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A Model for Finding Transition-Minors

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Abstract

The well known cycle double cover conjecture in graph theory is strongly related to the compatible circuit decomposition problem. A recent result by Fleischner et al. (2018) gives a sufficient condition for the existence of a compatible circuit decomposition in a transitioned 2-connected Eulerian graph, which is based on an extension of the definition of K_5-minors to transitioned graphs. Graphs satisfying this condition are called SUD-K_5-minor-free graphs. In this work we formulate a generalization of this property by replacing the K_5 by a 4-regular transitioned graph H, which is part of the input. Furthermore, we consider the decision problem of checking for two given graphs if the extended property holds. We prove that this problem is NP-complete and fixed parameter tractable with the size of H as parameter. We then formulate an equivalent problem, present a mathematical model for it, and prove its correctness. This mathematical model is then translated into a mixed integer linear program (MIP) for solving it in practice. Computational results show that the MIP formulation can be solved for small instances in reasonable time. In our computations we found snarks with perfect matchings whose contraction leads to SUD-K_5-minor-free graphs that contain K_5-minors. Furthermore, we verified that there exists a perfect pseudo-matching whose contraction leads to a SUD-K_5-minor-free graph for all snarks with up to 22 vertices.

Keywords: Transition Minor, Cycle Double Cover, Compatible Circuit Decomposition, Integer Programming

1. Introduction

The cycle double cover conjecture (CDCC) is a famous conjecture in graph theory. It states that for every bridgeless undirected graph there exists a multiset of cycles in the graph covering every edge exactly twice. This already over 40 years old conjecture is well studied and was originally posed by Szekeres [12] and Seymour [11]. Jaeger reduced the problem from general bridgeless graphs to the class of all snarks [6]. There are several similar definitions of snarks, but we will use the same definition as Jaeger does: A snark

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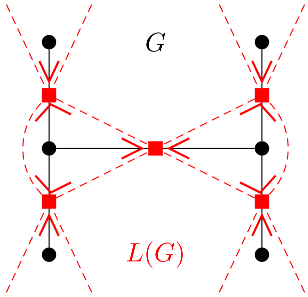

Figure 1: Transformation of a part of a 3-regular graph G into its line graph $L(G)$ and two transitions per vertex, represented by a red vee ($\vee$) between their two edges. The vertices of G are represented by black circles, the edges of G by black solid lines, the vertices of $L(G)$ by red squares, and the edges of $L(G)$ by red dashed lines.

is a simple cyclically 4-edge-connected cubic graph with chromatic index four. Note that a *cyclically k-edge connected graph* is a graph where at most one component contains cycles after removing fewer than k edges.

Finding a cycle double cover in a snark, or in general in a 3-regular graph, is correlated to the compatible circuit decomposition (CCD) problem, which is formulated on graphs with a transition system. A transition system is a collection of transitions, where each transition represents a set of two adjacent edges. For a formal definition see Definition 2 in Section 2.2 The CCD problem asks for a given 2-connected Eulerian graph G and a transition system $\mathcal{T}$ if there exists a set of circuits in G such that none of its circuits contains both edges of any transition T in $\mathcal{T}$.

Consider for a 3-regular graph G its 4-regular line graph $L(G)$ together with two transitions per vertex as shown in Figure 1. If the line graph together with the given transition system contains a compatible circuit decomposition, one can construct a cycle double cover of the original graph G.

Another correlation between the CCD problem and the CDC problem for 3-regular graphs can be seen by the following construction. Let G be a 3-regular graph and H the 4-regular graph obtained from G after contracting each edge of a perfect matching M of G. Now we define a transition system on H by adding transitions between two edges if and only if their corresponding edges in G are adjacent. Then, if H contains a compatible cycle decomposition, one can construct from it a cycle double cover in the original graph G containing the 2-factor $Q = E(G) - M$ as a subset. This construction can even be extended to contractions of perfect pseudo-matchings, which are spanning subgraphs, where each component is either a K_2 or a claw, which consists of a vertex of degree three connected to three vertices of degree one. In this case the resulting graph H has vertices of degree four or six.

Therefore, to prove the existence of a cycle double cover in a graph it suffices to find a perfect pseudo-matching such that its contraction leads to a transitioned graph containing a CCD. Since the number of vertices of the contracted graph is at most half of the number of vertices of the original graph, it may be faster to find a CCD in the much smaller graph than a CDC in the original graph. On the other hand, if a snark contains a CDC one cannot conclude that it also contains a perfect matching whose contraction leads to a graph with a CCD; the Peterson graph is a counter example. If we allow perfect pseudo-matchings, this direction is still an open problem.

Already in 1980 Fleischner [3] proved that every 2-connected planar Eulerian graph has a CCD regardless of the structure of the transition system. This result was generalized by Fan and Zhang [2] who proved that whenever the graph has no K_5-minor it has a CCD. These two results both only use the graph structure and ignore the transition system. In order to include the structure of the transition system, Fleischner et al. [4] extended the definition of transition minors to transitioned graphs and proved that if a transitioned 2-connected Eulerian graph is SUD-K_5-minor free, it contains a CCD, which is a generalization of Fan and Zhang's result. For a definition of SUD-K_5-minor free graphs see Definition 8 and Example 1 in the next Section.

This recent result of Fleischner et al. leads to the question of how to check for a graph if it contains a SUD-K_5-minor. This task is not easy because of the complex nature of the definition of a SUD-K_5-minor. Because of this difficulty Fleischner et al. could not provide a snark and a perfect pseudo-matching whose contraction leads to a graph that is SUD-K_5-minor free but has a K_5-minor, i.e. an example in the context of snarks where the new theorem is stronger than the old theorem.

In this article we analyze the problem of finding a SUD-K_5-minor and formulate a generalization that checks if a graph is sup-$(H, \mathcal{S})$-minor free. Furthermore, we prove some complexity results for this problem concerning NP-hardness and fixed parameter tractability. Finally, we describe a practical algorithm to solve it. With that algorithm we were able, among other things, to find a snark and a perfect matching whose contraction leads to a graph that is SUD-K_5-minor free but has a K_5-minor.

There is no literature yet that is concerned with finding SUD-K_5-minors, although the problem of finding K_5-minors is well analyzed. Robertson and Seymour proved that checking if a graph contains a K_5-minor can be done in polynomial time [10]. We will use the same proof-idea to prove that checking whether a graph contains a SUD-K_5-minor can be done in polynomial time. The polynomial algorithm described in the proof of Robertson and Seymour is not practically applicable since its computation time has large constants and polynomial factors. A more practical algorithm was provided by Reed and Li who proved that checking if a graph contains a K_5-minor can be done in linear time [9]. This algorithm heavily depends on the fact that a 4-connected graph contains no K_5-minor if and only if it is planar and that checking planarity can be done in linear time. Since there is no known extension of planarity to transitioned graphs that would lead to a connection between planarity and the existence of SUD-K_5-minors, this linear time approach cannot easily be extended to checking the existence of SUD-K_5-minors.

In the next section we present the graph theoretic definitions and notations we are using. In Section 3 we introduce the problem ESTM, its equivalent problem EBSRTM and prove the complexity results. A mathematical model for EBSRTM is then presented in Section 4.2, which also includes the correctness proof of the model. Based on the mathematical model a mixed integer linear programming model is presented in Section 5. The computational results of the MIP model are presented in Section 6. Finally, we conclude with Section 7 and sketch possible future work.

2. Preliminary Discussion

In this section we present all graph theoretical definitions needed for formulating our problem. Furthermore, we introduce some terminology and notation that will be used later on.

2.1. Terminology and Notation

In this article whenever we refer to a graph we mean an undirected multigraph without loops if not specified otherwise. Let $\mathcal{P}_2(X)$ denote the set of all unordered pairs of elements in X, i.e. $\mathcal{P}_2(X) = \{S \subseteq X \mid |S| = 2\}$. We denote a graph by $G = (V, E, r)$, where V is the vertex set, E the set of edges and $r \colon E \to \mathcal{P}_2(V)$ a function that maps each edge to its two incident vertices. Note that by this notation we exclude loops. We abbreviate $e = uv$ if $e \in E$ is incident to $u, v \in V$, i.e. $r(e) = \{u, v\}$. If $e = uv$ and $f = uv$ for $e, f \in E$ with $e \neq f$ we call e, f *parallel* edges. To distinguish between different graphs we also use the notation $G = (V_G, E_G, r_G)$ or we simply write $V(G) = V_G$ and $E(G) = E_G$. Furthermore, for a vertex $v \in V$ we write $E(v) = \{e \in E \mid v \in r(e)\}$ for the set of all edges incident to v and $N(v) = \{v' \in V \mid \exists e \in E : e = vv'\}$ for the set of all neighbors of v.

For a partial function $\alpha \colon A \nrightarrow B$ we will use the following notations. For a subset $X \subseteq A$ we write $\alpha[X] := \{b \in B \mid \exists a \in X : b = \alpha(a)\}$ for the image of X under α and for $a \in A$ we simply write $\alpha[a] := \alpha[\{a\}]$. Similarly, for a subset $Y \subseteq B$ we write $\alpha^{-1}[Y] := \{a \in A \mid \alpha(a) \in Y\}$ for the preimage of Y under α and for $b \in B$ we simply write $\alpha^{-1}[b] := \alpha^{-1}[\{b\}]$. Note that $\alpha^{-1}[b]$ may be empty if α is not surjective. If α is injective we denote by $\alpha^{-1} \colon B \nrightarrow A$ the inverse partial function of α and $\alpha^{-1}(b) = a$ if and only if $\alpha(a) = b$. Furthermore, we denote by $\mathrm{dom}(\alpha) = \alpha^{-1}[B] \subseteq A$ the domain of α.

2.2. Basic Definitions

Definition 1 (Eulerian Graph). A connected graph is called an *Eulerian graph* if every vertex has even degree.

We define a transition system as in [4] but use a different notation, which will be useful later on.

Definition 2 (Transition System). Let G be a graph. A *transition system* of G is a set $\mathcal{T} \subseteq V \times \mathcal{P}_2(E)$ of *transitions* that satisfies the following. Each transition $T \in \mathcal{T}$ with $T = (v, \{e_1, e_2\})$ has to satisfy $\{e_1, e_2\} \subseteq E(v)$. We use the projections $\pi_1(T) := v$ and $\pi_2(T) := \{e_1, e_2\}$ to denote the values of T.

Furthermore, we write $\mathcal{T}(v) := \{T \in \mathcal{T} \mid \pi_1(T) = v\}$ for the set of all transitions at vertex v. The transitions at a vertex v must all be edge-disjoint, i.e.

$$\pi_2(T_1) \cap \pi_2(T_2) = \emptyset \quad \forall v \in V, \forall T_1 \in \mathcal{T}(v), \forall T_2 \in \mathcal{T}(v) : T_1 \neq T_2$$

The graph G with a non-empty transition system $\mathcal{T}$ is called a *transitioned graph* and denoted by $(G, \mathcal{T})$. A *completely transitioned graph* is a transitioned graph where for each vertex each incident edge is in one transition of the vertex, i.e. $E(v) = \bigcup_{T \in \mathcal{T}(v)} \pi_2(T)$ for all $v \in V(G)$. For every subgraph H of G, $\mathcal{T}|_H = \{T \in \mathcal{T} \mid \pi_2(T) \subseteq E(H)\}$. Clearly, a connected completely transitioned graph is Eulerian.

The following two definitions are adopted from [4].

Definition 3 (Separator). Let G be a graph. A vertex subset U is a *separator of G separating G to G_1, G_2* if $E(G) = E(G_1) \cup E(G_2)$, $V(G_1) \cap V(G_2) = U$, and $E(G_1) \cap E(G_2) = \emptyset$. We call U a *t-separator* if $|U| = t$. We say a *separator U separating subgraphs X_1, X_2* of G if U is a separator of G separating G to G_1, G_2 with $X_i \subseteq G_i$, $i = 1, 2$.

Definition 4 (Bad-Cut-Vertex). Let $(G, \mathcal{T})$ be a transitioned graph. A 1-separator $\{v\}$ separating G to G_1, G_2 is a *bad-cut-vertex* if $(v, E(v) \cap E(G_1)) \in \mathcal{T}(v)$ implying that $|E(v) \cap E(G_1)| = 2$.

Definition 5 (Minor). Let G be a graph. H is a *minor* of G if and only if H can be derived from G by deletion of vertices, deletion of edges and contraction of edges.

For our purposes we will use the following equivalent representation of H. The vertices of H correspond to non-empty vertex-disjoint connected subgraphs of G. We can formalize this by a partial surjective function $\varphi\colon V(G) \nrightarrow V(H)$, which maps vertices in G to vertices in H. Note that φ is a partial function which means that there might be vertices in G that do not get mapped on vertices in H. Furthermore, the preimage $\varphi^{-1}[w] \subseteq V(G)$ for each $w \in V(H)$ must be connected in G.

The edges of H correspond to edges of G. We can again formalize this by a partial injective and surjective function $\kappa\colon E(G) \nrightarrow E(H)$, which maps an edge in G to its corresponding edge in H. The end vertices of an edge $\kappa(e) \in E(H)$ correspond to the connected subgraphs that contain the end vertices of the edge e in G. Formally this means

$$r_H(\kappa(e)) = \varphi\left[r_G(e)\right] \quad \forall e \in \mathrm{dom}(\kappa). \tag{1}$$

Note, that we also do not allow loops for minors, even if you could generate one by contracting some edges. Next we define a transition minor as in [4] but with a different notation.

Definition 6 (Transition Minor). Let $(G, \mathcal{T})$ be a transitioned graph and H a minor of G with correspondence maps φ and κ. We define a transition system $\mathcal{S}$ on H as follows. We keep all transitions whose edges do not get deleted or contracted; formally that means:

$$\mathcal{S}' := \left\{(\varphi(\pi_1(T)), \kappa[\pi_2(T)]) \mid T \in \mathcal{T}, \pi_1(T) \in \mathrm{dom}(\varphi), \pi_2(T) \subseteq \mathrm{dom}(\kappa)\right\}. \tag{2}$$

The transitioned graph $(H, \mathcal{S}')$ is called a *reduced transition minor of* $(G, \mathcal{T})$.

If $w \in V(H)$ is a vertex of degree four and there exists a transition between two of the incident edges, we also want to add a transition between the other two edges. Formally we get all in all the following transition system

$$\mathcal{S} := \mathcal{S}'(w) \cup \{(w, E(w) \setminus \pi_2(T)) \mid w \in V(H), \deg(w) = 4, T \in \mathcal{S}'(w)\}\,. \tag{3}$$

We call the transition graph $(H, \mathcal{S})$ a *transition-minor* of $(G, \mathcal{T})$.

The next two definitions are generalizations of the definitions of a SUD-K_5 and of SUD-K_5-transition-minor free from [4].

Definition 7 (Sup-$(H, \mathcal{S})$). Let $(H, \mathcal{S})$ be a completely transitioned 4-regular graph. A transitioned graph $(H', \mathcal{S}')$ is a *sup*-$(H, \mathcal{S})$ graph if the following holds.

The graph H' can be decomposed into $|E(H)| + |V(H)|$ connected edge-disjoint subgraphs

$$\{P_f \mid f \in E(H)\} \cup \{Q_w \mid w \in V(H)\}$$

as follows.

1. The graphs $\{Q_w \mid w \in V(H)\}$ are vertex-disjoint connected subgraphs of H'.
2. Each P_f for $f \in E(H)$ with $f = w_1 w_2$ is a path in H' joining $V(Q_{w_1})$ and $V(Q_{w_2})$, and all $\{P_f \mid f \in E(H)\}$ are internally disjoint.
3. Let Q_w^+ be the subgraph of H' induced by $E(Q_w)$ and the four adjacent paths P_f for $f \in E(w)$. Furthermore, let $\mathcal{S}(w) = \{S_w^1 = (w, \{f_1, f_2\}), S_w^2 = (w, \{f_3, f_4\})\}$. Then the subgraph Q_w^+ has a bad 1-separator $\{u_w\}$ separating H_w^1 and H_w^2 such that $P_{f_1} \cup P_{f_2} \subseteq H_w^i$ and $P_{f_3} \cup P_{f_4} \subseteq H_w^{3-i}$ for some $i \in \{1, 2\}$.

Definition 8 (Sup-$(H, \mathcal{S})$-Transition-Minor free). Let $(H, \mathcal{S})$ be a completely transitioned 4-regular graph. A transitioned graph $(G, \mathcal{T})$ is *sup-$(H, \mathcal{S})$-transition-minor free* if and only if it does not have any Eulerian transition-minor $(H', \mathcal{S}')$ that is a sup-$(H, \mathcal{S})$ graph.

