A Model for Checking Transition-Minors

Benedikt Klocker

Algorithms and Complexity Group
Institute of Computer Graphics and Algorithms
TU Wien

PHD Seminar, 15 May 2017

- Part of Herbert's FWF-project
- Problem was formulated during the research work of Behrooz and Herbert in the area of the cycle double cover conjecture

Definition (minor)

Let G be an undirected (maybe not simple) graph. A graph H which can be derived by deleting vertices, deleting edges and/or contracting edges in G is called a *minor* of G.

More Definitions

Definition (transition system)

Let $G = (V, E)$ be an undirected (maybe not simple) graph. A *transition system* $\mathcal{S}$ on G is a set

$$\mathcal{S} := \bigcup_{v \in V} S_v$$

where each $S_v \subseteq V \times E \times E$ is a set of triples consisting of the vertex v and an edge pair of edges which are incident to v such that each edge is at most in one triple of S_v.

Definition (exended transition system)

Let $G = (V, E)$ be an undirected (maybe not simple) graph. An *extended transition system* is simply a set $\mathcal{S} \subseteq V \times E \times E$, where for each triple $(v, e_1, e_2) \in \mathcal{S}$ the edges e_1 and e_2 are different and incident to v.

Definition (Subdivision)

Let $G = (V_G, E_G)$ be an undirected graph. A *subdivision* of G can be derived by subdividing edges of G.

Definition (Transition-Minor)

Let $G = (V_G, E_G, \mathcal{S})$ and $H = (V_H, E_H, \mathcal{T})$ be two undirected graphs with extended transition systems. We say that H is a *transition-minor* of G if the following holds.
There exists a minor H' of G which is a subdivision of H with the following property. For each transition $(v, e_1, e_2) \in \mathcal{T}$, there exists a transition $(w, f_1, f_2) \in \mathcal{S}$ such that w is a vertex in H' which gets mapped to v under the subdivision mapping and f_1 and f_2 are edges in H' which are part of two paths P_1 and P_2 which get mapped to the edges e_1 and e_2 under the subdivision mapping.

Problem — Checking transition-minors

Let $G = (V_G, E_G, \mathcal{S})$ and $H = (V_H, E_H, \mathcal{T})$ be two undirected graphs with extended transition systems.
The decision problem is now to decide if H is a transition-minor of G or not.

Towards a MIP model — $\tilde{H}$

- Paths from H' to H can be reduced to at most two edges
- To get H' decide for each edge in H if to divide it or not
- For each edge in H add a vertex and two new edges
  - Either the new vertex and edges can be used or the old edge, but not both
  - Call new graph $\tilde{H}$

Towards a MIP model — Variables

- $x_{g,h}$ for each $g \in V_G, h \in V_{\tilde{H}}$
- $y_{e,f}$ for each $e \in E_G, f \in E_{\tilde{H}}$
- $t_{a,b,h}$ for each $a \in V_G, b \in N_G(a), a < b, h \in V_{\tilde{H}}$
- $f_{a,b,h}$ for each $a \in V_G, b \in N_G(a), h \in V_{\tilde{H}}$
- v_h for each $h \in V_{\tilde{H}}$
- d_f for each $f \in E_{\tilde{H}}$
- o_{g_1,g_2,h_1,h_2} for each $g_1 g_2 \in E_G, g_1 < g_2, h_1 \in V_{\tilde{H}}, h_2 \in N_{\tilde{H}}(h_1)$
- $r_{t,s}$ for each $t \in \mathcal{T}, s \in \mathcal{S}$

MIP model — Part 1

$$\sum_{h \in V_{\tilde{H}}} x_{g,h} \leq 1 \qquad \forall g \in V_G \tag{1}$$

$$\sum_{f \in E_{\tilde{H}}} y_{e,f} \leq 1 \qquad \forall e \in E_G \tag{2}$$

$$t_{a,b,h} \leq x_{a,h} \qquad \forall a \in V_G, \forall b \in N_G(a), a < b, \forall h \in V_{\tilde{H}} \tag{3}$$

$$t_{a,b,h} \leq x_{b,h} \qquad \forall a \in V_G, \forall b \in N_G(a), a < b, \forall h \in V_{\tilde{H}} \tag{4}$$

$$\sum_{a \in V_G, b \in N_G(a), a<b} t_{a,b,h} = \sum_{g \in V_G} x_{g,h} - v_h \qquad \forall h \in V_{\tilde{H}} \tag{5}$$

$$f_{a,b,h} + f_{b,a,h} \geq t_{a,b,h} \qquad \forall h \in V_{\tilde{H}}, \forall a \in V_G, \forall b \in N_G(a), a < b \tag{6}$$

$$\sum_{a \in N_G(b)} f_{a,b,h} \leq 1 - \frac{1}{|V_G|} \qquad \forall b \in V_G, \forall h \in V_{\tilde{H}} \tag{7}$$

