Complexity of Positive Influence Domination on Partial Grids

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Abstract. The Positive Influence Domination (PID) problem asks to find a minimum cardinality subset of influencers among the vertices of an undirected graph such that at least half the number of neighbors of each vertex are influencers. The problem's underlying model can be used to determine an economical way of promoting (and keeping) good habits in society by interpreting vertices as individuals, neighbors as social contacts, and influencers as, for example, healthy eaters. We show that the problem is NP-hard even when restricted to planar subcubic graphs. The same result turns out to apply for the so-called double total domination problem, which exhibits a similar behavior on this graph class. We use this insight to derive NP-hardness of PID on the class of induced partial grids via a technique relying on orthogonal graph drawing. Finally, we derive bounds on the size of optimal solutions for arbitrarily dimensioned grids.

Keywords: Positive influence domination · NP-hardness · Partial grids

1 Introduction

The *Positive Influence Domination* (PID) problem [22] asks one to find a minimum cardinality subset of *influencers* of the vertex set of a given undirected graph such that for each vertex, the influencers in its neighborhood constitute a (not necessarily strict) majority. The problem, introduced in 2009 by Wang et al. [22], can be illustrated as follows: Suppose a public authority aims to establish a positive habit in society, e.g., the routine of eating healthily. Further assume that healthy eaters risk becoming unhealthy eaters when there are more unhealthy than healthy eaters in their circle of contacts. At this point the authority might wonder what is the minimum number of unhealthy eaters in the society who would need to be "turned" into healthy ones such that healthy eaters remain unaffected by the aforementioned social rebound effect.

While the latter application scenario is meaningful in graphs modeling a social network, the whole question can also be seen as a resource distribution problem in more general graphs. Here sufficient locally available neighboring resources must be provided as economically as possible. For practical applications like broadcasting problems, related problems (some considering also propagation

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aspects) have been studied on, e.g., unit disk graphs [5] but also more regular graph structures, e.g., hypercubes [1]. Optimal solutions on grid graphs are the subject of study for several variants [20] of the classical dominating set problem [10]. Remarkably, for the latter problem itself, although studied extensively, the process of finding and certifying optimal solutions was completed only in 2011 by Gonçalves et al. [10]. Motivated by these works, we point out properties on (partial) grids for the variant of positive influence domination.

Several closely related combinatorial optimization problems on graphs have been proposed in the form of *monopoly problems* [17], *α-domination* [8], and *signed domination* [7], to mention a few examples. Considering the aspect of temporal propagation of influence, the *Target Set Selection* problem [13] has also been studied. One can find comprehensive comparisons [4,9] of many such problems. This class of domination-type problems has been studied mainly from a theoretical viewpoint so far. For any scalar $\alpha \in (0, 1]$—steering the "strength" of a majority—the α-domination problem was shown to be NP-hard [8]. Its slight generalization *vector domination* [11] was shown to be fixed-parameter tractable on graphs excluding cycles of length 4 and on graphs of bounded degeneracy [19].

The more constrained problem *total vector domination* [4], which includes PID as special case, was shown to be solvable in polynomial time on graphs of bounded branchwidth and bounded treewidth [12]—the same was shown to hold for vector domination. Appearing under various names in the literature, *k-tuple total domination* (see, e.g., [14]) is a particularly uniform special case of total vector domination, for which NP-hardness on a special case of chordal graphs was shown [14]; NP-hardness was also shown on bipartite and split graphs [18].

Concerning complexity theoretic results on PID itself, it is shown in [23] that PID is APX-hard; this is also true more generally when, instead of a majority, at least a fixed proportion of neighbors is required to consist of influencers [4]. Recently, NP-hardness for the PID problem on bipartite and split graphs has been established [24]. NP-hardness was shown for the first time in [23] by a reduction from vertex cover (via a proof affirming this for graphs with a maximum degree of 6). The latter work is also our main motivation to identify more restrictive graph classes for which the hardness still applies.

