Introduction to Probabilistic Programming

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December 18, 2023

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Bayesian Learning

Learn, which model parameters are probable, given data.

Advantages

- Measure for uncertainty
- Use prior knowledge about model and parameters

$\rightarrow$ Sample efficient

Disadvantages

- Can be resource-heavy
- Probabilistic model required

Setting and Terminology

Given:

- Data x
- Probabilisic model $p(x \mid \theta)$
- Prior probability distribution $p(\theta)$

Compute: probability distribution $p(\theta \mid x)$

Bayes theorem

$$\underbrace{p(\theta \mid x)}_{\text{Posterior}} = \frac{\overbrace{p(x \mid \theta)}^{\text{Likelihood}}\ \overbrace{p(\theta)}^{\text{Prior}}}{\underbrace{p(x)}_{\text{Evidence}}}$$

$p(x)$ usually hard to compute, but x is known $\rightarrow$ $p(x)$ is constant

Therefore

$$p(\theta \mid x) \sim f(\theta) = p(x \mid \theta) p(\theta)$$

$f(\theta)$: Joint probability

How to specify f?
How to represent and compute posterior?

Running example: Coinflip

Flip (fair or unfair) coin n times
$\rightarrow$ k times head, $n-k$ times tail
Predict probability θ for head

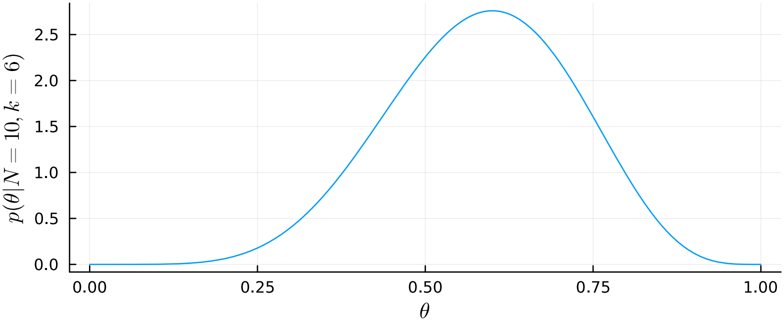

Figure: Analytical solution for $n=10$ and $k=6$

Probabilistic Programs

General framework to specify model via $f(\theta)$

Arbitrary program enriched by

$$\underbrace{A}_{\substack{\text{Name of}\\\text{random variable}}} \sim \underbrace{\mathrm{Normal}}_{\text{Distribution}} (\underbrace{\mu, \sigma^2}_{\text{Parameters}})$$

and the observed value for some random variables

Example: Coinflip in Turing.jl

```
@model function coinflip(heads::AbstractVector{Bool})
    θ ~ Uniform(0, 1)
    for i in eachindex(heads)
        heads[i] ~ Bernoulli(θ)
    end
end
```

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Prior:

$$p(\theta) = \mathrm{PDF}_{\mathrm{Uniform}}(\theta) = \begin{cases} 1 & \text{, if } 0 \leq \theta < 1 \\ 0 & \text{otherwise} \end{cases}$$

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Prior:

$$p(\theta) = \mathrm{PDF}_{\mathrm{Uniform}}(\theta) = \begin{cases} 1 & \text{, if } 0 \leq \theta < 1 \\ 0 & \text{otherwise} \end{cases}$$

Likelihood:

$$p(\mathrm{heads}|\theta) = \prod_{i=1}^{n} \mathrm{PDF}_{\mathrm{Bernoulli}}(\mathrm{heads}_i) = p^k p^{n-k}$$

where $n = |\mathrm{heads}|$ and $k = |\mathrm{heads}|_1$

→ Joint Probability is product of PDFs

Joint Probability, Trace

For each $\mathcal{V} \sim \mathcal{D}$, a function is called that

- determines and returns a value v for $\mathcal{V}$,
- appends $(\mathcal{V}, v)$ to the trace, and
- updates joint probability $f(\theta)$ (multiply with PDF).