Example 1 (SUD-K_5). The completely transitioned four-regular graph $(H, \mathcal{S})$, called *undecomposable* K_5 or short UD-K_5, is defined by $H = K_5$ and $\mathcal{S} = \{(v_i, \{v_{i-1}v_i, v_iv_{i+1}\}), (v_i, \{v_{i-2}v_i, v_iv_{i+2}\}) \mid i \in \mathbb{Z}_5\}$, see Figure 2. With this notation a sup-$(H, \mathcal{S})$ graph is called a *sup-undecomposable* K_5 or short SUD-K_5. If a graph is sup-$(H, \mathcal{S})$-transition-minor free it is called a *SUD-K_5-minor-free graph*.

3. Problem Formulation

We are focusing in this work primarily on the following question.

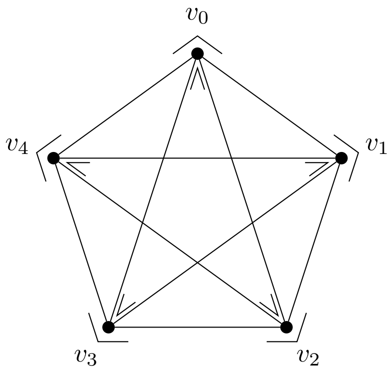

Figure 2: The completely transitioned graph UD-K_5 with transitions represented by a vee ($\vee$) between their two edges.

Problem 1 (Existence of Sup-Transition-Minors (ESTM)). Given a transitioned graph $(G, \mathcal{T})$ and a completely transitioned 4-regular graph $(H, \mathcal{S})$, does there exist an Eulerian transition minor of $(G, \mathcal{T})$ that is a sup-$(H, \mathcal{S})$ graph?

Note that ESTM is the inverse problem of asking if a transitioned graph $(G, \mathcal{T})$ is sup-$(H, \mathcal{S})$-transition-minor free. Next we will prove the following two complexity theorems.

Theorem 1. *ESTM is* NP*-complete.*

Theorem 2. *ESTM restricted to simple graphs is* NP*-complete.*

Formally it would be enough to prove that ESTM is in NP and that ESTM restricted to simple graphs is NP-hard. But since the NP-hardness proof for ESTM restricted to simple graphs is based on the same idea as the NP-hardness proof for the general ESTM, we first proof the more basic statement for the general ESTM.

We will use the following lemmas to proof Theorem 1.

Lemma 1. *If a graph H is a minor of a graph G with $|V(H)| = |V(G)|$, then H is a subgraph of G.*

Proof. We cannot contract any edges in G to get H, since that would reduce the number of vertices. Therefore, we only remove edges from G to get H, which results in a subgraph of G.

Lemma 2. *Let $(H, \mathcal{S})$ be a completely transitioned graph and $(H', \mathcal{S}')$ a sup-$(H, \mathcal{S})$ graph. Then H is a minor of H'.*

Proof. By contracting the paths P_f to one edge f and the connected subgraphs Q_w to one vertex w we get exactly H. Therefore, H is a minor of H'.

Lemma 3. *The minor relation is transitive, i.e. if H is a minor of H' and H' is a minor of G, then H is also a minor of G.*

Proof. A minor can be constructed by a finite number of edge removals, vertex removals and edge contractions. If we apply the steps from G to H' and then the steps from H' to H we still only applied a finite number of steps and got from G to H. Therefore, H is also a minor of G.

Proof of Theorem 1. First we prove that ESTM is in NP. As a solution representation we use a transitioned graph $(H', \mathcal{S}')$ with at most $|V(G)|$ vertices together with the minor correspondence maps $\varphi\colon V(G) \nrightarrow V(H')$ and $\kappa\colon E(G) \nrightarrow E(H')$, the decomposition $\{P_f \mid f \in E(H)\} \cup \{Q_w \mid w \in V(H)\}$ of H' and the sequence of 1-separator vertices $(u_w)_{w\in V(H)}$. The size of the solution representation is polynomial in the input size. Furthermore, checking for a given solution representation if $(H', \mathcal{S}')$ is by the embeddings φ and κ an Eulerian transition minor of $(G, \mathcal{T})$ can be done in polynomial time. Last but not least, using the given decomposition and 1-separator vertices it can be checked in polynomial time if $(H', \mathcal{S}')$ is a sup-$(H, \mathcal{S})$-graph. Therefore, ESTM is in NP.

To prove that ESTM is NP-hard we define a polynomial-time reduction from the NP-hard Hamiltonian cycle problem to ESTM. Let G be a simple graph for which we want to check if it contains a Hamiltonian cycle. We define a double cycle graph H with $n := |V(G)|$ vertices, i.e.

$$V(H) := \{1, ..., n\}.$$

The edges of H form two Hamiltonian cycles in H, i.e.

$$E(H) = \{e_1, ..., e_{2n}\}$$

with $r(e_i) = r(e_{i+n}) = \{i, i+1\}$ for $i < n$ and $r(e_n) = r(e_{2n}) = \{1, n\}$. Furthermore, we define a complete transition system $\mathcal{S}$ on H by adding transitions between all parallel edges of H, i.e.

$$\mathcal{S}(i) := \{(i, \{e_i, e_{i+n}\}), (i, \{e_{i-1}, e_{i-1+n}\})\} \quad \forall i > 1$$

and

$$\mathcal{S}(1) = \{(1, \{e_1, e_{n+1}\}), (1, \{e_n, e_{2n}\})\}.$$

Additionally, we define a transitioned graph $(G_2, \mathcal{T})$ by duplicating all edges in G and adding transitions between them, i.e.

$$V(G_2) = V(G), E(G_2) = \{e, e' \mid e \in E(G)\}, \mathcal{T}(v) = \{(v, \{e, e'\}) \mid e \in E(v)\} \quad \forall v \in V(G_2)$$

Together $(G_2, \mathcal{T})$ and $(H, \mathcal{S})$ form an instance of ESTM. What is left to prove is that $(G_2, \mathcal{T})$ has an Eulerian transition minor that is a sup-$(H, \mathcal{S})$ graph if and only if G has a Hamiltonian cycle.

If $(G_2, \mathcal{T})$ has an Eulerian transition minor $(H', \mathcal{S}')$ that is a sup-$(H, \mathcal{S})$ graph then we get by Lemma 2 that H is a minor of H' and by Lemma 3 that H is a minor of G_2. Since $V(H) = n = V(G_2)$ we get by

Lemma 1 that H is a subgraph of G_2. This implies that G_2 has a Hamiltonian cycle and therefore G has one, too.

On the other hand, if G has a Hamiltonian cycle C, taking the edges in C and the duplicates of them we get a subgraph H' of G_2 that is isomorphic to H. By adding all transitions between two parallel edges in H' to a transition system $\mathcal{S}'$ on H' we get that $(H', \mathcal{S}')$ is an Eulerian transition minor of G_2 and that the transition system $\mathcal{S}'$ corresponds to the transition system $\mathcal{S}$ on H. Therefore, $(H', \mathcal{S}')$ is trivially a sup-$(H, \mathcal{S})$ graph.

Proof of Theorem 2. In Theorem 1, we already proved that ESTM is in NP which implies that it is also in NP if we restrict it to simple graphs. Similar to the proof of Theorem 1 we prove hardness by reducing the Hamiltonian cycle problem to ESTM. Let G be an instance of a Hamiltonian cycle problem. Instead of replacing edges in G by double edges as we did in the proof of Theorem 1, we replace this time each edge e by a subgraph A_e. Each such subgraph A_e contains a K_4 of which two vertices are connected to the one end vertex of e and the other two edges are connected to the other end vertex of e, see Figure 3. Let G' be the resulting graph after replacing all edges e in G by the respective subgraphs A_e. We will call the vertices in G' that correspond to a vertex in G original vertices.

Furthermore, let H be a cycle of length $n = |V(G)|$ and let H' be the result after replacing all edges e in H by the subgraph A_e. We define the transition systems $\mathcal{T}$ and $\mathcal{S}$ on G' and H' by adding for each subgraph A_e the transitions as shown in Figure 3. The obtained graph H' is 4-regular and $(H', \mathcal{S})$ is completely transitioned. All in all we get that $((G', \mathcal{T}), (H', \mathcal{S}))$ is an instance of ESTM restricted to simple graphs.

We will prove now that G has a Hamiltonian cycle if and only if there exists an Eulerian transition minor of $(G', \mathcal{T})$ that is a sup $(H', \mathcal{S})$ graph, i.e. that $((G', \mathcal{T}), (H', \mathcal{S}))$ is a positive instance of ESTM.

$\Rightarrow$: Assume G has a Hamiltonian cycle C. We construct a corresponding subgraph H'' in G' by adding for each edge $e \in E(C)$ the subgraph A_e to H''. Since the length of the cycle C is also n we get that H'' is isomorphic to H'. Furthermore, the transitions in H'' correspond to the transitions in H' and therefore H'' is trivially a sup-$(H', \mathcal{S})$ graph. On the other hand, due to its construction H'' together with all transitions in H'' is an Eulerian transition minor of G'.

$\Leftarrow$: Let $(H'', \mathcal{S}')$ be an Eulerian transition minor of $(G', \mathcal{T})$ and a sup-$(H', \mathcal{S})$ graph. Note that there exists a Hamiltonian path in each A_e, $e \in E(H')$ from v_1 to v_2 and therefore H' has a Hamiltonian cycle. The graph H' has $|V(H)| + 4|E(H)| = n + 4n = 5n$ vertices and therefore the Hamiltonian cycle is of length $5n$. By definition of a sup-$(H', \mathcal{S})$ graph the cycle in H' corresponds to a cycle in H'' with at least the same number of vertices. Therefore, H'' has a cycle of length at least $5n$. Since H'' is a minor of G', we get that also G' contains a cycle C' of length at least $5n$.

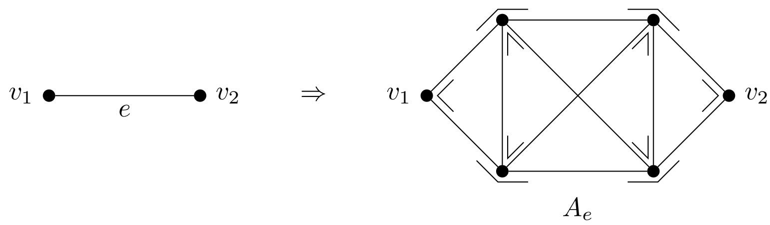

Figure 3: The graph A_e, which replaces an edge e in G and H. The transitions are represented by a vee ($\vee$) between their two edges.

Let $(v_1, ..., v_N)$ be the vertex sequence of C' with $N \geq 5n$. By construction of G' five consecutive vertices in C' contain at least one original vertex. Thus, C' contains at least n original vertices, i.e., all original vertices. Furthermore, if v_i and v_j are original vertices and all vertices between them in C', i.e. v_k for $i < k < j$, are not original, then v_i and v_j must both be part of a common subgraph A_e, and therefore they are connected by e in G. Thus, if we remove all non-original vertices from the vertex sequence $(v_1, ..., v_N)$ we get a new vertex sequence in G of length n in which all consecutive vertices are connected. Therefore, G is a Hamiltonian graph.

3.1. Problem Transformation

Definition 9 (Basic-Sup-$(H, \mathcal{S})$). Let $(H, \mathcal{S})$ be a completely transitioned 4-regular graph. A transitioned graph $(H', \mathcal{S}')$ is a *basic-sup*-$(H, \mathcal{S})$ graph if the following holds.

The graph H' can be decomposed into $|V(H)|$ many connected vertex-disjoint subgraphs

$$\{R_w \mid w \in V(H)\}$$

and $|E(H)|$ many edges $\{f' \mid f \in E(H)\}$. For each edge $f = w_1 w_2 \in E(H)$ the edge f' connects the subgraphs R_{w_1} and R_{w_2}. For a vertex $w \in V(H)$ let R_w^+ be the subgraph of H' induced by $E(R_w)$ and the four adjacent edges f' for $f \in E(w)$.

There exists an ordering of the outgoing edges of w by

$$E(w) = \{f_1, f_2, f_3, f_4\}$$

such that the following holds.

- The two transitions in $\mathcal{S}(w)$ are $S_w^1 = (w, \{f_1, f_2\})$ and $S_w^2 = (w, \{f_3, f_4\})$.
- There exists a transition S'_w in $\mathcal{S}'$ such that the form of R_w and the transition S'_w satisfy one of the four possibilities (see Figure 4):
  1. R_w is only one vertex w' and $S'_w = (w', \{f'_1, f'_2\}) \in \mathcal{S}'(w')$

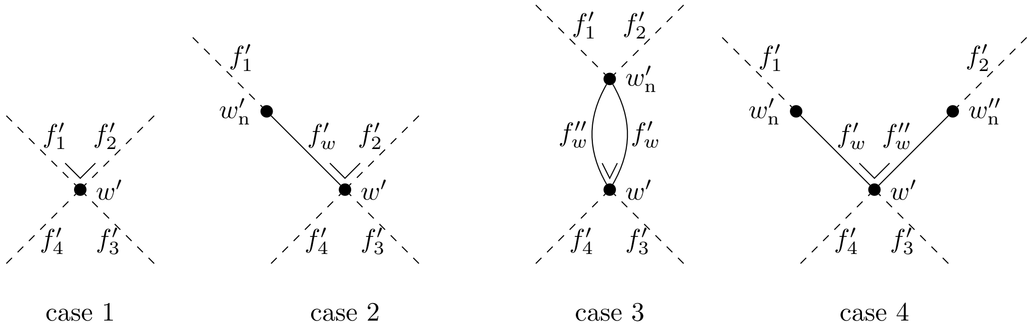

Figure 4: The four possibilities for R_w and R_w^+ including the transition S'_w. The solid edges represent the edges of R_w and the dashed edges the four additional edges of R'_w. The transition is represented by a vee ($\vee$) between its two edges.

2. R_w is a K_2 with two vertices w' and w'_{n}, where w'_{n} is of degree two with two incident edges f'_1 and f'_w. Moreover, $S'_w = (w', \{f'_w, f'_2\}) \in \mathcal{S}'(w')$.
3. R_w is a cycle of length two, i.e. two vertices w' and w'_{n} and two parallel edges f'_w and f''_w connecting them. Furthermore, w'_{n} has degree four and is incident to f'_1 and f'_2 and $S'_w = (w', \{f'_w, f''_w\}) \in \mathcal{S}'(w')$.
4. R_w consists of a vertex w' and two vertices w'_{n} and w''_{n} and two edges f'_w and f''_w connecting w' with w'_{n} and w' with w''_{n}. Furthermore, w'_{n} is incident to f'_1, w''_{n} is incident to f'_2, w' is incident to f'_3 and f'_4, and $S'_w = (w', \{f'_w, f''_w\}) \in \mathcal{S}'(w')$.

Note that we did not specify the order of the edges f_1, f_2, f_3, and f_4 except that there must be transitions $(w, \{f_1, f_2\})$ and $(w, \{f_3, f_4\})$ in $\mathcal{S}(w)$. Therefore, we could always interchange f_1 and f_2 with f_3 and f_4 in the second condition. The condition only has to hold for one of the two possibilities.

Definition 10 (Basic-Sup-$(H, \mathcal{S})$-Reduced-Transition-Minor free). Let $(H, \mathcal{S})$ be a completely transitioned 4-regular graph. A transitioned graph $(G, \mathcal{T})$ is *basic-sup-$(H, \mathcal{S})$-reduced-transition-minor free* if and only if it does not have any reduced-transition-minor $(H', \mathcal{S}')$ that is a basic-sup-$(H, \mathcal{S})$ graph.

Problem 2 (Existence of basic-sup-reduced-transition-minors (EBSRTM)). Given a transitioned graph $(G, \mathcal{T})$ and a completely transitioned 4-regular graph $(H, \mathcal{S})$, does there exist a reduced-transition minor of $(G, \mathcal{T})$ that is a basic-sup-$(H, \mathcal{S})$ graph?

Theorem 3. *EBSRTM is equivalent to ESTM.*

Proof. Let $((G, \mathcal{T}), (H, \mathcal{S}))$ be a positive instance of EBSRTM, i.e. there exists a reduced-transition minor $(H', \mathcal{S}')$ of $(G, \mathcal{T})$ that is a basic-sup-$(H, \mathcal{S})$ graph. We have to prove that this instance is also a positive instance of ESTM. First of all the reduced-transition minor can be extended to a transition minor $(H', \mathcal{S}'')$

of $(G, \mathcal{T})$ by adding opposite transitions for degree four vertices. For every vertex $w \in V(H)$ there exists a subgraph R_w of H' that is of one of the four forms described in Definition 9 and shown in Figure 4. In all four cases every vertex in this subgraph has degree two and four. Since every vertex in H' occurs in one R_w, we get that H' is Eulerian.