The paper is organized as follows. After providing relevant notation in Sect. 2, in Sect. 3 we show that PID and 2-tuple total domination are NP-hard problems on planar subcubic graphs. The result is lifted to NP-hardness of PID on induced partial grids in Sect. 4. Bounds on complete grids are given in Sect. 5, and Sect. 6 concludes the paper.

2 Notation and preliminaries

For two integers r and s, we use the notation $[r : s] := \{r, r + 1, \dots, s - 1, s\}$ and $[s] := [1 : s]$. For $a, b \in \mathbb{Z}$, $b \neq 0$, denote by $\mathrm{mod}(a, b)$ the binary modulo operator (returning the unique number $r \in [0 : b - 1]$ satisfying $a = kb + r$ for some $k \in \mathbb{Z}$).

In our discussion, we assume graphs $G = (V, E)$ to be simple and undirected. For a vertex $v \in V$ we denote its (open) neighborhood of adjacent vertices by $N(v)$ and its degree by $\deg(v) := |N(v)|$. A graph $G = (V, E)$ with $\max_{v \in V} \deg(v) \le k$ is called *k-subregular*—or *subcubic* for $k = 3$. If further $\min_{v \in V} \deg(v) = k$, it is called *k-regular*—or *cubic* for $k = 3$. We say that G is k-connected, if G remains connected when fewer than k vertices are removed. Denote by $\mathcal{G}_3^{\text{plan}}$ the class of *planar cubic graphs* and by $\mathcal{G}_{\le 3}^{\text{plan}}$ the class of *planar subcubic graphs*.

For the following we define a length-r path as a graph $P_r = (V, E)$ with $V = [r]$ and $E = \{\{i, i+1\} : i \in [r-1]\}$. Denote by $P_m \Box P_n$ the Cartesian product of P_m and P_n, henceforth also called *complete $m \times n$ grid (graph)*. Its vertices are given by all pairs in $[m] \times [n]$; two such vertices are defined as adjacent if their first coordinates or second coordinates form an adjacent pair in their respective path of origin P_m or P_n. Let us think of $P_m \Box P_n$ as embedded in the plane, with vertices on integer-valued coordinates and unit-length straight, axes-parallel edges; tacitly we assume the vertices to be oriented as the indices of an $m \times n$ matrix. By a *partial grid* we refer to an arbitrary (not necessarily induced) subgraph of a grid, in other words, a graph resulting from the removal of a nonnegative number of vertices and edges from $P_m \Box P_n$ for some specific $m, n \in \mathbb{N}$; here, a vertex-removal automatically removes all incident edges. A *unit disk graph* has vertices identifiable with points in the plane such that two vertices are adjacent if and only if the two compact unit disks centered at the respective points have nonempty intersection.

Although many of the subsequent variants of domination/independence are typically defined via subsets of vertices, let us state them for easier usability—see later Theorem 5—in a binary-encoded form: For a graph $G = (V, E)$, a function $f : V \to \{0, 1\}$ is *independent* if $f(v) + f(w) \le 1$ for all $\{v, w\} \in E$. A function $f : V \to \{0, 1\}$ is said to be

- *positive influence dominating* [22] if $\sum_{w \in N(v)} f(w) \ge \lceil \deg(v)/2 \rceil$ for $v \in V$,
- *k-tuple total dominating* (for some $k \in \mathbb{N}$) if $\sum_{w \in N(v)} f(w) \ge k$ for $v \in V$,
- *double total dominating* if it is 2-tuple total dominating.

