Homomorphisms of Conjunctive Normal Forms

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Abstract

We study homomorphisms of propositional formulas in CNF generalizing symmetries considered by Krishnamurthy. If $\varphi : H \to F$ is a homomorphism, then unsatisfiability of H implies unsatisfiability of F. Homomorphisms from F to a subset F' of F (endomorphisms) are of special interest, since in such case F and F' are satisfiability-equivalent. We show that smallest subsets F' of a formula F for which an endomorphism $F \to F'$ exists are mutually isomorphic. Furthermore, we study connections between homomorphisms and autark assignments.

We introduce the concept of "proof by homomorphism" which is based on the observation that there exist sets Γ of unsatisfiable formulas such that (i) formulas in Γ can be recognized in polynomial time, and (ii) for every unsatisfiable formula F there exist some $H \in \Gamma$ and a homomorphism $\varphi : H \to F$. We identify several sets Γ of unsatisfiable formulas satisfying (i) and (ii) for which proofs by homomorphism w.r.t. Γ and tree resolution proofs can be simulated by each other in polynomial time.

Key words: CNF formula, satisfiability problem, homomorphism, retraction, proof system, tree resolution, autark assignment, minimally unsatisfiable, p-simulation, category

1 Introduction

We consider propositional formulas in conjunctive normal form represented as sets of clauses. Let H and F be formulas and φ a map from the literals of H to the literals of F. We call φ a *homomorphism* from H to F if it preserves complements and clauses, i.e., $\varphi(\overline{\ell}) = \overline{\varphi(\ell)}$ for every literal ℓ of H,

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and $\{\varphi(\ell) : \ell \in C\} \in F$ for every clause C of H. It can be shown that homomorphisms preserve unsatisfiability (see Corollary 3):

(*) if there is a homomorphism from H to F, then unsatisfiability of H implies unsatisfiability of F.

Homomorphisms of formulas can be considered as a generalization of "symmetries with complementations" studied in [2,3,13,25].

1.1 *Reducing formulas by homomorphisms*

Consider a formula F and some subset F' of F; in general, it is hard to decide whether F and F' are equivalent w.r.t. satisfiability (take, for example, $F' = \emptyset$; then F is satisfiable if and only if F and F' are satisfiability-equivalent). In certain cases, however, we can give a "certificate" for equivalence—if the certificate is known, then equivalence can be checked efficiently; for example, autark assignments of F which satisfy all clauses in $F \setminus F'$ can be used as such certificate (see [14,17]).

Endomorphisms (i.e., homomorphisms from a formula to itself) can be used in a similar way. If φ is an endomorphism of F, then $\varphi(F) := \{\varphi(C) : C \in F\}$ is—by definition of homomorphisms—a subset of F. If φ is an isomorphism, then $\varphi(F) = F$. Otherwise, $\varphi(F)$ is a proper subset of F; in this case we can reduce F to $\varphi(F)$, since F is satisfiable if and only if $\varphi(F)$ is satisfiable (one direction follows by (*), the other direction follows trivially from $\varphi(F) \subseteq F$). In Section 4 we study such reductions in a more general framework; in particular, we consider subsets F' of F such that (i) $F' = \varphi(F)$ for some endomorphism φ of F, and (ii) $\varphi'(F') = F'$ for any endomorphism φ' of F' (i.e., F' is a minimal subset (w.r.t. set inclusion) of F to which F can be reduced by some endomorphism). In this case we call F' a *core* of F. Our main results about cores (see Section 4) are as follows.

- Cores of a formula are isomorphic; hence cores can be used as a normal form.
- To decide whether a formula F can be reduced by some endomorphism to a proper subset F' of F (i.e., F is not a core of itself) is NP-complete.

1.2 *Proof by homomorphism*

Assume that H is an unsatisfiable formula and, based on the specific nature of H, its unsatisfiability can be established in polynomial time. Given a homomorphism φ from H to some formula F, then, in view of (*), for showing unsatisfiability of F it suffices to verify that φ is in fact a homomorphism from H to F; evidently, the latter can be performed in polynomial time. Hence, the triple (H, φ, F) can be considered as a proof of the unsatisfiability of F.

Thus one might try to identify sets Γ of unsatisfiable formulas such that

- for every unsatisfiable formula F there exist $H \in \Gamma$ and a homomorphism φ from H to F (i.e., Γ is *homomorphically complete*);
- Γ can be recognized in polynomial time (i.e., Γ is *tractable*).

If Γ is a homomorphically complete and tractable set of unsatisfiable formulas, then $\Pi_\Gamma := \{ (H, \varphi, F) : H \in \Gamma \text{ and } \varphi \text{ is a homomorphism from } H \text{ to } F \}$ can be considered as a proof system.

We show that a well known tractable subclass of *minimally unsatisfiable formulas* is homomorphically complete. A formula F is minimally unsatisfiable, if it is unsatisfiable, and removing any clause from F makes it satisfiable. It is known that every minimally unsatisfiable formula has strictly more clauses than variables [1,5]. $\mathrm{MU}(k)$ denotes the set of minimally unsatisfiable formulas for which the number of clauses exceeds the number of variables exactly by k. Though recognition of minimally unsatisfiable formulas is a computationally hard problem ([20]), formulas in $\mathrm{MU}(k)$ can be recognized in polynomial time for every fixed $k \geq 1$ ([6,16]).

A main result of this paper is the homomorphic completeness of $\mathrm{MU}(1)$. We pinpoint exactly the efficiency of $\Pi_{\mathrm{MU}(1)}$ by showing that every proof $(H, \varphi, F) \in \Pi_{\mathrm{MU}(1)}$ can be transformed into a tree resolution proof of F in polynomial time, and vice versa. Hence, $\Pi_{\mathrm{MU}(1)}$ and tree resolution are *p-equivalent* (c.f. [4,24]). Clearly, for fixed $k \geq 1$, the set $\mathrm{MU}(\leq k) := \bigcup_{i=1}^{k} \mathrm{MU}(i)$ is homomorphically complete, since $\mathrm{MU}(1) \subseteq \mathrm{MU}(\leq k)$; it is conceivable that for $k \geq 2$ the proof system $\Pi_{\mathrm{MU}(\leq k)}$ is stronger than $\Pi_{\mathrm{MU}(1)}$. We show, however, that $\Pi_{\mathrm{MU}(\leq k)}$ and $\Pi_{\mathrm{MU}(1)}$ are p-equivalent. Further we show that for every fixed $k \geq 1$, the set $\mathrm{MU}'(k) := \mathrm{MU}(k) \cup \{\{\Box\}\}$ is homomorphically complete, and that the corresponding proof system $\Pi_{\mathrm{MU}'(k)}$ is p-equivalent with $\Pi_{\mathrm{MU}(1)}$. This result is due to Kleine Büning and Zhao [12].

