A Model for Finding Sup-Transition-Minors

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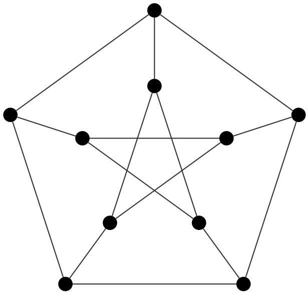
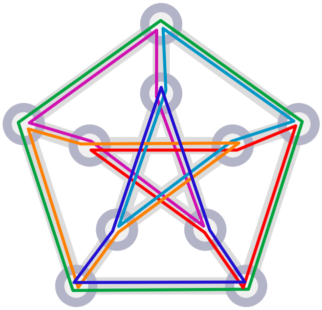
Circuit Double Cover Conjecture (CDCC)

Let G be a bridgeless undirected graph. Then, there exists a collection of circuits of G, such that each edge is contained in exactly two circuits.



$\Rightarrow$



Source: https://en.wikipedia.org/wiki/Cycle_double_cover#/media/File:Petersen_double_cover.svg

Reduction To Snarks

Definition

A *snark* is a simple, connected, bridgeless cubic undirected graph which has no 3-edge-coloring.

Theorem (Jaeger(1985))

Every minimal counter example to the CDCC must be a snark.

Corollary

If the CDCC holds on the graph class of all snarks it also holds on all bridgless graphs.

Definition

Let G be a graph. The line graph $L(G)$ is a simple graph with the edges of G as vertices. Two edges in G are adjacent in $L(G)$ if and only if they are incident in G.

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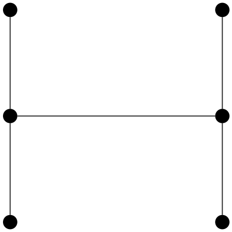

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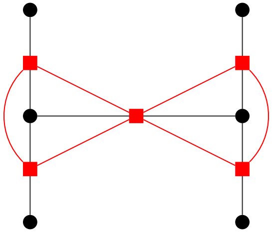
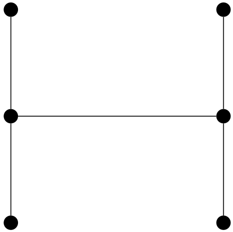
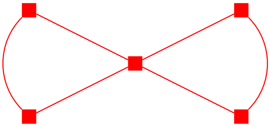


Definition (Cycle Decomposition)

Let G be a graph. A *circuit decomposition* of G is a set of circuits of G such that each edge is contained in exactly one circuit.

circuit decompositions in $L(G)$ correspond to double circuit covers in G:





Cycle Decomposition on Line Graphs

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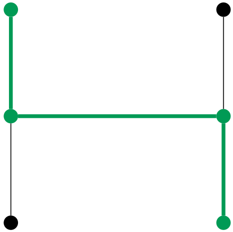

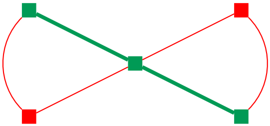

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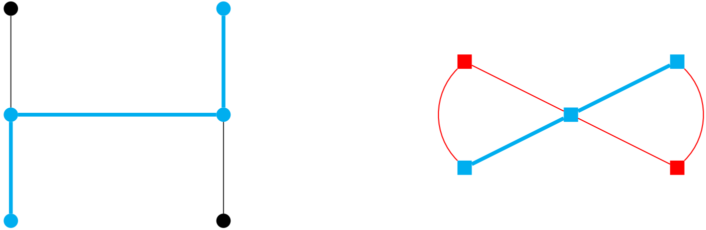

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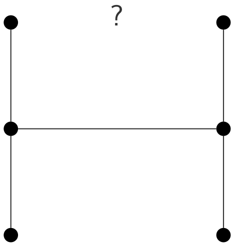
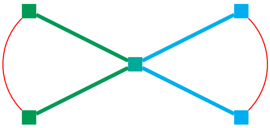
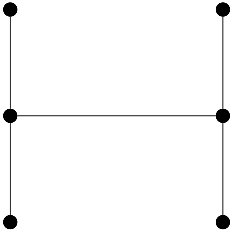


