, v_i\}$ containing the tuples $\langle 0, 1\rangle$ and $\langle 0, 0\rangle$. Since SOL($\mathbf{I}_\omega$) contains 2^ω tuples, it follows that every join decomposition with root r must have $|C(r)_{naive}| = (\sharp_{tup}(\mathbf{I}))^\omega = 2^\omega$ tuples, hence $\mathsf{jw}_{naive}(\mathbf{I}_\omega) \geq \omega$.

On the other hand, consider a linear join decomposition which introduces the constraints in an arbitrary order. Then for each inner node j, it holds that $S(j) = \{x\}$ and in particular $C_{proj}(j)$ contains a single tuple (0) over scope $\{x\}$. We conclude that $\mathsf{jw}_{proj}(\mathbf{I}_\omega) \leq 1$. □

The next proposition justifies the use of trees instead of just linear join decompositions. Its proof employs an interesting connection between branchwidth and joinwidth.

Proposition 9 *For every integer ω there exists a CSP instance $\mathbf{I}_\omega$ such that* $\mathsf{jw}(\mathbf{I}_\omega) \leq 1$ *but* $\mathsf{ljw}(\mathbf{I}_\omega) \geq \omega$*.*

The next proposition shows not only that pruning can significantly reduce the size of stored constraints, but also that without pruning (i.e., with projections alone) one cannot hope to generalize structural parameters such as fractional hypertree width.

Proposition 10 *For every integer ω there exists a CSP instance $\mathbf{I}_\omega$ with hypergraph H^ω such that* $\mathsf{jw}(\mathbf{I}_\omega) \leq 2$ *and* $\mathsf{fec}(H^\omega) \leq 2$ *(and hence also* $\mathsf{fhtw}(H^\omega) \leq 2$*), but* $\mathsf{jw}_{\mathrm{proj}}(\mathbf{I}_\omega) \geq \omega$*.*

We believe that the above results are of general interest, as they provide useful insights into how to best utilize the joining of constraints.

5 Tractable Classes

Here, we show that join decompositions of small width not only allow us to solve a wide range of CSP instances, but also provide a unifying reason for the tractability of previously established structural parameters and tractable classes.

5.1 Fractional Hypertree Width

We begin by showing that joinwidth is a strictly more general parameter than fractional hypertree width. We start with a simple example showing that the joinwidth of a CSP instance can be arbitrarily smaller than its fractional hypertree width. Indeed, this holds for any structural parameter ψ measured purely on the hypergraph representation, i.e., we say that ψ is a *structural parameter* if $\psi(\mathbf{I}) = \psi(H(\mathbf{I}))$ for any CSP instance $\mathbf{I}$. Examples for structural parameters include fractional and generalized hypertree width, but also *submodular width* [27].

Observation 11 *Let ψ be any structural parameter such that for every ω there is a CSP instance with $\psi(\mathbf{I}) = \psi(H(\mathbf{I})) \geq \omega$. Then for every ω there is a CSP instance $\mathbf{I}_\omega$ with* $\mathsf{jw}(\mathbf{I}_\omega) \leq 1$ *but $\psi(\mathbf{I}_\omega) \geq \omega$.*

The following theorem shows that, for the case of fractional hypertree width, the opposite of the above observation is not true.

Theorem 12 *For every CSP instance* $\mathbf{I}$*, it holds that* $\mathsf{jw}(\mathbf{I}) \leq \mathsf{fhtw}(\mathbf{I})$.

Proof. Let H be the hypergraph of the given CSP instance $\mathbf{I} = (V, D, C)$ and let $\mathcal{T} = (T, (B_t)_{t\in V(T)}, (\gamma_t)_{t\in V(T)})$ be an optimal fractional hypertree decomposition of H. We prove the theorem by constructing a join decomposition $\mathcal{J} = (J, \varrho)$ for $\mathbf{I}$, whose width is at most $\mathsf{fhtw}(H)$. Let $\alpha : E(H) \to V(T)$ be some function from the edges of H to the nodes of T such that $e \subseteq B_{\alpha(e)}$ for every $e \in E(H)$. Note that such a function always exists, because $(T, (B_t)_{t\in V(T)})$ is a tree decomposition of H. We denote by $\alpha^{-1}(t)$ the set $\{\, e \in E(H) : \alpha(e) = t \,\}$.

The construction of $\mathcal{J}$ now proceeds in two steps. First we construct a partial join decomposition $\mathcal{J}^t = (J^t, \varrho^t)$ for $\mathbf{I}$ that covers only the constraints in $\alpha^{-1}(t)$, for every $t \in V(T)$. Second, we show how to combine all the partial join decompositions into the join decomposition $\mathcal{J}$ for $\mathbf{I}$ of width at most $\mathsf{fhtw}(H)$.