It remains to show that $(H', \mathcal{S}'')$ is a sup-$(H, \mathcal{S})$ graph. For each edge $f \in E(H)$ we use the graph induced by the single edge $f' \in E(H')$ as path P_f. For each vertex $w \in V(H)$ we use the subgraph R_w as Q_w. All defined subgraphs are connected and edge-disjoint since the graphs R_w are vertex-disjoint and the edges f' are all different and always connect two different subgraphs R_{w_1} and R_{w_2}.

By definition of f' the path P_f only consists of f' and it connects the subgraphs $R_{w_1} = Q_{w_1}$ and $R_{w_2} = Q_{w_2}$ if $f = w_1w_2$. The paths P_f are also internally disjoint since the edges f' are all different. Furthermore, by definition the graphs $Q_w = R_w$ are vertex-disjoint connected subgraphs. The subgraphs $Q_w^+ = R_w^+$ have in all four cases a bad 1-separator $\{u_w := w'\}$ separating H_w^1 and H_w^2 where H_w^2 is the graph induced by $\{f_3', f_4'\}$ (in Figure 4 the lower part of the graphs below the vertex w') and H_w^1 is the rest of R_w^+, i.e. the graph induced by $E(R_w^+) \setminus \{f_3', f_4'\}$. It is a bad 1-separator since by definition we get in all four cases

$$(w', E(w') \cap E(H_w^1)) \in \mathcal{S}''(w').$$

Furthermore, we have $\{f_1', f_2'\} \subseteq H_w^1$ and $\{f_3', f_4'\} \subseteq H_w^2$ which concludes the proof that $(H', \mathcal{S}'')$ is a sup-$(H, \mathcal{S})$ graph. Therefore, the instance $((G, \mathcal{T}), (H, \mathcal{S}))$ is also a positive instance of ESTM.

Now, let $((G, \mathcal{T}), (H, \mathcal{S}))$ be a positive instance of ESTM, we want to show that it is also a positive instance of EBSRTM. Let $(H', \mathcal{S}')$ be an Eulerian transition-minor of $(G, \mathcal{T})$ that is a sup-$(H, \mathcal{S})$ graph. By removing the transitions in $\mathcal{S}'$ that do not correspond to a transition in $\mathcal{T}$, see (3), we get a reduced-transition-minor $(H', \mathcal{S}'')$ of $(G, \mathcal{T})$.

For each vertex $w \in V(H)$ the subgraph Q_w^+ of H' has a bad 1-separator u_w separating H_w^1 and H_w^2. Without loss of generality let $S_w^1 := (u_w, E(u_w) \cap E(H_w^1)) \in \mathcal{S}'(u_w)$. If $S_w^1 \notin \mathcal{S}''(u_w)$ we know that $\deg(u_w) = 4$ and that $S_w^2 := (u_w, E(u_w) \setminus S_w^1 = E(u_w) \cap E(H_w^2)) \in \mathcal{S}''(u_w)$. In this case we exchange S_w^1 and H_w^1 with S_w^2 and H_w^2 so that we always have

$$S_w^1 := (u_w, E(u_w) \cap E(H_w^1)) \in \mathcal{S}''(u_w).$$

Furthermore, we can assume without loss of generality that the paths P_f only consist of single edges, since otherwise we could move some edges from P_f to one of the subgraphs Q_w until P_f only consists of one edge. Let $Q_w^1 := Q_w \cap H_w^1$ and $Q_w^2 := Q_w \cap H_w^2$. Now we contract all edges in Q_w^2 such that only the vertex u_w remains. Furthermore, we contract all edges in $E(Q_w^1) \setminus S_w^1$, i.e. all edges in Q_w^1 that are not in the transition S_w^1. Starting from Q_w and applying those contractions we call the resulting graph R_w. Applying these contractions to all graphs Q_w for $w \in E(H)$ we call the whole resulting graph H'', which is a minor of H'. Let $\mathcal{S}'''$ be the corresponding transition system of H'' such that $(H'', \mathcal{S}''')$ is a reduced-transition-minor of $(H', \mathcal{S}'')$. We still have $S_w^1 \in \mathcal{S}'''$ since we did not contract any of the edges incident to u_w. Let R_w^+ be the graph R_w together with the four adjacent paths P_f, each consisting of only one edge, for $f \in E(w)$. To see that R_w^+ has one of the four forms shown in Figure 4, we only have to distinguish different cases for the degree of the vertex u_w in Q_w^1:

1. If u_w has degree 0 in Q_w^1, this would imply that Q_w only consists of the vertex u_w and therefore R_w only consists of the vertex u_w which results in case 1.
2. If u_w has degree 1 in Q_w^1, this would imply that R_w is a K_2 which results in case 2.
3. If u_w has degree 2 in Q_w^1, there are two possibilities. If u_w is not a cut vertex of Q_w^1, then R_w is a digon (two vertices and two parallel edges) which results in case 3. Otherwise, if u_w is a cut vertex of Q_w^1 then R_w is a path of length two which results in case 4.

Note that the degree of u_w in Q_w^1 is at most two, since $\pi_2(S_w^1) = E(u_w) \cap E(H_w^1)$ and $|\pi_2(S_w^1)| = 2$. In each of the four cases the transition S' is exactly the transition drawn in the figure. Therefore, we just proved that $(H'', \mathcal{S}''')$ is a basic-sup-$(H, \mathcal{S})$ graph. Since $(H', \mathcal{S}'')$ is a reduced-transition-minor of $(G, \mathcal{T})$ and $(H'', \mathcal{S}''')$ is a reduced-transition-minor of $(H', \mathcal{S}'')$, we get by transitivity that $(H'', \mathcal{S}''')$ is a reduced-transition-minor of $(G, \mathcal{T})$. All in all we proved that the given instance $((G, \mathcal{T}), (H, \mathcal{S}))$ is also a positive instance of EBSRTM.

With that result we can proof that EBSRTM and therefore also ESTM is fixed-parameter tractable with the parameter $k = |V(H)|$.

Theorem 4. *ESTM is fixed parameter tractable with the parameter $k = |V(H)|$.*

Proof. We prove this by showing that EBSRTM is fixed parameter tractable and by Theorem 3 it follows that also ESTM is fixed parameter tractable. The proof is similar to the one in [10].

Notice that every basic-sup-$(H, \mathcal{S})$ graph $(H', \mathcal{S}')$ satisfies $|V(H')| \leq 3|V(H)|$. This implies that for a fixed $k = |V(H)|$ there are only finite many possible graphs $(H', \mathcal{S}')$ that can be a basic-sup-$(H, \mathcal{S})$ graph for any graph $(H, \mathcal{S})$ with $|V(H)| = k$. We can formulate now an algorithm that first generates all possible graphs $(H', \mathcal{S}')$ and then checks for each of them if it is a basic-sup-$(H, \mathcal{S})$ graph and if it is a reduced-transition-minor of $(G, \mathcal{T})$. Since the size of the graphs H' and H are bounded, checking if $(H', \mathcal{S}')$ is a basic-sup-$(H, \mathcal{S})$ graph can be done in constant time. What is left is to prove that checking if $(H', \mathcal{S}')$ is a reduced-transition-minor of $(G, \mathcal{T})$ can be done in polynomial time in $m = |V(G)| + |E(G)|$ if $k' = |V(H')| \leq 3k$ is fixed. By definition $(H', \mathcal{S}')$ is a reduced-transition-minor of $(G, \mathcal{T})$ if and only if there exist partial mappings φ and κ as described in Definition 5 such that $\mathcal{S}'$ equals the set defined in (2).

Since κ must be a partial injective and surjective function and since each vertex of H' has at most degree four, there are at most

$$|E(G)|^{|E(H')|} \leq m^{2k'} \leq m^{6k}$$

many possibilities to define κ by choosing exactly one edge in G for each edge in H'. Since k is fixed this number is polynomial in m and therefore we can generate all those possibilities and check for each of them if there exists an adequate partial mapping φ satisfying all needed properties. Therefore, let κ be a fixed partial injective and surjective function from $E(G)$ to $E(H')$ for the rest of this proof.

Let

$$V_0 := \bigcup_{e \in \operatorname{dom}(\kappa)} r_G(e) \subseteq V(G)$$

be the set of all vertices in G that are incident to a mapped edge of κ. Since κ is injective and surjective, we have $|\operatorname{dom}(\kappa)| = |E(H')| \leq 2k'$ and therefore $|V_0| \leq 2|\operatorname{dom}(\kappa)| \leq 4k'$. Since φ must satisfy (1), φ must be defined on each vertex in V_0 and only two values for $\varphi(v)$ are possible for each $v \in V_0$. Thus, the number of possible definitions of φ on V_0 is smaller than or equal to

$$|V_0|^2 \leq (4k')^2$$

and is therefore constant in m. We can again generate all those possibilities and check for each of them if there exists an adequate extension of φ to the whole vertex set V. Let therefore be φ already fixed and defined on V_0.

By using κ and the partially defined φ on V_0 the transition set defined in (2) is already well-defined, and we can compute it and check if it equals the transition set $\mathcal{S}'$. If the two transition sets do not match we can discard this possibility. In the case when the two transition sets are equal we only have to check if we can extend φ to V in such a way that φ is a partial surjective function and that each preimage $\varphi^{-1}[w]$ is a connected vertex set in G for each $w \in V(H')$. Since φ as defined already on V_0 satisfies (1) we already know that it is surjective without extending it at all. Therefore, we only have to extend it in such a way that each preimage $\varphi^{-1}[w]$ is a connected vertex set in G for each $w \in V(H')$. The existence of such an extension is equivalent to the existence of disjoint spanning trees T_w in G for each $w \in V(H')$, such that the already defined sets $\varphi^{-1}[w] \cap V_0 \subseteq T_w$ for each $w \in V(H')$. Without loss of generality we can search for spanning trees T_w such that all leaf vertices of T_w are in V_0.

Let $w \in V(H')$ be an arbitrary vertex. For each incident edge f in $E(w)$ we know by (2) that exactly one vertex in G incident to $\kappa^{-1}(f)$ is mapped to w under the already defined part of φ on V_0. Therefore, $|\varphi^{-1}[w] \cap V_0| \leq \deg_{H'}(w) \leq 4$ for each $w \in V(H')$. Putting everything together we get that T_w has at most four leaf vertices for each $w \in V(H')$. Since T_w is a tree it holds that $|E(T_w)| = V(T_w) - 1$, which implies that T_w has at most two vertices that have degree three or larger in T_w. These vertices could be in $V \setminus V_0$. The number of possible selections of such vertices of degree three or larger in T_w outside of V_0 for all $w \in V(H')$ is bounded by $|V(G)|^{2(|V(H')|+1)} \leq m^{2k+2}$ and is therefore polynomial in m. We can again iterate through all such possible selections of vertices of degree three or larger in T_w and check for each of them if we can construct the trees T_w such that they satisfy all needed properties. Let $V_w \subseteq V \setminus V_0$ be the fixed selected set of vertices of degree three or larger in T_w outside of V_0 for each $w \in V(H')$.

We construct now all possible labeled simple trees with the vertices $V_w \cup \left(\varphi^{-1}[w] \cap V_0\right)$. Since those are at most six vertices, the number of such labeled trees is finite. We can again iterate through all combinations of labeled trees L_w for all $w \in V(H')$ and check if the following holds for at least one of them. For each $w \in V(H')$ and its current labeled tree L_w we check if we can replace the edges of L_w by paths in G to get a tree T_w in G. Since the trees T_w in G must be vertex disjoint for each $w \in V(H')$ all those paths in G replacing the edges of the labeled trees must be vertex disjoint. Therefore, what remains to check is if there exist vertex disjoint paths in G connecting all vertex pairs $r_{L_w}(e)$ for each edge $e \in L_w$ and each $w \in V(H')$. Note that the number of such pairs can be bounded by

$$\sum_{w \in V(H')} |E(L_w)| = \sum_{w \in V(H')} |V(L_w)| - 1 \leq \sum_{w \in V(H')} 5 = 5|V(H')| = 5k' \leq 15k,$$

which is assumed to be constant. This problem can be solved for a bounded number of pairs in polynomial time in the size of G and therefore in m, as proven in [10]. This concludes our proof.

4. Modeling

In this chapter we present a mathematical model for solving the problem EBSRTM that can be expressed as a mixed integer linear program, cf. Section 5. First we present the model, and then we prove extensively that the model really solves EBSRTM, since this is not obvious.

A naive model would need variables for representing the unknown graph $(H', \mathcal{S}')$ and constraints to ensure that $(H', \mathcal{S}')$ is a reduced-transition-minor of the given $(G, \mathcal{T})$ and that $(H', \mathcal{S}')$ is a basic-sup-$(H, \mathcal{S})$. Such a naive model would be quite large since we do not even know the size of $(H', \mathcal{S}')$. Furthermore, formulating the constraints would be nontrivial and likely need a lot of additional auxiliary variables. The model we will present does not describe the graph $(H', \mathcal{S}')$ directly. It just consists of mappings between $(G, \mathcal{T})$ and $(H, \mathcal{S})$ and constraints to ensure that a valid intermediate graph $(H', \mathcal{S}')$ exists.

4.1. Towards a Model

Let $(G, \mathcal{T})$ and $(H, \mathcal{S})$ be given as the input. Let us assume now that there exists a reduced transition minor $(H', \mathcal{S}')$ of $(G, \mathcal{T})$ that is a basic-sup-$(H, \mathcal{S})$ graph, i.e. that the given instance is a yes instance of EBSRTM. This situation and the notation which we will introduce in the following is visualized in Figure 5.

By the definition of a minor there exist partial mappings $\varphi' \colon V(G) \nrightarrow V(H')$ and $\kappa' \colon E(G) \nrightarrow E(H')$ where φ' is surjective and κ' is injective and surjective. By the definition of a basic-sup-$(H, \mathcal{S})$ graph we can define the following natural partial mappings. The vertex mapping $\varphi'' \colon V(H') \nrightarrow V(H)$ maps each vertex in the subgraph R_w to w for each $w \in V(H)$. Since each R_w is non-empty, φ'' is surjective. The edge mapping $\kappa'' \colon E(H') \nrightarrow E(H)$ maps each edge f' to its corresponding edge $f \in E(H)$ for each $f \in E(H)$ as defined in Definition 9. By definition this partial mapping is injective and surjective.

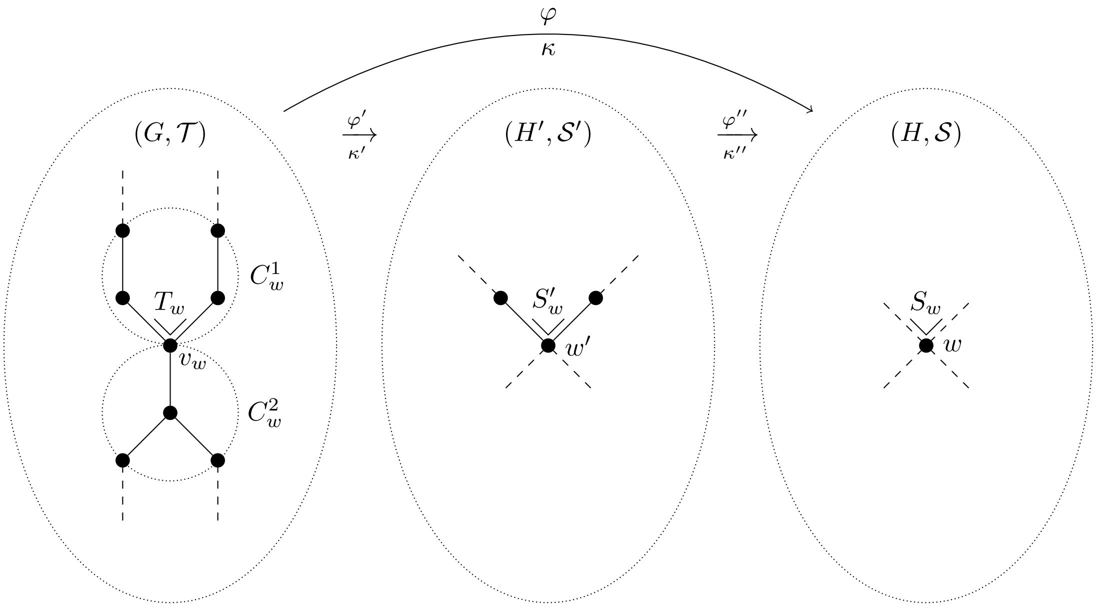

Figure 5: Exemplary visualization of the two input graphs together with the intermediate graph H'.