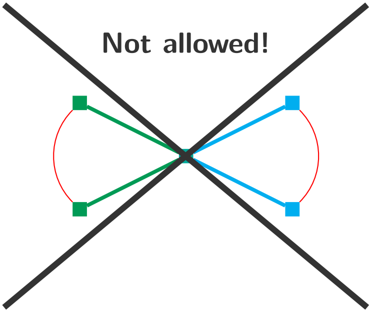


Definition (Cycle Decomposition)

Let G be a graph. A *circuit decomposition* of G is a set of circuits of G such that each edge is contained in exactly one circuit.

Some circuit decompositions in $L(G)$ correspond to double circuit covers in G:

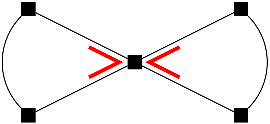




Definition (Transition)

A *transition* in a graph consists of a vertex v and two of its incident edges e_1, e_2.

Used to forbid circuits which contain the edges e_1 and e_2.



Definition (Transition System)

A *transition system* $\mathcal{T}$ is a set of transitions in a graph G. $(G, \mathcal{T})$ is called a *transitioned graph*.

Compatible Circuit Decomposition

Definition (Compatible Circuit Decomposition)

Let $(G, \mathcal{T})$ be a transitioned graph. A *compatible circuit decomposition of G* is a circuit decomposition $\mathcal{C}$ of G such that for all transitions in $\mathcal{T}$ there is no circuit in $\mathcal{C}$ which contains both edges of the transition.

Theorem

A snark G contains a circuit double cover if and only if its line graph $L(G)$ together with the transition system as described in the slides before has a compatible circuit decomposition.

Compatible Circuit Decomposition Problem

Let G be a 2-connected eulerian graph and $\mathcal{T}$ a transition system on G. Has G a compatible circuit decomposition?

Theorem (Fan and Zhang(2000))

Let $(G, \mathcal{T})$ be a 2-connectected eulerian graph associated with a transition system. If G is K_5-minor free, then $(G, \mathcal{T})$ has a compatible circuit decomposition.

Does not depend on actual transition system!

Fleischner et al. generalize this theorem by extending the definition of a minor to transitioned graphs:

Theorem (Fleischner et al.(2017))

Let $(G, \mathcal{T})$ be a 2-connectected eulerian graph associated with a transition system. If $(G, \mathcal{T})$ is SUD-K_5-minor-free, then it has a compatible circuit decomposition.

But what does SUD-K_5-minor-free mean?

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But what does SUD-K_5-minor-free mean?

Definition (Minor)

Let G be an undirected (maybe not simple) graph. A graph H which can be derived by deleting vertices, deleting edges and/or contracting edges in G is called a *minor* of G.

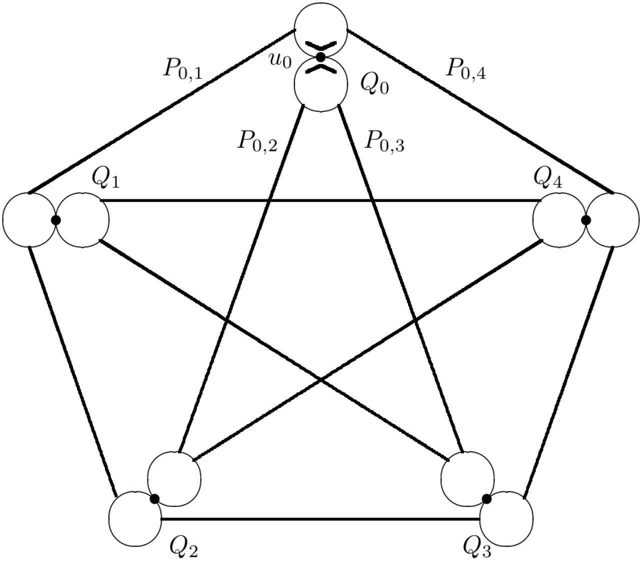
Definition (Separator)

Let G be a graph. A vertex subset U is a *separator of G separating G to G_1, G_2* if $E(G) = E(G_1) \cup E(G_2)$ and $V(G_1) \cap V(G_2) = U$ and $E(G_1) \cap E(G_2) = \emptyset$. U is a *t-separator* if $|U| = t$. We say a *separator U separating subgraphs X_1, X_2* if U is a separator of G separating G to G_1, G_2 with $X_i \subseteq G_i$ $(i = 1, 2)$.