When f is positive influence dominating we will often abbreviate this by saying that f is *PID-feasible* or a *PID-function*. For such binary functions f let us call the value $f(v)$ the *weight* or *label* of $v \in V$ and $\sum_{v \in V} f(v)$ the *(cumulative) weight* of f. Given a graph $G = (V, E)$ and $k \in \mathbb{N}$, the problem of determining the existence of an independent $g : V \to \{0, 1\}$ with $\sum_{v \in V} g(v) \ge k$ is known as the independent set problem. The problem of determining the existence of $f : V \to \{0, 1\}$ with $\sum_{v \in V} f(v) \le k$ being (i) positive influence dominating is denoted as PosInflDom(G, k); (ii) double total dominating is denoted as DblTotalDom(G, k). For fixed G, the minimum $k \in \mathbb{N}$ for which such a feasible function exists, is denoted as $\gamma_{\text{PID}}(G)$ for (i), and $\gamma_{\times 2,\text{t}}(G)$ for (ii).

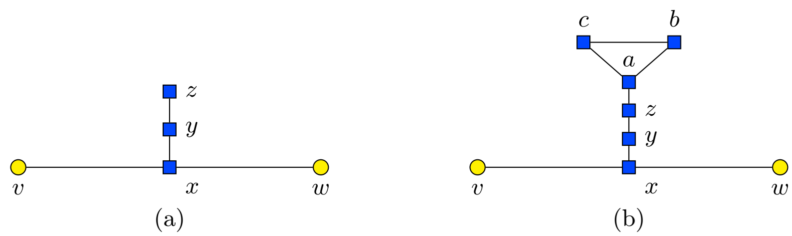

Fig. 1. Modifications of an edge $e = \{v, w\}$ used in the proof of (a) Theorem 2; and (b) Theorem 3. Subscripts e are omitted for the vertices x, y, z, a, b, and c.

3 Hardness on planar subcubic graphs

The results of the current and the next section are inspired by the works [6] and [25] which address the hardness of the so-called *signed dominating set* problem. In the latter problem, a minimum-weight function $f : V \to \{-1, 1\}$ must be found such that $f(v) + \sum_{w \in N(v)} f(w) \geq 1$ for $v \in V$. Although signed domination does not allow a zero-label (i.e., "neutral") and each vertex' own label contributes to meeting its threshold, we still obtain equivalent hardness results for positive influence domination. Unlike Damaschke [6] in 2001, we can now rely on a stronger NP-hardness result that permits assuming 2-connectedness. The latter is stated in the subsequent and underpins our later derived reductions.

Theorem 1 ([16, Theorem 4.1(a)]). *The maximum independent set problem is NP-hard when restricted to planar* 2*-connected cubic graphs.*

One can verify that Theorem 2 and Theorem 3 remain valid even when restricted to connected graphs. We omitted this observation which also follows from the fact that both concerned problems can always be solved independently on all connected components.

Consider the planarity-preserving, polynomial-time operation

$$\mathsf{SplApd} : \mathcal{G}_{\leq 3}^{\text{plan}} \to \mathcal{G}_{\leq 3}^{\text{plan}}$$

replacing each edge $e = \{v, w\} \in E$ of $G = (V, E)$ with new edges $\{v, x_e\}$, $\{w, x_e\}$, $\{x_e, y_e\}$, $\{y_e, z_e\}$ on top of fresh vertices x_e, y_e, z_e; see Fig. 1a. Define $\mathcal{SP}_{\leq 3}^{\text{plan}}$ as the class of all graphs that are—up to isomorphy—contained in $\{\mathsf{SplApd}(G) : G \in \mathcal{G}_3^{\text{plan}}\}$. Note that by detecting the vertices at (shortest-path) distance at most 2 from the leaves of $G' \in \mathcal{SP}_{\leq 3}^{\text{plan}}$, we can always determine the underlying cubic graph, even when no explicit specification of the fresh vertices is provided.

Theorem 2. *When restricted to the class of planar subcubic graphs, the problem* PosInflDom(G, k) *is NP-complete.*

Proof. More strongly we prove that the problem is NP-complete on $\mathcal{SP}^{\text{plan}}_{\leq 3}$, directly implying NP-completeness on $\mathcal{G}^{\text{plan}}_{\leq 3}$. The reduction is from maximum independent set on planar 2-connected cubic graphs; see Theorem 1. Let $G = (V, E)$ be a planar 2-connected cubic graph and let $G' := \mathsf{SplApd}(G)$, denoted as $G' = (V', E')$.