2 Basic Concepts and Notation

2.1 Formulas and Assignments

We think of *literals* as propositional variables with an assigned parity 0 or 1; a literal is called *positive* (*negative*) if its parity is 0 (1, respectively). Positive literals are called *variables*. For a literal ℓ we denote the literal with the opposite parity by $\overline{\ell}$. A set of literals is *tautological* if it contains literals ℓ and $\overline{\ell}$. A *clause* is a finite non-tautological set of literals. The empty clause is denoted by $\square$.

For a clause C, we denote the set of variables x such that x or $\overline{x}$ is in C by $\mathsf{var}(C)$, and we put $\mathsf{lit}(C) := \{\, x, \overline{x} : x \in \mathsf{var}(C) \,\}$. Similarly, for a formula F, we put $\mathsf{var}(F) := \bigcup_{C \in F} \mathsf{var}(C)$ and $\mathsf{lit}(F) := \bigcup_{C \in F} \mathsf{lit}(C)$. Following [7] we define $\delta(F) := |F| - |\mathsf{var}(F)|$ to be the *deficiency* of F. The *length* of a formula F is defined as $\sum_{C \in F} |C|$.

A *partial assignment* (or *assignment*, for short) of a formula F is a map $t : X_t \to \{0,1\}$ defined on a subset $X_t \subseteq \mathsf{var}(F)$. If $x \in X_t$ then we put $t(\overline{x}) := 1 - t(x)$. An assignment t of F is *total* if $X_t = \mathsf{var}(F)$. An assignment t *satisfies* a clause C if C contains a literal ℓ such that $t(\ell) = 1$; t satisfies a formula F if it satisfies all clauses of F. A formula F is *satisfiable* if it is satisfied by some partial assignment; otherwise, F is called *unsatisfiable*. A formula F is *minimally unsatisfiable* if F is unsatisfiable but every proper subset of F is satisfiable. The set of all minimally unsatisfiable formulas is denoted by MU; for an integer k we define

$$\begin{aligned} \mathrm{MU}(k) &:= \{\, F \in \mathrm{MU} : \delta(F) = k \,\}, \\ \mathrm{MU}(\leq k) &:= \{\, F \in \mathrm{MU} : \delta(F) \leq k \,\}. \end{aligned}$$

Note that $\mathrm{MU}(k) = \mathrm{MU}(\leq k) = \emptyset$ for $k \leq 0$, see [1,5].

Let t be an assignment of a formula F. We say that t *touches* a clause $C \in F$ if $\mathsf{var}(C) \cap X_t \neq \emptyset$. The assignment t is *autark* if t satisfies all clauses which it touches. Following [17] we call a formula *lean* if it has no autark assignment t such that $X_t \neq \emptyset$. Note that if t is an autark assignment of F, then $\{\, C \in F : \mathsf{var}(C) \cap X_t = \emptyset \,\}$ and F are satisfiability-equivalent; it follows that minimally unsatisfiable formulas are lean. Autark assignments were introduced by Monien and Speckenmeyer [19] and have been studied in depth by Kullmann [14,17].

2.2 Proof Systems

Cook and Reckhow [4] introduced a general concept of propositional proof systems in terms of functions on sets of strings. We use a more informal concept based on the discussion in [24].

A *proof* of a formula F is a finite object x which certifies unsatisfiability of F in the sense that, if x is given, then unsatisfiability of F can be verified in polynomial time (proofs of unsatisfiability are also called *refutations*). A *proof system* Π is a set of proofs such that (i) elements of Π can be recognized in polynomial time, and (ii) a formula F is unsatisfiable if and only if Π contains a proof of F.

Let Π, Π' be proof systems. We say that Π' *p-simulates* Π if every proof $x \in \Pi$ can be transformed into a proof $x' \in \Pi'$ in polynomial time such that x and x' prove the same formula. If Π and Π' p-simulate each other, then we say that they are *p-equivalent*.

The efficiency of (propositional) proof systems is closely related to the NP = co-NP question; this relationship is a main motivation for a systematic study of proof systems and their relative strength in terms of p-simulation ([4,24]).

2.3 Resolution

If C_1 and C_2 are clauses and there is exactly one variable x such that $x \in C_1$, $\overline{x} \in C_2$, then the clause $C := (C_1 \cup C_2) \setminus \{x, \overline{x}\}$ is called the *resolvent* of C_1 and C_2. We also say that C is obtained by *resolving on* x. A *tree resolution proof* T is a binary rooted tree where the vertices v of T are labeled by clauses $\lambda_T(v)$ such that (i) whenever a vertex v has two parents v_1, v_2, then $\lambda_T(v)$ is the resolvent of $\lambda_T(v_1)$ and $\lambda_T(v_2)$, and (ii) the root of T is labeled by the empty clause. In case (i) we call the variable on which $\lambda_T(v_1)$ and $\lambda_T(v_2)$ are resolved the *resolution variable* of v. A tree resolution proof T is *literal-once* (cf. [22]) if distinct non-leaves v, v' always have distinct resolution variables, i.e., $\lambda_T(v)$ and $\lambda_T(v')$ are not obtained by resolving on the same variable. If v is a leaf of T then we call $\lambda_T(v)$ a *premise* of T; the set of all premises of T is denoted by $\mathsf{pre}(T)$. We say that T is a tree resolution proof *of* a formula F if $\mathsf{pre}(T) \subseteq F$.

It is well-known that a formula is unsatisfiable if and only if there is a tree resolution proof of it; thus tree resolution is a proof system in the above sense.

3 Homomorphisms

Let H, F be formulas and $\varphi : \mathsf{lit}(H) \rightarrow \mathsf{lit}(F)$ a map. We call φ a *homomorphism* from H to F if

(1) $\varphi(\overline{\ell}) = \overline{\varphi(\ell)}$ for every literal $\ell \in \mathsf{lit}(H)$, and
(2) $\varphi(C) \in F$ for every clause $C \in H$

where $\varphi(C) := \{ \varphi(\ell) : \ell \in C \}$. We simply write $\varphi : H \to F$ if φ is a homomorphism from H to F. For a homomorphism $\varphi : H \to F$ we call the formula $\varphi(H) := \{ \varphi(C) : C \in H \}$ the *homomorphic image* of H under φ.

It is immediate that if $\varphi : F_1 \to F_2$ and $\psi : F_2 \to F_3$ are homomorphisms, then their *composition* $\psi \circ \varphi$, defined by $\psi(\varphi(\ell))$ for $\ell \in \mathsf{lit}(F_1)$, is a homomorphism from F_1 to F_3.