Definition (Bad-Cut-Vertex)

Let $(G, \mathcal{T})$ be a transitioned graph. A 1-separator v separating G to G_1, G_2 is a *bad-cut-vertex* if $(v, E(v) \cap E(G_1)) \in \mathcal{T}(v)$.

Sup-un-decomposable K_5 (SUD-K_5)



Definition (Reduced Transition Minor)

Let $(G, \mathcal{T})$ be a transitioned graph and H a minor of G. Let the transition system $\mathcal{S}'$ consist of all transitions in G whose edges didn't get removed or contracted. Then $(H, \mathcal{S}')$ is called a *reduced transition minor* of $(G, \mathcal{T})$.

Definition (Sup-un-decomposable-K_5-transition-minor free)

$(G, \mathcal{T})$ is *sup-un-decomposable-K_5-transition-minor free* (or short SUD-K_5-minor free) if it does not have any eulerian reduced transition minor which is a SUD-K_5.

Two open problems:

Problem

Does there exist a snark G such that its line graph $L(G)$ has a K_5-transition minor but is SUD-K_5-minor free?

Problem

Does there exist an eulerian transitioned graph which has a compatible circuit decomposition and is not SUD-K_5-minor free?

Theorem

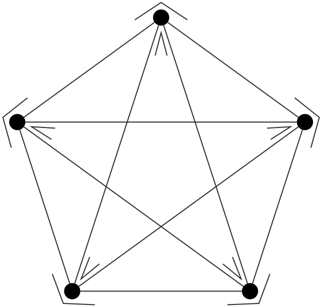
SUD-K_5-TM is solvable in polynomial time.

Proof.

Similar to the proof I presented the last time. □

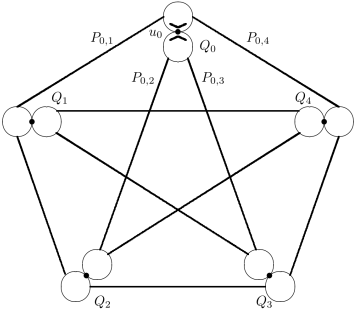
$\Rightarrow$ consider more general problem

sup-un-decomposable $K_5 \Rightarrow$ sup-$(H, \mathcal{S})$



un-decomposable K_5

$\Rightarrow$

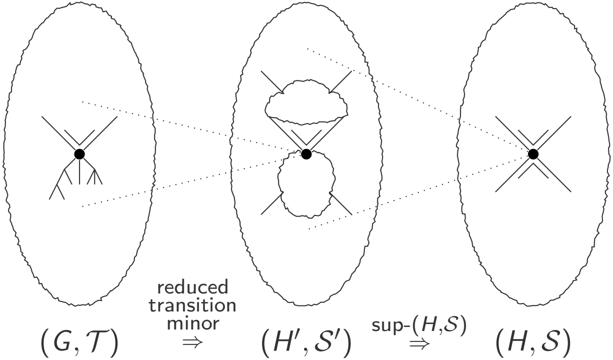


sup-un-decomposable K_5

completely transitioned 4-regular graph $(H, \mathcal{S})$ $\Rightarrow$ sup-$(H, \mathcal{S})$

Problem (Existence of sup-transition-minors (ESTM))

Given a transitioned graph $(G, \mathcal{T})$ and a completely transitioned 4-regular graph $(H, \mathcal{S})$, does there exist an Eulerian transition minor of $(G, \mathcal{T})$ which is a sup-$(H, \mathcal{S})$ graph?



