Finding Smooth Graphs with Small Independence Numbers*

Benedikt Klocker, Herbert Fleischner, Günther R. Raidl

Institute of Computer Graphics and Algorithms, TU Wien,
Favoritenstraße 9–11/186-1, 1040 Vienna, Austria
{klocker,fleischner,raidl}@ac.tuwien.ac.at

Abstract. In this paper we formulate an algorithm for finding smooth graphs with small independence numbers. To this end we formalize a family of satisfaction problems and propose a branch-and-bound-based approach for solving them. Strong bounds are obtained by exploiting graph-theoretic aspects including new results obtained in cooperation with leading graph theorists. Based on a partial solution we derive a lower bound by computing an independent set on a partial graph and finding a lower bound on the size of possible extensions.
The algorithm is used to test conjectured lower bounds on the independence numbers of smooth graphs and some subclasses of smooth graphs. In particular for the whole class of smooth graphs we test the lower bound of $2n/7$ for all smooth graphs with at least $n \geq 12$ vertices and can proof the correctness for all $12 \leq n \leq 24$. Furthermore, we apply the algorithm on different subclasses, such as all triangle free smooth graphs.

Keywords: branch and bound, smooth graphs, combinatorial optimization

1 Introduction

In graph theory independent sets are well studied objects and the independence number of a graph is a central characteristic which is strongly related to many important properties. One natural research subject is to find lower and upper bounds for the independence number for general graphs, see for example [6], or for specific subclasses of graphs, see for example [12].

In this paper we focus on the independence number of smooth graphs, a subclass of 4-regular Hamiltonian graphs. For a complete definition of smooth graphs see Section 2. This work is motivated by the works of Fleischner, Sabidussi and Sarvanov [2,3], three renowned graph theorists, who already studied smooth graphs and their independence number in depth from a graph-theoretic perspective.

We are interested in lower bounds on the independence number of smooth graphs. Sarvanov conjectured that every smooth graph G with $n > 11$ vertices

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has independence number $\alpha(G) \geq \frac{2}{7}n$ [11]. The main goal of this work is to design an algorithm which can check lower bounds on the independence number for smooth graphs and either prove them for all graphs with a given number of vertices or disprove them by finding a graph with a smaller independence number.

By using Brooks' Theorem [1] we get a lower bound on the independence number for all 4-regular graphs. It states that every 4-regular graph with n vertices that is not the K_5 can be colored with 4 colors which implies that it has an independent set of size at least $n/4$. This property, together with the fact that we only consider graphs containing a Hamiltonian cycle and therefore having an independence number of at most $n/2$, give us an interval of interesting possible lower bounds.

We will describe a branch-and-bound algorithm which heavily depends on the graph-theoretic results and bounds to search through the space of possible graphs in an efficient way [10]. The main idea is to use a heuristic to compute a large independent set together with the graph-theoretic bounds to detect infeasible subproblems as early as possible. For complete solutions we use an integer linear programming (ILP) model to compute their independence number and to check if they are feasible.

In the next section we will formally define smooth graphs and state the problem framework. In Section 4 we will infer some useful bounds and properties using already existing graph-theoretic results, and in Section 5 we will describe how to use those bounds and properties to compute a usually very tight bound on the independence number of a partial solution in order to detect infeasibility as early as possible. In Section 6 we will present some computational results for four different problem variants. Finally, we will conclude with Section 7 and propose promising further work.

2 Problem Formulation

In the context of this paper we only consider loopless undirected graphs, which may contain multiple edges, and just write graph for this type of graphs. A graph is called *r-regular* if every vertex has degree r. We are interested in 4-regular Hamiltonian graphs $G = (V, E)$, in which a Hamiltonian cycle $H \subseteq E$ exists. If we consider the graph $G \setminus H$ after removing the cycle H we get a 2-regular graph which consists of a set of cycles. We call the cycles of $G \setminus H$ the *inner cycles* of G. Such a graph is called *smooth* if the inner cycles are "non-selfcrossing" in the sense that the cyclic order of its vertices agrees with their cyclic order of H. An example for a smooth graph is given in Figure 1.

