On Theorems Equivalent with Kotzig's Result on Graphs with Unique 1-Factors

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Abstract

We show that several known theorems on graphs and digraphs are equivalent. The list of equivalent theorems include Kotzig's result on graphs with unique 1-factors, a lemma by Seymour and Giles, theorems on alternating cycles in edge-colored graphs, and a theorem on semicycles in digraphs.

We consider computational problems related to the quoted results; all these problems ask whether a given (di)graph contains a cycle satisfying certain properties which runs through p prescribed vertices. We show that all considered problems can be solved in polynomial time for $p < 2$ but are NP-complete for $p \geq 2$.

1. Introduction

We consider several results on graphs and digraphs which all have been shown separately with considerable efforts:

1. Kotzig's theorem on graphs with unique 1-factors [8] (Theorem 1);
2. the main lemma in Seymour's 'Sums of Circuits'-paper [10] (Theorem 2);
3. theorems on properly colored cycles in edge-colored graphs [6, 15] (Theorems 3 and 4);
4. a theorem on semicycles in digraphs [12] (Theorem 5).

We show that all these theorems are equivalent[1]. Up to now, no pair of these theorems was shown equivalent; in particular, it was believed that Theorem 4 cannot be obtained from Theorems 3 directly (see [15]). However, the following implications have been observed elsewhere:

[1] by "equivalent" we mean that the theorems can be deduced from each other.

– Theorem 2 implies Theorem 1 ([10]);

– Theorem 3 implies Theorem 1 ([6]);

– Theorem 2 implies Theorem 3 (attributed in [14] to B. Jackson).

Moreover, Theorem 4 clearly implies Theorem 3, since the latter is a special case of the former.

Computational problems which arise from the quoted results ask whether (di)graphs contain cycles which satisfy certain properties and run through p specified vertices. We show that these problems can be reduced to each other in polynomial time. Moreover, we show that all these problems can be solved in polynomial time if $p = 0$ or $p = 1$, and are NP-complete for $p \geq 2$.

2. Notation

All graphs and digraphs considered are finite, simple and contain at least one vertex. For a graph G and $v \in V(G)$ we denote by $E_G(v)$ the set of edges of G which are incident with v. Graph theoretic terminology not defined here may be found in [4, 5].

3. Graphs with unique 1-factors

A 1-*factor* (or *perfect matching*) of a graph G is a set of independent edges $F \subseteq E(G)$ such that every vertex of G is incident with some edge in F. The following is a well-known theorem due to Kotzig [8] (for generalizations, see [2, 7]).

Theorem 1 (Kotzig). *If a graph G has a unique 1-factor F, then G has a bridge which belongs to F.*

Let G be a graph, and let F, F' be 1-factors of G. The symmetric difference $F \triangle F'$ is a vertex disjoint union of F-alternating cycles (i.e., of cycles whose edges are alternately in and out of F), cf. [3]. Hence Theorem 1 can be stated as follows:

> *Let G be a graph and F a 1-factor of G. If no edge in F is a bridge, then G contains an F-alternating cycle.*

By means of Kotzig's theorem one can decide efficiently whether a graph G with given 1-factor F contains some F-alternating cycle (see the proof of Lemma 1 below). It is natural to consider the following generalization of this problem (here, and in all problems presented in the sequel, the integer p is considered as some fixed parameter).

Problem 1. *Given a graph G, a 1-factor F of G, and a set X of p vertices; is there an F-alternating cycle in G which runs through all vertices of X?*

We will show that this problem (and several other problems formulated in the sequel) can be solved efficiently if $p < 2$, and are NP-complete for $p \geq 2$.

Lemma 1. *Problem 1 can be solved in polynomial time for $p < 2$.*

Proof. For the case $p = 1$, let $X = \{v\}$ and let e be the unique edge in $E_G(v) \cap F$. Observe that there is an F-alternating cycle C which runs through v if and only if $G - e$ has a 1-factor F_e; the latter can be checked in polynomial time by matching algorithms (see, e.g., [9]). If $G - e$ has a 1-factor F_e, then we choose the unique cycle C in $F_e \triangle F$ such that $e \in E(C)$.