Let $t \in V(T)$ and let $\mathcal{J}^t = (J^t, \varrho^t)$ be an arbitrary partial join decomposition for $\mathbf{I}$ that covers the constraints in $\alpha^{-1}(t)$. Let us consider an arbitrary node $j \in V(J^t)$. By Lemma 5, we know that $C(j) = \pi_{S(j)}(\mathrm{SOL}(\mathbf{I}[V(j)]))$. Moreover, the fact that $\bigcup_{e\in\alpha^{-1}(t)} e \subseteq B_t$ implies $V(j) \subseteq B_t$. Since $|\pi_{S(j)}(\mathrm{SOL}(\mathbf{I}[V(j)]))| \leq |\mathrm{SOL}(\mathbf{I}[V(j)])|$, by invoking Proposition 1 we obtain that $|C(j)| \leq |\mathrm{SOL}(\mathbf{I}[V(j)])| \leq \sharp_{tup}(\mathbf{I})^{\mathsf{fhtw}(\mathbf{I})}$. Hence we conclude that $\mathsf{jw}(j) \leq \mathsf{fhtw}(H)$.

Next, we show how to combine the partial join decompositions $\mathcal{J}^t$ into the join decomposition $\mathcal{J}$ for $\mathbf{I}$. We will do this via a bottom-up algorithm that computes a (combined) partial join decomposition $\mathcal{F}^t = (F^t, \rho^t)$ (for every node $t \in V(T)$) that covers all constraints in $\alpha^{-1}(T_t) = \bigcup_{t\in V(T)} \alpha^{-1}(t)$. Initially, we set $\mathcal{F}^l = \mathcal{J}^l$ for every leaf $l \in L(T)$. For a non-leaf $t \in V(T)$ with children $t_1, \dots, t_\ell$ in T, we obtain $\mathcal{F}^t$ from the already computed partial join decompositions $\mathcal{F}^{t_1}, \dots, \mathcal{F}^{t_\ell}$ as follows. Let P be a path on the new vertices $p_1, \dots, p_\ell$ and let r_t and $r_{t_1}, \dots, r_{t_\ell}$ be the root nodes of J^t and $F^{t_1}, \dots, F^{t_\ell}$, respectively. Then we obtain F^t from the disjoint union of P, J^t, $F^{t_1}, \dots, F^{t_\ell}$ after adding an edge between r_t and p_1 and an edge between r_{t_i} and p_i for every i with $1 \leq i \leq \ell$ and setting p_ℓ to be the root of F^t. Moreover, ρ^t is obtained as the combination (i.e., union) of the functions ϱ^t, $\rho^{t_1}, \dots, \rho^{t_\ell}$. Observe that because α assigns every hyperedge to precisely one bag of T, it holds that every constraint assigned to T_t is mapped to precisely one leaf of $\mathcal{F}^t$. At this point, all that remains is to show that $\mathcal{F}^t$ has joinwidth at most $\mathsf{fhtw}(H)$.

Since we have already argued that $|\mathrm{SOL}(\mathbf{I}[V(j)])| \leq \sharp_{tup}(\mathbf{I})^{\mathsf{fhtw}(H)}$ for every node j of $\mathcal{J}^t$ and moreover we can assume that the same holds for every node j of $\mathcal{F}^{t_1}, \dots, \mathcal{F}^{t_\ell}$ by the induction hypothesis, it only remains to show that the same holds for the nodes $p_1, \dots, p_\ell$. First, observe that since $(T, (B_t)_{t\in V(T)})$ is a tree decomposition of H, it holds that $S(r_t), S(r_{t_1}), \dots, S(r_{t_\ell}) \subseteq B_t$. Indeed, consider for a contradiction that, w.l.o.g., there exists a variable $x \in S(r_{t_1}) \setminus B_t$. Then there must exist a hyperedge $e_1 \ni x$ mapped to t_1 or one of its descendants, and another hyperedge $e_i \ni x$ mapped to some node t' that is neither t_1 nor one of its descendants. But then both $B_{t'}$ and B_{t_1} must contain x, and so B_t must contain x as well. Moreover, since $S(p_i) \subseteq (S(r_t) \cup S(r_{t_1}) \cup \dots \cup S(r_{t_\ell}))$ for every $i \in [\ell]$, it follows that $S(p_i) \subseteq B_t$ as well.