A homomorphism φ from F to itself is called an *endomorphism* of F. Note that the set of endomorphisms of a formula F is a monoid under composition. We denote the unit element of this monoid by id_F.

A homomorphism φ from H to F is a *bimorphism* if the underlying map $\varphi : \mathsf{lit}(H) \to \mathsf{lit}(F)$ is bijective. In contrast to group theory, the inverse map $\varphi^{-1} : \mathsf{lit}(F) \to \mathsf{lit}(H)$ of a bimorphism is not necessarily a homomorphism from F to H (for example, every homomorphism from $H = \{\{x\}\}$ to $F = \{\{y\}, \{\overline{y}\}\}$ is a bimorphism, but there is no homomorphism from F to H). A bimorphism $\varphi : H \to F$ is called *isomorphism* if φ^{-1} is a homomorphism; φ is called *automorphism* if it is an isomorphism and $H = F$. Obviously, an endomorphism φ of H is an automorphism if and only if $\varphi(H) = H$.

Note that a *renaming* of a formula F (in the sense of [18]) is nothing but an automorphism φ of F such that $\varphi(\ell) \in \{\ell, \overline{\ell}\}$ for all literals $\ell \in \mathsf{lit}(F)$.

Lemma 1 *Let $\varphi : H \to F$ and t an autark assignment of $\varphi(H)$. Then t' defined by $t'(x) := t(\varphi(x))$ for $x \in X_{t'} := \varphi^{-1}(X_t)$ is an autark assignment of H.*

PROOF. Consider a clause $C' \in H$ such that $X_{t'} \cap \mathsf{var}(C') \neq \emptyset$. Hence $\mathsf{var}(\varphi(C')) \cap X_t \neq \emptyset$. Since t is autark, we have $t(\ell) = 1$ for some literal $\ell \in \varphi(C')$. Choose $\ell' \in C'$ such that $\varphi(\ell') = \ell$. Consequently, $t'(\ell') = t(\varphi(\ell')) = t(\ell) = 1$. $\square$

Example 2 Let $H = \{\{x, \overline{y}, z\}, \{\overline{x}, y\}\}$ and $F = \{\{\overline{u}\}, \{u\}, \{u, v\}\}$. We define a homomorphism $\varphi : H \to F$ by setting $\varphi(x) = \varphi(\overline{y}) = \varphi(z) = \overline{u}$; the values for the remaining literals $\overline{x}, y, \overline{z}$ of H are determined uniquely by the condition $\varphi(\overline{\ell}) = \overline{\varphi(\ell)}$. The homomorphic image $\varphi(H)$ of H under φ is $\{\{\overline{u}\}, \{u\}\}$. Another homomorphism $\psi : H \to F$ can be defined by $\psi(x) = \psi(\overline{y}) = u$, $\psi(z) = v$. We have $\psi(H) = \{\{\overline{u}\}, \{u, v\}\}$. The partial assignment t of $\psi(H)$ with $X_t = \{v\}$ and $t(v) = 1$ satisfies the clause $\{u, v\}$. Since $\{u, v\}$ is the only clause of $\varphi(H)$ touched by t, t is autark. Applying Lemma 1 we obtain an autark assignment t' of H with $X_{t'} = \psi^{-1}(X_t) = \{z\}$ and $t'(z) = t(v) = 1$.

Corollary 3 *Let $\varphi : H \to F$ be a homomorphism. If H is unsatisfiable, then F is unsatisfiable.*

The above result is key for our subsequent considerations. From Lemma 1 the following is also immediate.

Corollary 4 *The homomorphic image of a lean formula is lean.*

4 Retracts and Cores

A homomorphism $\varphi : H \to F$ is a *retraction* if there exists a homomorphism $\psi : F \to H$ such that $\varphi \circ \psi = id_F$. In this case we call ψ a *co-retraction* and F a *retract* of H. A formula H is a *core* if every retract F of H is isomorphic to H. A retract F of H is a *core of* H if F is a core. The following observations are direct consequences of this concept.

(1) The composition of retractions is a retraction; hence, a retract of a retract of H is a retract of H.
(2) Every retract of a formula H is isomorphic to a subset H' of H, and there is a retraction $\varphi : H \to H'$ whose restriction to H' equals $id_{H'}$.
(3) If F is a retract of H, then F is satisfiable if and only if H is satisfiable; consequently, minimally unsatisfiable formulas are cores.

Example 5 Let $F = \{\{x_0, x_1, y_1\}, \{\overline{x_1}, y_0, y_1\}, \{\overline{x_0}, \overline{x_2}, \overline{y_1}\}, \{x_2, \overline{y_0}, \overline{y_1}\}, \{x_0, y_0\}, \{x_0, \overline{y_0}\}, \{\overline{x_0}, y_0\}, \{\overline{x_0}, \overline{y_0}\}\}$. Setting $\varphi(x_0) = \varphi(x_1) = \varphi(x_2) = x_0$ and $\varphi(y_0) = \varphi(y_1) = y_0$ defines a retraction of F with retract $F' := \{\{x_0, y_0\}, \{x_0, \overline{y_0}\}, \{\overline{x_0}, y_0\}, \{\overline{x_0}, \overline{y_0}\}\}$ and co-retraction $id_{F'}$. Since F' is minimally unsatisfiable, it follows that F' is a core of F.

If we know an endomorphism of a formula, then we can find a retraction efficiently:

Lemma 6 *Let φ be an endomorphism of F. Then there exists an integer $n \in \{1, \ldots, |F|\}$ such that φ^n is a retraction. Consequently, a formula is a core if and only if each of its endomorphisms is an automorphism.*

PROOF. Note that for every $i \geq 1$ we have $\varphi^{i+1}(F) \subseteq \varphi^i(F)$; thus $|\varphi^{i+1}(F)| \leq |\varphi^i(F)|$. Therefore, there exists an integer $n \in \{1, \ldots, |F|\}$ such that $|\varphi^n(F)| = |\varphi^{n+1}(F)|$. Since $\varphi^{n+1}(F) \subseteq \varphi^n(F)$, it follows that $F' := \varphi^n(F) = \varphi^{n+1}(F)$; thus φ^n acts as an automorphism on F'. Let ψ denote the automorphism of F' which is inverse to φ^n. We have $\varphi^n \circ \psi = id_{F'}$; thus φ^n is indeed a retraction. If F is a core, we have $\varphi^n(F) = F$, thus $\varphi(F) = F$; i.e., φ is an automorphism. □

Example 7 Consider $F = \{\{x_1, x_2\}, \{y\}, \{z\}\}$. Setting $\varphi(x_1) = \varphi(x_2) = y$ and $\varphi(y) = \varphi(z) = z$ defines an endomorphism of F with $F' := \varphi(F) = \{\{y\},$

$\{z\}\}$. Note that φ is not a retraction, since $\varphi \circ \psi = id_{F'}$ implies that $\psi(y) \in \{x_1, x_2\}$, but then $\psi(\{y\}) \notin F$. (Nevertheless, F' is a retract of F with respect to the retraction defined by $\varphi'(x_1) = \varphi'(x_2) = y$, $\varphi'(y) = y$, and $\varphi'(z) = z$.) However, φ^2 is a retraction of F with retract $F'' := \{\{z\}\}$; as co-retraction we can take either $\psi(z) = y$ or $\psi(z) = z$. Evidently, F'' is a core, but F' is not.

