A Model for Finding Sup-Transition-Minors

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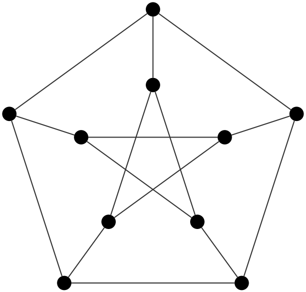
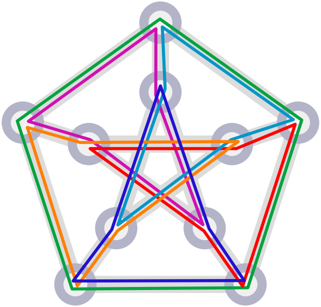
Circuit Double Cover Conjecture (CDCC)

Let G be a bridgeless undirected graph. Then, there exists a collection of circuits of G, such that each edge is contained in exactly two circuits.



$\Rightarrow$



Source: https://en.wikipedia.org/wiki/Cycle_double_cover#/media/File:Petersen_double_cover.svg

Definition

A *snark* is a simple, connected, bridgeless cubic undirected graph which has no 3-edge-coloring.

Theorem (Jaeger(1985))

Every minimal counter example to the CDCC must be a snark.

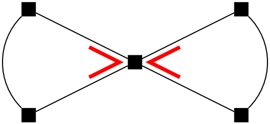
Corollary

If the CDCC holds on the graph class of all snarks it also holds on all bridgless graphs.

Definition (Transition)

A *transition* in a graph consists of a vertex v and two of its incident edges e_1, e_2.

Used to forbid circuits which contain the edges e_1 and e_2.



Definition (Transition System)

A *transition system* $\mathcal{T}$ is a set of transitions in a graph G. $(G, \mathcal{T})$ is called a *transitioned graph*.

Definition (Compatible Circuit Decomposition)

Let $(G, \mathcal{T})$ be a transitioned graph. A *compatible circuit decomposition of* G is a circuit decomposition $\mathcal{C}$ of G such that for all transitions in $\mathcal{T}$ there is no circuit in $\mathcal{C}$ which contains both edges of the transition.

Theorem

A snark G *contains a circuit double cover if and only if its line graph* $L(G)$ *together with the transition system as described in the slides before has a compatible circuit decomposition.*

Compatible Circuit Decomposition Problem

Let G be a 2-connected eulerian graph and $\mathcal{T}$ a transition system on G. Has G a compatible circuit decomposition?

Theorem (Fan and Zhang(2000))

Let $(G, \mathcal{T})$ be a 2-connected eulerian graph associated with a transition system. If G is K_5-minor free, then $(G, \mathcal{T})$ has a compatible circuit decomposition.

Does not depend on actual transition system!

Fleischner et al. generalize this theorem by extending the definition of a minor to transitioned graphs:

Theorem (Fleischner et al.(2017))

Let $(G, \mathcal{T})$ be a 2-connected eulerian graph associated with a transition system. If $(G, \mathcal{T})$ is SUD-K_5-minor-free, then it has a compatible circuit decomposition.

But what does SUD-K_5-minor-free mean?

Compatible Circuit Decomposition

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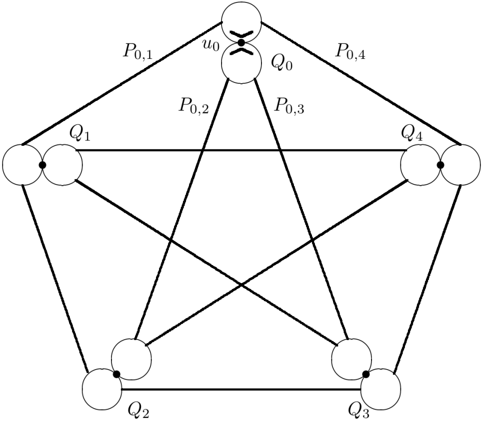
But what does SUD-K_5-minor-free mean?

Definition (Minor)

Let G be an undirected (maybe not simple) graph. A graph H which can be derived by deleting vertices, deleting edges and/or contracting edges in G is called a *minor* of G.



Definition (Reduced Transition Minor)

Let $(G, \mathcal{T})$ be a transitioned graph and H a minor of G. Let the transition system $\mathcal{S}'$ consist of all transitions in G whose edges didn't get removed or contracted. Then $(H, \mathcal{S}')$ is called a *reduced transition minor* of $(G, \mathcal{T})$.