The problem is in NP, as a guessed function can be checked for feasibility in linear time (in the size of G).

We claim that G has an independent $g : V \to \{0, 1\}$ with $\sum_{v \in V} g(v) \geq k$ iff G' has a PID-feasible $f : V' \to \{0, 1\}$ with $\sum_{v' \in V'} f(v') \leq k'$ for some k' depending on G and k. In fact, if an independent g with

$$\sum_{v \in V} g(v) \geq k \tag{1}$$

exists, then the function

$$f(v') := \begin{cases} 1 - g(v') & \text{if } v' \in V, \\ 1 & \text{if } v' \in \{x_e, y_e : e \in E\}, \\ 0 & \text{if } v' \in \{z_e : e \in E\}, \end{cases}$$

is positive influence dominating on G': each vertex in V has three 1-labeled x-neighbors; all vertices in $\{x_e, y_e, z_e : e \in E\}$ have one 1-labeled neighbor; and, finally, the absence of an assigned 1-label by g to one of v and w (according to the independence of g) guarantees at least one additional 1-label assigned by f to one of the neighbors v or w of $x_{\{v,w\}}$.

We conclude that (below obtaining estimate (2) via (1))

$$\begin{aligned} \sum_{v' \in V'} f(v') &= \sum_{v \in V} f(v) + \sum_{v' \in V' \setminus V} f(v') = |V| - \sum_{v \in V} g(v) + \sum_{v' \in V' \setminus V} f(v') \\ &\leq |V| + \sum_{v' \in V' \setminus V} f(v') - k = |V| + 2|E| - k =: k'. \end{aligned} \tag{2}$$

The converse proof direction requires us to show that for a PID-function f on G' the circumstance $\sum_{v' \in V'} f(v') \leq k'$ allows constructing an independent g on G of cumulative weight at least $k = |V| + 2|E| - k'$. We need the intermediate observation that each positive influence dominating $f : V' \to \{0, 1\}$ can be updated on $V' \setminus V$ in a weight-preserving—or even weight-reducing—manner such that for each $e \in E$ we eventually have more canonically $f(x_e) = f(y_e) = 1$ and $f(z_e) = 0$.

This simple claim can be justified as follows: We recognize that always $f(y_e) = 1$ due to $\deg(z_e) = 1$. In case now $f(z_e) = 1$, we can overwrite $f(z_e) \leftarrow 0$ and if not already $f(x_e) = 1$, we update $f(x_e) \leftarrow 1$ to maintain the influence threshold of y_e—the cumulative weight does not increase by doing so. On the other hand, if initially $f(z_e) = 0$, necessarily we have already $f(x_e) = 1$.

We can therefore assume that a PID-feasible f with $\sum_{v' \in V'} f(v') \leq k'$, further satisfying the above canonicity, exists. Setting $g := 1 - f(\cdot)$ yields independence for g, as $f(v) + f(w) \geq 1$ implies $g(v) + g(w) \leq 1$. Furthermore, we have

$$\sum_{v\in V} g(v) = |V| - \sum_{v\in V} f(v) = |V| - \sum_{v\in V} f(v) - \sum_{v'\in V'\setminus V} f(v') + \sum_{v'\in V'\setminus V} f(v')$$
$$= |V| - \sum_{v'\in V'} f(v') + 2|E| \geq |V| - k' + 2|E|.$$

□

Corollary 1. PosInflDom(G, k) *is NP-complete on planar subcubic* bipartite *graphs.*

Proof. The vertices of $G' \in \mathcal{SP}^{\text{plan}}_{\leq 3}$ with an underlying cubic graph $G = (V, E)$ are bi-partitioned by $V \cup \{y_e : e \in E\}$ and $\{x_e, z_e : e \in E\}$. □

Similarly, we obtain the following result.