Lemma 8 *Cores of a formula are mutually isomorphic.*

PROOF. Let F_1, F_2 be cores of a formula H and let φ_i be a retraction $H \to F_i$, $i = 1, 2$. By the above observation (2) we may assume that F_1, F_2 are subsets of H. Consider the restriction φ_1' of φ_1 to F_2, and the restriction φ_2' of φ_2 to F_1. The composition $\varphi_1' \circ \varphi_2'$ is an endomorphism of F_1, and by Lemma 6 it is an automorphism. Hence φ_1' and φ_2' are isomorphisms. $\square$

In view of Lemma 8, a core of a formula can be considered as a normal form. Unfortunately, cores are difficult to recognize. To show this we deploy the following construction.

Let F be a formula. For each $x \in \mathsf{var}(F)$ we take two new variables $x[1]$, $x[2]$, and for every clause $C \in F$ we define

$$C^\circ \quad := \quad \{\, \overline{x[1]} : \overline{x} \in C \,\} \;\cup\; \{\, \overline{x[2]} : x \in C \,\}.$$

We put

$$F^\circ \quad := \quad \{\, C^\circ : C \in F \,\} \;\cup\; \{\, \{x[1], x[2]\} : x \in \mathsf{var}(F) \,\}.$$

Note that each clause C of F° is either positive (all literals in C are positive) or negative (all literals in C are negative). The following can be verified easily (cf. [5, Lemma 2]).

Lemma 9 *For every formula F*

(1) F is satisfiable if and only if F° is satisfiable, and
(2) $F \in \mathrm{MU}(k)$ if and only if $F^\circ \in \mathrm{MU}(k)$, for every $k \geq 1$.

In the proof of Theorem 12 below we use a simple concept of connectedness: We say that clauses C, C' of a formula F are *connected* in F if there exists a sequence of clauses $D_1, \ldots, D_r$ ($r \geq 1$ and $D_i \in F$ for $i \in \{1, \ldots, r\}$) such that $D_1 = C$, $D_r = C'$, and $\mathsf{var}(D_i) \cap \mathsf{var}(D_{i+1}) \neq \emptyset$ for $1 \leq i < r$. We call F *connected* if every pair of clauses of F is connected.

Lemma 10 *Minimally unsatisfiable formulas are connected.*

PROOF. Let F be a minimally unsatisfiable formula and suppose to the contrary that F is not connected. Consequently, there is a proper subset $F' \neq \emptyset$ of F such that (i) F' is connected and (ii) there is no connected F'' such that $F' \subsetneq F'' \subseteq F$. Being a proper subset of a minimally unsatisfiable formula, F' is satisfiable. Let t be a satisfying total assignment of F'. Note that $X_t = \mathsf{var}(F') \neq \emptyset$. By (ii), $\mathsf{var}(F') \cap \mathsf{var}(F \setminus F') = \emptyset$; hence F' contains all clauses of F which are touched by t. We conclude that t is an autark assignment of F. Since $X_t \neq \emptyset$, F is not lean. However, every minimally unsatisfiable formula is lean, a contradiction. □

Lemma 11 *The homomorphic image of a connected formula is connected.*

PROOF. Let H, F be formulas and let $\varphi : H \to F$ be a homomorphism. We assume that $\varphi(H) \neq \emptyset$; otherwise the lemma is vacuously true. Choose $C, C' \in \varphi(H)$ arbitrarily, and let $C_0, C_0' \in H$ with $\varphi(C_0) = C$ and $\varphi(C_0') = C'$. Since H is connected, there is a sequence $D_1, \dots, D_r$ ($r \geq 1$ and $D_i \in H$ for $i \in \{1, \dots, r\}$) such that $D_1 = C_0$, $D_r = C_0'$, and $\mathsf{var}(D_i) \cap \mathsf{var}(D_{i+1}) \neq \emptyset$ for $1 \leq i < r$. It follows that $\mathsf{var}(\varphi(D_i)) \cap \mathsf{var}(\varphi(D_{i+1})) \neq \emptyset$ for $1 \leq i < r$. Hence the sequence $\varphi(D_1), \dots, \varphi(D_r)$ certifies that C and C' are connected in $\varphi(H)$. Since C, C' where chosen arbitrarily, the lemma follows. □

Theorem 12 *Recognition of cores is* co-NP*-complete.*

PROOF. If a formula F is not a core, then by Lemma 6 there must be an endomorphism φ of F which is not an automorphism; i.e., $\varphi(F) \neq F$. If such φ is guessed, then $\varphi(F) \neq F$ can be verified in polynomial time. Hence, recognition of cores is in co-NP.

To demonstrate co-NP-completeness, we use the following construction. In [20] it is shown that for every formula F one can construct in polynomial time a formula $f(F)$ such that

- F is satisfiable if and only if $f(F)$ is satisfiable;
- if $f(F)$ is unsatisfiable, then $f(F)$ is minimally unsatisfiable.

Let $F \neq \emptyset$ be an arbitrary formula and put $H := f(F)$. Furthermore, let

$$Y := \{\{y[1], y[2]\}, \{\overline{y[1]}, \overline{y[2]}\}, \{\overline{y[2]}\}\}$$

and observe that Y is a satisfiable core. Consider $H^* := H^\circ \cup Y$ (we assume that $\mathsf{var}(H^\circ)$ and $\mathsf{var}(Y)$ are disjoint). We show that F is unsatisfiable if and only if H^* is a core; the theorem will then follow from the co-NP-completeness of unsatisfiability.

Assume that F is unsatisfiable. Now H and (by Lemma 9) H° are minimally unsatisfiable; thus H° is a core (see observation (3) above). Since Y is satisfiable, we conclude by Corollary 3 that

(*) there is no homomorphism from H° to Y.