Finally, recall that $C(p_i) = \pi_{S(p_i)}(\text{SOL}(\mathbf{I}[V(p_i)]))$ by Lemma 5, and observe that $|\pi_{S(p_i)}(\text{SOL}(\mathbf{I}[V(p_i)]))| \leq |\text{SOL}(\mathbf{I}[S(p_i)])|$. Then by Proposition 1 combined with the fact that $S(p_i) \subseteq B_t$, we obtain $|C(p_i)| \leq |\text{SOL}(\mathbf{I}[S(p_i)])| \leq \sharp_{tup}(\mathbf{I})^{\mathsf{fhtw}(H)}$, which implies that the width of p_i is indeed at most $\mathsf{fhtw}(H)$. $\square$

5.2 Functionality and Root Sets

Consider a CSP instance $\mathbf{I} = \langle V, D, C\rangle$ with $n = |V|$. We say that a constraint $c \in C$ is *functional* on variable $v \in V$ if c does not contain two tuples that differ *only* at variable v; more formally, for every t and $t' \in R(c)$ it holds that if $t[v] \neq t'[v]$, then there exists a variable $z \in S(c)$ distinct from v such that $t[z] \neq t'[z]$. The instance $\mathbf{I}$ is then called *functional* if there exists a variable ordering $v_1 < \cdots < v_n$ such that, for each $i \in [n]$, there exists a constraint $c \in C$ such that $\pi_{\{v_1,\dots,v_i\}}(c)$ is functional on v_i. Observe that every CSP instance that is functional can admit at most 1 solution [5]; this restriction can be relaxed through the notion of *root sets*, which can be seen as variable sets that form "exceptions" to functionality. Formally, a variable set Q is a root set if there exists a variable ordering $v_1 < \cdots < v_n$ such that, for each $i \in [n]$ where $v_i \notin Q$, there exists a constraint $c \in C$ such that $\pi_{\{v_1,\dots,v_i\}}(c)$ is functional on v_i; we say that Q is *witnessed* by the variable order $v_1 < \cdots < v_n$.

Functionality and root sets were studied for Boolean CSP [10,9]. Cohen et al. [5] later extended these notions to the CSP with larger domains. Our aim in this section is twofold: (1) generalize root sets through the introduction of *constraint root sets* and (2) show that bounded-size constraint root sets (and also root sets) form a special case of bounded joinwidth. Before we proceed, it will be useful to show that one can always assume the root set to occur at the beginning of the variable ordering.

Observation 13 *Let Q be a root set in $\mathbf{I}$ witnessed by a variable order α, assume a fixed arbitrary ordering on Q, and let the set $V' = V(\mathbf{I}) \setminus Q$ be ordered based on the placement of its variables in α. Then Q is also witnessed by the variable order $\alpha' = Q \circ V'$.*

For ease of presentation, we will say that $\mathbf{I}$ is *k-rooted* if k is the minimum integer such that $\mathbf{I}$ has a root set of size k. It is easy to see, and also follows from the work of David [9] and Cohen et al. [5], that for every fixed k the class of k-rooted CSP instances is polynomial-time solvable: generally speaking, one can first loop through and test all variable-subsets of size at most k to find a root Q, and then loop through all assignments $Q \to D$ to get a set of functional CSP instances, each of which can be solved separately in linear time.

While even 1-rooted CSP instance can have unbounded fractional hypertree width (see also the discussion of Cohen et al. [5]), the class of k-rooted CSP instances for a fixed value k is, in some sense, not very robust. Indeed, consider the CSP instance $\mathcal{W} = \langle\{v_1,\dots,v_n\},\{0,1\},\{c\}\rangle$ where c ensures that precisely a single variable is set to 1 (i.e., its relation can be seen as an $n \times n$ identity matrix). In spite of its triviality, it is easy to verify that $\mathcal{W}$ is not k-rooted for any $k < n-2$.

Let us now consider the following alternative to measuring the size of root sets in a CSP instance $\mathbf{I}$. A constraint set P is a *constraint-root set* if $\bigcup_{c\in P} S(c)$ is a root set, and $\mathbf{I}$ is then called k-constraint-rooted if k is the minimum integer such that $\mathbf{I}$ has a constraint-root set of size k. Since we can assume that each variable occurs in at least one constraint, every k-rooted CSP also has a constraint-root set of size at most k; on the other hand, the aforementioned example of $\mathcal{W}$ shows that an instance can be 1-constraint-rooted while not being k-rooted for any small k. The following result, which we prove by using join decompositions and joinwidth, thus gives rise to strictly larger tractable classes than those obtained via root sets:

Proposition 14 *For every fixed $k \in \mathbb{N}$, every k-constraint-rooted CSP instance has joinwidth at most k and can be solved in time $|\mathbf{I}|^{\mathcal{O}(k)}$.*

