A Lagrangian Relax-and-Cut Approach for the Bounded Diameter Minimum Spanning Tree Problem

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Abstract. We consider the problem of finding for a given weighted graph a minimum cost spanning tree whose diameter does not exceed a specified upper bound. This problem is NP-hard and has several applications, e.g. when designing communication networks and quality of service is of concern. We model the problem as an integer linear program (ILP) using so-called jump inequalities. As the number of these constraints grows exponentially with the problem size, directly solving this ILP is no viable option. Instead, we relax them in a Lagrangian fashion and further apply a cutting plane algorithm for only separating violated inequalities on demand. This relax-and-cut approach yields relatively tight lower bounds also for larger problem instances where existing exact techniques are not applicable anymore. High quality feasible solutions, i.e. upper bounds, are obtained by a repair heuristic in combination with a powerful variable neighborhood descent strategy.

Keywords: Bounded diameter minimum spanning tree, Lagrangian relaxation, relax-and-cut, variable neighborhood descent
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INTRODUCTION

We are given an undirected graph $G = (V,E)$ with node set V and edge set E; each edge $(u,v) \in E$ has associated costs $c_{u,v} \geq 0$. Our objective is to find a spanning tree $T = (V,E_T)$, $E_T \subseteq E$, whose diameter (i.e., the number of edges on any path) does not exceed a specified upper bound D ($4 \leq D < |V|-1$) and whose total costs $c(T) = \sum_{(u,v)\in E_T} c_{u,v}$ are minimal. This problem is known as the *bounded diameter minimum spanning tree* (BDMST) problem, and it has been shown to be NP-hard [5]. Applications exist in communication network design when quality of service is of concern and the number of intermediate nodes on communication routes should be limited, in ad-hoc wireless networks, but also in the areas of data compression and distributed mutual exclusion algorithms.

The BDMST problem therefore already received much attention. Existing exact solution approaches primarily utilize (mixed) *integer linear programming* (ILP). In particular, strong hop-indexed multi-commodity network flow models have been described in [6, 7]. Their linear programming (LP) relaxations yield tight lower bounds, but due to the huge number of variables, they are only applicable to relatively small instances. Recently, we described a branch-and-cut algorithm that scales better but also has its limits w.r.t. problem size [8]. To approach larger instances, rather simple greedy construction heuristics and more sophisticated metaheuristics have been suggested. Among the leading metaheuristics are an ant colony optimization and an evolutionary algorithm which are both based on a special level encoding of candidate solutions and strong local improvement procedures [9].

Related to the BDMST problem is the *hop-constrained minimum spanning tree* (HCMST) problem, in which a root node is fixed and the number of hops (edges) on the paths to each other node is limited [3]. Recently, a promising Lagrangian relax-and-cut algorithm has been described for this problem [2]. We adapt and extend this approach for the BDMST problem and further combine it with the local improvement procedures from [9] in order to obtain reasonable lower bounds together with high-quality feasible solutions also for larger BDMST instances in acceptable running times. Further information on our work including detailed experimental results can be found in the master thesis of Ferdinand Zaubzer [12], who was mainly responsible for implementing our approach.

THE JUMP MODEL FOR THE BDMST PROBLEM

We first define an extended directed graph $G^+ = (V^+, A^+)$. The node set V^+ corresponds to V except that we now have an additional artificial root node r, i.e. $V^+ = V \cup \{r\}$. Arc set A^+ is derived by including for each undirected edge $(u,v) \in E$ two reversely directed arcs (u,v) and (v,u) with the same costs $c_{u,v} = c_{v,u}$; furthermore, the root r is connected to each other node $v \in V$ by an arc (r,v) with costs $c_{r,v} = 0$.

Let us first consider the case when the diameter bound D is even. Our model uses decision variables $x_{u,v} \in \{0,1\}$, $\forall (u,v) \in A^+$, indicating whether or not arc (u,v) is part of the solution. An edge of the original graph G is finally chosen iff one of its corresponding directed arcs of G^+ is selected. Furthermore, let AR be the set of all incidence vectors on A^+ corresponding to feasible spanning arborescences (i.e. outgoing spanning trees) rooted at r.