Theorem 3. DblTotalDom(G, k) *is NP-complete even when restricted to planar subcubic graphs.*

Proof. We fall back on a leaves-free class of subcubic graphs resulting from the replacement of the edges of a cubic graph by the gadgets described in Fig. (1b) yielding six additional fresh vertices per edge. Analogously to the proof of Theorem 2, we can here carry out the argumentation with $k' := |V| + 6|E| - k$ and $f(v') := 1 - g(v')$ if $v' \in V$ and otherwise, if $v' \in \{x_e, y_e, z_e, a_e, b_e, c_e : e \in E\}$, $f(v') := 1$. For the converse proof direction the degree of freedom for the choice of f-values is now considerably lower than in the last proof, as, to be feasible for DblTotalDom, all fresh vertices necessarily must be 1-labeled. □

4 Hardness on induced partial grids

In this section we lift the result in Theorem 2 to the class of induced partial grids. We start with the following preparatory observation.

Lemma 1. *Consider an undirected path* $P = (v, x_1, x_2, \ldots, x_{4\ell-1}, x_{4\ell}, w)$ *for some* $\ell \in \mathbb{N} \cup \{0\}$ *and* $f : \{v, w\} \cup \{x_j : j = 1, \ldots, 4\ell\} \to \{0, 1\}$ *such that each vertex* x_j, $j = 1, \ldots, 4\ell$, *has at least one* 1*-labeled neighbor. Suppose further that* $f(v) = f(w) = 1$. *Then,* $\sum_{j=1}^{4\ell} f(x_j) \geq 2\ell$, *where the lower bound* 2ℓ *is attainable.*

Proof. Examine f on the induced subpaths $(x_1, x_2, x_3, x_4), \ldots, (x_{4\ell-3}, x_{4\ell-2}, x_{4\ell-1}, x_{4\ell})$. By exhaustion it turns out that each of them, independently of the f-assignment on their neighboring paths, must have a cumulative f-weight of at least 2. Thus, $\sum_{j=1}^{4\ell} f(x_j) \geq 2\ell$. This weight is attainable by setting $f(x_j) := 1$, when $\mathrm{mod}(j-1, 4) \in \{1, 2\}$; and $f(x_j) := 0$, otherwise. □

We anticipate an auxiliary result, suppressing its proof following from a simple canonization strategy comparable to the proof of Theorem 2.

Lemma 2. *Let $k \in \mathbb{N} \cup \{0\}$. Further, let $G = (V, E)$ be a subcubic graph, and $L_e \in \mathbb{N} \cup \{0\}$ be divisible by 4, for $e \in E$. Consider $G' = (V', E')$ resulting from replacing each $e = \{v, w\} \in E$ by an undirected path $(v, p_e^1, \ldots, p_e^{L_e}, w)$ of length $L_e + 1$, i.e., the former edge e is split by L_e fresh vertices. Let $U = \{(0,1,1,0), (0,0,1,0), (0,0,1,1), (0,1,1,1)\}$. If there exists a PID-feasible f' with $\sum_{v \in V'} f'(v) = K$, then there exists a PID-feasible $f'' : V' \to \{0,1\}$ satisfying $(f''(v), f''(p_e^1), f''(p_e^{L_e}), f''(w)) \notin U$, for $e = \{v, w\} \in E$, and $\sum_{v' \in V'} f''(v') \leq K$.*

We will consider inter-vertex paths as in Lemma 1 for any two adjacent vertices v and w. The paths' lengths will again be zero or multiples of 4 but can vary depending on the edge $e = \{v, w\}$. The following lemma affirms that, after inserting a number of splitting vertices divisible by 4 on each edge, a PID-function of the same quality can be achieved (up to an additional constant originating from the number of splitting vertices). The core argument of the proof is that the labels on the splitting vertices can be chosen such that the constellations of neighboring labels remains invariant around all original vertices. The fresh labels can further be feasibly chosen such that the weight does not exceed the half of the count of splitting vertices.