Theorem

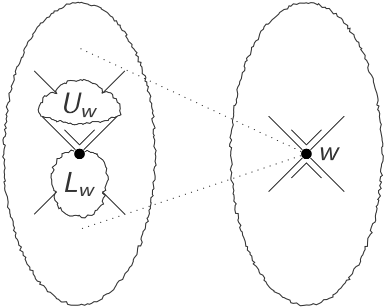
ESTM is NP*-complete.*

Proof.

- **Containment**: Use a graph $(H', \mathcal{S}')$ together with associations between G and H' and H' and H as solution representation (polynomial size). Check if $(H', \mathcal{S}')$ is a reduced transition minor of G with the given associations and if $(H', \mathcal{S}')$ is a sup-$(H, \mathcal{S})$ with the given associations in polynomial time.
- **Hardness**: Polynomial time reduction from Hamiltonian cycle problem to ESTM: Duplicate the edges of an instance G of the Hamiltonian cycle problem and add transitions between the duplicates. Apply ESTM with $(H, \mathcal{S})$ a cycle of length $|V(G)|$ and duplicated edges with transitions between the duplicates.

□

Let $(H', \mathcal{S}')$ be a sup-$(H, \mathcal{S})$.



$$(H', \mathcal{S}') \overset{\text{sup-}(H,\mathcal{S})}{\Rightarrow} (H, \mathcal{S})$$

Contract U_w and L_w as far as possible by contracting all edges and removing self loops.

$\Rightarrow$ L_w contracts to one vertex, four possibilities for U_w:

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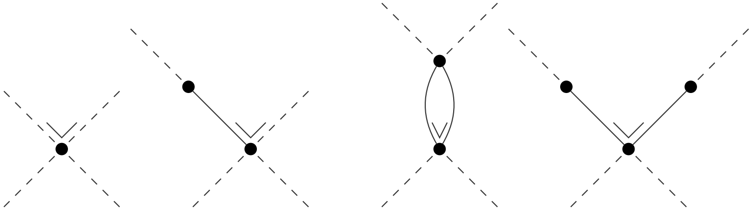

case 1 case 2 case 3 case 4

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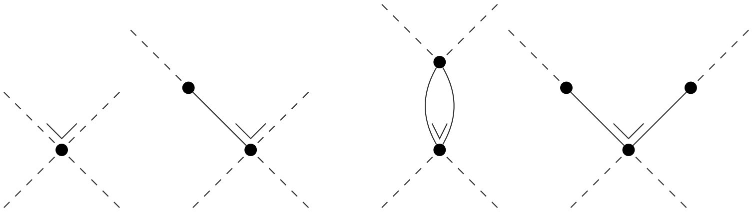

case 1 case 2 case 3 case 4

Do this for every $w \in V(H) \Rightarrow$ get a new transitioned graph $(H'', \mathcal{S}'')$. Furthermore, contract all paths between components of different vertices to one edge. Call such a graph *basic-sup*-$(H, \mathcal{S})$.

Problem (Existence of basic-sup-transition-minors (EBSTM))

Given a transitioned graph $(G, \mathcal{T})$ and a completely transitioned 4-regular graph $(H, \mathcal{S})$, does there exist a transition minor of $(G, \mathcal{T})$ which is a basic-sup-$(H, \mathcal{S})$ graph?

Theorem

ESTM and EBSTM are equivalent.

Proof.

Every basic-sup-$(H, \mathcal{S})$ graph is Eulerian and also a sup-$(H, \mathcal{S})$ graph.
On the other hand for an Eulerian sup-$(H, \mathcal{S})$ graph $(H', \mathcal{S}')$ there exists a reduced transition minor $(H'', \mathcal{S}'')$ which is a basic-sup-$(H, \mathcal{S})$ graph. $\Rightarrow$ $(H'', \mathcal{S}'')$ is also a reduced transition minor of $(G, \mathcal{T})$. □

Idea: Model EBSTM without using explicit variables for the intermediate graph $(H', \mathcal{S}')$. The model consists of

1. a partial surjective function $\iota : V(G) \nrightarrow V(H)$,
2. a partial injective and surjective function $\kappa : E(G) \nrightarrow E(H)$,
3. a partial injective function $\lambda : \mathcal{T} \nrightarrow \mathcal{S}$,
4. for each $w \in V(H)$ two simple trees C_w^1 and C_w^2 with $V(C_w^i) \subseteq V(G)$ for $i = 1, 2$.