Using the above mappings we can define partial mappings from G to H by composing them. We define the two functions

$$\varphi := \varphi'' \circ \varphi' \colon V(G) \nrightarrow V(H),$$

and

$$\kappa := \kappa'' \circ \kappa' \colon E(G) \nrightarrow E(H).$$

Since φ' and φ'' are surjective also φ is surjective and since κ' and κ'' are injective and surjective also κ is injective and surjective.

We denote for each $w \in V(H)$ the transition $S_w := S_w^1 \in \mathcal{S}$ as defined in Definition 9. Each such transition S_w is associated via Definition 9 with a transition S'_w in $\mathcal{S}'$. Furthermore, by this definition we have $\pi_1(S'_w) =: w' \in \varphi''^{-1}[w]$. Moreover, since S'_w is in $\mathcal{S}'$ we get by definition of a reduced transition minor that

$$T_w := (v_w, \kappa'^{-1}[\pi_2(S'_w)]) \in \mathcal{T}$$

for some $v_w \in \varphi'^{-1}[w'] \subseteq \varphi^{-1}[w]$.

To model the fact that each vertex set $\varphi'^{-1}[x]$ must be connected in G for each $x \in V(H')$ and to model the four different possible structures of each R_w, we further introduce for each $w \in V(H)$ two trees C_w^1 and C_w^2. The two trees cover the connected subgraph $\varphi^{-1}[w]$ and share exactly one common vertex $v_w = \pi_1(T_w)$.

The tree C_w^2 represents a spanning tree of $\varphi'^{-1}[w']$ in G, where w' is as specified in Definition 9 for each $w \in V(H)$. The tree C_w^1 represents a spanning tree of $\{v_w\}$, $\{v_w\} \cup \varphi'^{-1}[w'_{\mathrm{n}}]$, or $\{v_w\} \cup \varphi'^{-1}[\{w'_{\mathrm{n}}, w''_{\mathrm{n}}\}]$ depending on the possible four cases for the form of R_w. Since the two trees share one vertex this ensures that the vertex set $\varphi^{-1}[w] = V(C_w^1) \cup V(C_w^2)$ is connected in G. We have to distinguish the two trees to formulate the constraint that the edges of T_w must be part of $C_w^1 \cup \kappa^{-1}[\pi_2(S_w)]$. Since we are only concerned with connectivity for the trees C_w^1 and C_w^2 we define them not as subtrees of G but as simple trees with vertices $V(C_w^i) \subseteq V(G)$ and for each edge in $E(C_w^i)$, which is represented as a set of two vertices, there must exist at least one edge in G whose end vertices are exactly those two vertices.

In case 3 of Definition 9 it may be that $r_G(\kappa'^{-1}(f'_w)) \neq r_G(\kappa'^{-1}(f''_w))$. In this case only one of those two edges will be included in the tree C_w^1. Therefore, for this case we have to reformulate the condition that the edges of T_w must be part of $C_w^1 \cup \kappa^{-1}[\pi_2(S_w)]$. To do this we introduce a new partial injective mapping $\theta\colon E(G) \nrightarrow V(H)$ to our model, which only maps an edge to the vertex w if R_w is of the form of case 3 of Definition 9 and $r_G(\kappa'^{-1}(f'_w)) \neq r_G(\kappa'^{-1}(f''_w))$. With that mapping we can reformulate the condition such that the edges of T_w must be part of $C_w^1 \cup \kappa^{-1}[\pi_2(S_w)] \cup \theta^{-1}[w]$.

With the above motivation we are ready to formulate our model.

4.2. The Model

Let $(G, \mathcal{T})$ and $(H, \mathcal{S})$ be given as the input. We will use the following additional notation in the model. If C_w^i is a simple tree with vertices in G we define

$$E_w^i := \left\{ e \in E(G) \mid r(e) \in E(C_w^i) \right\}$$

as the set of all edges in G whose end vertices are connected in C_w^i by an edge. The model is now defined as finding

1. a partial surjective function $\varphi\colon V(G) \nrightarrow V(H)$,
2. a partial injective and surjective function $\kappa\colon E(G) \nrightarrow E(H)$,
3. a partial injective function $\theta\colon E(G) \nrightarrow V(H)$,
4. for each $w \in V(H)$ a pair (T_w, S_w) of transitions with $T_w \in \mathcal{T}$ and $S_w \in \mathcal{S}(w)$,
5. for each $w \in V(H)$ two simple trees C_w^1 and C_w^2 with $V(C_w^i) \subseteq V(G)$ for $i = 1, 2$,

such that

$$E(C_w^i) \subseteq r_G[E(G)] \qquad \forall w \in V(H), \forall i \in \{1, 2\} \tag{4}$$

$$\kappa(e) = f \Rightarrow \varphi[r_G(e)] = r_H(f) \qquad \forall e \in E(G), \forall f \in E(H) \tag{5}$$

$$V(C_w^1) \cup V(C_w^2) = \varphi^{-1}[w] \qquad \forall w \in V(H) \tag{6}$$

$$\{\pi_1(T_w)\} = V(C_w^1) \cap V(C_w^2) \qquad \forall w \in V(H) \tag{7}$$

$$\pi_2(T_w) \subseteq \kappa^{-1}[\pi_2(S_w)] \cup \theta^{-1}[w] \cup E_w^1 \qquad \forall w \in V(H) \tag{8}$$

$$\left(\kappa^{-1}[\pi_2(S_w)] \cap E(\pi_1(T_w))\right) \cup \theta^{-1}[w] \subseteq \pi_2(T_w) \qquad \forall w \in V(H) \tag{9}$$

$$e \in \mathrm{dom}(\kappa) \wedge \kappa(e) \in \pi_2(S_w) \Rightarrow r_G(e) \cap V(C_w^1) \neq \emptyset \qquad \forall w \in V(H), \forall e \in E(G) \tag{10}$$

$$e \in \operatorname{dom}(\kappa) \wedge \kappa(e) \in E(w) \setminus \pi_2(S_w) \Rightarrow r_G(e) \cap V(C_w^2) \neq \emptyset \qquad \forall w \in V(H), \forall e \in E(G) \tag{11}$$

$$\begin{aligned} &v \in V(C_w^1) \setminus \{\pi_1(T_w)\} \wedge \deg_{C_w^1}(v) = 1 \wedge v \notin \bigcup r_G[\theta^{-1}[w]] \\ &\quad \Rightarrow E(v) \cap \kappa^{-1}[\pi_2(S_w)] \neq \emptyset \end{aligned} \qquad \forall w \in V(H), \forall v \in V(G) \tag{12}$$

$$E_{C_w^1}(\pi_1(T_w)) \subseteq r_G[\pi_2(T_w)] \qquad \forall w \in V(H) \tag{13}$$

$$\theta(e) = w \Rightarrow r_G(e) \subseteq V(C_w^1) \qquad \forall e \in E(G), \forall w \in V(H) \tag{14}$$

$$\theta(e) = w \Rightarrow r_G(e) \notin E(C_w^1) \qquad \forall e \in E(G), \forall w \in V(H) \tag{15}$$

Constraints (4) ensure that the trees C_w^i are simple trees where each simple edge corresponds to at least one edge of G. Constraints (5) couple the edge map κ with the vertex map φ, similar to condition (1). Furthermore, conditions (6) guarantee that the two trees C_w^1 and C_w^2 together really cover $\varphi^{-1}[w]$ and together with constraints (7) this ensures that $\varphi^{-1}[w]$ is connected in G. Constraints (7) furthermore enforce that $\pi_1(T_w)$ gets mapped to w under φ. Constraints (8) ensure that the edges of a transition T_w get either mapped directly to edges of S_w (see f_1', f_2' in case 1 and f_2' in case 2 of Definition 9), or correspond to edges of C_w^1 (see f_w' in case 2 or f_w', f_w'' in case 3 or 4), or get mapped to w by θ (f_w'' in case 3 of Definition 9 if $r_G(\kappa'^{-1}(f_w')) \neq r_G(\kappa'^{-1}(f_w''))$). On the other hand, constraints (9) guarantee that the only edges that are incident to $\pi_1(T_w)$ and get mapped under κ to an edge in $\pi_2(S_w)$ are in $\pi_2(T_w)$. Furthermore, those constraints ensure that only edges in $\pi_2(T_w)$ may be mapped to w under θ.

An edge in G that gets mapped under κ to $\pi_2(S_w)$ must be incident to a vertex in C_w^1, which is ensured by conditions (10). This forces edges like f_1' or f_2' in Definition 9 to be incident to a vertex of the upper tree. On the other hand, conditions (11) guarantees that edges that get mapped under κ to $E(w) \setminus \pi_2(S_w)$ are incident to a vertex of C_w^2. This forces edges like f_3' and f_4' in Definition 9 to be incident to a vertex of the lower tree.

Constraints (12) are the most complex ones of the model. They are needed to avoid situations like one similar to case 4 of Definition 9 where f_2' is not incident to w_n'' but to w_n'. In this case a leaf vertex of the upper tree would have no incident edge in $\operatorname{dom}(\kappa)$. On the other hand, it may happen in case 3 if $r_G(\kappa'^{-1}(f_w')) \neq r_G(\kappa'^{-1}(f_w''))$ that one leaf vertex of the upper tree also has no incident edge but is still representing a valid case. But then we can ensure that this leaf vertex is incident to the edge that gets mapped to w by θ. Therefore, to exclude the unwanted situation described above, which is similar to case 4, but to include the wanted cases in case 3, we need constraints (12).

Constraints (13) guarantee that all edges in the upper tree that are incident to $\pi_1(T_w)$ are represented in G by edges in $\pi_2(T_w)$. Furthermore, constraints (14) enforce that if an edge e gets mapped by θ to a vertex w that both incident vertices of e are in the upper tree C_w^1 and (15) ensures that the edge e itself is not represented by an edge in C_w^1.

Theorem 5. *For the above model exists a valid solution if and only if $((G,\mathcal{T}),(H,\mathcal{S}))$ is a yes instance of EBSRTM.*

We split this theorem into two parts, proofing each direction independently. Theorem 5 then directly follows Proposition 1 and Proposition 2.

Proposition 1. *If $((G,\mathcal{T}),(H,\mathcal{S}))$ is a yes instance of EBSRTM there exists a valid solution in the above model.*

Proof. Let $((G,\mathcal{T}),(H,\mathcal{S}))$ be a yes instance of EBSRTM and $(H',\mathcal{S}')$ a reduced-transition-minor of $(G,\mathcal{T})$ that is a basic-sup-$(H,\mathcal{S})$ graph. In the first part of the proof we will derive the partial mappings φ,κ,θ, the transitions S_w and T_w for $w \in V(H)$, and the trees C_w^i for $w \in V(H), i \in \{1,2\}$. In the second part we will prove that the constraints (4)–(15) are satisfied. Let φ' and κ' be the correspondence functions of the minor H' of G, see the second part of Definition 5. W.l.o.g. we assume that $\mathrm{dom}(\varphi')$ is minimal, i.e. after removing any vertex from $\mathrm{dom}(\varphi')$ the conditions of Definition 5 are not satisfied anymore. We will need the following property for the minimal φ'.

Lemma 4. *Every vertex $v \in \mathrm{dom}(\varphi')$ is either incident to an edge $e \in \mathrm{dom}(\kappa')$ or a cut vertex of $\varphi'^{-1}[\varphi'(v)]$.*

Proof. Assume there is a vertex $v \in \mathrm{dom}(\varphi')$ that is not incident to any edge in $\mathrm{dom}(\kappa')$ and is no cut vertex of $\varphi'^{-1}[\varphi'(v)]$. We prove now that removing the vertex v from φ' still leads to a valid vertex mapping for Definition 5, which then is a contradiction to the minimality of φ'. Let $\overline{\varphi'} = \varphi' \setminus \{(v,\varphi'(v))\}$ be the reduced mapping. We need to prove all conditions of Definition 5 for $\overline{\varphi'}$. First of all $\overline{\varphi'}^{-1}[\varphi'(v)]$ is non-empty since H' is four regular and therefore there exists an edge $f \in E(H')$ that is incident to $\varphi'(v)$. Then we have

$$\varphi'(v) \in r_H(f) = \varphi'[r_G(\kappa'^{-1}(f))] = \overline{\varphi'}[r_G(\kappa'^{-1}(f))],$$

which implies that $\overline{\varphi'}^{-1}[\varphi'(v)]$ is non-empty. Furthermore, $\overline{\varphi'}^{-1}[\varphi'(v)] = \varphi'^{-1}[\varphi'(v)] \setminus \{v\}$ is connected since v was not a cut vertex of $\varphi'^{-1}[\varphi'(v)]$. Last but not least (1) is also valid for $\overline{\varphi'}$ since v was not incident to any edges in $\mathrm{dom}(\kappa')$.

Let the subgraphs R_w for $w \in V(H)$ and the edges f' of H' for $f \in E(H)$ be as described in Definition 9. We define a new surjective function $\varphi'' \colon V(H') \to V(H)$ and a partial injective and surjective function $\kappa'' \colon E(H') \nrightarrow E(H)$ by

$$\varphi''(x) = w \quad \forall x \in R_w, \forall w \in V(H)$$

and

$$\kappa''(f') = f \quad \forall f \in E(H).$$

Function φ'' is well-defined and surjective since the vertex sets of the non-empty subgraphs R_w partition the vertex set $V(H')$. The partial function κ'' is well-defined and injective since the edge f' is different for each $f \in E(H)$. This can be seen by the fact that $|\{f' \mid f \in E(H)\}| = |E(H)|$ in Definition 9. Furthermore, κ'' is surjective since there exists an edge f' for every $f \in E(H)$.

Now we can concatenate the given functions and get the surjective partial functions $\varphi\colon V(G) \twoheadrightarrow V(H)$ and $\kappa\colon E(G) \twoheadrightarrow E(H)$ by $\varphi = \varphi'' \circ \varphi'$ and $\kappa = \kappa'' \circ \kappa'$. Since both κ' and κ'' are injective κ is also injective. Moreover, since the four maps φ', φ'', κ', and κ'' are surjective the maps φ and κ are also surjective.

In the following we use the notation in Definition 9, which is possible since we know that $(H', \mathcal{S}')$ is a sup-$(H, \mathcal{S})$ graph. For each vertex $w \in V(H)$ let $S'_w \in \mathcal{S}'(w')$ be the transition at vertex w' defined in Definition 9. Since the transition S'_w is in the reduced-transition minor $(H', \mathcal{S}')$ of $(G, \mathcal{T})$ there must exist a transition $T_w \in \mathcal{T}$ with $\kappa'[\pi_2(T_w)] = \pi_2(S'_w)$ and $\varphi'(\pi_1(T_w)) = \pi_1(S'_w) = w'$. Furthermore, for $w \in V(H)$, let $S_w := S^1_w = (w, \{f_1, f_2\})$ as specified in Definition 9. With this we defined for each vertex $w \in V(H)$ the pair (T_w, S_w). For a vertex $w \in V(H)$ we define the simple graph $G_w = (V_w, E_w)$ by $V_w := \varphi'^{-1}[V(R_w)]$ and

$$E_w := \left\{ r_G(e) \mid e \in \kappa'^{-1}[E(R_w)] \vee \exists x \in V(R_w) : r_G(e) \subseteq \varphi'^{-1}[x] \right\}.$$

We know that $\varphi'^{-1}[x]$ is connected in G for every $x \in V(R_w)$, and for each edge $f^* \in E(R_w)$ with $r_{H'}(f^*) = \{x_1, x_2\}$ the edge $\kappa'^{-1}(f^*)$ connects $\varphi'^{-1}[x_1]$ with $\varphi'^{-1}[x_2]$. From that follows that G_w is connected. We can now define C^2_w to be a spanning tree of the vertex set $\varphi'^{-1}[w']$ in G_w.

To define C^1_w let V^1_w be the vertex set containing $v_w := \pi_1(T_w)$ and $\varphi'^{-1}[w'_{\mathrm{n}}]$ if the vertex w'_{n} exists together with $\varphi'^{-1}[w''_{\mathrm{n}}]$ if the vertex w''_{n} exists. The vertex set V^1_w is connected in G_w since $\varphi'^{-1}[w'_{\mathrm{n}}]$ is connected if existent, $\varphi'^{-1}[w''_{\mathrm{n}}]$ is connected if existent, the edge $\kappa'^{-1}(f'_w)$ connects v_w with $\varphi'^{-1}[w'_{\mathrm{n}}]$ if existent, and if $\varphi'^{-1}[w''_{\mathrm{n}}]$ exists it is connected to v_w by $\kappa'^{-1}(f''_w)$. We define now C^1_w to be a spanning tree of V^1_w in G_w. In case 3 of Definition 9 we require from C^1_w w.l.o.g. that v_w is a leaf vertex in C^1_w connected by $r_G(\kappa'^{-1}(f'_w))$. This is possible since $V^1_w \setminus \{v_w\} = \varphi'^{-1}[w'_{\mathrm{n}}]$ is still connected.