Lemma 3. *Let $k \in \mathbb{N} \cup \{0\}$. Further, let $G = (V, E)$ be a subcubic graph, and $L_e \in \mathbb{N} \cup \{0\}$ be divisible by 4, for $e \in E$. Consider $G' = (V', E')$ resulting from replacing each $e = \{v, w\} \in E$ by an undirected path $(v, p_e^1, \ldots, p_e^{L_e}, w)$ of length $L_e + 1$, i.e., the former edge e is split by L_e fresh vertices. Then, the following assertions are equivalent.*

(i) A PID-feasible f on G with $\sum_{v \in V} f(v) \leq k$ exists.
(ii) A PID-feasible f' on G' with $\sum_{v \in V'} f'(v) \leq k + \sum_{e \in E} L_e/2$ exists.

Proof. On all undirected paths we will consider the values of f'. Formally, as each undirected path can be traversed in two directions, and as labels have to be assigned to the vertices on the path, let us encode the labels via $F'_{(v,w)} \in \{0,1\}^{L_{\{v,w\}}}$ determining hence the chronological ordering of the vertex labels (excluding those of v and w) encountered when traversing the path from v to w in G'. Implicitly we assume that $F'_{(w,v)}$ is automatically coherently specified by the reversal of $F'_{(v,w)}$. With a slight abuse of notation, by writing $F'_{\{v,w\}}$ we refer to $F'_{(v,w)}$.

(i) $\Longrightarrow$ (ii): Starting from a feasible f on G, defining $f'(v) := f(v)$, for $v \in V$, the idea is then, for all $e = \{v, w\} \in E$, to choose the remaining f'-values on $(v, p_e^1, \ldots, p_e^{L_e}, w)$ ensuring feasibility, having cumulative weight $L_e/2$, and satisfying

$$f'(v) = f'(p_e^{L_e}) \text{ and } f'(p_e^1) = f'(w). \tag{3}$$

The latter property requires for vertices $v \in V$, seen as vertices from G', that the neighboring label constellation around v corresponds to the neighboring label constellation around v according to f on G.

The choice of the following length-L_e labeled paths ensures such a behavior (note that together with the labeled endpoints $f(v)$, $f(w)$ PID-feasibility is guaranteed on such a path, i.e., at least one 1-labeled neighbor is present for the path's vertices):

$$F'_{(v,w)} := \begin{cases} (0,1,1,0,\dots,0,1,1,0,\dots,0,1,1,0) & \text{if } f(v)=0 \wedge f(w)=0, \\ (1,1,0,0,\dots,1,1,0,0,\dots,1,1,0,0) & \text{if } f(v)=0 \wedge f(w)=1, \\ (0,0,1,1,\dots,0,0,1,1,\dots,0,0,1,1) & \text{if } f(v)=1 \wedge f(w)=0, \\ (1,0,0,1,\dots,1,0,0,1,\dots,1,0,0,1) & \text{if } f(v)=1 \wedge f(w)=1. \end{cases}$$

The definition of f' on V results hence from the older values of f on V while on the paths' fresh vertices the labels are inferrable from the definition of F'_e according to the above case distinction, which also shows that a weight of only $\sum_{e\in E} L_e/2$ is added to the cumulative weight of f on V.

(ii) $\Longrightarrow$ (i): Due to Lemma 2 we can not only assume f' to be PID-feasible but also without loss of generality $(f'(v), f'(p_e^1), f'(p_e^{L_e}), f'(w)) \notin U$. For all edges $e = \{v,w\}$, up to symmetry breaking (reversed quadruples) we can therefore only have $(f'(v), f'(p_e^1), f'(p_e^{L_e}), f'(w)) \in T \,\dot\cup\, W$ with $T := \{(1,1,1,1), (0,1,0,1), (0,0,0,0)\}$ and $W := \{(1,0,1,1), (1,0,0,1), (0,0,0,1)\}$. The ultimate goal will be to modify f' towards a more useful version f'' being as well PID-feasibile, having a weight not greater than f' on G' and satisfying for all edges $\{v,w\} \in E$ the special property

$$f''(v) = f''(p_e^{L_e}) \text{ and } f''(p_e^1) = f''(w). \tag{4}$$

Initially set $f'' := f'$. Immediately we notice that all edges whose behavior is captured by one of the scenarios in T satisfies (4); no updates will hence be needed.