Determine v:

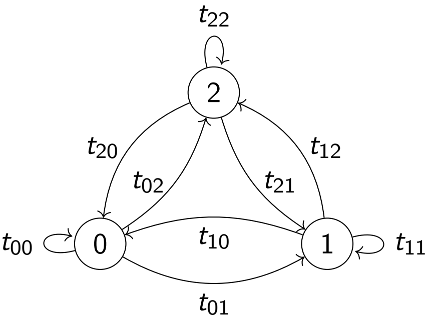
- $\mathcal{V}$ is observed: use observed value
- Otherwise: sampler-dependent, e.g. use proposal distribution or reuse from previous trace

Run results in:

- trace θ
- joint probability $f(\theta)$

Markov chains

Next state only depends on current state



Markov chains

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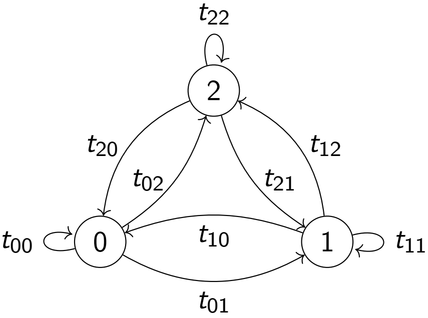

s: probability distribution over states

Definition (Steady state)

s is a steady state iff $sT = s$

Theorem

$s_i t_{ij} = s_j t_{ji}$ (aka reversibility) $\Rightarrow$ s is steady state

Monte-Carlo Markov Chains

Bayes Theorem: $p(\theta \mid x) = \frac{p(x|\theta)p(\theta)}{p(x)} = Cf(\theta)$

Problems:

- C unknown
- many dimensions possible, therefore hard to handle

$\rightarrow$ sample from $p(\theta \mid x)$
But: no direct way

Idea: create Markov chain with steady state $p(\theta \mid x)$

Metropolis Hastings

State space: parameter configurations θ
Transitions: use proposal distribution $q(\theta'|\theta)$
Make $p(\theta|x)$ reversible by rejecting proposals with probability $1 - A(\theta'|\theta)$
Reversible if

$$p(\theta|x)q(\theta'|\theta)A(\theta'|\theta) = p(\theta'|x)q(\theta|\theta')A(\theta|\theta')$$

Fulfilled with

$$A(\theta'|\theta) = \min\left(1, \frac{p(\theta'|x)q(\theta|\theta')}{p(\theta|x)q(\theta'|\theta)}\right)$$

Metropolis Hastings

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Fulfilled with

$$A(\theta'|\theta) = \min\left(1, \frac{Cf(\theta')q(\theta|\theta')}{Cf(\theta)q(\theta'|\theta)}\right)$$

Metropolis Hastings

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$$A(\theta'|\theta) = \min\left(1, \frac{f(\theta')q(\theta|\theta')}{f(\theta)q(\theta'|\theta)}\right)$$

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Algorithm 4:

Data: Distributions f, q, initial parameters θ, #Iterations n
Result: n samples
$S \leftarrow \{\}$;
for $i \in \{1, \ldots, n\}$ **do**
  sample $\theta' \sim q(\cdot|\theta)$;
  sample $a \sim \mathrm{Uniform}(0, 1)$;
  if $a < \min\left(1, \frac{f(\theta')q(\theta|\theta')}{f(\theta)q(\theta'|\theta)}\right)$ **then**
    $\theta \leftarrow \theta'$
  end
  append θ to S;
end
return S

Metropolis Hastings

State space: parameter configurations θ
Transitions: use proposal distribution $q(\theta'|\theta)$
Make $p(\theta|x)$ reversible by rejecting proposals with probability $1 - A(\theta'|\theta)$
Reversible if

$$p(\theta|x)q(\theta'|\theta)A(\theta'|\theta) = p(\theta'|x)q(\theta|\theta')A(\theta|\theta')$$