The *independence number* of a graph is the size of its largest independent set. Based on Sarvanov's conjecture [11] we formulate the following problem. Given $n \in \mathbb{N}$ as input, does there exist a smooth graph with n vertices and independence number smaller than $\frac{2}{7}n$? This problem can be generalized to the following family of problems. Given $n \in \mathbb{N}$ as input, does there exist a smooth graph with n vertices that satisfies some properties $\mathcal{P}$ and has independence number smaller

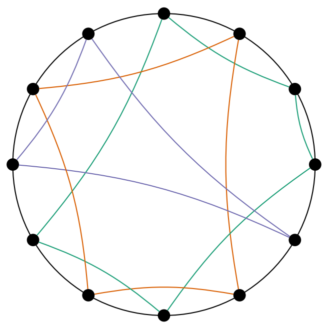

Fig. 1: Smooth graph with twelve vertices and three inner cycles in different colors

than qn for some given factor $q \in \left(\frac{1}{4}, \frac{1}{2}\right]$? We call this problem *Existence of Smooth Graphs with Small Independence Number* or short ESSI$(q, \mathcal{P})$.

3 Algorithmic Approach

In this section we present a branch-and-bound approach that solves ESSI$(q, \mathcal{P})$, i.e. it checks for a given $n \in \mathbb{N}$ if there exists a smooth graph with n vertices and independence number smaller than qn that satisfies the conditions $\mathcal{P}$. The conditions of $\mathcal{P}$ can get added to the branch-and-bound approach in a problem-specific manner.

3.1 Solution Representation

If we assume that the Hamiltonian cycle and therefore the order of the vertices in the Hamiltonian cycle is given, every inner cycle of a smooth graph is already uniquely determined if we only know the set of its vertices. W.l.o.g. we assume the vertex set $V = \{1, \ldots, n\}$ to be ordered so that the Hamiltonian cycle $\{\{1, 2\}, \{2, 3\}, \ldots, \{1, n\}\}$ is fixed. Therefore, we only have to partition the vertex set $\{1, \ldots, n\}$ into sets of size at least three and the result represents a smooth graph. For the rest of the algorithmic description section we will use a partitioning of the ordered vertex set $\{1, \ldots, n\}$ into sets of size at least three as a solution representation.

3.2 Core Algorithm

The core algorithm is based on the branch-and-bound principle. The branching is done by assigning the next not yet assigned vertex in the order of the Hamiltonian cycle to an already existing partition or to a new partition. The start solution is the solution where no vertex is assigned. After assigning a vertex to a partition we check if the resulting partial solution satisfies all bounds and if there is a theoretical possibility to complete it to a solution that satisfies the wanted conditions. We call a partial solution that fails this check an infeasible partial solution. If the current partial solution is infeasible, we can cut off this branch and continue with the next partial solution. The infeasibility check of partial solutions is described in more detail in Section 5.

Whenever the branching reaches a complete solution, where all vertices are assigned to partitions, we compute its independence number and check the conditions $\mathcal{P}$. Note that computing the independence number is NP-hard for the class of smooth graphs [2]. We compute it by solving the integer linear program

$$\max\left\{\sum_{v\in V} x_v \middle| x_v \in \{0,1\}\ \forall v \in V \wedge x_v + x_w \leq 1\ \forall\{v,w\} \in E\right\}.$$

As search strategy we use depth first search. Although for searching through the whole tree in order to obtain all feasible graphs, the search strategy is irrelevant since we are not reusing information of found solutions, it may be relevant for finding a feasible solution as fast as possible.

4 Bounds and Other Useful Properties

To reduce the search space for our problem we first derive some bounds and other properties for smooth graphs that may have an independence number smaller than qn. We will mainly use the results of Fleischner, Sabidussi and Sarvanov to infer bounds and other properties [2,3]. Those will then be useful for checking infeasibility and recognizing infeasible partial solutions as early as possible.

We consider the problem ESSI$(q,\mathcal{P})$ and we assume that the satisfaction properties $\mathcal{P}$ and the factor q are fixed. For the rest of this section we will assume that G^* is a smooth graph with n vertices that satisfies the properties $\mathcal{P}$ and has independence number $\alpha(G^*) < qn$, i.e. G^* is a solution to the problem ESSI$(q,\mathcal{P})$. Let r^* be the number of inner cycles of G^*.