$$v_h = 1 \qquad \forall h \in V_H \tag{8}$$

$$\sum_{g \in V_G} x_{g,h} \geq v_h \qquad \forall h \in V_{\tilde{H}} \tag{9}$$

$$x_{g,h} \leq v_h \qquad \forall g \in V_G, h \in V_{\tilde{H}} \tag{10}$$

MIP model — Part 2

$$\sum_{e \in E_G} y_{e,f} = d_f \qquad \forall f \in E_{\tilde{H}} \qquad (11)$$

$$d_f \leq v_{h_1} \qquad \forall h_1 \in V_{\tilde{H}}, \forall f \sim h_1 h_2 \in E_{\tilde{H}} \qquad (12)$$

$$d_{e^f,h_1} + d_{e^f,h_2} = 2v_{h^f} \qquad \forall f \sim h_1 h_2 \in E_H \qquad (13)$$

$$d_f + \frac{1}{2}\left(d_{e^f,h_1} + d_{e^f,h_2}\right) = 1 \qquad \forall f \sim h_1 h_2 \in E_H \qquad (14)$$

$$\frac{1}{2}(x_{g_1,h_1} + x_{g_2,h_2}) \geq o_{g_1,g_2,h_1,h_2} \qquad \forall g_1 \in V_G, \forall g_2 \in N_G(g_1), g_1 < g_2, \forall h_1 \in \tilde{H}, \forall h_2 \in N_{\tilde{H}}(h_1) \qquad (15)$$

$$o_{g_1,g_2,h_1,h_2} + o_{g_1,g_2,h_2,h_1} \geq y_{e,f} \qquad \forall e \sim g_1 g_2 \in E_G, g_1 < g_2, \forall f \sim h_1 h_2 \in E_{\tilde{H}} \qquad (16)$$

$$\sum_{t \in \mathcal{T}} r_{t,s} = 1 \qquad \forall s \in \mathcal{S} \qquad (17)$$

$$\begin{aligned} &y_{e_1,f_1} + y_{e_1,e^{f_1},h} \\ &+ y_{e_1,f_2} + y_{e_1,e^{f_2},h} \end{aligned} \geq r_{t,s} \qquad \forall t = (g, e_1, e_2) \in \mathcal{T}, \forall s = (h, f_1, f_2) \in \mathcal{S} \qquad (18)$$

$$\begin{aligned} &y_{e_2,f_1} + y_{e_2,e^{f_1},h} \\ &+ y_{e_2,f_2} + y_{e_2,e^{f_2},h} \end{aligned} \geq r_{t,s} \qquad \forall t = (g, e_1, e_2) \in \mathcal{T}, \forall s = (h, f_1, f_2) \in \mathcal{S} \qquad (19)$$

$$x_{g,h} \geq r_{t,s} \qquad \forall t = (g, e_1, e_2) \in \mathcal{T}, \forall s = (h, f_1, f_2) \in \mathcal{S} \qquad (20)$$

Theorem

Checking if a graph H is a transition-minor of a graph G is fixed parameter tractable by the size of H.

Proof is motivated by the proof of fixed parameter tractability of checking if H is a minor of G.[RS95]

Neil Robertson and Paul D. Seymour.
Graph minors. XIII. The disjoint paths problem.
Journal of combinatorial theory, Series B, 63(1):65–110, 1995.

Proofing fixed parameter tractability — cont.

Definition (Disjoin Paths Problem (DP))

Given a graph G and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ of vertices of G. Do there exist paths $P_1, \cdots, P_k$ of G mutually vertex-disjoint, such that P_i joins s_i and t_i?

Theorem (Robertson and Seymor)

If k is fixed there is a polynomial algorithm to solve DP.

Corollary

Checking if there is a subgraph of G which is a subdivision of H is solvable in polynomial time for fixed H.

If we find a polynomial algorithm for a fixed H we also have one for all H with fixed or bounded size.

Let H be fixed!

Only fixed (constant) many possibilities for $H' \Leftrightarrow$

Let H' be fixed!

Adapt extended transition system $\mathcal{T}$ of H to H'.

Determine finite list of graphs H_k such that G contains H' as minor if and only if there exists a k such that G contains a subdivision of H_k as a subgraph. $\Rightarrow$ blackboard

Let H_k be fixed!

Adapt extended transition system $\mathcal{T}$ of H' to H_k if possible (otherwise discard H_k).

Proofing fixed parameter tractability — cont.

For a given G there are only a polynomial amount of possibilities how to map the vertices of H_k to the vertices of G.

Let the mapping between the vertices of H_k and the vertices of G be fixed!

For a given transition in H_k there is only a polynomial amount of possibilities how to map the transition to two edges in G.

Let the mapping between the transitions in H_k and the edges in G be fixed!

For each edge in H' we search a path in G. If the edge is part of a transition the first edge (on one or maybe on both sides) is already given.

We remain with the problem of searching vertex disjoint paths for each edge in H_k. This is solvable in polynomial time since H_k is fixed and therefore the number of edges in H_k are fixed.