Definition (Sup-un-decomposable-K_5-transition-minor free)

$(G, \mathcal{T})$ is *sup-un-decomposable-K_5-transition-minor free* (or short SUD-K_5-minor free) if it does not have any eulerian reduced transition minor which is a SUD-K_5.

Two open problems:

Problem

Does there exist a snark G such that its line graph $L(G)$ has a K_5-minor but is SUD-K_5-minor free?

Problem

Does there exist a snark G and a perfect (pseudo-) matching of G such that the transitioned graph after contracting all edges in the matching is SUD-K_5-minor free but not K_5-minor free?

Problem

Does there exist an eulerian transitioned graph which has a compatible circuit decomposition and is not SUD-K_5-minor free?

Theorem

SUD-K_5-TM is solvable in polynomial time.

$\Rightarrow$ consider more general problem

Problem (Existence of sup-transition-minors (ESTM))

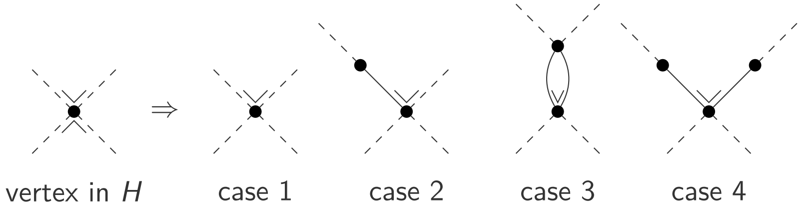
Given a transitioned graph $(G, \mathcal{T})$ and a completely transitioned 4-regular graph $(H, \mathcal{S})$, does there exist an Eulerian transition minor of $(G, \mathcal{T})$ which is a sup-$(H, \mathcal{S})$ graph?

Theorem

ESTM is NP*-complete.*

Basic-sup-$(H, \mathcal{S})$)

A transitioned graph $(H', \mathcal{S}')$ is a basic-sup-$(H, \mathcal{S})$ graph iff we can get $(H', \mathcal{S}')$ by replacing each vertex v in H by one of the four following subgraphs:



Problem (Existence of basic-sup-transition-minors (EBSTM))

Given a transitioned graph $(G, \mathcal{T})$ *and a completely transitioned 4-regular graph* $(H, \mathcal{S})$*, does there exist a transition minor of* $(G, \mathcal{T})$ *which is a basic-sup-*$(H, \mathcal{S})$ *graph?*

Theorem

ESTM and EBSTM are equivalent.

A Model for EBSTM

Idea: Model EBSTM without using explicit variables for the intermediate graph $(H', \mathcal{S}')$. The model consists of

1. a partial surjective function $\iota : V(G) \nrightarrow V(H)$,
2. a partial injective and surjective function $\kappa : E(G) \nrightarrow E(H)$,
3. a partial injective function $\lambda : \mathcal{T} \nrightarrow \mathcal{S}$,
4. a partial injective function $\theta : E(G) \nrightarrow V(H)$,
5. for each $w \in V(H)$ two simple trees C_w^1 and C_w^2 with $V(C_w^i) \subseteq V(G)$ for $i = 1, 2$.

$$E(C_w^i) \subseteq \mathrm{sig}_G[E(G)] \qquad \forall w \in V(H), \forall i = 1,2$$

$$\kappa(e) = f \Rightarrow \iota[\mathrm{sig}_G(e)] = \mathrm{sig}_H(f) \qquad \forall e \in E(G), \forall f \in E(H)$$

$$V(C_w^1) \cup V(C_w^2) = \iota^{-1}(w) \qquad \forall w \in V(H)$$

$$\lambda(T) = S \Rightarrow \iota(\pi_1(T)) = \pi_1(S) \qquad \forall T \in \mathcal{T}, \forall S \in \mathcal{S}$$

$$\lambda(T) = S \Rightarrow \pi_2(T) \subseteq \kappa^{-1}[\pi_2(S)] \cup \theta^{-1}[\pi_1(S)] \cup \left\{ e \in E(\pi_1(T)) : \mathrm{sig}_G(e) \in E(C^1_{\pi_1(S)}) \right\} \qquad \forall T \in \mathcal{T}, \forall S \in \mathcal{S}$$

$$\lambda(T) = S \Rightarrow \left(\kappa^{-1}[\pi_2(S)] \cap E(\pi_1(T))\right) \cup \theta^{-1}[\pi_1(S)] \subseteq \pi_2(T) \qquad \forall T \in \mathcal{T}, \forall S \in \mathcal{S}$$