Furthermore, for a vertex $w \in V(H)$ in case 3 of Definition 9 we define $\theta(\kappa'^{-1}(f''_w)) = w$ if $r_G(\kappa'^{-1}(f''_w)) \neq r_G(\kappa'^{-1}(f'_w))$. If $\kappa'^{-1}(f''_w)$ is a parallel edge to $\kappa^{-1}(f'_w)$ and in the other three cases of Definition 9 no edge maps to w, i.e. $\theta^{-1}[w] = \emptyset$. By definition θ is injective since the edges $\kappa^{-1}(f''_w)$ are incident to two vertices in $\varphi'^{-1}[R_w]$ and the sets $\varphi'^{-1}[R_w]$ are disjoint for different vertices w. What remains is now to show that the constraints (4)–(15) hold, which we will prove in the following.

(4) Constraints (4) are satisfied since C^1_w and C^2_w are subtrees of G_w and since $E(G_w) = E_w \subseteq r_G[E(G)]$ by definition.

(5) Let $e \in E(G)$ and $f \in E(H)$ with $\kappa(e) = f$, i.e. $\kappa''(\kappa'(e)) = f$. By definition of κ'' we get $\kappa'(e) = f'$ and by (1) we get $r'_H(f') = \varphi'[r_G(e)]$. Let $f = w_1 w_2$ then we know by Definition 9 that f' connects

the subgraphs R_{w_1} and R_{w_2} in H', i.e. $f' = x_1x_2$ with $x_1 \in V(R_{w_1})$ and $x_2 \in V(R_{w_2})$. By definition of φ'' we get $\varphi''(x_1) = w_1$ and $\varphi''(x_2) = w_2$ and plugging everything together we get

$$\varphi[r_G(e)] = \varphi''[\varphi'[r_G(e)]] = \varphi''[r'_H(f')] = \varphi''[\{x_1, x_2\}] = \{w_1, w_2\} = r_H(f).$$

(6) By the definition of C_w^1 and C_w^2 we have

$$V(C_w^1) \cup V(C_w^2) = V(G_w) = \varphi'^{-1}[V(R_w)] = \varphi'^{-1}[\varphi''^{-1}[w]] = \varphi^{-1}[w].$$

(7) Let $w \in V(H)$ then by definition of C_w^1 and C_w^2 we get (7) by the fact that $\varphi'^{-1}[w'_{\mathrm{n}}]$ and $\varphi'^{-1}[w''_{\mathrm{n}}]$ are disjoint to $\varphi'^{-1}[w']$ and because $\varphi'(\pi_1(T_w)) = w'$ by definition of T_w.

(8) Let $w \in V(H)$ and $e \in \pi_2(T_w)$. Then we know by the definition of T_w that $\kappa'(e) = f^* \in \pi_2(S'_w)$. In all four cases in the Definition 9 the edges of $\pi_2(S'_w)$ are either f'_i for some $f_i \in \pi_2(S_w)$, f'_w, or f''_w. If $f^* = f'_i$ for some $f_i \in \pi_2(S_w)$ we get $e = \kappa^{-1}(f_i) \in \kappa^{-1}[\pi_2(S_w)]$.

Next we show that $r_G(\kappa^{-1}(f'_w))$ is in $E(C_w^1)$ in the cases 2 to 4 of Definition 9. In case 2 the only edge between $\pi_1(T_w)$ and $\varphi'^{-1}[w'_{\mathrm{n}}]$ in C_w^1 is $r_G(\kappa^{-1}(f'_w))$ and therefore it must be part of C_w^1 since C_w^1 is a spanning tree. In case 3 we directly forced $r_G(\kappa^{-1}(f'_w))$ to be a part of C_w^1. Furthermore, in case 4 both edges $r_G(\kappa^{-1}(f'_w))$ and $r_G(\kappa^{-1}(f''_w))$ must be part of C_w^1 since otherwise the subgraphs $\varphi'^{-1}[w'_{\mathrm{n}}]$ and $\varphi'^{-1}[w''_{\mathrm{n}}]$ would not be connected. Therefore, if $f^* = f'_w$ we directly get $e \in E_w^1$. We already proved for the case 4 that $r_G(\kappa'^{-1}(f''_w)) \in E(C_w^1)$ and therefore also in case 4 if $f^* = f''_w$ we get $e \in E_w^1$.

The only remaining case is case 3 and if $f^* = f''_w$. Then either $\kappa'^{-1}(f''_w)$ is parallel to $\kappa'^{-1}(f'_w)$ which would imply again $r_G(\kappa'^{-1}(f''_w)) \in E(C_w^1)$ or $\theta(\kappa'^{-1}(f''_w)) = \theta(e) = w$ and therefore $e \in \theta^{-1}[w]$. All in all we just proved (8).

(9) Let $w \in V(H)$ and $e \in \kappa^{-1}[\pi_2(S_w)] \cap E(\pi_1(T_w))$. Let $f = \kappa(e) = \kappa''(\kappa'(e))$, then we know by definition of κ'' that $\kappa'(e) = f'$. Since e is incident to $\pi_1(T_w)$, we get by (1) that f' is incident to $\varphi'(\pi_1(T_w)) = w'$. We know now that $f \in \pi_2(S_w)$ and that f' is incident to w', which implies that $f' \in \pi_2(S'_w)$. This can easily be checked in all four cases of Definition 9. But since $\pi_2(S'_w) = \kappa'[\pi_2(T_w)]$ we get that $e \in \kappa'^{-1}[\pi_2(S'_w)] = \pi_2(T_w)$.

On the other hand, if $e \in \theta^{-1}[w]$ this means that we are in case 3 of Definition 9 and $e = \kappa'^{-1}(f''_w) \in \kappa'^{-1}[\pi_2(S'_w)] = \pi_2(T_w)$. All in all we proved (9).

(10) Let $w \in V(H)$, $e \in \mathrm{dom}(\kappa)$, and $f = \kappa(e) \in E(w)$. Then we know by definition of κ that $\kappa'(e) = f'$ and since f is incident to w we get that f' is incident to a vertex in R_w. If $f \in \pi_2(S_w)$, this implies by the notation of Definition 9 that $f = f_i$ with $i \in \{1, 2\}$. In all four cases f'_i is either in $\pi_2(S'_w)$ or

is incident to w'_{n} or w''_{n} for $i \in \{1,2\}$. If $f'_i \in \pi_2(S'_w)$ we get $e = \kappa'^{-1}(f'_i) \in \pi_2(T'_w)$, which implies that e is incident to $v_w \in V(C^1_w)$. On the other hand, if f'_i is incident to w'_{n} or w''_{n}, we also get that $e = \kappa'^{-1}(f'_i)$ is incident to a vertex in $\varphi'^{-1}[\{w'_{\mathrm{n}}, w''_{\mathrm{n}}\}] \subseteq V(C^1_w)$. With that we proved (10).

(11) With the same notation as for constraints (10) if $f \in E(w) \setminus \pi_2(S)$ we get $f' = f'_i$ with $i \in \{3,4\}$ and in all four cases of Definition 9 those edges are incident to w'. This implies that $e = \kappa'^{-1}(f')$ is incident to an edge in $\varphi'^{-1}[w'] = V(C^2_w)$ and therefore we showed (11).

(12) Let $w \in V(H)$, $v_w := \pi_1(T_w)$, and $v \in V(C^1_w) \setminus \{v_w\}$ with $|E_{C^1_w}(v)| = 1$ and $v \notin \bigcup r_G[\theta^{-1}[w]]$. Now we use Lemma 4 and obtain that $v \in V(C^1_w) \subseteq V_w = \varphi'^{-1}[V(R_w)]$ is either incident to an edge $e \in \mathrm{dom}(\kappa')$ or a cut vertex of $\varphi'^{-1}[\varphi'(v)]$. Since $v \in V(C^1_w) \setminus \{v_w\}$, we know $\varphi'(v) \neq w'$ and therefore that $\varphi'^{-1}[\varphi'(v)] \subseteq C^1_w$. If v would be a cut vertex of $\varphi'^{-1}[\varphi'(v)]$ this would imply that $\deg_{C^1_w}(v) \geq 2$, which is a contradiction to our assumption. Therefore, v is not a cut vertex of $\varphi'^{-1}[\varphi'(v)]$ and we get that v is incident to an edge $e \in \mathrm{dom}(\kappa')$. There are now two possibilities for $f' = \kappa'(e)$, either $f' \in \mathrm{dom}(\kappa'')$ or $f' \in \{f'_w, f''_w\}$. In the first case we know $f' \in \mathrm{dom}(\kappa'')$ and that f' is incident to $\varphi'(v) \subseteq \{w'_{\mathrm{n}}, w''_{\mathrm{n}}\}$. We can check in all four cases of Definition 9 that this implies either $f' = f'_1$ or $f' = f'_2$, which further implies $f \in \{f_1, f_2\} = \pi_2(S_w)$. In this case we are done since $e \in E(v) \cap \kappa^{-1}[\pi_2(S_w)] \neq \emptyset$.

On the other hand, the second case if $f' \in \{f'_w, f''_w\}$ implies $e \in \pi_2(T_w)$. Since $v \neq v_w$ we get that v is the other incident vertex of e, i.e. $e = vv_w$. Let $x = \varphi'(v)$, then $r_{H'}(f') = \varphi'[r_G(e)] = \{x, w'\}$. We know in the case $f' = f'_w$ that f'_1 or in the case that $f' = f''_w$ that f'_2 is incident to x; let f'_i denote this incident edge in both cases. From this we get that $e_i := \kappa'^{-1}(f'_i)$ is incident to a vertex in $\varphi'^{-1}[x]$. By the definition of $E(G_w) = E_w$ all edges in G_w that connect $\varphi'^{-1}[w']$ with $\varphi'^{-1}[x]$ are edges in $r_G[\kappa'^{-1}[E(R_w)]]$. Since $v_w \in \varphi'^{-1}[w']$ and $v \in \varphi'^{-1}[x]$ are vertices of the connected subtree C^1_w of G_w there must exist an edge $\tilde{e} \in \kappa'^{-1}[E(R_w)]$ that connects $\varphi'^{-1}[w']$ with $\varphi'^{-1}[x]$ such that $r_G(\tilde{e}) \in E(C^1_w)$. In all four cases of the Definition 9 the only two possibilities for $\tilde{e}$ are $\kappa'^{-1}(f'_w)$ and $\kappa'^{-1}(f''_w)$. If $\kappa'(\tilde{e}) = f'$ we get by the injectivity of κ' that $\tilde{e} = e$ and therefore $r_G(e) \in E(C^1_w)$, but then we know that there is no other edge in C^1_w that is incident to v, which implies $\varphi'^{-1}[x] = \{v\}$ and therefore that e_i is incident to v, which implies $e_i \in E(v) \cap \kappa^{-1}[\pi_2(S_w)]$.

The last case we need to check is if $\kappa'(\tilde{e}) \neq f'_i$. Since $r_{H'}(\kappa'(\tilde{e})) = \varphi'[r_G(\tilde{e})] = \{w', x\} = r_{H'}(f'_i)$ this can only happen in case 3 of Definition 9 and if $f' = f''_w$ and $\tilde{e} = \kappa'^{-1}[f'_w]$. But in this case we ensured that $w = \theta(\kappa'^{-1}(f''_w)) = \theta(\kappa'^{-1}(f')) = \theta(e)$, which implies $e \in \theta^{-1}[w]$, which is a contradiction since $v \in r_G(e) = \bigcup [r_G[\theta^{-1}[w]]]$. All in all we showed that (12) holds.

(13) Let $w \in V(H)$ and $\{v_w, v\} \in E_{C^1_w}(\pi_1(T_w))$. By definition of C^1_w this implies that there exists an edge $e \in E(G)$ with $r_G(e) = \{v_w, v\}$. By definition of $E_w \supseteq E_{C^1_w}(v_w)$ either $e \in \kappa'^{-1}[E(R_w)]$ or

$r_G(e) \subseteq \varphi'^{-1}[x]$ for some $x \in V(R_w)$. Since v_w is the only vertex of $\varphi'^{-1}[w']$ in C^1_w we get that $\varphi'[v] \neq \varphi'[v_w]$ and therefore that $e \in \kappa'^{-1}[E(R_w)]$. Either $\kappa'(e) = f'_w$ or $\kappa'(e) = f''_w$ but in both cases we know that $e \in \pi_2(T_w)$. Therefore, we always get that $\{v_w, v\} = r_G(e) \in r_G[\pi_2(T_w)]$, which implies (13).

(14) Let $w \in V(H)$, and $e \in E(G)$ with $\theta(e) = w$. By definition of θ this implies that the structure of R_w equals the one of case 3 of Definition 9 and $e = \kappa'^{-1}(f''_w)$, which implies $e \in \pi_2(T_w)$. Since $r_{H'}(f''_w) = \{w', w''_{\mathrm{n}}\}$, we know that e is incident to a vertex in $\varphi^{-1}[w''_{\mathrm{n}}]$. Since e is incident to v_w and $\varphi^{-1}[w''_{\mathrm{n}}] \subseteq V(C^1_w)$ we get $r_G(e) \subseteq V(C^1_w)$, which finishes the proof for (14).

(15) These constraints follow directly from our definition of θ. We only define it in case 3 and only if $r_G(\kappa'^{-1}(f''_w)) \neq r_G(\kappa'^{-1}(f'_w))$. Since we also define C^1_w in this case in such a way that v_w has degree 1 with the only incident edge $r_G(\kappa'^{-1}(f'_w))$, we know that $r_G(\kappa'^{-1}(f''_w))$, which is also incident to v_w, is not in $E(C^1_w)$.

Proposition 2. *If there exists a valid solution in the model given at the beginning of this section the instance $((G, \mathcal{T}), (H, \mathcal{S}))$ is a yes instance of EBSRTM.*

Proof. Let

$$(\varphi, \kappa, \theta, (T_w)_{w \in V(H)}, (S_w)_{w \in V(H)}, (C^1_w)_{w \in V(H)}, (C^2_w)_{w \in V(H)})$$

be a valid solution of the model. The proof consists of the following steps. We define a transitioned graph H', proof that H' is a minor of G, define a transition system $\mathcal{S}'$ on H' in such a way that $(H', \mathcal{S}')$ is a reduced transition minor of $(G, \mathcal{T})$ and finally prove that $(H', \mathcal{S}')$ is a basic-sup-$(H, \mathcal{S})$-graph.

Before we define the graph H', we introduce some notation that we will use during the proof. For a vertex $w \in V(H)$ we define $v_w := \pi_1(T_w)$ and denote a possible extension of E^1_w by

$$\overline{E^1_w} := E^1_w \cup \theta^{-1}[w].$$

Lemma 5. *For an edge $e \in \overline{E^1_w}$ we get $r_G(e) \subseteq V(C^1_w)$.*

Proof. For $e \in E^1_w$ the lemma follows by definition of E^1_w and for $e \in \theta^{-1}[w]$ it follows from (14).

We describe now for each vertex $w \in V(H)$ a graph R_w. These graphs will then be used to form the graph H'.

Definition 11 (R_w). To describe R_w we distinguish the following four cases.

1. $\deg_{C^1_w}(v_w) = 0$: In this case R_w consists only of one vertex w' (see case 1 in Figure 4).
2. $\deg_{C^1_w}(v_w) = 1 \wedge |\kappa[\pi_2(T_w)]| = 1$: In this case R_w consists of two vertices w' and w'_{n} and an edge f'_w connecting them (see case 2 in Figure 4).

3. $\deg_{C_w^1}(v_w) = 1 \wedge |\kappa[\pi_2(T_w)]| = 0$: In this case R_w consists of two vertices w' and w'_{n} and two parallel edges f'_w and f''_w connecting them (see case 3 in Figure 4).
4. $\deg_{C_w^1}(v_w) = 2$: In this case R_w consists of three vertices w', w'_{n}, and w''_{n} and two edges f'_w connecting w' and w'_{n} and f''_w connecting w' and w''_{n} (see case 4 in Figure 4).

Note that $\deg_{C_w^1}(v_w) \leq 2$, which follows from (13) by

$$\deg_{C_w^1}(v_w) = |E_{C_w^1}(v_w)| \leq |r_G[\pi_2(T)]| \leq |\pi_2(T)| = 2.$$

To conclude that the above cases cover all possible cases we need to show that $\kappa[\pi_2(T_w)] \leq 1$ holds when $\deg_{C_w^1}(v_w) = 1$. In this case we know that there exists an edge $e \in E_w^1 \cap E(v_w)$, which implies by (13) that

$$r_G(e) \in E_{C_w^1}(v_w) \subseteq r_G[\pi_2(T_w)]$$

and therefore there must exist an edge $\tilde{e} \in \pi_2(T_w)$ with $r_G(\tilde{e}) = r_G(e) \subseteq V(C_w^1)$. Furthermore, this implies by (5), (6), and the fact that we have no loops in H that $\tilde{e} \notin \operatorname{dom}(\kappa)$. Since $|\pi_2(T_w)| = 2$, this gives us $\kappa[\pi_2(T_w)] \leq 1$.