To find any F-alternating cycle ($p = 0$), we proceed similarly as in the case $p = 1$: we consider all $e \in E(G)$ and check whether $G - e$ has a 1-factor. If, however, only existence of F-alternating cycles should be decided, then we can use Kotzig's theorem and proceed as follows. Denote the set of bridges of G by $B(G)$. We put $G_0 := G$, and for $i > 0$ we obtain G_i from G_{i-1} by deletion of the vertices which are incident with edges in $B(G_{i-1}) \cap F$. We stop as soon as we have either (i) no vertex of G_n is incident with some edge in $B(G_n) \cap F$ (i.e., $B(G_n) \cap F = \emptyset$) or (ii) every vertex of G_n is incident with some edge in $B(G_n) \cap F$; evidently $n \leq |V(G)|$. In case (i) we conclude by Kotzig's theorem that G_n (and so G) contains some F-alternating cycle. In case (ii), G_n certainly has no F-alternating cycle, and so G has no F-alternating cycle (for, if G_{i-1} has some F-alternating cycle C $(1 \leq i < n)$, then no vertex of G_{i-1} which is incident with edges in $B(G_{i-1}) \cap F$ can lie on C; consequently, C is also an F-alternating cycle of G_i). $\square$

We will show in the final section of this paper that Problem 1 is NP-complete for $p \geq 2$.

4. Seymour and Giles' Theorem

Next we consider a result which is stated as a lemma in Seymour's famous paper on sums of circuits [10]; Seymour attributes this result to him and Giles, and he calls it "the most tricky step" in the proof of the main theorem of [10]. Consider a graph G and map $\varphi : V(G) \to E(G)$ such that $\varphi(v) \in E_G(v)$ for all $v \in V(G)$. We call a cycle C of G *φ-conformal* if $\varphi(v) \in E(C)$ for every $v \in V(C)$.

Theorem 2 (Seymour and Giles). *Let G be a bridgeless graph and let $\varphi : V(G) \to E(G)$ be a map such that $\varphi(v) \in E_G(v)$ for all $v \in V(G)$. Then G has a φ-conformal cycle.*

We strengthen this theorem slightly as follows (observe that $0 \leq |\varphi^{-1}(e)| \leq 2$ holds for all edges $e \in E(G)$).

Corollary 1. *Let G be a graph and let $\varphi : V(G) \to E(G)$ be a map such that $\varphi(v) \in E_G(v)$ for all $v \in V(G)$. If $\varphi^{-1}(e) = \emptyset$ for every bridge e of G, then G has a φ-conformal cycle.*

Proof. We assume that $\varphi^{-1}(e) = \emptyset$ for all bridges e of G; we remove all bridges from G and obtain a graph G'. Now G' is bridgeless and $\varphi(v) \in E_{G'}(v)$ holds for all $v \in V(G')$. Hence Theorem 2 applies. Thus G' has a φ-conformal cycle C, which is clearly a φ-conformal cycle of G as well. □

Problem 2. *Given a graph G, a map $\varphi : V(G) \to E(G)$ with $\varphi(v) \in E_G(v)$ for all $v \in V(G)$, and a set X of p vertices; is there a φ-conformal cycle C which runs through all vertices in X?*

Proposition 1. *(1) Theorem 1 implies Theorem 2. (2) For every $p \geq 0$, Problem 2 can be reduced to Problem 1 in polynomial time.*

Proof. Let G and φ as stated in Problem 2. Consider an edge $e = uv \in E(G)$ and put $k_e := |\varphi^{-1}(e)|$. If $k_e = 1$, then we subdivide e by introducing a new vertex v_e; if $\varphi(u) = e$ then we mark the edge uv_e, otherwise we mark the edge vv_e. If $k_e = 2$, then we replace e by a path u, u_e, v_e, v (u_e and v_e are new vertices); we mark the edges uu_e and vv_e. Finally, if $k_e = 0$, then we replace e by a path u, u_e, v_e, v and mark the edge $u_e v_e$. Applying this construction to all edges of G we obtain a graph G' with $V(G) \subseteq V(G')$. It can be verified easily that the set F of marked edges is a 1-factor of G'.