Fleischner and Sarvanov proved in [3] the following theorem.

Theorem 1. *Let G be a smooth graph with n vertices and r the number of inner cycles. Then the following holds.*

$$\alpha(G) \geq \frac{n-r}{3} \tag{1}$$

We use this theorem to compute a lower bound of r^*.

Corollary 1. *For G^* and r^* the following holds.*

$$r^* \geq n - 3\lceil qn \rceil + 3 \tag{2}$$

Proof. Since the independence number $\alpha(G^*)$ is integral we get from (1) that $\alpha(G^*) \geq \left\lceil \frac{n-r^*}{3} \right\rceil$.

$$\alpha(G^*) < qn \Rightarrow \left\lceil \frac{n-r^*}{3} \right\rceil < qn \Leftrightarrow \left\lceil \frac{n-r^*}{3} \right\rceil \leq \lceil qn \rceil - 1$$
$$\Leftrightarrow \frac{n-r^*}{3} \leq \lceil qn \rceil - 1 \Leftrightarrow r^* \geq n - 3\lceil qn \rceil + 3$$

Inequality (1) can be strengthened if we exclude one special graph, which we call $G^{(2)}$. $G^{(2)}$ is defined for even n and is the unique simple smooth graph with only two inner cycles. $G^{(2)}$ is unique since the only possibility to being simple and having only two inner cycles is if all even vertices are in one inner cycle and all odd vertices are in another inner cycle. By excluding $G^{(2)}$ Fleischner and Sarvanov [3] proved the following stronger inequality.

Theorem 2. *Let G be a smooth graph with n vertices that is not isomorphic to $G^{(2)}$ and let r be the number of inner cycles. Then the following holds.*

$$\alpha \geq \frac{n - r + 1}{3} \tag{3}$$

Fleischner and Sarvanov stated this theorem with another equivalent condition. They proved Theorem 2 first for multigraphs and then showed that it also holds for simple graphs that have three consecutive vertices in different inner cycles. Putting this two conditions together we get that two consecutive vertices lie in different cycles, since the graph must be simple. Therefore, if three consecutive vertices never lie in three different inner cycles it must hold that vertex k and vertex $k + 2$ always lie in the same inner cycle. This further implies that all even vertices form one inner cycle and so do all odd vertices. Therefore, the only graph that does not satisfy both conditions is $G^{(2)}$.

As before we can use this theorem to compute a stronger lower bound for r^*.

Corollary 2. *If G^* is not isomorphic to $G^{(2)}$ the following holds.*

$$r^* \geq n - 3\lceil qn \rceil + 4 \tag{4}$$

Proof. The proof is analogous to the proof of Corollary 1 by replacing (1) with (3).

Another useful theorem is the following from [4].

Theorem 3 (Cycle-Plus-Triangles Theorem). *Let G be a smooth graph where all inner cycles are triangles, i.e. have length three. Then G is 3-colorable.*

In [3] the following corollary of the cycle-plus-triangle theorem is stated.

Corollary 3. *Let G be a smooth graph with n vertices where all inner cycles have length smaller than or equal to four. Let r be the number of inner cycles and r_3 be the number of inner cycles of length three. Then the following holds.*

$$\alpha(G) \geq \frac{n - (r - r_3)}{3} \tag{5}$$

Let for the following corollary r_3^* be the number of inner cycles of length three of G^*.

Corollary 4. *G^* has either an inner cycle with length greater than four or the following holds.*

$$r^* \geq n - 3\lceil qn \rceil + 3 + r_3^* \tag{6}$$

Proof. The proof is analogue to the proof of Corollary 1 by replacing (1) with (5).

Until now, we only provided lower bounds for r^*, but by using Theorem 3 we can also compute the following upper bound.