Proof. Consider a CSP instance $\mathbf{I}$ with a constraint-root set P of size k. We argue that $\mathbf{I}$ has a linear join decomposition of width at most k where the elements of P occur as the leaves farthest from the root. Indeed, consider the linear join decomposition (J, ϱ) constructed in a bottom-up manner, as follows. First, we start by gradually adding the constraints in P as the initial leaves. At each step after that, consider a node j which is the top-most constructed node in the join decomposition. By definition, there must exist a variable v and a constraint c such that $\pi_{\bigcup_{c\in P} S(c)}(c)$ is functional on v. Moreover, this implies that $|\pi_{\bigcup_{c\in P} S(c)}(c) \bowtie C(j)| \leq |C(j)|$, and thus $|c \bowtie C(j)| \leq |C(j)|$. Hence this procedure does not increase the size of constraints at nodes after the initial k constraints, immediately resulting in the desired bound of k on the width of (J, ϱ).

To complete the proof, observe that a join decomposition with the properties outlined above can be found in time at most $|\mathbf{I}|^{\mathcal{O}(k)}$: indeed, it suffices to branch over all k-element subsets of $C(\mathbf{I})$ and test whether the union of their scopes is functional using, e.g., the result of Cohen et al. [5, Corollary 1]. Once we have such a join decomposition, we can solve the instance by invoking Theorem 6. $\square$

As a final remark, we note that the class of k-constraint rooted CSP instances naturally includes all instances which contain k constraints that are in conflict (i.e., which cannot all be satisfied at the same time).

5.3 Other Tractable Classes

Here, we identify some other classes of tractable CSP instances with bounded joinwidth. First of all, we consider CSP instances such that introducing their variables in an arbitrary order always results in a subinstance with polynomially many solutions. In particular, we call a CSP instance $\mathbf{I}$ *hereditarily k-bounded* if for every subset V' of its variables it holds that $|\text{SOL}(\mathbf{I}[V'])| \leq \sharp_{tup}(\mathbf{I})^k$. Examples of hereditarily k-bounded CSP instances include k-Turan CSPs [5, page 12] and CSP instances with fractional edge covers of weight k [20].

Proposition 15 *The class of hereditarily k-bounded CSP instances has joinwidth at most k and can be solved in time at most $\mathcal{O}(|\mathbf{I}|^k)$.*

Proof. Consider an arbitrary linear join decomposition (J, ϱ). By definition, for each $j \in V(J)$ it holds that $|\text{SOL}(\mathbf{I}[V(j)])| \leq \sharp_{tup}(\mathbf{I})^k$. Then $|\pi_{\overline{V}(j)}(\text{SOL}(\mathbf{I}[V(j)]))| \leq \sharp_{tup}(\mathbf{I})^k$, and by Lemma 5 we obtain $|C(j)| \leq \sharp_{tup}(\mathbf{I})^k$, as required. $\square$

Another example of a tractable class of CSP instances that we can solve using joinwidth are instances where all constraints interact in a way which forces a unique assignment of the variables. In particular, we say that a CSP $\mathbf{I} = \langle V, D, C\rangle$ is *unique at depth* k if for each constraint $c \in C$ there exists a *fixing set* $C' \subseteq C$ such that $c \in C'$, $|C'| \leq k$, and $|(\bowtie_{c' \in C'} c')| \leq 1$.

Proposition 16 *The class of CSP instances which are unique at depth k has joinwidth at most k and can be solved in time at most $|\mathbf{I}|^{\mathcal{O}(k)}$.*

6 Solving Bounded-Width Instances

This section investigates the tractability of CSP instances whose joinwidth is bounded by a fixed constant ω. In particular, one can investigate two notions of tractability. The first one is the classical notion of *polynomial-time tractability*, which asks for an algorithm of the form $|\mathbf{I}|^{\mathcal{O}(1)}$. In this setting, the complexity of CSP instances of bounded joinwidth remains an important open problem. Note that the NP-hardness of the ω-JOIN DECOMPOSITION problem established in Theorem 7 does not exclude polynomial-time tractability for CSP instances of bounded joinwidth. For instance, tractability could still be obtained with a suitable approximation algorithm for computing join decompositions (as it is the case for fractional hypertreewidth [25]) or by using an algorithm that does not require a join decomposition of bounded width as input.