Let C' be an F-alternating cycle of G'. We observe that C' is a subdivision of a φ-conformal cycle C in G; we put $\pi(C') = C$. On the other hand, if C is a φ-conformal cycle of G, then there is a unique F-alternating cycle C' in G' with $\pi(C') = C$. Thus π is a bijection between F-alternating cycles of G' and φ-conformal cycles of G. Moreover, since $V(\pi(C')) \subseteq V(C')$ for every F-alternating cycle of G', it follows that $X \subseteq V(C')$ if and only if $X \subseteq V(\pi(C'))$ for any given set $X \subseteq V(G)$. Thus Problem 2 reduces to Problem 1. Evidently, the reduction can be carried out in polynomial time, hence part (2) of the proposition follows.

To show Theorem 2, we assume that G is bridgeless, and we suppose to the contrary that G has no φ-conformal cycle. As shown above, this implies that G' has no F-alternating cycle. By Theorem 1, G' has a bridge e. Since every edge of G' can be considered as the result of subdivision of an edge of G, we conclude that G must have a bridge as well, a contradiction. Whence Theorem 2 follows from Theorem 1. □

5. Properly colored cycles in edge-colored graphs

An *edge-colored graph* is a graph G with an associated map χ_G which assigns to every edge $e \in E(G)$ a positive integer $\chi_G(e)$, the *color* of e (note that the coloring is possibly 'improper'). If $r \geq 2$ is an integer such that $\chi_G(e) \leq r$ for each $e \in E(G)$, then we say that G is *r-edge-colored*. For a vertex v of an edge-colored graph G we write $c_G(v)$ for the number of different colors occurring on edges incident with v. We say that v is *monochromatic* if $c_G(v) \leq 1$, i.e., if all incident edges have the same color. A cut vertex v *separates colors* if v is monochromatic in all blocks of G to which v belongs. A cycle C in an edge-colored graph is *properly colored* if consecutive edges of C have different colors. The following Theorem is due to Grossman and Häggkvist [6].

Theorem 3 (Grossman and Häggkvist). *Let G be a 2-edge-colored graph without monochromatic vertices. Then either G has a cut vertex separating colors, or G has a properly colored cycle.*

Again, we state a related problem.

Problem 3. *Given a 2-edge-colored graph G and a set X of p vertices; is there a properly colored cycle C which runs through all vertices in X?*

Proposition 2. *(1) Theorem 2 implies Theorem 3. (2) For every $p \geq 0$, Problem 3 can be reduced to Problem 3 in polynomial time.*

Proof. Let G be a 2-edge-colored graph. We transform G into a graph G' by splitting each vertex $v \in V(G)$ into vertices v_1 and v_2 such that the edges e incident with v are incident with $v_{\chi_G(e)}$, and by joining each pair of such vertices v_1, v_2 by an edge e_v. Let $\varphi : V(G') \to E(G')$ be the map defined by $\varphi(v_1) = v_1v_2 = \varphi(v_2)$ for $v \in V$.

Consider a properly colored cycle C in G. We observe that the set $\{ v_1, v_2 \mid v \in V(C) \}$ defines a φ-conformal cycle C' in G'. We put $\pi(C) = C'$. On the other hand, consider a φ-conformal cycle C' of G'. It follows that $v_1 \in V(C')$ if and only if $v_2 \in V(C')$, for every $v \in V(G)$. Hence $\{ v \in V(G) \mid v_1, v_2 \in V(C') \}$ induces a properly colored cycle C in G such that $\pi(C) = C'$. Thus π is a bijection between properly colored cycles in G and φ-conformal cycles in G', and for any set $X \subseteq V(G)$ we have $X \subseteq V(C)$ if and only if $X' := \{ v_1 \mid v \in X \} \subseteq V(\pi(C))$ (clearly $|X| = |X'|$). Since the above construction can be carried out in polynomial time, part (2) of the proposition follows.