On the other hand, every homomorphic image of Y is either isomorphic to Y or to $Y' = \{\{z\}, \{\overline{z}\}\}$. However, no subset of H° is isomorphic to Y or Y' by construction. Hence

(**) there is no homomorphism from Y to H°.

Let φ be any endomorphism of H^*; we show that φ is an automorphism. Since Y is evidently connected, and since H° is connected by Lemma 10, it follows by Lemma 11 that $\varphi(Y)$ and $\varphi(H^\circ)$ are connected subsets of H^*. However, every connected subset of H° is either a subset of Y or a subset of H°, since $\mathsf{var}(Y) \cap \mathsf{var}(H^\circ) = \emptyset$. Therefore, (*) implies $\varphi(Y) \subseteq Y$, and (**) implies $\varphi(H^\circ) \subseteq H^\circ$. Since Y and H° are cores, $\varphi(Y) = Y$ and $\varphi(H^\circ) = H^\circ$ follows. Thus $\varphi(H^*) = H^*$, and so φ is an automorphism of H^*. In view of the second statement of Lemma 6, we conclude that H^* is a core.

Conversely, assume that F is satisfiable; thus H and H° are satisfiable. Let t be a satisfying total assignment of H°. Observe that $t(x[1]) = 1$ or $t(x[2]) = 1$ for every $x \in \mathsf{var}(H)$; we assume, w.l.o.g., that always $t(x[1]) = 1$ prevails. It follows that every negative clause of H° must contain $\overline{x[2]}$ for some $x \in \mathsf{var}(H)$. Define $\varphi : \mathsf{lit}(H^\circ) \to \mathsf{lit}(Y)$ by setting $\varphi(x[i]) := y[i]$ for all $x \in \mathsf{var}(F)$ and $i = 1, 2$. It follows now that all positive clauses of H° are mapped to $\{y[1], y[2]\}$, and all negative clauses of H° are mapped to $\{\overline{y[1]}, \overline{y[2]}\}$ or $\{\overline{y[2]}\}$. Thus φ is a homomorphism from H° to Y. The union of φ and id_Y yields an endomorphism φ' of H^* with $\varphi'(H^*) = Y \subsetneq H^*$; hence H^* is not a core. $\square$

5 The Concept of Proof by Homomorphism

Let Γ be a set of unsatisfiable formulas. We say that Γ is *tractable* if Γ can be recognized in polynomial time, and that Γ is *homomorphically complete* (or *h-complete*, for short) if for every unsatisfiable formula F there exist some $H \in \Gamma$ and a homomorphism $\varphi : H \to F$. We call a triple (H, φ, F) a *proof of F by homomorphism* (with respect to Γ) if $H \in \Gamma$ and φ is a homomorphism from H to F. The set of all proofs by homomorphism w.r.t. Γ is denoted by Π_Γ.

The next result is a direct consequence of these newly introduced concepts and Corollary 3, and is key for the subsequent considerations.

Proposition 13 *If a set Γ of unsatisfiable formulas is both tractable and h-complete, then Π_Γ is a proof system.*

Example 14 The tractable set Γ_{horn} of unsatisfiable Horn formulas is *not* h-complete: For, every unsatisfiable Horn formula contains at least one clause C with $|C| \leq 1$ (see, e.g., [11, p. 205]); thus, if F is an unsatisfiable formula with $|C| \geq 2$ for all $C \in F$ (e.g., $F = \{\{x, y\}, \{x, \overline{y}\}, \{\overline{x}, y\}, \{\overline{x}, \overline{y}\}\}$), then there cannot be some $H \in \Gamma_{\text{horn}}$ with $\varphi : H \to F$ being a homomorphism.

One may ask whether there exists some tractable and h-complete set of unsatisfiable formulas at all. Goldstern [8] observed that a trivial set Γ_{triv} with such property can be obtained by adding an irrelevant clause C_F of exponential cardinality (w.r.t. the length of F) to every unsatisfiable formula F. This can be done in such a way that

- there is a homomorphism from $F \cup \{C_F\}$ to F;
- $F \cup \{C_F\}$ is unsatisfiable.

Thus $\Gamma_{\text{triv}} := \{\, F \cup \{C_F\} : F \text{ is unsatisfiable } \}$ is h-complete. Now, the unsatisfiability of F can be tested in polynomial time w.r.t. the length of $F \cup \{C_F\}$; hence Γ_{triv} is tractable.

We are going to identify non-trivial sets of unsatisfiable formulas which are both tractable and h-complete.

The next lemma, which is due to an observation by Kullmann [15], follows from the fact that if $\varphi : H \to F$ is a homomorphism and C is a resolvent of clauses C_1, C_2 such that $\mathsf{lit}(C_1), \mathsf{lit}(C_2) \subseteq \mathsf{lit}(H)$, then either (i) $\varphi(C)$ is the resolvent of $\varphi(C_1)$ and $\varphi(C_2)$ or (ii) $\varphi(C)$ is a tautological set of literals. However, by standard transformations one can efficiently eliminate tautological sets of literals from resolution proofs (see [11]). An explicit proof of Lemma 15 can be found in [21].

Lemma 15 *Let $\varphi : H \to F$ be a homomorphism. Then every tree resolution proof of H can be transformed into a tree resolution proof of F in polynomial time.*

Proposition 16 *Let Γ be a set of unsatisfiable formulas. Tree resolution p-simulates Π_Γ if and only if for every formula in Γ a tree resolution proof can be found in polynomial time.*

PROOF. Assume that tree resolution p-simulates Π_Γ. Choose some $H \in \Gamma$ and observe that $(H, id_H, H) \in \Pi_\Gamma$. By assumption we can obtain a tree resolution proof T of H in polynomial time.

Conversely, let $H \in \Gamma$ and $\varphi : H \to F$ be given. By assumption we can find a tree resolution proof of H in polynomial time. In view of Lemma 15 we find efficiently a tree resolution proof of F. Hence tree resolution p-simulates Π_Γ. □

6 Proofs by Homomorphism w.r.t. MU(1)

Lemma 17 *Let T be a tree resolution proof. Then we can find in polynomial time a tree resolution proof T' and a homomorphism $\varphi : \mathsf{pre}(T') \to \mathsf{pre}(T)$ such that*

(1) T' differs from T at most in its labeling;
(2) $\varphi(\lambda_{T'}(v)) = \lambda_T(v)$ for all vertices of T, consequently $\varphi(\mathsf{pre}(T')) = \mathsf{pre}(T)$;
(3) T' is literal-once (c.f. Section 2.3).