Corollary 5. *Let G^* and r^* be as described at the beginning of the section. Then $r^* < qn$ holds.*

Proof. We remove vertices for each inner cycle with length greater than three until every inner cycle has length three. For each removed vertex we connect the two neighbors in the inner cycle and the two neighbors in the Hamiltonian cycle. The result is a smooth graph G' with $n' = 3r^*$ where all inner cycles are triangles. Removing vertices and adding edges can only decrease the independence number since every independent set in the transformed graph is also an independent set in the original graph. Therefore, we know $\alpha(G') \leq \alpha(G^*)$ and we can conclude the proof using Theorem 3 as follows.

$$qn > \alpha(G) \geq \alpha(G') = \frac{n'}{3} = r^*$$

5 Checking Infeasibility

To check if a given partial solution is infeasible, we use the bounds and properties from Section 4, and compute an as tight lower bound for the independence number of any completion of the partial solution as possible. Let S be a partial solution, i.e. S is a partitioning of a subset of the vertices of G.

To be able to use the lower bound from Corollary 2 for r, we need to exclude the graph $G^{(2)}$. To do this we check the conditions $\mathcal{P}$ for the unique graph $G^{(2)}$ and compute the independence number of it before we execute the branch and bound algorithm. Let r^{LB} be the lower bound for the number of inner cycles r which we get from (4). Furthermore, let $r^{\mathrm{UB}} = \lfloor qn \rfloor$ be the upper bound for the number of inner cycles r which we get from Corollary 5.

If $|S| > r^{\mathrm{UB}}$ the given partial solution is infeasible. Let $k = \sum_{P \in S} |P|$ be the number of fixed vertices in S and

$$\ell := \sum_{P \in S : |P| < 3} 3 - |P|$$

the number of vertices that are at least needed to complete all partitions of S. Furthermore, let $R_i := |\{P \in S : |P| \geq i\}|$ be the number of partitions in S with at least i vertices. Now we can show the following theorem.

Theorem 4. *Let S be a partial solution and r^{UB}, r^{LB}, k, ℓ and $(R_i)_{i \geq 3}$ be as described above. Furthermore, let*

$$c := \max(3, \lfloor 1/q \rfloor + 1, \min\{n \in \mathbb{N} : q(n-c) \notin \mathbb{N}, 2 - qc \geq \lceil qn \rceil - qn\}).$$

With that we can define the following value.

$$m := \max\left[0, \min\left(5 - \max\left(3, \max_{P\in S}|P|\right), n - 3\lceil qn\rceil + 3 - R_4\right)\right].$$

If there exists a feasible completion of S the following holds.

$$k + \ell + m + 3\max(0, r^{\mathrm{LB}} - |S|) \leq n \tag{7}$$

Proof. First of all every completion of S must complete all partitions $P \in S$ with $|P| < 3$, which implies that at least ℓ vertices must be added to the k existing ones. If $|S| < r^{\mathrm{LB}}$ we know that a completion of S with the desired properties must have at least r^{LB} different partitions and therefore $3(r^{\mathrm{LB}} - |S|)$ additional vertices must be added.

By Corollary 4 either the completion must contain a partition of size at least five or (6) must hold. To get a partition of size five we can add $\max(0, 5 - \max(3, \max_{P\in S}|P|))$ additional vertices to the largest partition. Otherwise, to satisfy (6) we need to have $n-3\lceil qn\rceil+3$ many partitions of size at least four. We have at the moment R_4 many inner cycles with length at least four and therefore we need $\max(0, n-3\lceil qn\rceil+3-R_4)$ many additional vertices to get enough inner cycles of length four.

Plugging everything together and considering that in total we have n vertices we get (7).

If (7) is violated we know that S is infeasible.

We covered now the cases where we can determine that S is infeasible without even computing an independent set. Now we compute an independent set on the partial graph of S, which is the graph induced by all fixed vertices $V_S = \bigcup_{P\in S} P$. By the branching rules we know that $V_S = \{1, \dots, k\}$ for some $k \leq n$.

The partial graph $G_S = (V_S, E_S)$ consists of the fixed vertices and all possible edges between those vertices. Since we do not know if a partition $P \in S$ with $|P| \geq 3$ is already complete or not, we also do not know if the vertices $\min(P)$ and $\max(P)$ are connected or not. We want that every independent set in G_S is also an independent set in G and therefore we have to add those edges to E_S.

$$E_S := \{\{a, b\} \in E_G : a, b \in V_S\} \cup \{\{\min(P), \max(P)\} : P \in S, |P| \geq 3\}$$

To compute an independent set on G_S we use the minimum-degree greedy algorithm [8]. In each iteration this algorithm adds a vertex with the minimum degree to the independent set and removes the vertex and all its neighbors from the graph. Besides good approximation ratios the greedy algorithm is also fast, it can be implemented in $\mathcal{O}(n)$ time.