Now assume that (i) G has no monochromatic vertices and (ii) G has no properly colored cycle; we show that G has a cut vertex separating colors. From (i) it follows that the minimum degree of G is at least 2, and from (ii) it follows—as shown above—that G' has no φ-conformal cycle. Hence, by Theorem 2, G' has a bridge e. In view of Corollary 1, we may assume that $\left|\varphi^{-1}(e)\right| \geq 1$. By definition of G' and φ, we conclude that $e = v_1v_2$ for some $v \in V(G)$. Consequently, v is a color separating cut vertex of G (note that $d_{G'}(v_1), d_{G'}(v_2) \geq 2$). Thus Theorem 2 implies Theorem 3, and the proposition is shown true. $\square$

A generalization of Theorem 3 to r-edge-colored graphs for arbitrary $r \geq 2$ has been shown by Yeo [15].

Theorem 4 (Yeo). *Let G be an r-edge-colored graph, $r \geq 2$, without monochromatic vertices. Then either G has a color separating cut vertex, or a properly colored cycle.*

We generalize Problem 3 respectively:

Problem 4. *Given an r-edge-colored graph G, $r \geq 2$, and a set X of p vertices; is there a properly colored cycle C which runs through all vertices in X?*

In the proof of Theorem 4, Yeo proceeds quite differently than the authors of [6]; moreover he remarks "it appears that Grossman and Häggkvist's result cannot be used to obtain the desired extension."

Nevertheless, in the proof of the next proposition we provide a simple construction by which one can derive Yeo's extension from Grossman and Häggkvist's result directly (another application of this construction can be found in [13]).

Proposition 3. *(1) Theorems 3 and 4 are equivalent. (2) For every $p \geq 0$, Problems 3 and 4 can be reduced to each other in polynomial time.*

Proof. Since Theorem 4 is a generalization of Theorems 3, and Problem 4 is a generalization of Problem 3, we only have to show one direction of (1) and (2), respectively. Let $r \geq 2$ and G an r-edge-colored graph. Choose $v \in V(G)$ and put $s := c_G(v)$ ($s \geq 2$ if G has no monochromatic vertices). W.l.o.g., we assume that $\{\, \chi_G(e) \mid e \in E_G(v) \,\}$ equals $\{1, \dots, s\}$. We apply the following local transformation (see Figure 2 for an illustration). We split v into vertices $v_1, \dots, v_s$ such that edges e incident with v in G become incident with $v_{\chi_G(e)}$ in G'. We add new vertices $u_1, \dots, u_s$ and edges u_iv_i for $1 \leq i \leq s$. Finally, we add new vertices w_1 and w_2, the edge w_1w_2 and the edges u_iw_j for all $1 \leq i \leq s$ and $1 \leq j \leq 2$. We mark the edge w_1w_2 and the edges u_iv_i. We put $S_v = \{\, u_i, v_i \mid 1 \leq i \leq s \,\}$ and $w_v = w_1$. Applying this construction to all $v \in V(G)$ we obtain a graph G'. Note that S_v and $S_{v'}$ are disjoint for $v \neq v'$. We define a 2-edge-coloring of G' by putting $\chi_{G'}(e) := 1$ if e is a marked edge, and $\chi_{G'}(e) := 2$ otherwise.

Let C' be properly colored cycle in G'. We observe that $\{\, v \in V(G) \mid S_v \cap V(C') \neq \emptyset \,\}$ induces a properly colored cycle C in G. We put

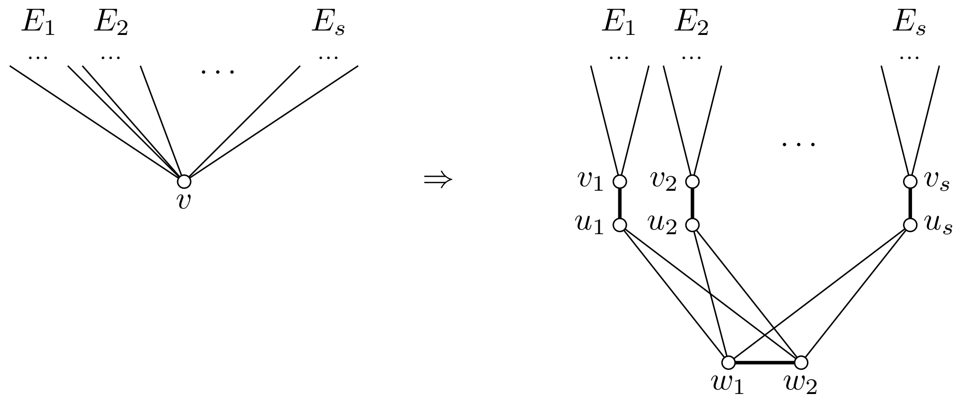

Figure 1: Illustration for the proof of Proposition 3. E_i denotes the set of edges of color i incident with v.