PROOF. We proceed by induction on the number n of vertices of T. If $n = 1$ then we put $T' := T$. Now assume $n > 1$ and choose a non-leaf v of T such that the predecessors v_1, v_2 of v are leaves. Let x be the resolution variable of v and assume, w.l.o.g., that $x \in \lambda_T(v_1)$ and $\overline{x} \in \lambda_T(v_2)$. Denote by T_0 the tree resolution proof obtained from T by removing v_1 and v_2. Let T_0', φ_0 as supplied by induction hypothesis with respect to T_0. We take a new variable y and obtain from T a tree resolution proof T' by replacing λ_T by $\lambda_{T'}$ defined as follows. We put $\lambda_{T'}(w) := \lambda_{T_0'}(w)$ if $w \notin \{v_1, v_2\}$ and

$$\lambda_{T'}(v_1) := \varphi_0\Big(\lambda_T(v_1) \setminus \{x\}\Big) \cup \{y\},$$
$$\lambda_{T'}(v_2) := \varphi_0\Big(\lambda_T(v_2) \setminus \{\overline{x}\}\Big) \cup \{\overline{y}\}.$$

Evidently, T' satisfies the claimed properties. We extend φ_0 to the required homomorphism φ by setting $\varphi(y) := x$. □

Based on structural properties of MU(1) established in [5], it is shown in [22, Proposition 3] that $F \in \mathrm{MU}(1)$ if and only if there is a literal-once resolution proof T with $\mathsf{pre}(T) = F$. Hence Lemma 17 implies the following (see also [16, Lemma C.5]).

Proposition 18 *To every tree resolution proof T one can find in polynomial time a formula $H \in \mathrm{MU}(1)$ and a homomorphism $\varphi : H \to \mathsf{pre}(T)$ such that*

(1) $\varphi(H) = \mathsf{pre}(T)$, and
(2) $|H|$ equals the number of leaves of T.

Corollary 19 MU(1) *is homomorphically complete.*

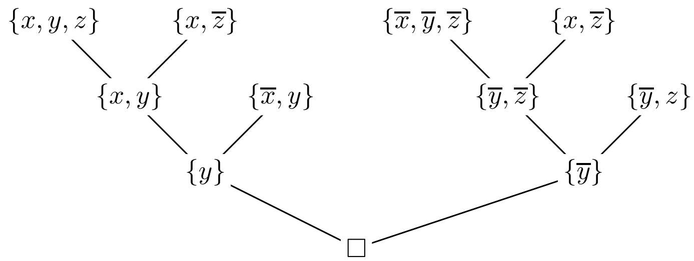

Fig. 1.

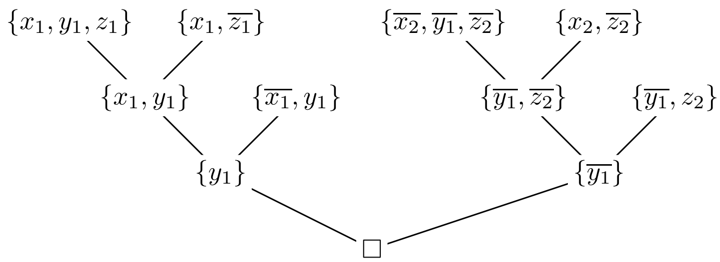

Fig. 2.

Example 20 Figure 1 shows a tree resolution proof T of the formula $F = \{\{x, y, z\}, \{x, \overline{z}\}, \{\overline{x}, y\}, \{\overline{x}, \overline{y}, \overline{z}\}, \{\overline{y}, z\}\}$ (the clause $\{x, \overline{z}\}$ appears at two leaves of T). By the construction presented in the proof of Lemma 17 we obtain the literal-once tree resolution proof T' depicted in Figure 2 with $H := \mathsf{pre}(T') = \{\{x_1, y_1, z_1\}, \{x_1, \overline{z_1}\}, \{\overline{x_1}, y_1\}, \{\overline{x_2}, \overline{y_1}, \overline{z_2}\}, \{x_2, \overline{z_2}\}, \{\overline{y_1}, z_2\}\} \in$ MU(1) and a homomorphism $\varphi : H \to F$ defined by $\varphi(x_1) = \varphi(x_2) = x$, $\varphi(y_1) = y$, and $\varphi(z_1) = \varphi(z_2) = z$.

If $F \in$ MU(1) then a tree resolution proof of F can be found in polynomial time (formulas in MU(1) can be refuted by unit resolution, [5]). Hence MU(1) satisfies the hypothesis of Proposition 16; together with Proposition 18 the next result follows. (Observe that, if T is literal-once, then $|\mathsf{pre}(T)|$ equals the number of leaves of T.)

Theorem 21 *Tree resolution and* $\Pi_{\mathrm{MU}(1)}$ *are p-equivalent.*

By means of Proposition 18 we can generalize the following characterization of lean formulas which is due to Kullmann [17].

Theorem 22 ([17]) *A formula F is lean if and only if for every clause $C \in F$ there is a tree resolution proof T such that $C \in \mathsf{pre}(T) \subseteq F$.*

Corollary 23 *A formula F is lean if and only if for every h-complete set Γ of unsatisfiable formulas the following holds. For every clause $C \in F$ there is some formula $H \in \Gamma$ and a homomorphism $\varphi : H \to F$ such that $C \in \varphi(H)$.*

PROOF. Assume that F is lean. Let Γ be a h-complete set of unsatisfiable formulas and choose some $C \in F$. By Theorem 22 there is a tree resolution proof T with $C \in \mathsf{pre}(T) \subseteq F$. Hence, by Proposition 18, there is a formula $H_1 \in \mathrm{MU}(1)$ and a homomorphism $\varphi_1 : H_1 \to F$ such that $\mathsf{pre}(T) = \varphi_1(H_1)$; thus $C \in \varphi_1(H_1)$. Since Γ is h-complete, there exist $H_2 \in \Gamma$ and a homomorphism $\varphi_2 : H_2 \to H_1$. However, since H_1 is minimally unsatisfiable, $\varphi_2(H_2) = H_1$ follows. Putting $\varphi := \varphi_1 \circ \varphi_2$ yields a homomorphism from H_2 to F such that $C \in \varphi(H_2)$.