We can now define the vertex set of the graph H' as the disjoint union of the vertex sets of all graphs R_w. Formally this means

$$V(H') := \biguplus_{w \in V(H)} V(R_w).$$

We define one more notation, which we will use in the rest of the proof. In case 4 of Definition 11 we know that $\deg_{C_w^1} = 2$ and therefore the tree C_w^1 decomposes after removing the vertex v_w into two subtrees, which we denote by $C_w^{1,1}$ and $C_w^{1,2}$. Before we define edges between the subgraphs R_w we describe a partial vertex mapping $\varphi' \colon V(G) \nrightarrow V(H')$.

Definition 12 ($\varphi' \colon V(G) \nrightarrow V(H')$). For a vertex $w \in V(H)$ we distinguish again the four cases (see Definition 11).

1. $\varphi'(v) := w' \quad \forall v \in V(C_w^2)$.

2. + 3. $\varphi'(v) := w' \quad \forall v \in V(C_w^2), \quad \varphi'(v) := w'_{\mathrm{n}} \quad \forall v \in V(C_w^1) \setminus \{v_w\}$.

4. $\varphi'(v) := w' \quad \forall v \in V(C_w^2), \quad \varphi'(v) := w'_{\mathrm{n}} \quad \forall v \in V(C_w^{1,1}), \quad \varphi'(v) := w''_{\mathrm{n}} \quad \forall v \in V(C_w^{1,2})$.

To prove that φ' is well-defined we need to show that no vertex maps to more than one vertex, that means that all occurring sets in the universal quantifiers are disjoint. For two vertices $w_1, w_2 \in V(H)$ with $w_1 \neq w_2$ and $i, j \in \{1, 2\}$ we get from (6) that

$$V(C_{w_1}^i) \cap V(C_{w_2}^j) \subseteq \varphi^{-1}[w_1] \cap \varphi^{-1}[w_2] = \emptyset.$$

Therefore, it remains to check that the universal quantifiers are disjoint within the cases of one vertex $w \in V(H)$. In the case 1 there is only one universal quantifier. In the cases 2 and 3 the two sets $V(C_w^2)$ and $V(C_w^1) \setminus \{v_w\}$ are disjoint since $V(C_w^1) \cap V(C_w^2) = \{v_w\}$ by (7). Furthermore, in case 4 we know by definition of $C_w^{1,1}$ and $C_w^{1,2}$ that its vertex sets are disjoint. Since for $i \in \{1,2\}$ the tree $C_w^{1,i}$ is a subtree of C_w^1 that does not contain v_w, its vertex set is disjoint with $V(C_w^2)$ with the same argumentation as in the cases 2 and 3.

Lemma 6. *The above defined partial function φ' is surjective.*

Proof. We prove that for all $w \in V(H)$ all vertices of R_w are in the image of φ', this is enough since $V(H') = \biguplus_{w \in V(H)} V(R_w)$. In all four cases w' is in the image of φ' since $v_w \in V(C_w^2)$ by (7) and therefore $\varphi'(v_w) = w'$. In the cases 2 and 3 there exists a vertex in $v \in V(C_w^1) \setminus \{v_w\}$ since the degree of v_w in C_w^1 is one and therefore $\varphi'(v) = w'_{\mathrm{n}}$. In case 4 both $C_w^{1,1}$ and $C_w^{1,2}$ are by construction non-empty and therefore w'_{n} and w''_{n} are both in the image of φ'.

Lemma 7. *For each vertex $x \in V(H')$ the vertex set $\varphi'^{-1}[x]$ is connected in G.*

Proof. We distinguish again the four cases. In all four cases $\varphi'^{-1}[w'] = V(C_w^2)$ is connected in G since C_w^2 is a tree and (4) holds. In the cases 2 and 3 we have $\varphi'^{-1}[w'_{\mathrm{n}}] = V(C_w^1) \setminus \{v_w\}$, which is connected since C_w^1 is a tree, the vertex v_w has degree one in C_w^1, which implies that after removing the vertex the result is still a tree, and (4). Furthermore, in case 4 we have $\varphi'^{-1}[w'_{\mathrm{n}}] = V(C_w^{1,1})$ and $\varphi'^{-1}[w''_{\mathrm{n}}] = V(C_w^{1,2})$, which are both vertex sets of connected subtrees of C_w^1. The connectedness in G follows again by (4).

Lemma 8. *It holds that* $\mathrm{dom}(\varphi') = \mathrm{dom}(\varphi)$ *and for* $v_1, v_2 \in \mathrm{dom}(\varphi)$ *it holds that* $\varphi(v_1) \neq \varphi(v_2) \Rightarrow \varphi'(v_1) \neq \varphi'(v_2)$.

Proof. By the definition of φ', (6), and (7) we get

$$\mathrm{dom}(\varphi') = \bigcup_{w \in V(H)} V(C_w^1) \cup V(C_w^2) \overset{(6)}{=} \bigcup_{w \in V(H)} \varphi^{-1}[w] = \varphi^{-1}[V(H)] = \mathrm{dom}(\varphi).$$

The second statement follows from the fact that $\varphi'[\varphi^{-1}[w]] = V(R_w)$ for all $w \in V(H)$ and that $V(R_{w_1}) \cap V(R_{w_2}) = \emptyset$ for two different vertices $w_1, w_2 \in V(H)$.

Using φ' we can now define edges that connect the subgraphs R_w. For each edge $f \in E(H)$ we add an edge f' to H' with

$$r_{H'}(f') := \varphi'[r_G(\kappa^{-1}(f))].$$

First of all since κ is injective and surjective the inverse function κ^{-1} is well-defined. Furthermore, we have to prove that $|\varphi'[r_G(\kappa^{-1}(f))]| = 2$. By using (5) we get $\varphi[r_G(\kappa^{-1}(f))] = r_H(f)$ and therefore

$r_G(\kappa^{-1}(f)) \subseteq \mathrm{dom}(\varphi)$. By Lemma 8, we get $r_G(\kappa^{-1}(f)) \subseteq \mathrm{dom}(\varphi')$. Since $r_G(\kappa^{-1}(f))$ has two elements and $\varphi[r_G(\kappa^{-1}(f))] = r_H(f)$ has still two elements we get by Lemma 8 that $\varphi'[r_G(\kappa^{-1}(f))]$ has also two elements.

All in all we can define now the edge set of H' by

$$E(H') := \{f' \mid f \in E(H)\} \mathbin{\dot\cup} \biguplus_{w \in V(H)} E(R_w).$$

The graph H' is therefore fully defined. To prove that H' is a minor of G we will use the vertex mapping φ' and a surjective edge mapping $\kappa' \colon E(G) \twoheadrightarrow E(H')$, which we will define in the following.

Definition 13 ($\kappa' \colon E(G) \twoheadrightarrow E(H')$). To define the preimages of edges in R_w we distinguish for each vertex $w \in V(H)$ the four cases of Definition 11.

1. There are no edges in R_w.
2. From (13) we know $E_{C^1_w}(v_w) \subseteq r_G(\pi_2(T_w))$. Since $E_{C^1_w}(v_w)$ has one element if follows that there exists an edge $e \in \pi_2(T_w)$ such that $r_G(e) \in E_{C^1_w}(v_w)$, which implies $e \in E^1_w \cap \pi_2(T_w)$. Using this edge e we define $\kappa'(e) := f'_w$.
3. In this case we have $|\kappa[\pi_2(T_w)]| = 0$, which is equivalent to $\kappa^{-1}[E(H)] \cap \pi_2(T_w) = \emptyset$. This implies $\kappa^{-1}[\pi_2(S_w)] \cap \pi_2(T_w) = \emptyset$. Using (8) we can derive
$$\begin{aligned}(8) \Rightarrow & \pi_2(T_w) \subseteq \kappa^{-1}[\pi_2(S)] \cup \theta^{-1}[w] \cup (E(v_w) \cap E^1_w)\\ \Rightarrow & \pi_2(T_w) \subseteq \underbrace{\left(\kappa^{-1}[\pi_2(S)] \cap \pi_2(T_w)\right)}_{\emptyset} \cup \theta^{-1}[w] \cup (E(v_w) \cap E^1_w) = \theta^{-1}[w] \cup (E(v_w) \cap E^1_w).\end{aligned}$$
Since θ is injective $\theta^{-1}[w]$ contains at most one edge and therefore we can write $\pi_2(T_w) = \{e_1, e_2\}$ with $e_1 \in E^1_w$ and e_2 is either in $\theta^{-1}[w]$ or in E^1_w. We define now $\kappa'(e_1) := f'_w$ and $\kappa'(e_2) := f''_w$.
4. Again by (13) we know $E_{C^1_w}(v_w) \subseteq r_G(\pi_2(T_w))$. Since $E_{C^1_w}(v_w)$ has in this case two elements it follows that $E_{C^1_w}(v_w) = r_G(\pi_2(T_w))$. Let $\pi_2(T_w) = \{e_1, e_2\}$. Since $r_G(\pi_2(T_w)) = E_{C^1_w}(v_w)$ contains two edges where one is connecting v_w to $C^{1,1}_w$ and the other is connecting v_w to $C^{1,2}_w$ we can say w.l.o.g. that e_i connects v_w and a vertex in $V(C^{1,i}_w)$ for $i \in \{1, 2\}$. Using this notation we define $\kappa'(e_1) := f'_w$ and $\kappa'(e_2) := f''_w$.

Furthermore, for an edge $f \in E(H)$ we define $\kappa'(\kappa^{-1}(f)) := f'$, which is well-defined since κ is injective and surjective.

We have to prove now that κ' is well-defined, i.e. that no edge in $E(G)$ has more than one image under κ'. First of all we note that in all four cases only edges from $\overline{E^1_w} := E^1_w \cup \theta^{-1}[w]$ get mapped to an edge f'_w or f''_w.

By Lemma 5 an edge $e \in \overline{E^1_w}$ always has $r_G(e) \subseteq V(C^1_w)$. For two different vertices $w_1, w_2 \in V(H), w_1 \neq w_2$ we always have $\overline{E^1_{w_1}} \cap \overline{E^1_{w_2}} = \emptyset$ since by (6) we get $V(C^1_{w_1}) \cap V(C^1_{w_2}) \subseteq \varphi^{-1}[w_1] \cap \varphi^{-1}[w_2] = \emptyset$.

Furthermore, by the definition of κ' an edge never gets mapped to both f'_w and f''_w for a vertex w. It remains to show that an edge in $\mathrm{dom}(\kappa)$ gets not mapped to any edges f'_w or f''_w for any $w \in V(H)$. By (5) we get for an edge $e \in \mathrm{dom}(\kappa)$ that $\varphi[r_G(e)] = r_H(\kappa(e))$ and therefore e is not in $\overline{E^1_w}$ since all edges in $\overline{E^1_w}$ are incident to two vertices in $V(C^1_w) \subseteq \varphi^{-1}[w]$, which would imply $\varphi[r_G(e)] = \{w\} \neq r_H(\kappa(e))$, which is a contradiction. Therefore, we get $\overline{E^1_w} \cap \mathrm{dom}(\kappa) = \emptyset$ for all $w \in V(H)$. This finishes the proof that κ' is well-defined.

Lemma 9. *κ' is injective and surjective*

Proof. In the definition of κ' we defined in all cases for each edge in $E(H')$ exactly one preimage that maps under κ' to this edge. This implies directly both, that κ' is injective and surjective.

Lemma 10. *The graph H' is a minor of G.*

Proof. We already defined a partial vertex mapping $\varphi' \colon V(G) \nrightarrow V(H')$ and a partial edge mapping $\kappa' \colon E(G) \nrightarrow E(H')$. In Lemma 6 we showed that φ' is surjective, in Lemma 7 we showed that all preimages $\varphi'^{-1}[x]$ are connected in G for all $x \in V(H')$, and in Lemma 9 we showed that κ' is injective and surjective. The only remaining condition of H' being a minor of G is (1). To prove this we distinguish again the four cases as we did in Definition 13.

1. E^1_w is empty
2. Let $e = \kappa'^{-1}(f'_w)$, then $e \in E^1_w \cap \pi_2(T_w)$, which implies $r_G(e) = \{v_w, v\}$ for some $v \in V(C^1_w) \setminus \{v_w\}$. Therefore, we get

$$\varphi'[r_G(e)] = \{\varphi'(v_w), \varphi'(v)\} = \{w', w'_{\mathrm{n}}\} = r_{H'}(f'_w) = r_{H'}(\kappa'(e)).$$

3. Let e_1 and e_2 be as in case 3 of the definition of κ'. For $i \in \{1, 2\}$ we have by Lemma 5 $r_G(e_i) = \{v_w, v\}$ with $v \in V(C^1_w)$ and $v \neq v_w$. This implies

$$\varphi'[r_G(e_i)] = \{\varphi'(v_w), \varphi'(v)\} = \{w', w'_{\mathrm{n}}\} = r_{H'}(f'_w) = r_{H'}(f''_w) = r_H(\kappa'(e_i)) \quad \forall i \in \{1, 2\}\,.$$

4. Let e_1 and e_2 be as in case 4 of the definition of κ' . Then we get $r_G(e_i) = \{v_w, v_i\}$ with $v_i \in V(C^{1,i}_w)$ and $v_i \neq v_w$ for $i \in \{1, 2\}$. With that we can follow (1) for $i = 1, 2$.

$$\varphi'[r_G(e_1)] = \{\varphi'(v_w), \varphi'(v_1)\} = \{w', w'_{\mathrm{n}}\} = r_{H'}(f'_w) = r_H(\kappa'(e_1))$$

$$\varphi'[r_G(e_2)] = \{\varphi'(v_w), \varphi'(v_2)\} = \{w', w''_{\mathrm{n}}\} = r_{H'}(f''_w) = r_H(\kappa'(e_2))$$

Finally for $e \in \mathrm{dom}(\kappa)$ with $\kappa(e) = f$ we get

$$r_{H'}(\kappa'(e)) = r_{H'}(f') = \varphi'[r_G(\kappa^{-1}(f))] = \varphi'[r_G(e)].$$

All in all we proved (1) for all $e \in \mathrm{dom}(\kappa')$.

To get a reduced transition minor of $(G, \mathcal{T})$ we define $\mathcal{S}'$ as in (2) using φ' and κ'. Therefore, we get by definition that $(H', \mathcal{S}')$ is a reduced transition minor of $(G, \mathcal{T})$.

In the second part we want to prove that $(H', \mathcal{S}')$ is a basic-sup-$(H, \mathcal{S})$-graph. We already defined the connected subgraphs R_w for each $w \in V(H)$, they are by definition all vertex disjoint. We also defined the edges $f' \in E(H')$ for each $f \in E(H)$.

Lemma 11. *For each edge $f \in E(H)$ with $r_H(f) = \{w_1, w_2\}$ the edge f' connects the subgraphs R_{w_1} and R_{w_2} of H'.*

Proof. Let $e = \kappa^{-1}(f) \in E(G)$ and $r_G(e) = \{v_1, v_2\}$ then by definition we get

$$r_{H'}(f') = \varphi'[r_G(\kappa^{-1}(f))] = \varphi'[\{v_1, v_2\}].$$

By using (5) we get $\varphi[\{v_1, v_2\}] = r_H(f) = \{w_1, w_2\}$ and therefore w.l.o.g. $v_i \in \varphi^{-1}[w_i]$, which implies by definition of φ' that $\varphi'(v_i)$ is defined and in R_{w_i}. Therefore, f' connects the vertices $\varphi'(v_1) \in V(R_{w_1})$ and $\varphi'(v_2) \in V(R_{w_2})$.

For the rest of the proof let $w \in V(H)$ be fixed. Furthermore, let $\{f_1, f_2\} = \pi_2(S_w)$ and let $\{f_3, f_4\} = E_H(w) \setminus \{f_1, f_2\}$ be the remaining edges (note, that w has degree four, since H is 4-regular). Since H is completely transitioned we get that $(w, \{f_3, f_4\})$ is also a transition in $\mathcal{S}(w)$. Let $e_i = \kappa^{-1}(f_i)$, which is well-defined since κ is surjective. By definition of κ' we get $\kappa'(e_i) = f'_i$. Before we prove the final lemma, we need the following lemmas.