Model Constraints

$$
\begin{aligned}
E(C_w^i) &\subseteq \mathrm{sig}_G[E(G)] && \forall w \in V(H), \forall i = 1,2 \\
\kappa(e) = f \Rightarrow \iota[\mathrm{sig}_G(e)] &= \mathrm{sig}_H(f) && \forall e \in E(G), \forall f \in E(H) \\
V(C_w^1) \cup V(C_w^2) &= \iota^{-1}(w) && \forall w \in V(H) \\
\lambda(T) = S \Rightarrow \iota(\pi_1(T)) &= \pi_1(S) && \forall T \in \mathcal{T}, \forall S \in \mathcal{S} \\
\lambda(T) = S \Rightarrow \{\pi_1(T)\} &= V(C^1_{\pi_1(S)}) \cap V(C^2_{\pi_1(S)}) && \forall T \in \mathcal{T}, \forall S \in \mathcal{S} \\
\lambda(T) = S \Rightarrow \pi_2(T) &\subseteq \kappa^{-1}[\pi_2(S)] \cup \\
&\left\{ e \in E(G)(\pi_1(T)) : \mathrm{sig}_G(e) \in E(C^1_{\pi_1(S)}) \right\} && \forall T \in \mathcal{T}, \forall S \in \mathcal{S} \\
\lambda(T) = S \Rightarrow \kappa^{-1}[\pi_2(S)] \cap E(\pi_1(T)) &\subseteq \pi_2(T) && \forall T \in \mathcal{T}, \forall S \in \mathcal{S} \\
|\lambda[\mathcal{T}] \cap \mathcal{S}(w)| &= 1 && \forall w \in V(H) \\
\lambda(T) = S \Rightarrow \kappa[\pi_2(T)] &\subseteq \pi_2(S) && \forall T \in \mathcal{T}, \forall S \in \mathcal{S} \\
\kappa(e) \in \pi_2(S) \Rightarrow \mathrm{sig}_G(e) \cap V(C^1_{\pi_1(S)}) &\neq \emptyset && \forall S \in \mathcal{S}, \forall e \in E(G) \\
\kappa(e) \in E(\pi_1(S)) \setminus \pi_2(S) \Rightarrow \mathrm{sig}_G(e) \cap V(C^2_{\pi_1(S)}) &\neq \emptyset && \forall S \in \mathcal{S}, \forall e \in E(G) \\
\lambda(T) = S \Rightarrow E_{C^1_{\pi_1(S)}}(\pi_1(T)) &\subseteq \mathrm{sig}_G[\pi_2(T)] && \forall T \in \mathcal{T}, \forall S \in \mathcal{S}
\end{aligned}
$$

Last but not least ...

$$\Big(\lambda(T) = S \wedge v \in V(C^1_{\pi_1(S)}) \wedge E_{C^1_{\pi_1(S)}}(v) = 1 \wedge$$
$$\Big(\Big|N_G(v) \cap V(C^1_{\pi_1(S)})\Big| = 1 \vee \{\pi_1(T), v\} \notin E(C^1_{\pi_1(S)})\Big)\Big) \Rightarrow$$
$$E(G)(v) \cap \kappa^{-1}[\pi_2(S)] \neq \emptyset$$
$$\forall T \in \mathcal{T}, \forall S \in \mathcal{S}, \forall v \in V(G) \setminus \{\pi_1(T)\}$$

Future Work

- Is ESTM still NP-hard if G and H are restricted to simple graphs?
- Finish formal proof that model solves EBSTM
- Implement and test MIP for model
  - Instance Generation
- Implement and test a CP
- Implement and test a SAT
- Compare the results