Since minimally unsatisfiable formulas are lean, the converse follows from Corollaries 4 and 19 by putting $\Gamma := \mathrm{MU}(1)$. $\square$

7 Proofs by Homomorphism w.r.t. $\mathrm{MU}(k)$ and $\mathrm{MU}(\leq k)$

It is natural to consider proof systems based on $\mathrm{MU}(k)$ and $\mathrm{MU}(\leq k)$ for some fixed $k \geq 2$. Note that $\mathrm{MU}(\leq k)$ is both tractable ([16,6]) and h-complete ($\mathrm{MU}(1) \subseteq \mathrm{MU}(\leq k)$) for every fixed $k \geq 1$; thus $\Pi_{\mathrm{MU}(\leq k)}$ is a proof system by Proposition 13. The question arises whether, for $k > 1$, $\Pi_{\mathrm{MU}(\leq k)}$ is stronger than $\Pi_{\mathrm{MU}(1)}$. In [10] it is shown that formulas in $\mathrm{MU}(k)$ have short resolution proofs. Moreover, in [16] it is shown that tree resolution proofs of formulas in $\mathrm{MU}(k)$ (and so tree resolution proofs of formulas in $\mathrm{MU}(\leq k)$) can be found in polynomial time. Hence, the next result follows by Proposition 16 and Theorem 21.

Theorem 24 *Tree resolution and $\Pi_{\mathrm{MU}(\leq k)}$ are p-equivalent, for fixed $k \geq 1$.*

Note that every h-complete set Γ of unsatisfiable formulas must contain the trivial formula $F_0 = \{\square\}$, since otherwise unsatisfiability of F_0 cannot be established by a homomorphism from an element of Γ. Thus, given any set Γ of unsatisfiable formulas, we consider $\Gamma' := \Gamma \cup \{F_0\}$.

In a preliminary version of this article we asked whether the sets $\mathrm{MU}'(k)$ for $k > 1$ are homomorphically complete. A recent result by Kleine Büning and Zhao [12] answers this question positively. Below, we present a proof based on the proof given in [12], using the following construction.

Lemma 25 *For every $k \geq 1$ there is a formula $F_k \in \mathrm{MU}(k)$ and a homomorphism φ_k with $\varphi_k(F_k) = \{\{x\}, \{\overline{x}\}\}$; F_k and φ_k can be obtained in polynomial time depending on k.*

PROOF. First we obtain a formula $H_k \in \mathrm{MU}(k)$ by the following recursive construction. Set

$$H_1 := \{\{x\}, \{\overline{x}\}\},$$
$$H_2 := \{\{x, y\}, \{\overline{x}, y\}, \{x, \overline{y}\}, \{\overline{x}, \overline{y}\}\}.$$

Clearly $H_1 \in \mathrm{MU}(1)$ and $H_2 \in \mathrm{MU}(2)$. For $k \geq 3$ construct H_{k-1} and take a formula H_2' isomorphic to H_2 such that $\mathsf{var}(H_{k-1}) \cap \mathsf{var}(H_2') = \emptyset$. Choose clauses $C \in H_{k-1}$, $D \in H_2'$, and a new variable $z \notin \mathsf{var}(H_{k-1}) \cup \mathsf{var}(H_2')$. Put $C' := C \cup \{z\}$, $D' := D \cup \{\overline{z}\}$, and

$$H_k \;:=\; (H_{k-1} \setminus \{C\}) \;\cup\; (H_2' \setminus \{D\}) \;\cup\; \{C', D'\}.$$

It can be easily verified that $H_k \in \mathrm{MU}(k)$.

Now consider $F_k := H_k^\circ$ for some $k \geq 1$. Note that it takes only polynomial time to construct H_k and F_k. By Lemma 9, $F_k \in \mathrm{MU}(k)$. We define a homomorphism $\varphi_k : F_k \to H_1$ by setting $\varphi_k(x[1]) = \varphi_k(x[2]) = x$ for each $x \in \mathsf{var}(H_k)$. Since each clause C of F_k is either negative or positive, it follows that $\varphi_k(F_k) = H_1$. Trivially, φ_k is obtained in linear time for given F_k. $\square$

Lemma 26 ([12]) *For each formula $\{\square\} \neq F \in \mathrm{MU}(1)$ and every $k \geq 1$ there is some $H_k \in \mathrm{MU}(k)$ such that $F = \varphi(H_k)$ for some homomorphism $\varphi : H_k \to F$. H_k and φ can be obtained in polynomial time depending on the length of F.*

PROOF. Consider $\{\square\} \neq F \in \mathrm{MU}(1)$. By [5, Theorem 12] there is a variable x such that F contains exactly one clause C_1 with $x \in C_1$ and exactly one clause C_2 with $\overline{x} \in C_2$. Consider F_k and $\varphi_k : F_k \to \{\{x\}, \{\overline{x}\}\}$ as defined in Lemma 25 (we assume that F_k and F have no variables in common). Now put

$$\begin{aligned} H_k \quad := \quad & F \setminus \{C_1, C_2\} \cup \\ & \{\, C \cup C_1 \setminus \{x\} : C \in F_k,\ \varphi_k(C) = \{x\}\,\} \cup \\ & \{\, C \cup C_2 \setminus \{\overline{x}\} : C \in F_k,\ \varphi_k(C) = \{\overline{x}\}\,\}. \end{aligned}$$

It can be verified by a straight forward argument that H_k is minimally unsatisfiable. Moreover, $|H_k| = |F| - 2 + |F_k|$ and $|\mathsf{var}(H_k)| = |\mathsf{var}(F)| - 1 + |\mathsf{var}(F_k)|$. Thus $\delta(H_k) = k$ and so $H_k \in \mathrm{MU}(k)$. Setting

$$\varphi(y) := \begin{cases} \varphi_k(y) & \text{if } y \in \mathsf{var}(F_k); \\ y & \text{otherwise (i.e., } y \in \mathsf{var}(F) \setminus \{x\}) \end{cases}$$

for $y \in \mathsf{var}(H_k)$ evidently defines a homomorphism $\varphi : H_k \to F$ with $\varphi(H_k) = F$. $\square$

Example 27 Consider the formula $F = \{\{\overline{v}, w\}, \{\overline{w}\}, \{v, w, x\}, \{v, \overline{x}\}\} \in \mathrm{MU}(1)$. We look for a formula $H_2 \in \mathrm{MU}(2)$ and a homomorphism $\varphi : H_2 \to F$. According to Lemma 25 we construct $F_2 = \{\{x[1], x[2]\}, \{y[1], y[2]\}\} \cup \{\, \{\overline{x[i]}, \overline{y[j]}\} : 1 \leq i, j \leq 2 \,\}$ and $\varphi_2 : F_2 \to \{\{x\}, \{\overline{x}\}\}$ with $\varphi_2(x[i]) = \varphi_2(y[i]) = x$, $i = 1, 2$. Observe that $C_1 = \{v, w, x\}$ and $C_2 = \{v, \overline{x}\}$ are the only clauses of F containing x and $\overline{x}$, respectively. By the construction presented in the proof of Lemma 26 we get $H_2 = \{\{\overline{v}, w\}, \{\overline{w}\}, \{v, w, x[1], x[2]\}, \{v, w, y[1], y[2]\}\} \cup \{\, \{v, \overline{x[i]}, \overline{y[j]}\} : 1 \leq i, j \leq 2 \,\}$ and the homomorphism $\varphi : H_2 \to F$ defined by $\varphi(v) = v$, $\varphi(w) = w$, $\varphi(x[i]) = \varphi(y[i]) = x$, $i = 1, 2$.