$\pi(C') := C$. On the other hand, if C is a properly colored cycle in G, then $C = \pi(C')$ for some properly colored cycle C' in G' (note, however, that π is not 1-to-1). Let $X \subseteq V(G)$; we define $X' := \{ w_v \mid v \in X \}$ (note that $|X| = |X'|$). Evidently, G has a properly colored cycle C running through all vertices in X if and only if the properly colored cycle C' with $\pi(C') = C$ runs trough all vertices in X'. Obviously, the above constructions can be carried out efficiently. Thus Problem 4 reduces to Problem 3, and part (2) of the proposition is shown.

For part (1) we assume to the contrary that G has no monochromatic vertices, no properly colored cycles, and no color separating cut vertex. Furthermore, we may assume, w.l.o.g., that no r-edge-colored graph with fewer vertices than G has this property. As shown above, the 2-edge-colored graph G' obtained from G has no properly colored cycles as well (and no monochromatic vertices by construction); thus by Theorem 3, G' has a color separating cut vertex x. By construction of G', $x \in S_v$ for some unique $v \in V(G)$. For letting $S_v = \{u_1, v_1, \ldots, u_s, v_s\}$, $s = c_G(v) \geq 2$, it follows that some edge $u_i v_i$, $1 \leq i \leq s$, is a bridge of G'; thus v is a cut vertex of G. Let $K_1, \ldots, K_k$ be the components of $G - v$, $k \geq 2$. Since v is not color separating by assumption, there is some $i \in \{1, \ldots, k\}$ such that v is joined to vertices in K_i by edges of different colors; w.l.o.g., assume $i = 1$. Let $G_1 := G - \bigcup_{i=2}^{k} V(K_i)$. Since $|V(G_1)| < |V(G)|$, and since G_1 neither contains monochromatic vertices (in particular, v is not monochromatic by the choice of i) nor a properly colored cycle, if follows by the minimal choice of G that G_1 has a color separating cut vertex v_1. Since the blocks of G_1 are also blocks of G, we conclude that v_1 is a color separating cut vertex of G, a contradiction. Whence, Theorem 4 follows from Theorem 3, and the proposition is shown true. $\square$

6. Semicycles in digraphs

Let D be a digraph. A subdigraph C of D is a *semicycle* if the (undirected) graph underlying C is a cycle. A vertex v of a semicycle C is a *turning vertex* of C if v is either a source or a sink of C (i.e., the arcs of C incident with v are either both outgoing from or both incoming to v).

The following theorem is due to Shoesmith and Smiley [12].

Theorem 5 (Shoesmith and Smiley). *If a nonempty set S of vertices of a digraph D contains a turning vertex of each semicycle of D, then S contains a vertex which is a turning vertex of every semicycle it belongs to.*

Let D be a digraph and $S \subseteq V(D)$. We call a semicycle C of D an *S-semicycle* if no turning vertex of C belongs to S (thus directed cycles and $V(D)$-semicycles coincide). Further, we call a cut vertex v of D a *strong cut vertex* if there is no (weakly connected) component of $D - v$ containing vertices u, w (possibly $u = w$) such that both $(u, v), (v, w) \in A(D)$. Using these definitions, Theorem 5 can be stated as follows:

> *Let D be a digraph and $\emptyset \neq S \subseteq V(D)$ such that D has no S-semicycles. Then S contains some source or some sink or some strong cut vertex.*

Below we will refer to this formulation of Theorem 5.

The appendant problem reads as follows.