Lemma 12. *The only edges in H' that are incident to at least one vertex in R_w are the edges f'_1, f'_2, f'_3, f'_4 and the edges f'_w and f''_w if they exist. Furthermore, for $i \in \{1, 2, 3, 4\}$ the edges f'_i are incident to exactly one vertex x_i in R_w and $x_i = w'$ if $i \in \{3, 4\}$.*

Proof. Let E'_w be the set of all edges that are incident to at least one vertex in R_w. Since f'_w and f''_w are by definition incident to two vertices of R_w if they exist and since $V(R_{w_1})$ and $V(R_{w_2})$ are disjoint for $w_1 \neq w_2$ we know that f'_w and f''_w are in E'_w if they exist and f'_{w_1} and f''_{w_1} are not in E'_w if $w_1 \neq w$. Furthermore, from Lemma 11 follows that f' is incident to a vertex of R_w if and only if $w \in r_H(f)$. Therefore, we get

$$E'_w \cap \{f' \mid f \in E(H)\} = \{f' \mid f \in E_H(w)\} = \{f'_1, f'_2, f'_3, f'_4\}.$$

Lemma 11 also gives us that the edges f_i are incident to exactly one vertex in R_w, since the other vertex must be in another subgraph R_{w_2} with $w_2 \neq w$ since we do not allow loops in H. Let $x_i \in R_w$ be the unique incident vertex of f'_i. By (11) we get $r_G(e_i) \cap V(C_w^2) \neq \emptyset$ for $i \in \{3, 4\}$. Let $v_i \in r_G(e_i) \cap V(C_w^2)$ for $i \in \{3, 4\}$, which implies $\varphi'(v_i) = w'$ and all in all we get

$$r_{H'}(f'_i) = \varphi'[r_G(\kappa^{-1}(f_i))] = \varphi'[r_G(e_i)] \ni \varphi'(v_i) = w'.$$

Therefore, we know that w' is incident to f'_i and since there exists exactly one vertex in R_w that is incident to f'_i we get $x_i = w'$ for $i \in \{3,4\}$.

Lemma 13. *The edges f'_i are incident to w' if and only if $e_i \in \pi_2(T_w)$ for $i \in \{1,2\}$.*

PROOF. If $e_i \in \pi_2(T_w)$ we get

$$r_{H'}(f'_i) = \varphi'[r_G(\kappa^{-1}(f_i))] = \varphi'[r_G(e_i)] \ni \varphi'(v_w) = w'.$$

For the other direction we first note that by (10) we get that e_i is incident to a vertex in $V(C_w^1)$ for $i \in \{1,2\}$. If f'_i is incident to w' we get

$$w' \in r_{H'}(f'_i) = \varphi'[r_G(\kappa^{-1}(f_i))] = \varphi'[r_G(e_i)],$$

which implies that e_i is incident to a vertex v that is in $\varphi'^{-1}[w'] = V(C_w^2)$. By (4) we get that e_i is only incident to one vertex in $\varphi^{-1}[w] = V(C_w^1) \cup V(C_w^2)$. Therefore, we know that e_i is incident to a vertex that is in $V(C_w^1)$ and in $V(C_w^2)$, which can only be v_w by (7). All in all we get $e_i \in \kappa^{-1}[\pi_2(S_w)] \cap E(\pi_1(T_w)) \subseteq \pi_2(T_w)$ by (9).

What remains to prove that $(H', \mathcal{S}')$ is a sup-$(H, \mathcal{S})$-graph is the following lemma.

Lemma 14. *There exists a transition S'_w in $\mathcal{S}'$ such that the form of R_w and the transition S'_w satisfy one of the four possibilities (see Figure 4):*

1. *R_w is only one vertex w' and $S'_w = (w', \{f'_1, f'_2\}) \in \mathcal{S}'(w')$.*
2. *R_w is a K_2 with two vertices w' and w'_n, where w'_n is of degree two with two incident edges f'_1 and f'_w. Moreover, $S'_w = (w', \{f'_w, f'_2\}) \in \mathcal{S}'(w')$.*
3. *R_w is a cycle of length two, i.e. two vertices w' and w'_n and two parallel edges f'_w and f''_w connecting them. Furthermore, w'_n has degree four and is incident to f'_1 and f'_2 and $S'_w = (w', \{f'_w, f''_w\}) \in \mathcal{S}'(w')$.*
4. *R_w consists of a vertex w' and two vertices w'_n and w''_n and two edges f'_w and f''_w connecting w' with w'_n and w' with w''_n. Furthermore, w'_n is incident to f'_1, w''_n is incident to f'_2, w' is incident to f'_3 and f'_4, and $S'_w = (w', \{f'_w, f''_w\}) \in \mathcal{S}'(w')$.*

PROOF. First we define the transition S'_w by $S'_w := (w', \kappa'[\pi_2(T_w)])$. In the next step we prove that S'_w is in $\mathcal{S}'$. By (7), which implies $v_w \in V(C_w^2)$, and the definition of φ' in all four cases we get $v_w \in \mathrm{dom}(\varphi')$ and $\varphi'(v_w) = w'$. By definition of $\mathcal{S}'$ it only remains to show

$$\pi_2(T_w) \subseteq \mathrm{dom}(\kappa'). \tag{16}$$

to imply that $S'_w \in \mathcal{S}'$. We will show (16) together with the rest of the lemma by case distinction of the four cases of Definition 11.

1. In this case E_w^1 is empty. By Lemma 12, we get that f_i' is incident to w' for $i \in \{1,2\}$ and by Lemma 13 this implies that $e_i \in \pi_2(T_w)$ for $i \in \{1,2\}$. Since $\pi_2(T_w)$ has two elements, we get $\pi_2(T_w) = \{e_1, e_2\} \subseteq \mathrm{dom}(\kappa')$. By definition of κ' this implies $\pi_2(S_w') = \kappa'[\pi_2(T_w)] = \{\kappa'(e_1), \kappa'(e_2)\} = \{f_1', f_2'\}$. With that we proved the lemma for case 1.
2. Let e be the only edge in $\pi_2(T_w) \cap \mathrm{dom}(\kappa)$. Since $e \in \mathrm{dom}(\kappa)$ we get $|\varphi(r_G(e))| \overset{(5)}{=} |r_H(\kappa(e))| = 2$, and therefore $r_G(e) \nsubseteq V(C_w^1)$. This implies by (14) that $e \notin \theta^{-1}[w]$ and that $e \notin E_w^1$. From (8) we then get $e \in \kappa^{-1}[\pi_2(S_w)]$. Therefore, $\kappa(e) \in \pi_2(S_w) = \{f_1, f_2\}$ and by exchanging f_1 and f_2 if needed we get $\kappa(e) = f_2$, which implies $e = e_2$.

   Let now $\tilde{e} := \kappa'^{-1}(f_w')$. By definition of κ' this implies $\tilde{e} \in \pi_2(T_w)$. Furthermore, by definition of κ' we know $\tilde{e} \in E_w^1$, which implies $\tilde{e} \neq e_2$. Therefore, we found the two edges of $\pi_2(T_w) = \{\tilde{e}, e_2\}$ and both of them are in $\mathrm{dom}(\kappa')$ (note that $e_2 \in \mathrm{dom}(\kappa) \subseteq \mathrm{dom}(\kappa')$), which implies (16).

   For the edges of S_w' we get $\pi_2(S_w') = \kappa'[\pi_2(T_w)] = \kappa'[\{\tilde{e}, e_2\}] = \{\kappa'(\tilde{e}), \kappa'(\kappa^{-1}(f_2))\} = \{f_w', f_2'\}$. It remains to show that w_n' has degree two and is incident to f_1' and f_w'. By definition of f_w' it is incident to w_n', so we have to show that the only other incident edge to w_n' is f_1'. By Lemma 12, we get that the only possible other incident edges are f_1' and f_2' and by Lemma 13 we get that f_2' is incident to w' and therefore not to w_n'.

   Assume f_1' would not be incident to w_n'. By Lemma 12, we get that f_1' is incident to w' and by Lemma 13 that $e_1 \in \pi_2(T_w)$. But then $f_1 = \kappa(e_1) \in \kappa[\pi_2(T)] \ni \kappa(e_2) = f_2$, which is a contradiction to the fact that $|\kappa[\pi_2(T)]| = 1$ and that κ is injective.
3. By definition of κ' in this case we get directly (16) and also $\kappa'[\pi_2(T_w)] = \{f_w', f_w''\}$. By definition of R_w in this case it is a cycle of length two consisting of two parallel edges f_w' and f_w'' connecting two nodes w' and w_n'. Therefore, it only remains to show that w_n' has degree four and is connected by f_1' and f_2'. By Lemma 12, we get that the only other possible incident edges to w_n' are f_1' and f_2'.

   To show that f_i' is incident to w_n' for $i \in \{1,2\}$ we show that it is not incident to w', which is enough by Lemma 12. This is equivalent to showing that $e_i \notin \pi_2(T_w)$ for $i \in \{1,2\}$ by Lemma 13. For $i \in \{1,2\}$ the fact $e_i \notin \pi_2(T_w)$ follows directly from the fact that $e_i \in \mathrm{dom}(\kappa)$ and $|\kappa[\pi_2(T_w)]| = 0$ in this case. All in all we showed that the incident edges of w_n' are exactly f_w', f_w'', f_1', and f_2'.
4. By definition of κ' in this case we get directly (16) and also $\kappa'[\pi_2(T_w)] = \{f_w', f_w''\}$. By definition of R_w in this case it consists of a vertex w' and two vertices w_n' and w_n'' and two edges f_w' and f_w'' connecting w' with w_n' and w' with w_n''. By Lemma 12, we get that f_3' and f_4' are incident to w' and therefore it only remains to show that f_1' is incident to w_n' and f_2' is incident to w_n''.

   As shown in case 4 of the definition of κ' we know that the edges of $\pi_2(T_w)$ are in E_w^1 and therefore $e_i \notin \pi_2(T_w)$ for $i \in \{1,2\}$. By Lemma 13, we get that f_i' is not incident to w' and therefore that f_i' is incident to w_n' or to w_n'' for $i \in \{1,2\}$.

   We know in this case that the two trees $C_w^{1,i}$ are both nonempty subtrees of C_w^1 for $i \in \{1,2\}$. If

$C_w^{1,i}$ consists only of one vertex, we know that this vertex has degree 1 in C_w^1. On the other hand, if $C_w^{1,i}$ has more than one vertex we know that $C_w^{1,i}$ has at least two leaf vertices of degree 1 and at least one of them has also degree 1 in C_w^1. Therefore, we know that in any case there exists a vertex $v_i \in V(C_w^{1,i})$ that has degree 1 in C_w^1 for $i \in \{1, 2\}$.

Assume that there exists an edge $e \in \theta^{-1}[w]$ then by (9) we know $e \in \pi_2(T_w)$ but we already know that both edges of $\pi_2(T_w)$ are in E_w^1, which is a contradiction to (15). Therefore, we know $\theta^{-1}[w] = \emptyset$ in this case.

Putting everything together we get

$$v_i \in V(C_w^1) \setminus \{\pi_1(T_w)\} \wedge \deg_{C_w^1}(v_i) = 1 \wedge v \notin \bigcup r_G[\theta^{-1}[w]],$$

which implies by (12) that $E(v_i) \cap \kappa^{-1}[\pi_2(S_w)] \neq \emptyset$. Since $E(v_1) \cap E(v_2) \cap \mathrm{dom}(\kappa) = \emptyset$ and the fact that $\kappa^{-1}[\pi_2(S_w)] = \{e_1, e_2\}$ we get by exchanging e_1 with e_2 if needed that $e_i \in E(v_i) \cap \kappa^{-1}[\pi_2(S_w)]$ for $i \in \{1, 2\}$. But this implies

$$r_{H'}(f_i') = \varphi'[r_G(\kappa^{-1}(f_i))] = \varphi'[r_G(e_i)] \ni \varphi'(v_i) \quad \forall i \in \{1, 2\}$$

and therefore f_1' is incident to $\varphi'(v_1) = w_\mathrm{n}'$ and f_2' is incident to $\varphi'(v_2) = w_\mathrm{n}''$.

With Lemma 14 we finished the proof that $(H', \mathcal{S}')$ is a sup-$(H, \mathcal{S})$-graph and all together that $(G, \mathcal{T})$ has a basic-sup-$(H, \mathcal{S})$-reduced-transition-minor.

5. Mixed Integer Linear Programming Model

In this section we derive a mixed integer linear programming model (MIP) from the mathematical model described in Section 4.2 in order to practically solve it. Let E_G^S denote the set of simple edges $r_G[E(G)]$.

To model the partial functions φ, κ, and λ we use boolean variables x_v^w for $v \in V(G)$ and $w \in V(H)$, which are true if and only if $\varphi(v) = w$, boolean variables y_e^f for $e \in E(G)$ and $f \in E(H)$, which are true if and only if $\kappa(e) = f$, and boolean variables z_e^w for $e \in E(G)$ and $w \in V(H)$, which are true if and only if $\theta(e) = w$. Furthermore, to describe the transitions T_w and S_w we use boolean variables a_T^w for $w \in V(H)$ and $T \in \mathcal{T}$ representing $T_w = T$ and boolean variables b_S^w for $w \in V(H)$ and $S \in \mathcal{S}(w)$ representing $S_w = S$.

To describe the trees C_w^i we use boolean variables $h_v^{i,w}$ for $v \in V(G)$, $i \in \{1, 2\}$, and $w \in V(H)$ that encode $v \in V(C_w^i)$ and boolean variables $t_{v_1,v_2}^{i,w}$ for $\{v_1, v_2\} \in r_G[E(G)]$, $i \in \{1, 2\}$, and $w \in V(H)$ that encode that the simple edge $\{v_1, v_2\}$ is in $E(C_w^i)$. Note that the variables $t_{v_1,v_2}^{i,w}$ are directed, i.e. for $\{v_1, v_2\} \in r_G[E(G)]$ there exist two variables $t_{v_1,v_2}^{i,w}$ and $t_{v_2,v_1}^{i,w}$. To eliminate subtours in the trees C_w^i, which form together a forest, we use a Miller-Tucker-Zemlin (MTZ) formulation using continuous variables u_v for $v \in V(G)$ [8].

The following first part of the constraints is concerned with ensuring the properties of the mappings φ, κ, λ, and θ, and the tree structure of the trees C^i_w.

$$\sum_{w \in V(H)} x^w_v \leq 1 \qquad \forall v \in V(G) \tag{17}$$

$$\sum_{v \in V(G)} x^w_v \geq 1 \qquad \forall w \in V(H) \tag{18}$$

$$\sum_{f \in E(H)} y^f_e \leq 1 \qquad \forall e \in E(G) \tag{19}$$

$$\sum_{e \in E(G)} y^f_e = 1 \qquad \forall f \in E(H) \tag{20}$$

$$\sum_{w \in V(H)} z^w_e \leq 1 \qquad \forall e \in E(G) \tag{21}$$

$$\sum_{e \in E(G)} z^w_e \leq 1 \qquad \forall w \in V(H) \tag{22}$$

$$\sum_{T \in \mathcal{T}} a^w_T = 1 \qquad \forall w \in V(H) \tag{23}$$

$$\sum_{S \in \mathcal{S}(w)} b^w_S = 1 \qquad \forall w \in V(H) \tag{24}$$

$$t^{i,w}_{v_1,v_2} + t^{i,w}_{v_2,v_1} \leq h^{i,w}_{v_j} \qquad \forall \{v_1, v_2\} \in E^{\mathrm{S}}_G, \forall w \in V(H), \forall i, j \in \{1, 2\} \tag{25}$$

$$\sum_{v_1 \in N_G(v)} t^{i,w}_{v_1,v} = h^{i,w}_v - \sum_{T \in \mathcal{T}(v)} a^w_T \qquad \forall v \in V(G), \forall i \in \{1, 2\}, \forall w \in V(H) \tag{26}$$

$$u_v \leq (|V_G| - |V_H|)(1 - \sum_{\substack{T \in \mathcal{T}(v) \\ w \in V(H)}} a^w_T) \qquad \forall v \in V(G) \tag{27}$$

$$u_{v_1} - u_{v_2} + 1 \leq (|V_G| - |V_H| + 1)(1 - \sum_{\substack{w \in V(H) \\ i \in \{1,2\}}} t^{i,w}_{v_1,v_2}) \qquad \forall \{v_1, v_2\} \in E^{\mathrm{S}}_G \tag{28}$$

Constraints (17), (19), and (21) ensure that φ, κ, and θ are partial functions. Furthermore, constraints (18) guarantee that φ is surjective, (20) ensure that κ is injective and surjective, and (22) guarantee that θ is injective. The fact that there should be exactly one transition $T_w \in \mathcal{T}$ and $S_w \in \mathcal{S}(w)$ for each $w \in V(H)$ is ensured by constraints (23) and (24). Note the difference that T_w can be any transition in $\mathcal{T}$ but S_w must be a transition at the vertex w, i.e. in $\mathcal{S}(w)$.