Fulfilled with

$$A(\theta'|\theta) = \min\left(1, \frac{f(\theta')q(\theta|\theta')}{f(\theta)q(\theta'|\theta)}\right)$$

$\rightarrow$ Jupyter-Notebook

Algorithm 5:

Data: Distributions f, q, initial parameters θ, #Iterations n
Result: n samples

$S \leftarrow \{\}$;
for $i \in \{1, \ldots, n\}$ **do**
  sample $\theta' \sim q(\cdot|\theta)$;
  sample $a \sim \mathrm{Uniform}(0, 1)$;
  if $a < \min\left(1, \frac{f(\theta')q(\theta|\theta')}{f(\theta)q(\theta'|\theta)}\right)$ **then**
    $\theta \leftarrow \theta'$
  end
  append θ to S;
end
return S

Other inference algorithms

- Hamiltonian Monte Carlo (HMC) and No U-Turn Sampler (NUTS)
- Variational Inference (VI)
- Gibbs sampling

Visualization: `https://chi-feng.github.io/mcmc-demo/app.html`

Predicting User Availabilities

→ Jupyter-Notebook

Captchas

procedure CAPTCHA
  $\nu \sim p(\nu)$ ▷ sample number of letters
  $\kappa \sim p(\kappa)$ ▷ sample kerning value
Generate letters:
  $\Lambda \leftarrow \{\}$
  for $i = 1, \ldots, \nu$ **do**
    $\lambda \sim p(\lambda)$ ▷ sample letter identity
    $\Lambda \leftarrow \text{append}(\Lambda, \lambda)$
Render:
  $\gamma \leftarrow \text{render}(\Lambda, \kappa)$
  $\pi \sim p(\pi)$ ▷ sample noise parameters
  $\gamma \leftarrow \text{noise}(\gamma, \pi)$
  return γ

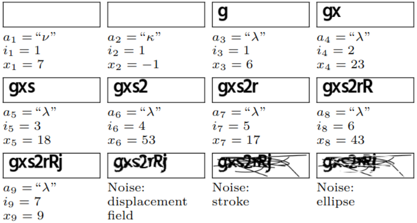
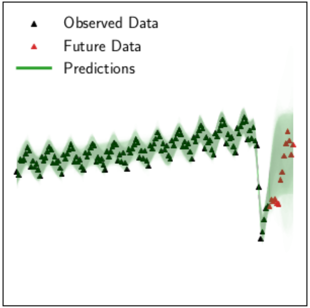


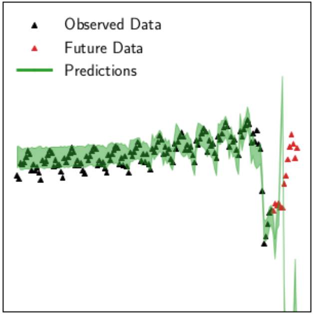
Table 1: Captcha recognition rates.

	Baidu 2011	Baidu 2013	eBay	Yahoo	reCaptcha	Wikipedia	Facebook
Our method	99.8%	99.9%	99.2%	98.4%	96.4%	93.6%	91.0%
Bursztein et al. (2014)	38.68%	55.22%	51.39%	5.33%	22.67%	28.29%	
Starostenko et al. (2015)				91.5%	54.6%		
Gao et al. (2014)	34%			55%	34%		
Gao et al. (2013)		51%		36%			
Goodfellow et al. (2014)					99.8%		
Stark et al. (2015)					90%		

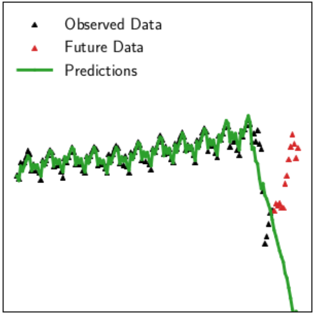
Airline passenger forecast



(a) Gaussian Process DSL
100 Synthesized Structures



(c) Facebook Prophet
With ChangePoint (Custom)



(e) Neural Prophet
With ChangePoint (Custom)

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