We can now formulate the even-diameter BDMST problem as follows:

$$\text{minimize} \quad \sum_{(u,v)\in A^+} c_{u,v} \cdot x_{u,v} \tag{1}$$

$$\text{subject to} \quad x \in AR \tag{2}$$

$$\sum_{v\in V} x_{r,v} = 1 \tag{3}$$

$$\sum_{(u,v)\in J(P)} x_{u,v} \geq 1 \qquad \forall\, P \in P(V^+) \mid r \in S_0. \tag{4}$$

The objective is to minimize the total costs of chosen arcs (1), while vector x must describe a feasible arborescence (2). Equation (3) additionally ensures that there is exactly one outgoing arc from the root r to the node that will form the *center* of the BDMST (i.e., a node with minimum eccentricity). Constraints (4) are the so-called *jump inequalities* and impose a hop-limit of $H = D/2+1$ on the paths from r to any other node $v \in V$ [2]. They work as follows: Consider a partitioning P of V^+ into $H+2$ pairwise disjoint non-empty sets S_0 to S_{H+1} with $r \in S_0$. Let $\sigma(v)$ denote the index of the partition a node v is assigned to. Jump $J(P)$ is defined as the set of arcs $(u,v) \in A^+$ with $\sigma(u) < \sigma(v) - 1$, i.e. $J(P)$ contains all arcs leading from a partition to a higher indexed one and skipping at least one in-between. The jump inequality associated with this partitioning states that at least one of these arcs in $J(P)$ must appear. Otherwise, there would be a path connecting the root contained in S_0 to a node in S_{H+1} with length $H+1$ violating the hop constraint. Such jump inequalities must hold for all possible partitionings $P(V^+)$ of V^+ with $r \in S_0$.

In case the diameter bound D is odd, a BDMST has a *center edge* instead of a single center node. We define further variables $z_{u,v} \in \{0,1\}$, $\forall (u,v) \in E$, indicating which edge forms this center, and adapt our model as follows:

$$\text{minimize} \quad \sum_{(u,v)\in A^+} c_{u,v} \cdot x_{u,v} + \sum_{(u,v)\in E} c_{u,v} \cdot z_{u,v} \tag{5}$$

$$\text{subject to} \quad x \in AR \tag{6}$$

$$\sum_{v\in V} x_{r,v} = 2 \tag{7}$$

$$\sum_{v\mid(u,v)\in E} z_{u,v} = x_{r,u} \qquad \forall u \in V \tag{8}$$

$$\sum_{(u,v)\in J(P)} x_{u,v} \geq 1 \qquad \forall\, P \in P(V^+) \mid r \in S_0. \tag{9}$$

The objective function (5) is extended by also considering the costs of the center edge. Now, two arcs must lead from the root node to two center nodes (7), and these two nodes must be incident to the center edge (8). The arborescence condition (6) and the jump inequalities (9) remain unchanged, but now $H = (D+1)/2$.

LAGRANGIAN RELAXATION AND THE RELAX-AND-CUT ALGORITHM

We approach the described formulations by relaxing the jump inequalities in a Lagrangian fashion. Thus, these constraints are removed and the objective function is augmented by corresponding penalty terms. For the even diameter case this yields

$$\text{minimize} \quad \sum_{(u,v)\in A^+} c_{u,v} \cdot x_{u,v} + \sum_{P\in P(V^+)} \lambda_P \left(1 - \sum_{(u,v)\in J(P)} x_{u,v}\right) = \sum_{(u,v)\in A^+} \alpha_{u,v} \cdot x_{u,v} + \sum_{P\in P(V^+)} \lambda_P, \tag{10}$$

where $\alpha_{u,v} = c_{u,v} - \sum_{P\in P(V^+)\mid(u,v)\in J(P)} \lambda_P$ are the *reduced arc costs*. For any setting of Lagrange factors $\lambda_P \geq 0$, the optimal solution value of the relaxed problem corresponds to a lower bound of our original BDMST problem, and our aim is to find Lagrange factors that maximize this lower bound.

Considering a specific choice of Lagrange factors, the relaxed problem is not NP-hard anymore but can be solved efficiently. When D is even, we temporarily increase the reduced costs of all arcs incident to the root by a sufficiently large value M and apply Edmonds' algorithm to find a minimum spanning arborescence [4]. Due to the large arc costs, the root node will only have one outgoing arc and condition (3) will be satisfied. In case of an odd diameter, our subproblem involves finding a minimum cost *triangle tree*, i.e. an arborescence with two arcs emerging from the root node plus the (undirected) central edge connecting the two target nodes. We iteratively try each edge $(u,v) \in E$ as center, temporarily shrink it with r into a new "super" root node inducing base costs $c_{u,v} + \alpha_{r,u} + \alpha_{r,v}$, and apply Edmonds' algorithm to obtain the remaining arborescence. The overall cheapest triangle tree is returned. Of course, this approach is significantly more expensive than the method for the even diameter case. It can, however, be sped up substantially by processing the potential center edges in decreasing base cost order and aborting Edmonds' algorithm as soon as a lower bound of the total costs exceeds the so far best solution's costs. An alternative for handling the odd diameter case is to also relax equations (7) and (8) in a Lagrangian fashion. Then, only a simple minimum arborescence must be found as subproblem, but the obtainable lower bound is in general weaker.