Problem 5. *Given a digraph D, a set $S \subseteq V(D)$, and a set X of p vertices; is there an S-semicycle which runs through all vertices in X?*

In [1, Propositions 9.2.1 and 9.2.2] it is shown that the problem whether a digraph contains a (directed) cycle which runs through two prescribed vertices is NP-complete. This problem, however, is a special case of Problem 5, choosing S to be the set of all vertices of the given digraph. Hence we have the following:

Lemma 2. *Problem 5 is* NP*-complete for $p \geq 2$.*

Lemma 3. *For proving Theorem 5 and for solving Problem 5 it suffices to consider bipartite digraphs D with bipartition (S, T).*

Proof. Let D be a digraph and $S \subseteq V(D)$; we put $T := V(D) \setminus S$. If D has some bridge $b = (t, t')$ with $t, t' \in T$, then we can remove b from D without effecting validity of Theorem 5 or solutions of Problem 5. Hence we assume, w.l.o.g., that D does not contain bridges of this type. By subdivision of arcs which join vertices in S or vertices in T we transform D into a bipartite digraph D' with bipartition (S', T') such that $S \subseteq S', T \subseteq T'$

Observe that S-semicycles of D and S'-semicycles of D' correspond to each other in a natural way; thus, S-semicycles of D which run trough $X \subseteq V(D)$ correspond to S'-semicycles of D' which run through X. Consequently, for solving Problem 5 it suffices to consider bipartite digraphs D with bipartition (S, T).

Now assume $\emptyset \neq S$ and that D has no S-semicycles; consequently, D' has no S'-semicycles. We apply Theorem 5 to D', and conclude that some $x \in S'$ is a source or a sink or a strong cut vertex. If x is a source or a sink, then $x \in S$ by construction of D'; hence x is also a source or a sink of D. Now assume that x is a strong cut vertex of D'. Since D has no bridges (t, t') with $t, t' \in T$ by assumption, no cut vertex of D' belongs to $S' \setminus S$; thus $x \in S$ follows. By construction of D', x is also a strong cut vertex of D. Hence, for proving Theorem 5 it suffices to consider bipartite digraphs with bipartition (S, T). $\square$

In fact, a proof of Theorem 5, restricted to bipartite digraphs where S is one of the bipartition sets, can already be found in [11, Lemma 10.6].

Proposition 4. *(1) Theorem 4 implies Theorem 5. (2) For every $p \geq 0$, Problem 5 can be reduced to Problem 4 in polynomial time.*

Proof. Let D be a digraph and $S \subseteq V(D)$. In view of Lemma 3 we may assume that D is bipartite with bipartition (S, T). We construct a 2-edge-colored graph G' as follows (for an example see Figure 2). For each $v \in$

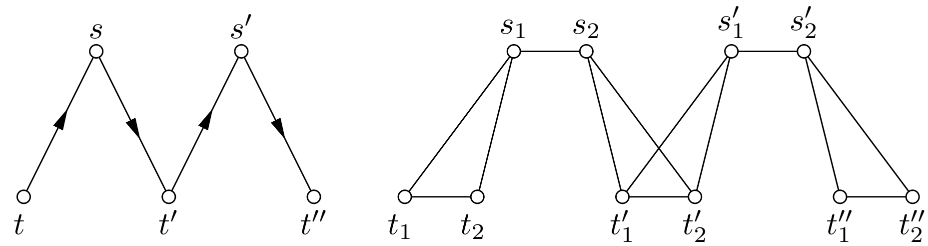

Figure 2: Example for the construction in the proof of Proposition 4.

$V(D)$ we take two new vertices v_1, v_2 and join them by an edge e_v; we put $\chi_{G'}(e_v) = 1$. For each arc $(s, t) \in A(D)$, $s \in S$, $t \in T$, we add edges $s_2 t_1$ and $s_2 t_2$, and for each arc $(t, s) \in A(D)$, $s \in S$, $t \in T$, we add edges $s_1 t_1$ and $s_1 t_2$; we put $\chi_{G'}(t_i s_j) = 2$.

Let C' be a properly colored cycle in G'. Observe that for every $v \in V(G)$, $v_1 \in V(C')$ if and only if $v_2 \in V(C')$; furthermore, observe that $\{\, v \in V(D) \mid v_1, v_2 \in V(C') \,\}$ induces an S-semicycle C in D; we put $\pi(C') := C$.