Let I be the independent set found by the minimum-degree greedy on the graph G_S. Our goal is now to find a good lower bound on how many additional vertices can be added to I in each completion of S.

Theorem 5. *Let S be a partial solution and I an independent set on the graph G_S. Furthermore, let k, ℓ, m and r^{UB} be as described above and let*

$$V_I^{\max} := |I \cap 1, k| + |I \cap \{\min P : P \in S\}| + |I \cap \{\max P : P \in S\}|.$$

Then there exists for every completion G of S an independent set I_G with

$$|I_G| \geq |I| + \frac{\left\lfloor n - k - V_I^{\max} - \min\left(r^{\mathrm{UB}} - |S|, \frac{n-k-\ell-m}{3}\right)\right\rfloor}{3}. \tag{8}$$

Proof. Let G be an arbitrary completion of S. First of all we upper bound the number of inner cycles r of G. Clearly we know $r \leq r^{\mathrm{UB}}$. Furthermore, by using the same reduction as in the proof of Theorem 4 we get

$$k + \ell + m + 3\max(0, r - |S|) \leq n \Rightarrow r \leq \frac{n - k - \ell - m}{3} + |S|. \tag{9}$$

Now we can compute a lower bound on the independence number of G. Let $V_I \subseteq V_G \setminus V_S$ be the set of all vertices in G that are not in V_S and are adjacent to one of the vertices in I. The vertices of V_I are either connected to I via the Hamiltonian cycle, which is only possible if the vertex 1 or the vertex k is in I, or via an inner cycle, which is only possible for the end vertices $\min P$ and $\max P$ of an inner cycle $P \in S$. Therefore we can bound the size of V_I by

$$|V_I| \leq |I \cap \{1, k\}| + |I \cap (\{\min P : P \in S\}| + |I \cap \{\max P : P \in S\})| = V_I^{\max}.$$

We consider now the residual graph G^{rem} after removing the vertices V_S and V_I from G, which is a graph with $n - k - |V_I|$ vertices. We complete the independent set I by an algorithm that is similar to the minimum-degree greedy algorithm. Instead of always taking a vertex with the minimum degree we take the minimum remaining vertex, i.e. the first vertex in the order of the Hamiltonian cycle that is not adjacent to any vertex in the independent set so far.

Let $I_0 = I$ be the start set and I_i the set after iteration i and let v_i be the vertex added in iteration i. Furthermore, let P_i be the partition in G of the vertex v_i and G_i be the remaining graph in iteration i, $G_0 = G^{\mathrm{rem}}$. We distinguish two cases, the case if $v_i = \min(P_i)$ is the first vertex in P_i or not. Since we selected v_i as the first vertex in the order of the Hamiltonian cycle which is still in G_{i-1} we know that the preceding neighbor of v_i in the Hamiltonian cycle is not in G_{i-1} and therefore we obtain that the degree $d_{G_{i-1}}(v_i)$ of v_i in G_{i-1} is smaller than or equal to three. If $v_i \neq \min(P_i)$ we also know that one neighbor in the inner cycle containing v_i is a predecessor of v_i in the Hamiltonian cycle and therefore it is also not in G_{i-1}, which gives us $d_{G_{i-1}}(v_i) \leq 2$. Summing up over all iterations we get

$$\begin{aligned} n - k - |V_I| &= \sum_{i=1}^{x} d_{G_{i-1}}(v_i) + 1 \leq x + 3(r - |S|) + 2(x - r + |S|) \\ \Rightarrow x &\geq \frac{n - k - |V_I| - r + |S|}{3} \geq \frac{n - k - |V_I| - \min\left(r^{\mathrm{UB}} - |S|, \frac{n-k-\ell-m}{3}\right)}{3}. \end{aligned}$$

In total, we constructed a new independent set I_G with $|I| + x$ elements and therefore (8) holds.

If $\mathcal{P}$ is not empty we can calculate problem specific bounds for those constraints and check them. To summarize this section Algorithm 1 describes the whole procedure for checking infeasibility.