$$|\lambda[\mathcal{T}] \cap \mathcal{S}(w)| = 1 \qquad \forall w \in V(H)$$

$$S \in \lambda[\mathcal{T}] \wedge e \in \mathrm{dom}(\kappa) \wedge \kappa(e) \in \pi_2(S) \Rightarrow \mathrm{sig}_G(e) \cap V(C^1_{\pi_1(S)}) \neq \emptyset \qquad \forall S \in \mathcal{S}, \forall e \in E(G)$$

$$S \in \lambda[\mathcal{T}] \wedge e \in \mathrm{dom}(\kappa) \wedge \kappa(e) \in E(\pi_1(S)) \setminus \pi_2(S) \Rightarrow \mathrm{sig}_G(e) \cap V(C^2_{\pi_1(S)}) \neq \emptyset \qquad \forall S \in \mathcal{S}, \forall e \in E(G)$$

$$\lambda(T) = S \Rightarrow E_{C^1_{\pi_1(S)}}(\pi_1(T)) \subseteq \mathrm{sig}_G[\pi_2(T)] \qquad \forall T \in \mathcal{T}, \forall S \in \mathcal{S}$$

$$\theta(e) = w \Rightarrow \mathrm{sig}_G(e) \subseteq V(C_w^1) \qquad \forall e \in E(G), \forall w \in V(H)$$

$$\theta(e) = w \Rightarrow \mathrm{sig}_G(e) \notin E(C_w^1) \qquad \forall e \in E(G), \forall w \in V(H)$$

Model Constraints cont.

Last but not least ...

$$\left(\lambda(T) = S \wedge v \in V(C^1_{\pi_1(S)}) \wedge \left|E_{C^1_{\pi_1(S)}}(v)\right| = 1 \wedge v \notin \bigcup \mathrm{sig}_G[\theta^{-1}[w]]\right)$$
$$\Rightarrow E(v) \cap \kappa^{-1}[\pi_2(S)] \neq \emptyset \quad \forall T \in \mathcal{T}, \forall S \in \mathcal{S}, \forall v \in V(G) \setminus \{\pi_1(T)\}$$

The MIP Model - Variables

- x_v^w for $v \in V(G)$, $w \in V(H)$ ... 1 iff $\iota(v) = w$
- y_e^f for $e \in E(G)$, $f \in E(H)$ ... 1 iff $\kappa(e) = f$
- z_T^S for $T \in \mathcal{T}$, $S \in \mathcal{S}$ ... 1 iff $\lambda(T) = S$
- a_e^w for $e \in E(G)$, $w \in V(H)$ ... 1 iff $\theta(e) = w$
- $u_v^{i,w}$ for $v \in V(G)$, $i \in \{1,2\}$, $w \in V(H)$ ... 1 iff $v \in C_w^i$
- $t_{\{v_1,v_2\}}^{i,w}$ for $\{v_1, v_2\} \in \text{sig}_G[E(G)]$, $i \in \{1,2\}$, $w \in V(H)$ ... 1 iff $\{v_1, v_2\} \in E(C_w^i)$
- $f_{v_1,v_2}^{i,w}$ for $\{v_1, v_2\} \in \text{sig}_G[E(G)]$, $i \in \{1,2\}$, $w \in V(H)$ ... helper variable representing a directed flow on edge $\{v_1, v_2\} \in E(C_w^i)$

The MIP Model - Constraints 1

$$\sum_{w \in V(H)} x_v^w \leq 1 \qquad \forall v \in V(G)$$

$$\sum_{v \in V(G)} x_v^w \geq 1 \qquad \forall w \in V(H)$$

$$\sum_{f \in E(H)} y_e^f \leq 1 \qquad \forall e \in E(G)$$

$$\sum_{e \in E(G)} y_e^f = 1 \qquad \forall f \in E(H)$$

$$\sum_{S \in \mathcal{S}} y_S^T \leq 1 \qquad \forall T \in \mathcal{T}$$

$$\sum_{T \in \mathcal{T}} y_S^T \leq 1 \qquad \forall S \in \mathcal{S}$$

$$\sum_{w \in V(H)} a_e^w \leq 1 \qquad \forall e \in E(G)$$

$$\sum_{e \in E(G)} a_e^w \leq 1 \qquad \forall w \in V(H)$$

The MIP Model - Constraints 2

$$t^{i,w}_{\{v_1,v_2\}} \leq u^{i,w}_{v_j} \qquad \forall \{v_1, v_2\} \in E^S_G, \forall w \in V(H), \forall i,j \in \{1,2\}$$