Knowing now how to solve the Lagrangian subproblems efficiently, the next challenge is to find suitable Lagrange factors λ_P yielding a largest possible lower bound. As this task is a concave piecewise linear optimization problem, a subgradient algorithms is traditionally used for this purpose [1].

There is, however, a fundamental remaining problem: The number of jump inequalities depends exponentially on $|V|$, and therefore it is in practice not possible to directly consider all of them in our objective function. Instead, we apply the principle of relax-and-cut [11] similarly as it has been described for the HCMST in [2]. Our algorithm only considers a very restricted pool of partitionings $P'(V^+) \subset P(V^+)$ and corresponding jump inequalities. New jump inequalities that are violated by the current solution after solving the relaxed problem for a particular set of Lagrange factors are determined by a separation procedure and added to the pool $P'(V^+)$. On the other hand, when $P'(V^+)$ exceeds a given size, jump constraints that have not been violated for at least a certain number of iterations are removed. The separation of violated jump inequalities is based on the detection of paths that exceed the hop limit, which is straight-forward. An initial pool of jump inequalities is derived by a dual ascent heuristic. In contrast to [2], we found that a non-delayed relax-and-cut scheme [11] is more efficient in our case than a delayed approach.

FINDING PRIMAL SOLUTIONS

Lagrangian relaxation is primarily a technique for obtaining lower bounds; feasible solutions to the original problem corresponding to upper bounds are often not directly obtained. In our case, the relax-and-cut algorithm occasionally finds solutions that are also feasible for the original BDMST problem. Usually, these solutions are, however, only of moderate quality when compared to final solutions of the leading metaheuristics. Therefore, we decided to incorporate local improvement strategies from [9] into our relax-and-cut. More precisely, a so-called *variable neighborhood descent* (VND) algorithm is performed on the trees obtained as intermediate results in regular intervals. VND is an extension of local search in which several different neighborhood structures are utilized in a systematic way [10] in order to obtain a solution that is locally optimal w.r.t. all these neighborhood structures.

First of all, eventually infeasible solutions containing paths from the root longer than $\lceil D/2 \rceil + 1$ arcs are repaired and improved by a greedy heuristic that reconnects nodes in the order of their original depths to their locally best feasible predecessors. Then, VND is applied with a best improvement strategy using the following neighborhood structures [9]:

Arc exchange neighborhood: Neighboring solutions are all feasible arborescences that differ from the current solution in exactly one arc.

Node swap neighborhood: This neighborhood contains all solutions that are obtained by exchanging in the tree structure the position of a node with one of its direct successors.

Level change neighborhood: Neighboring solutions are obtained by increasing or decreasing the depth of a node by one. The node as well as all its former direct successors are reconnected in a locally optimal way by choosing cheapest arcs.

Center exchange neighborhood: In neighboring solutions, the center nodes are replaced by other nodes. Former center nodes are newly connected by cheapest possible arcs.

Improved primal solutions also have a positive impact on the subgradient algorithm: As it updates the Lagrange factors in dependence of an estimation of the optimal solution value, tighter upper bounds (which are obtained from the improved primal solutions) speed up the convergence.

EXPERIMENTAL RESULTS

We tested the described approach on standard BDMST benchmark instances including pure random and Euclidean instances involving sparse and complete graphs with even diameter bounds $D \in \{4,6,8,10\}$. Detailed results can be found in the master thesis of Zaubzer [12]. For smaller instances with up to 30 nodes and 200 edges and known optima, the algorithm was almost always able to find these optimal solutions. For larger complete graphs on 81 nodes the percentage gaps between lower and upper bounds ranged between 10% and 16%. Tests with odd diameter bounds are ongoing work.

CONCLUSIONS AND FUTURE WORK

In this work we proposed a Lagrangian relax-and-cut algorithm based on a strong ILP formulation for the BDMST problem. Besides lower bounds high quality feasible solutions can be found also for problem instances that are too large to be solved by today's leading exact techniques. An essential feature is the integration of the relax-and-cut method with the variable neighborhood descent metaheuristic for improving candidate solutions. Both algorithms benefit from each other: VND is started with promising initial solutions obtained from the Lagrangian relaxation, while relax-and-cut benefits from the improved upper bounds.

In currently ongoing work we investigate even more sophisticated hybrid strategies: For example, the ant colony optimization from [9] is performed with the subgradient algorithm in an intertwined way and the (intermediate) reduced costs of the Lagrangian relaxation are exploited in the solution construction process of the ant colony optimization as heuristic information. Such concepts of hybridizing ILP techniques with metaheuristics are also highly promising for other combinatorial optimization problems.

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