Algorithm 1 Checking Infeasibility

INPUT: n, q, $\mathcal{P}$ and a partial solution S
Compute r^{LB}, r^{UB}, k, ℓ, m
if $|S| > r^{\text{UB}}$ **then**
  return *infeasible*
end if
if (7) is not satisfied **then**
  return *infeasible*
end if
Construct G_S and apply minimum-degree greedy to get independent set I
Compute $V_I^{\max}$
if $|I| + \frac{\left[n-k-V_I^{\max}-\min\left(r^{\text{UB}}-|S|,\frac{n-k-\ell-m}{3}\right)\right]}{3} \geq qn$ **then**
  return *infeasible*
end if
if Problem specific bound check for $\mathcal{P}$ fails **then**
  return *infeasible*
end if
return *possibly feasible*

5.1 Symmetry Breaking

Until now the branch and bound procedure will consider many isomorphic graphs, such as all rotations alongside the Hamiltonian cycle and their reversals. In this section we will describe how we break those symmetries.

To this end we define the gap sequence of a complete solution. Let S be a complete solution, i.e., a partitioning of the vertex set $V = \{1, \ldots, n\}$. Let $P_i \in S$ be the partition of vertex i and let g_i be the gap between vertex i and its successor j in the partition P_i, i.e., let $j = \min\{j \in P_i : j > i\}$ if this set is not empty or $j = \min\{j \in P_i : j < i\}$ otherwise and $g_i = j - i$ if $j > i$ or $j + n - i$ otherwise. We call the sequence $(g_i)_{i=1}^n$ the gap sequence of S.

If two S have the same gap sequence they are not only isomorphic but also exactly the same according to the vertex labeling. We break those symmetries by ensuring that the gap sequence is minimal according to the lexicographical order under all rotations alongside the Hamiltonian cycle and their reversals. Be aware that rotating alongside the Hamiltonian cycle simply means shifting the gap sequence, but reversing the Hamiltonian cycle is a non-trivial change in the gap sequence.

We can compute the gap sequence not only for complete solutions but also for partial solutions. In some cases the next gap is not yet known and instead of calculating a gap we can calculate a lower bound and an upper bound for the gap. With the lower and upper bounds we can check if there is a rotation that always leads to a smaller gap sequence. We can also compute lower and upper bounds for the reversed gap sequence and also check if reversing leads to a smaller gap sequence.

If we found a rotation or a reversed rotation that always leads to a smaller gap sequence, we can fathom the current branch and continue with the next one. The motivations behind the choices of $\mathcal{P}$ are explained subsequently.

6 Computational Results

In this section we will present computational results for instances to four different problems. Our algorithm is implemented in C++ and compiled with g++ 4.8.4. To solve the ILP model for finding a maximum independent set we used Gurobi 7.0.1 [7]. All tests were performed on a single core of an Intel Xeon E5540 processor with 2.53 GHz and 2 GB RAM.

We consider four different variants of the problem. The first and original variant is with $q_1 = \frac{2}{7}$ and with an empty constraint set $\mathcal{P}_1 = \emptyset$. The second problem is also with $q_2 = \frac{2}{7}$ but with the additional constraint that all inner cycles have length at most four, i.e. $\mathcal{P}_2 = \{(R_5 = 0)\}$. The third problem is with $q_3 = \frac{5}{16}$ and $\mathcal{P}_3 = \{$(all inner cycles have length 4)$\}$. The fourth problem is with $q_4 = 0.334$ and $\mathcal{P}_4 = \{$(G contains no triangles)$\}$.

6.1 Problem 1

We tested the implementation for $n \in \{6, \dots, 29\}$. The algorithm found for $n = 8$ one feasible solution and $n = 11$ two feasible solutions. For all larger n it could not find any feasible solutions. Furthermore, the algorithm was able to finish the branch-and-bound search for all $n \le 24$, which proves that for $n = 8$ and $n = 11$ the found feasible solutions are the only ones and for all other $n \le 24$ there does not exist any feasible solution. For $n > 24$ it could not finish the search within 5,000,000 seconds.