$$\sum_{\{v_1,v_2\} \in E^S_G} t^{i,w}_{\{v_1,v_2\}} = \sum_{v \in V(G)} u^{i,w}_{v} - 1 \qquad \forall i \in \{1,2\}, \forall w \in V(H)$$

$$f^{i,w}_{v_1,v_2} + f^{i,w}_{v_2,v_1} \geq t^{i,w}_{\{v_1,v_2\}} \qquad \forall \{v_1, v_2\} \in E^S_G, \forall w \in V(H), \forall i \in \{1,2\}$$

$$\sum_{v_1 \in N_G(v_2)} f^{i,w}_{v_1,v_2} \leq 1 - \frac{1}{|V_G| - |V_H| + 1} \qquad \forall v_2 \in V(G), \forall w \in V(H), i \in \{1,2\}$$

The MIP Model - Constraints 3

Base-Model:

$$\kappa(e) = f \Rightarrow \iota[\text{sig}_G(e)] = \text{sig}_H(f) \quad \forall e \in E(G), \forall f \in E(H)$$

MIP-Model:

$$y_e^f \leq x_v^{w_1} + x_v^{w_2} \qquad \forall e \in E(G), \forall v \in \text{sig}_G(e), \forall f \in E(H), \text{sig}_H(f) = \{w_1, w_2\}$$

$$y_e^f \leq x_{v_1}^w + x_{v_2}^w \qquad \forall e \in E(G), \text{sig}_G(e) = \{v_1, v_2\}, \forall f \in E(H), \forall w \in \text{sig}_H(f)$$

The MIP Model - Constraints 4

Base-Model:

$$V(C_w^1) \cup V(C_w^2) = \iota^{-1}(w) \quad \forall w \in V(H)$$

$$\lambda(T) = S \Rightarrow \{\pi_1(T)\} = V(C^1_{\pi_1(S)}) \cap V(C^2_{\pi_1(S)}) \quad \forall T \in \mathcal{T}, \forall S \in \mathcal{S}$$

MIP-Model:

$$u_v^{1,w} + u_v^{2,w} = x_v^w + \sum_{\substack{T \in \mathcal{T}(v) \\ S \in \mathcal{S}(w)}} z_T^S \quad \forall v \in V(G), \forall w \in V(H)$$

$$z_T^S \leq x_{\pi_1(T)}^{\pi_1(S)} \quad \forall T \in \mathcal{T}, \forall S \in \mathcal{S}$$

The MIP Model - Constraints 5

Base-Model:

$$\lambda(T) = S \Rightarrow \pi_2(T) \subseteq \kappa^{-1}[\pi_2(S)] \cup \theta^{-1}[\pi_1(S)] \cup \left\{ e \in E(\pi_1(T)) : \mathrm{sig}_G(e) \in E(C^1_{\pi_1(S)}) \right\}$$

$$\forall T \in \mathcal{T}, \forall S \in \mathcal{S}$$

MIP-Model:

$$z_T^S \leq \sum_{f \in \pi_2(S)} y_e^f + a_e^{\pi_1(S)} + t_{\mathrm{sig}_G(e)}^{1,\pi_1(S)} \quad \forall T \in \mathcal{T}, \forall S \in \mathcal{S}, \forall e \in \pi_2(T)$$

The MIP Model - Constraints 6

Base-Model:

$$\lambda(T) = S \Rightarrow \left(\kappa^{-1}[\pi_2(S)] \cap E(\pi_1(T))\right) \cup \theta^{-1}[\pi_1(S)] \subseteq \pi_2(T)$$

$$\forall T \in \mathcal{T}, \forall S \in \mathcal{S}$$

MIP-Model:

$$z_T^S \leq 1 - \sum_{e \in E(\pi_1(T)) \setminus \pi_2(T)} y_e^f \quad \forall T \in \mathcal{T}, \forall S \in \mathcal{S}, \forall f \in \pi_2(S)$$

$$z_T^S \leq 1 - \sum_{e \in E(G) \setminus \pi_2(T)} a_e^{\pi_1(S)} \quad \forall T \in \mathcal{T}, \forall S \in \mathcal{S}$$

The MIP Model - Constraints 7

Base-Model:

$$|\lambda[\mathcal{T}] \cap \mathcal{S}(w)| = 1 \quad \forall w \in V(H)$$

MIP-Model:

$$\sum_{\substack{S \in \mathcal{S}(w) \\ T \in \mathcal{T}}} z_T^S = 1 \quad \forall w \in V(H)$$