Constraints (25) couple the vertex variables h with the directed edge variables t for each tree C^i_w. To enforce the tree structure of all C^i_w we ensure that each vertex in the tree has exactly one incoming arc except the root vertex, which has no incoming arc and that there are no cycles. Together, this is enough to guarantee the tree structure. As root vertex for each tree C^i_w we use the vertex $v_w = \pi_1(T_w)$, which must be part of both trees by (7). Constraints (26) specify that each vertex in a tree except the root has exactly one incoming arc and that the root has no incoming arcs. To enforce connectivity we use the MTZ formulation, which is realized by the constraints (27) and (28). The second part of the MIP is concerned with ensuring constraints (4)–(15).

$$y_e^f \leq x_v^{w_1} + x_v^{w_2} \qquad \forall e \in E(G), \forall v \in r_G(e), \quad \forall f \in E(H), r_H(f) = \{w_1, w_2\} \tag{29}$$

$$y_e^f \leq x_{v_1}^w + x_{v_2}^w \qquad \forall e \in E(G), r_G(e) = \{v_1, v_2\}, \quad \forall f \in E(H), \forall w \in r_H(f) \tag{30}$$

$$h_v^{1,w} + h_v^{2,w} = x_v^w + \sum_{T \in \mathcal{T}(v)} a_T^w \qquad \forall v \in V(G), \forall w \in V(H) \tag{31}$$

$$\sum_{T \in \mathcal{T}(v)} a_T^w \leq x_v^w \qquad \forall v \in V(G), \forall w \in V(H) \tag{32}$$

$$a_T^w + b_S^w - 1 \leq \sum_{f \in \pi_2(S)} y_e^f + z_e^w + t_{v_1,v_2}^{1,w} + t_{v_2,v_1}^{1,w} \qquad \forall T \in \mathcal{T}, \forall e = v_1v_2 \in \pi_2(T) \quad \forall w \in V(H), \forall S \in \mathcal{S}(w), \tag{33}$$

$$a_T^w + b_S^w - 1 \leq 1 - \sum_{e \in E(\pi_1(T)) \setminus \pi_2(T)} y_e^f \qquad \forall T \in \mathcal{T}, \forall w \in V(H), \forall S \in \mathcal{S}(w), \forall f \in \pi_2(S) \tag{34}$$

$$a_T^w \leq 1 - \sum_{e \in E(G) \setminus \pi_2(T)} z_e^w \qquad \forall T \in \mathcal{T}, \forall w \in V(H) \tag{35}$$

$$b_S^w + \sum_{f \in \pi_2(S)} y_e^f - 1 \leq \sum_{v \in r_G(e)} h_v^{1,w} \qquad \forall w \in V(H), \forall S \in \mathcal{S}(w), \forall e \in E(G) \tag{36}$$

$$b_S^w + \sum_{f \in E(w) \setminus \pi_2(S)} y_e^f - 1 \leq \sum_{v \in r_G(e)} h_v^{2,w} \qquad \forall w \in V(H), \forall S \in \mathcal{S}(w), \forall e \in E(G) \tag{37}$$

$$\begin{aligned} b_S^w + h_v^{1,w} - 1 - \sum_{T \in \mathcal{T}(v)} a_T^w \leq & \frac{1}{2} \sum_{v_2 \in N_G(v)} t_{v,v_2}^{1,w} + t_{v_2,v}^{1,w} + \\ & \sum_{e \in E(v)} z_e^w + \sum_{\substack{e \in E_G(v) \\ f \in \pi_2(S)}} y_e^f \end{aligned} \qquad \forall w \in V(H), \forall S \in \mathcal{S}(w), \forall v \in V(G) \tag{38}$$

$$a_T^w \leq 1 - t_{v,\pi_1(T)}^{1,w} - t_{\pi_1(T),v}^{1,w} \qquad \forall T \in \mathcal{T}, \forall w \in V(H), \quad \forall v \in N(\pi_1(T)) \setminus \bigcup r_G[\pi_2(T)] \tag{39}$$

$$z_e^w \leq \frac{1}{2} \sum_{v \in r_G(e)} h_v^{1,w} \qquad \forall e \in E(G), \forall w \in V(H) \tag{40}$$

$$z_e^w \leq 1 - t_{v_1,v_2} - t_{v_2,v_1} \qquad \forall e = v_1v_2 \in E(G), \forall w \in V(H) \tag{41}$$

$$x_v^w \in \{0, 1\} \qquad \forall v \in V(G), \forall w \in V(H) \tag{42}$$

$$y_e^f \in \{0, 1\} \qquad \forall e \in E(G), \forall f \in E(H) \tag{43}$$

$$z_e^w \in \{0, 1\} \qquad \forall e \in E(G), \forall w \in V(H) \tag{44}$$

$$a_T^w \in \{0, 1\} \qquad \forall T \in \mathcal{T}, \forall w \in V(H) \tag{45}$$

$$b_S^w \in \{0, 1\} \qquad \forall w \in V(H), \forall S \in \mathcal{S}(w) \tag{46}$$

$$h_v^{i,w} \in \{0,1\} \qquad \forall v \in V(G), \forall i \in \{1,2\}, \forall w \in V(H) \tag{47}$$

$$t_{v_1,v_2}^{i,w} \in \{0,1\} \qquad \forall \{v_1,v_2\} \in r[E(G)], \forall i \in \{1,2\}, \forall w \in V(H) \tag{48}$$

$$0 \leq u_v \leq |V_G| - |V_H| \qquad \forall v \in V(G) \tag{49}$$

By the index range of the variables $t_{v_1,v_2}^{i,w}$ constraints (4) are implicitly satisfied. Constraints (29) and (30) ensure constraints (5). Furthermore, constraints (31) and (32) together enforce (6) and (7). Constraint (8) are realized by (33) and constraints (9) by (34) and (35). Moreover, (36) ensure (10) and (37) ensures (11). Furthermore, (12) are guaranteed by (38) and (13) by (39). Last but not least (40) ensure (14) and (41) ensure (15).

6. Computational Results

To test the MIP model from Section 5 computationally we implemented it in Python 3$\dot{5}$ using Gurobi 7.0.1 [5]. All tests were performed on a single core of an Intel Xeon E5-2640 v4 processor with 2.40GHz and 8GB RAM.

6.1. Instance Sets

We consider three different instance sets S1, S2, and G1, two of them represent graph theoretic use cases and one represents the most general case. As discussed in Section 1 there are two correlations between the CCD and CDC, one via the line graph of a 3-regular graph and the other via contractions of a perfect pseudo-matching of a 3-regular graph. The first correlation is not interesting for us, since the line graph of a 3-regular graph is larger than the original graph, the second correlation, however, gives us in general a graph with at most half the number of vertices and is therefore the use case we test in S1 and S2. Note that we can restrict our instances to contractions of perfect pseudo-matchings of snarks as discussed in Section 1. For both sets S1 and S2 we use a snark together with a perfect pseudo-matching of the snark to build a transitioned graph as described already in Section 1. We contract all components of the matching and add then a transition between the two remaining edges of each original vertex of the snark, which gives us a 4-regular completely transitioned graph.

To generate instance set S1 we use all snarks with up to 26 vertices plus 1000 snarks with 28 vertices and compute for each of them three random perfect matchings. For each snark and each of its perfect matchings we build the resulting transitioned graph as described above and use it as $(G, \mathcal{T})$, and as transitioned graph $(H, \mathcal{S})$ we use the graph UD-K_5 as defined in Example 1. As source for the snarks with up to 28 vertices we used the lists published by Brinkmann et al. [1].

For instance set S2 we consider all possible perfect pseudo-matchings of each snark with up to 22 vertices. Again we construct the transitioned graph $(G, \mathcal{T})$ as described above. Note that by the cyclically 4-edge

Table 1: Computation results for instance set S1.

			Output Types		
$\lvert V \rvert$	instances	time[s]	infeasible	feasible	time limit
10	4	0.18	0	4	0
18	8	7.33	0	8	0
20	24	4.20	0	24	0
22	124	14.42	2	121	1
24	620	16.02	12	604	4
26	5188	20.73	22	5124	42
28	4004	34.77	11	3973	20

connectedness and the 3-regularity of a snark it follows that a snark contains no cycles of length three and therefore that after contracting a K_2 or a claw there will be no loop. As $(H, \mathcal{S})$ we use again the UD-K_5 from Example 1. When we construct all possible perfect pseudo-matchings of a snark we first compute its automorphism group using the nauty tool from McKay et al. [7], which we then also apply to filter out all perfect pseudo-matchings that lead after contraction to isomorphic transitioned graphs.

The instances of the first two instance sets always used for $(H, \mathcal{S})$ the graph UD-K_5. To also test the more general problem for a general $(H, \mathcal{S})$ we created the third instance set G1. There we use a randomly generated completely transitioned 4-regular (multi-)graph with n vertices as $(G, \mathcal{T})$ and a random completely transitioned 4-regular (multi-)graph with k vertices as $(H, \mathcal{S})$. To randomly generate a 4-regular graph with l vertices we start with l vertices and no edges and add random edges until the graph is 4-regular. Then we randomly partition the four edges incident to each vertex into two partitions of size two to define the two transitions. For our tests we computed 30 instances for all combinations of $9 \leq n \leq 15$ and $5 \leq k \leq 7$, which gives us 630 instances.

6.2. Results

Each run has three different output states: infeasible, feasible, or time limit reached. The time limit for all our runs is 12 hours. In Table 1 we list the results of our algorithm for instance set S1 grouped by the number of vertices $|V|$ of the snark from which the instance was constructed. Column $|V|$ gives the number of vertices of the snarks and column *instances* states the number of instances originating from a snark of this vertex size. Furthermore, column *time[s]* states the median computation times in seconds, and columns *infeasible*, *feasible* and *time limit* list the number of instances with the corresponding output states.

As we can see most instances of set S1 are feasible, which shows us that at least for small graphs a snark together with a random perfect matching leads with high probability to a contracted graph containing

Table 2: Computation results for instance set S2.

			Output Types	
$\|V\|$	instances	time[s]	infeasible	feasible
18	98	186.78	83	15
20	1116	255.68	700	416
22	10694	355.58	5821	4873

SUD-K_5 minors. This implies that to prove that a snark has a cycle double cover one has to test many perfect matchings until one finds one whose contraction leads to a SUD-K_5 minor free graph.

In total, we found 31 SUD-K_5 minor free graphs, which we further tested if they contain a K_5-minor. We found several examples in them that are SUD-K_5 minor free and contain a K_5-minor. As mentioned in the introduction such a graph resulting from a contraction of a perfect matching of a snark was not known before and proves that the theorem by Fleischner et al. [4] is indeed stronger than the older theorem by Fan and Zhang [2].

Since instance set S2 contains also perfect pseudo-matchings, we hope that this leads to more infeasible graphs, i.e. to more SUD-K_5 minor free graphs. Table 2 lists the results for instances set S2.

For this instance set the algorithm terminated for all instances before the time limit of 12 hours was reached. All other columns are the same as in Table 1. We can see that when using perfect pseudo-matchings more contracted graphs are SUD-K_5 minor free than when we only use perfect matchings. In fact our results prove that for every snark with up to 22 vertices there exists a perfect pseudo-matching such that the contracted graph is SUD-K_5-minor free, which implies the already known fact that all snarks with up to 22 vertices have a cycle double cover.

This raises the question if it is possible to find for every snark a perfect pseudo-matching that leads to a SUD-K_5-minor free graph, which would imply the cycle double cover. Interestingly there are some snarks with up to 22 vertices for which for every perfect matching the contraction of the graph is not SUD-K_5-minor free, which shows us that we really need perfect pseudo-matchings.

For our third instance set G1 results are listed in Table 3. Since we do not construct the instances based on snarks, we do not have column $|V|$ in this table but have two new columns $|V_G|$ and $|V_H|$, which list the numbers of vertices of the input graphs G and H, respectively. All other columns have the same meaning as in Table 1.

We can see that the computation times depend heavily on the size of graph H. Furthermore, if H is much smaller than G the probability for finding a sup-$(H, \mathcal{S})$ transition minor in G is high, which also leads to fast computation times. This effect leads to the result that the median computation time of the instances

Table 3: Computation results for instance set G1.

				Output Types		
$\lvert V_G \rvert$	$\lvert V_H \rvert$	instances	time[s]	infeasible	feasible	time limit
9	5	30	232.29	15	15	0
9	6	30	3448.82	26	4	0
9	7	30	8233.49	30	0	0
10	5	30	210.95	11	19	0
10	6	30	7492.30	26	4	0
10	7	30	33104.28	21	0	9
11	5	30	147.85	5	25	0
11	6	30	15633.03	14	9	7
11	7	30	43200.00	5	1	24
12	5	30	113.21	2	28	0
12	6	30	1643.45	1	21	8
12	7	30	43200.00	0	1	29
13	5	30	118.06	0	28	2
13	6	30	2182.05	0	20	10
13	7	30	43200.00	0	3	27
14	5	30	43.39	0	30	0
14	6	30	753.11	0	27	3
14	7	30	43200.00	0	5	25
15	5	30	43.93	0	30	0
15	6	30	654.87	0	28	2
15	7	30	43200.00	0	14	16

with $|V_G| = 15$ and $|V_H| = 5$ was smaller than the median computation time of the instances with $|V_G| = 14$ and $|V_H| = 5$. In both cases all instances were feasible, i.e. $(G, \mathcal{T})$ contained a sup-$(H, \mathcal{S})$ transition minor. For $|V_H| = 7$ and $|V_G| \geq 11$ most of the instances run into the time limit of 12 hours.

7. Conclusion and Future Work

In this work we formulated the new problem ESTM for checking if a transitioned graph contains a sup-$(H, \mathcal{S})$ transition minor, which is a generalization of sup-K_5-minors defined by Fleischner et al [4]. A complexity analysis of the problem showed that it is NP-complete, even if restricted to simple graphs. Furthermore, it is fixed-parameter tractable with the size of graph H as parameter. We also formulated an equivalent problem EBSRTM, which is used as base problem for our models.

In the next step we formulated a mathematical model, which uses simple mathematical objects such as partial functions and trees together with a set of constraints in logical form. It does not directly model the intermediate graph, which needs to be a transition minor of the input graph $(G, \mathcal{T})$ and a sup-$(H, \mathcal{S})$ graph, but ensures with its constraints on the two input graphs the existence of such an intermediate graph. Since it is not trivial that this model solves the problem EBSRTM we provided a thorough proof of the equivalence in Section 3.

From the mathematical model we derived in a more or less straight forward manner a mixed integer linear program with which we can solve the problem EBSRTM in practice. We tested the MIP model on three different instance sets. Two of them are motivated by the cycle double cover problem and are based on contractions of perfect pseudo-matchings of snarks. The instances of the third set are randomly generated to consider the whole scope of the problem.

Results show that the algorithm can solve the problem for small instances in reasonable time. The tested instances consisted of a graph G with up to 15 vertices and a graph H with up to 7 vertices, but for the larger graphs some instances could not get solved within the time limit of 12 hours.

Through our tests we were able to find graphs that are contractions of perfect matchings of snarks and contain a K_5-minor but no SUD-K_5-minor. Such graphs were not known before and show that the theorem about the existence of compatible circuit decompositions by Fleischner et al. [4] is really stronger than the theorem by Fan and Zhang [2]. Furthermore, we could verify that there exists a perfect pseudo-matching whose contraction leads to a SUD-K_5-minor free graph for all snarks with up to 22 vertices. This motivates the question whether there exists such a perfect pseudo-matching for all snarks. If we ask the same question for perfect matchings we were able to find counter examples for this.

In future work the presented mathematical model may be used to develop other algorithms for this problem, which may be able to solve the problem faster. Since the mathematical model is not MIP specific constraint programming or SAT-based approaches might be promising. If we want to test large graphs, also a heuristic approach for finding SUD-$(H, \mathcal{S})$ transition minors in reasonable time would be interesting, although such an algorithm would not be able to prove that there does not exist such a transition minor.

In the context of snarks, in this work we only considered solving the problem for a given snark and a given perfect pseudo-matching by contracting the snark and using the resulting graph as input graph G. Of course a desirable extension would be to effectively search for a perfect pseudo-matching in a given snark such that the resulting contracted graph is SUD-K_5 transition minor free. Adding this additional level of searching for a perfect pseudo-matching adds a new dimension of complexity, especially since we want to find a perfect pseudo-matching for which our model is infeasible.

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