Theorem 28 ([12]) *Tree resolution and* $\Pi_{\mathrm{MU}'(k)}$ *are p-equivalent, for every fixed* $k \geq 1$.

PROOF. In view of Proposition 16, and since tree resolution proofs of formulas in $\mathrm{MU}(k)$ can be found in polynomial time (see the discussion at the beginning of this section), it suffices to show that $\Pi_{\mathrm{MU}'(k)}$ p-simulates tree resolution. Let $F \neq \{\square\}$ be an arbitrary unsatisfiable formula and T a tree resolution proof of F. By Theorem 21 we can obtain a formula $F_1 \in \mathrm{MU}(1)$ and a homomorphism $\varphi : F_1 \to F$ in polynomial time. Applying Lemma 26 we obtain $H_k \in \mathrm{MU}(k)$ and a homomorphism $\psi : H_k \to F_1$ with $\psi(H_k) = F_1$ in polynomial time with respect to the length of F. Now $\varphi \circ \psi$ is the required homomorphism from H_k to F. $\square$

8 Concluding remarks

Our results do not imply that tree resolution p-simulates Π_Γ for every tractable h-complete set Γ of unsatisfiable formulas. For example, one could consider the set $\mathrm{MU}(1) \cup \mathrm{PH}$ where PH denotes the set of so called "pigeonhole formulas." Since pigeonhole formulas require (tree) resolution proofs of exponential size [9], it follows that tree resolution cannot p-simulate $\Pi_{\mathrm{MU}(1)\cup\mathrm{PH}}$. Recently we showed that for *every* proof system Π there is a tractable and h-complete set Γ of unsatisfiable formulas such that Π_Γ and Π are p-equivalent [23].

Formulas in CNF, together with our notion of homomorphism, form a category, and there are several adjunctions which naturally arise within this framework. It is conceivable that an in-depth study of this category and its adjunctions will provide new insights into the structure of formulas in CNF and the satisfiability problem.

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References

[1] R. Aharoni and N. Linial. Minimal non-two-colorable hypergraphs and minimal unsatisfiable formulas. *Journal of Combinatorial Theory, Series A*, 43:196–204, 1986.

[2] N. H. Arai and A. Urquhart. Local symmetries in propositional logic. In R. Dyckhoff, editor, *Automated Reasoning with Analytic Tableaux and Related Methods (Proc. TABLEAUX 2000)*, volume 1847 of *Lecture Notes in Computer Science*, pages 40–51. Springer Verlag, 2000.

[3] B. Benhamou and L. Sais. Tractability through symmetries in propositional calculus. *Journal of Automated Reasoning*, 12:89–102, 1994.

[4] S. A. Cook and R. A. Reckhow. The relative efficiency of propositional proof systems. *The Journal of Symbolic Logic*, 44(1):36–50, Mar. 1979.

[5] G. Davidov, I. Davydova, and H. Kleine Büning. An efficient algorithm for the minimal unsatisfiability problem for a subclass of CNF. *Annals of Mathematics and Artificial Intelligence*, 23:229–245, 1998.

[6] H. Fleischner, O. Kullmann, and S. Szeider. Polynomial-time recognition of minimal unsatisfiable formulas with fixed clause-variable difference. To appear in *Theoretical Computer Science*.

[7] J. Franco and A. Van Gelder. A perspective on certain polynomial time solvable classes of satisfiability. To appear in *Discrete Applied Mathematics*.

[8] M. Goldstern. Personal communication.

[9] A. Haken. The intractability of resolution. *Theoretical Computer Science*, 39:297–308, 1985.

[10] H. Kleine Büning. An upper bound for minimal resolution refutations. In G. Gottlob, E. Grandjean, and K. Seyr, editors, *CSL'98*, volume 1584 of *Lecture Notes in Computer Science*, pages 171–178. Springer Verlag, 1999.

[11] H. Kleine Büning and T. Lettman. *Propositional Logic: Deduction and Algorithms*. Cambridge University Press, Cambridge, 1999.

[12] H. Kleine Büning and X. Zhao. Homomorphisms and MU(k). Unpublished Manuscript, Mar. 2001.

[13] B. Krishnamurthy. Short proofs for tricky formulas. *Acta Informatica*, 22:327–337, 1985.

[14] O. Kullmann. Lean clause-sets: Generalizations of minimally unsatisfiable clause-sets. To appear in *Discrete Applied Mathematics*.

[15] O. Kullmann. Personal communication.

[16] O. Kullmann. An application of matroid theory to the SAT problem. In *Fifteenth Annual IEEE Conference of Computational Complexity*, pages 116–124, 2000.

[17] O. Kullmann. Investigations on autark assignments. *Discrete Applied Mathematics*, 107(1-3):99–137, 2000.

[18] B. Meltzer. Theorem-proving for computers: some results on resolution and renaming. *The Computer Journal*, 8:341–343, 1966.

[19] B. Monien and E. Speckenmeyer. Solving satisfiability in less than 2^n steps. *Discrete Applied Mathematics*, 10:287–295, 1985.

[20] C. H. Papadimitriou and D. Wolfe. The complexity of facets resolved. *Journal of Computer and System Sciences*, 37(1):2–13, 1988.

[21] S. Szeider. *Conjunctive Normal Forms with Bounded Deficiency*. PhD thesis, University of Vienna, Sept. 2001.

[22] S. Szeider. NP-completeness of refutability by literal-once resolution. In R. Gore, A. Leitsch, and T. Nipkow, editors, *IJCAR 2001, Proceedings of the International Joint Conference on Automated Reasoning*, volume 2083 of *Lecture Notes in Artificial Intelligence*, pages 168–181. Springer Verlag, 2001.

[23] S. Szeider. The universality of refutation by homomorphism. Preprint, 2001.

[24] A. Urquhart. The complexity of propositional proofs. *The Bulletin of Symbolic Logic*, 1(4):425–467, Dec. 1995.

[25] A. Urquhart. The symmetry rule in propositional logic. *Discrete Applied Mathematics*, 96/97:177–193, 1999.