The second notion of tractability we consider is called *fixed-parameter tractability* and asks for an algorithm of the form $f(k) \cdot |\mathbf{I}|^{\mathcal{O}(1)}$, where k is a numerical parameter capturing a certain natural measure of $\mathbf{I}$. Prominently, Marx investigated the fixed-parameter tractability of CSP and showed that CSP instances whose hypergraphs have bounded *submodular width* [27] are fixed-parameter tractable when k is the number of variables. Moreover, Marx showed that submodular width is the most general structural property *among those measured purely on hypergraphs* with this property.

Here, we obtain two single-exponential fixed-parameter algorithms for CSP instances of bounded joinwidth (i.e., algorithms with a running time of $2^{\mathcal{O}(k)} \cdot |\mathbf{I}|^{\mathcal{O}(1)}$): one where k is the number of variables, and the other where k is the number of constraints. Since there exist classes of instances of bounded joinwidth and unbounded submodular width (see Observation 11), this expands the frontiers of (fixed-parameter) tractability for CSP.

Parameterization by Number of Constraints. To solve the case where k is the number of constraints, our primary aim is to obtain a join decomposition of width at most ω, i.e., solve the ω-JOIN DECOMPOSITION problem defined in Section 3. Indeed, once that is done we can solve the instance by Theorem 6.

Theorem 17 ω-JOIN DECOMPOSITION *can be solved in time $\mathcal{O}(4^{|C|} + 2^{|C|}|\mathbf{I}|^{2\omega+1})$ and is hence fixed-parameter tractable parameterized by $|C|$, for a CSP instance $\mathbf{I} = \langle V, D, C\rangle$.*

From Theorem 17 and Theorem 6 we immediately obtain:

Corollary 18 *A CSP instance $\mathbf{I}$ with k constraints and joinwidth at most ω can be solved in time $2^{\mathcal{O}(k)} \cdot |\mathbf{I}|^{\mathcal{O}(\omega)}$.*

Parameterization by Number of Variables. Note that Corollary 18 immediately establishes fixed-parameter tractability for the problem when k is the number of variables (instead of the number of constraints), because one can assume that $|C| \leq 2^{|V|}$ for every CSP instance $\mathbf{I} = (V, D, C)$. However, the resulting algorithm would be double-exponential in $|V|$. The following theorem shows that this can be avoided by designing a dedicated algorithm for CSP parameterized by the number of variables. The main idea behind both algorithms is dynamic programming, however, in contrast to the algorithm for $|C|$, the table entries for the fpt-algorithm for $|V|$ correspond to subsets of V instead of subsets of C. Interestingly, the fpt-algorithm for $|V|$ does not explicitly construct a join decomposition, but only implicitly relies on the existence of one.

Theorem 19 *A CSP instance* $\mathbf{I}$ *with* k *variables and joinwidth at most* ω *can be solved in time* $2^{\mathcal{O}(k)} \cdot |\mathbf{I}|^{\mathcal{O}(\omega)}$.

7 Beyond Join Decompositions

Due to their natural and "mathematically clean" definition, one might be tempted to think that join decompositions capture all the algorithmic power offered by join and projection operations. It turns out that this is not the case, i.e., we show that if one is allowed to use join and projections in an arbitrary manner (instead of the more natural but also more restrictive way in which they are used within join decompositions) one can solve CSP instances that are out-of-reach even for join decompositions. This is interesting as it points towards the possibility of potentially more powerful parameters based on join and projections than joinwidth.

Theorem 20 *For every* ω*, there exists a CSP instance* $\mathbf{I}_\omega$ *that can be solved in time* $\mathcal{O}(|\mathbf{I}|^4)$ *using only join and projection operations but* $\mathsf{jw}(\mathbf{I}_\omega) \geq \omega$.

8 Conclusions and Outlook

The main contribution of our paper is the introduction of the notion of a join decomposition and the associated parameter joinwidth (Definitions 2 and 3). These notions are natural as they are entirely based on fundamental operations of relational algebra: joins, projections, and pruning (which can equivalently be stated in terms of semijoins). It is also worth noting that our algorithms seamlessly extend to settings where each variable has its own domain (this can be modeled, e.g., by unary constraints).

Our results give rise to several interesting directions for future work. We believe that Theorem 6 can be generalized to other problems, such as #CSP or the FAQ-Problem [23]. Theorem 7 gives rise to the question of whether there exists a polynomial-time approximation algorithm for computing join decompositions of suboptimal joinwidth, similar to Marx's algorithm for fractional hypertree-width [25].

Observation 11 shows that submodular width is not more general than joinwidth. We conjecture that also the converse direction holds, i.e., that the two parameters are actually incomparable. Motivated by Theorem 20, one could try to define a natural parameter that captures the full generality of join and projection operations, or to at least define a parameter that is more general than join decompositions without sacrificing the simplicity of the definition.

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