The interesting values of n are the ones where $2n/7$ is only a little bit larger than $\lfloor 2n/7 \rfloor$, since then it may be easier to find a graph with independence number $\lfloor 2n/7 \rfloor$. Therefore, we are especially interested in the values $n \equiv 1 \pmod 7$ and $n \equiv 4 \pmod 7$. Table 1 summarizes the results and running times for those values and compares them with the results of Problem 2. Column $t[s]$ shows the

	Problem 1		Problem 2	
n^{**}	$t[s]$	candidates	$t[s]$	candidates
8	0	1	0	1
11	0	3	0	1
15	0.5	5	0.2	0
18	94.2	2,298	32.8	259
22	25,443	5,795	5,047	145
25	> 5,000,000	> 330,000	4,868,324	160,556
29	> 5,000,000	> 1,463	> 5,000,000	> 60,713

Table 1: Results for selected values of n for Problem 1 and Problem 2

run time in seconds and column *candidates* the number of complete solutions that got checked by the ILP solver.

6.2 Problem 2

Problem 2 is a more restricted variant of Problem 1 and was tested to check if the restriction helps speeding up the search. Especially the bound corresponding to the value m can be improved through this restriction. We tested again all inputs $n \in \{6, \ldots, 29\}$. For $n = 8$ and $n = 11$ the algorithms found one solution, the second solution of $n = 11$ contains an inner cycle of length five. For all larger n it also could not find any feasible solution.

Through the speedup compared to Problem 1 the algorithm was able to finish the search for all $n \leq 28$ and therefore proves for all $11 < n \leq 28$ that there does not exist a feasible solution. For $n = 29$ it could not finish the search within 5,000,000 seconds. Table 1 summarizes the results and running times and compares them with Problem 1.

6.3 Problem 3

Fleischner conjectured that smooth graphs only containing inner cycles of length four with at least 12 vertices have independence number at least $5n/16$ [5]. This was the motivation to consider this problem with $q_3 = \frac{5}{16}$. Our algorithm was able to disprove the conjecture by finding 36 smooth graphs with 20 vertices and independence number $6 < qn = 20 \cdot 5/16$ containing only inner cycles of length four. Furthermore, it could find feasible graphs with 24 vertices and independence number $7 < qn = 24 \cdot 5/16$.

Clearly we only have to consider values for n with $n \equiv 0 \pmod 4$. For $n = 8$ we found the same graph as in Problem 1 and 2, for $n = 12$ and $n = 16$ the algorithm could prove that there are no feasible graphs. For $n = 20$ it could finish the search and prove that the found 36 feasible graphs are the only ones but for $n = 24$ the search did not finish in under 5,000,000 seconds.

The run time for $n = 20$ was 11 minutes and for $n = 24$ it was 11 hours. For $n = 28$ the algorithm could not finish in reasonable time and also did not find a feasible solution in the first 5,000,000 seconds run time.

6.4 Problem 4

For triangle-free smooth graphs it is proven that $4n/13$ is a valid lower bound for the independence number [9]. This raises the question if it is possible to reach this lower bound or if there exists a stronger lower bound. We use $q = 0.334$ since we want to check if there exist triangle-free smooth graphs with independence number smaller than or equal to $n/3$ and therefore we could use for q any value $1/3 + \varepsilon$ with a small $\varepsilon > 0$. The algorithm was not able to find a graph with independence number smaller than $n/3$ but it was able to find graphs with independence number $n/3$. It could solve the instances up to $n = 26$ in under 5,000,000 seconds.

7 Conclusion and Further Work

In this paper we formalized a family of problems for finding smooth graphs with small independence numbers. We proposed an algorithm for solving problems of this family which is based on branch and bound. To increase the efficiency of the algorithm by computing good bounds, we used graph-theoretic results to obtain properties and bounds for the number of inner cycles and their sizes. Using those results we proposed a procedure for computing a strong lower bound on the independence number of partial solutions to detect infeasibility as early as possible. We applied our algorithm to four different problems and reported the results and the running times for different graph sizes. Doing this we could disprove one conjecture and find more support for other conjectures for small graphs.

Further work may be to compare different heuristics for computing independent sets for partial solutions. Furthermore, one idea could be to search for a minimal feasible graph, which may enable some reduction properties and therefore some stronger bounds. Additionally, it would be interesting to use a metaheuristic to solve our problems, which would allow to search larger smooth graphs with small independence numbers heuristically.

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