Conversely, let C be an S-semicycle in D. It follows that the subgraph of G induced by $\{ v_1, v_2 \mid v \in V(C) \}$ contains a properly colored cycle C' such that $\pi(C') = C$. For $X \subseteq V(D)$ let $X' = \{ v_1 \mid v \in X \}$ ($|X| = |X'|$ follows). Clearly, D contains an S-semicycle C with $X \subseteq V(C)$ if and only if there is some properly colored cycle C' in G' such that $X' \subseteq V(C')$. Whence Problem 5 reduces to Problem 4 in polynomial time.

Assume $S \neq \emptyset$ and that (i) D contains no S-semicycles, and (ii) S contains no sources or sinks. We show that S contains a strong cut vertex of D. From (i) it follows (as shown above) that G' has no properly colored cycles; from (ii) and the construction of G', it follows that G' has no monochromatic vertices. Hence we conclude by Theorem 4 that G' has some color separating cut vertex v_i for some $v \in V(D)$ and $i \in \{1, 2\}$. Moreover, v must belong to S, since for every $t \in T$, t_1 and t_2 lie on a triangle. However, if v_i is a cut vertex, then the very construction of G' implies that $v_1 v_2$ is a bridge of G', and consequently, v is a strong cut vertex of D. Thus Theorem 5 follows from Theorem 4. □

Proposition 5. *(1) Theorem 1 follows from Theorem 5. (2) For every $p \geq 0$, Problem 5 can be reduced to Problem 1 in polynomial time.*

Proof. Let G be a graph, F a 1-factor of G, and set $S := V(G)$. We obtain a bipartite Graph G' with bipartition (S, T) from G by subdividing each edge e of G by some new vertex $t_e \in T$. We define an orientation D' of G' by replacing edges $st_e \in E(G')$, by (t_e, s) if $e \in F$, and by (s, t_e) otherwise. Observe that every vertex in T is either source or sink of D'; namely, t_e is a source if $e \in F$ and a sink otherwise.

Consider an F-alternating cycle C in G. Observe that C is a subdivision of an S-semicycle cycle C' of D'; we put $\pi(C) = C'$. On the other hand, for any S-semicycle C' of D', $V(C') \cap S$ defines an F-alternating cycle C in G with $\pi(C) = C'$. Whence π is a bijection between F-alternating cycles of G and S-semicycles of D'. Since $V(C) \subseteq V(\pi(C))$, we conclude that Problem 1 can be reduced to Problem 5 in polynomial time.

Assume that no bridge of G belongs to F; we show that G contains an F-alternating cycle. No $s \in S$ is a strong cut vertex of D'; otherwise, the unique edge $e \in E_G(s) \cap F$ would be a bridge of G. Moreover, since every $s \in S$ is incident in G with some $e \in F$, s is not a source of D; and since no $e \in F$ is a bridge of G, s is not a sink of D. Consequently, S contains no sources or sinks, and so we conclude by Theorem 5 that D has some S-semicycle C; hence $\pi^{-1}(C)$ is an F-alternating cycle of G. Whence Theorem 1 follows from Theorem 5, and the proposition is shown true. □

7. Conclusion

Putting together Propositions 1–5, we get the following result.

Theorem 6. *Theorems 1–5 are all mutually equivalent.*

Moreover, since Problems 1–5 can all be reduced to each other in polynomial time, Lemmas 1 and 2 imply the following (which has been noted w.r.t. Problem 4 in [1, Propositions 11.1.1 and 11.1.9]).

Theorem 7. *Problems 1–5 can be solved in polynomial time for $p < 2$ and are* NP*-complete for $p \geq 2$.*

Note that by the procedure described in Lemma 1, one cannot only decide *existence* of an F-alternating cycle through < 2 prescribed vertices, but such cycle can be *found* in polynomial time (if it exists). Since the reductions in the proofs of Propositions 1–5 transform cycles which are solutions w.r.t. one problem to cycles which are solutions w.r.t. an other problem, we can actually find solutions for Problems 1–5 ($p < 2$) in polynomial time (if such exist).

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