The MIP Model - Constraints 8

Base-Model:

$$S \in \lambda[\mathcal{T}] \wedge e \in \mathrm{dom}(\kappa) \wedge \kappa(e) \in \pi_2(S) \Rightarrow \mathrm{sig}_G(e) \cap V(C^1_{\pi_1(S)}) \neq \emptyset$$

$$\forall S \in \mathcal{S}, \forall e \in E(G)$$

MIP-Model:

$$\sum_{T \in \mathcal{T}} z_T^S + \sum_{f \in \pi_2(S)} y_e^f \leq 1 + \sum_{v \in \mathrm{sig}_G(e)} u_v^{1,\pi_1(S)} \quad \forall S \in \mathcal{S}, \forall e \in E(G)$$

The MIP Model - Constraints 9

Base-Model:

$$S \in \lambda[\mathcal{T}] \wedge e \in \mathrm{dom}(\kappa) \wedge \kappa(e) \in E(\pi_1(S)) \setminus \pi_2(S)$$

$$\Rightarrow \mathrm{sig}_G(e) \cap V(C^2_{\pi_1(S)}) \neq \emptyset$$

$$\forall S \in \mathcal{S}, \forall e \in E(G)$$

MIP-Model:

$$\sum_{T \in \mathcal{T}} z_T^S + \sum_{f \in E(\pi_1(S)) \setminus \pi_2(S)} y_e^f \leq 1 + \sum_{v \in \mathrm{sig}_G(e)} u_v^{2,\pi_1(S)}$$

$$\forall S \in \mathcal{S}, \forall e \in E(G)$$

The MIP Model - Constraints 10

Base-Model:

$$\theta(e) = w \Rightarrow \mathrm{sig}_G(e) \subseteq V(C^1_w) \quad \forall e \in E(G), \forall w \in V(H)$$

$$\theta(e) = w \Rightarrow \mathrm{sig}_G(e) \notin E(C^1_w) \quad \forall e \in E(G), \forall w \in V(H)$$

MIP-Model:

$$a^w_e \leq \frac{1}{2} \sum_{v \in \mathrm{sig}_G(e)} u^{1,w}_v \quad \forall e \in E(G), \forall w \in V(H)$$

$$a^w_e \leq 1 - t^{1,w}_{\mathrm{sig}_G(e)} \quad \forall e \in E(G), \forall w \in V(H)$$

The MIP Model - Constraints 11

Base-Model:

$$\left(\lambda(T) = S \wedge v \in V(C^1_{\pi_1(S)}) \wedge \left| E_{C^1_{\pi_1(S)}}(v) \right| = 1 \wedge v \notin \bigcup \mathrm{sig}_G[\theta^{-1}[w]]\right)$$
$$\Rightarrow E(v) \cap \kappa^{-1}[\pi_2(S)] \neq \emptyset$$

$$\forall T \in \mathcal{T}, \forall S \in \mathcal{S}, \forall v \in V(G) \setminus \{\pi_1(T)\}$$

MIP-Model:

$$z^S_T + u^{1,\pi_1(S)}_v - 1 \leq \frac{1}{2} \sum_{v_2 \in N_G(v)} t^{1,\pi_1(S)}_{\{v,v_2\}} + \sum_{e \in E(v)} a^w_e + \sum_{\substack{e \in E_G(v) \\ f \in \pi_2(S)}} y^f_e$$

$$\forall T \in \mathcal{T}, \forall S \in \mathcal{S}, \forall v \in V(G) \setminus \{\pi_1(T)\}$$

Testing Instances

- Contractions of snarks: For each snark up to 28 vertices (complete lists up to 36 vertices are downloadable) find 4 random perfect matchings and for each of them use the contracted transitioned graph as G together with $H = \text{SUD} - K_5$ as instance $\Rightarrow$ 56036 instances
- Line graphs of snarks: use the line graph of each snark as G and $H = \text{SUD} - K_5 \Rightarrow$ at the moment not practical for our algorithm since the second smallest instance already has 27 vertices and 54 edges, which is too much to solve
- Instances of the general problem: Randomly generate transitioned (4-regular) graphs $(G, \mathcal{T})$, and a completely transitioned 4-regular graphs $(H, \mathcal{S})$ with $|V(H)| = |V(G)| \cdot \alpha$ for some $\alpha < 1$

- No complete instance set tested yet, only manual (debug) tests.
- For $H = \text{SUD} - K_5$ solvable in under 12 hours for instances up to 14 vertices.