On the other hand, an update leaving the values of f'' unchanged on v and w suffices for the scenarios covered by W:

$$F''_e \leftarrow \begin{cases} (1,0,0,1,\dots,1,0,0,1) & \text{if } (f'(v), f'(p_e^1), f'(p_e^{L_e}), f'(w)) = (1,0,1,1), \\ (1,0,0,1,\dots,1,0,0,1) & \text{if } (f'(v), f'(p_e^1), f'(p_e^{L_e}), f'(w)) = (1,0,0,1), \\ (1,1,0,0,\dots,1,1,0,0) & \text{if } (f'(v), f'(p_e^1), f'(p_e^{L_e}), f'(w)) = (0,0,0,1). \end{cases} \tag{5}$$

Note that the updated values on $p_e^1, \dots, p_e^{L_e}$ meet the bound $L_e/2$ (see Lemma 1) and yield an on-par or better weight than the initial values of f'. After carrying out the updates on all edges, we end up with a labeling f'' fulfilling (4). Therefore, if we contract the inter-vertex paths to the original edges of G, the constellation of neighboring f''-labels will remain unchanged for all vertices $v \in V$. Consequently, the restriction $f := f''|_V$ is the claimed existing labeling. As the weight of f does not include an additional weight $L_e/2$ per edge e, we finally conclude

$$\sum_{v\in V} f(v) \le \sum_{v'\in V'} f''(v) - \sum_{e\in E} L_e/2 \le \sum_{v'\in V'} f'(v) - \sum_{e\in E} L_e/2 \le k.$$

□

Next, let us show how the previous result is related to partial grids. Recall that a *simple rectilinear polyline* (SRP) is an injectively parameterizable polygonal chain made up of axes-parallel line segments. However, for all what follows, we assume a strengthened definition of SRP requiring each of its atomic line segments to be of *integer-valued length*. The argumentation relies on the following auxiliary Lemma due to Valiant [21].

Lemma 4 (adapted from [21, Theorem 2]). *There is a polynomial-time algorithm taking as input a planar* 4*-subregular graph* G *and returning a planar embedding of* G *into the plane using* $O(|V|^2)$ *area such that embedded vertices possess integer-valued coordinates, and embedded edges are SRPs.*

Theorem 4. *The problem* $\textsc{PosInflDom}(G,k)$ *is NP-complete on the class of induced partial grids.*

Proof. From the proof of Theorem 1 we know that $\textsc{PosInflDom}(G,k)$ is NP-complete on $\mathcal{SP}^{\text{plan}}_{\leq 3}$. We will rely on a reduction from the latter.

Starting from $G' = (V', E') \in \mathcal{SP}^{\text{plan}}_{\leq 3}$ consider also its underlying cubic graph $G = (V, E)$, i.e., $\mathsf{SplApd}(G) = G'$. The first goal will be to find a particular embedding for G' by adapting an embedding for G. This is accompanied by an illustration in Fig. 2 and works as follows:

Using Lemma 4, planarly draw in $\mathbb{R}^2$ the embedding of G such that embedded vertices have integer-valued coordinates and all embedded edges are SRPs; to keep the result polynomially-sized, make sure the embedded vertices and edges are bounded as in Lemma 4. Afterwards, subject the drawing in $\mathbb{R}^2$ to the geometric scaling $(x, y) \mapsto (8x, 8y)$. For each edge $\{v, w\}$ of G, prolong by two units the associated (now scaled) SRP $\rho_{\{v,w\}}$ via the following procedure consisting of two steps:

Step 1. Fix an arbitrary endpoint $p = p_{\{v,w\}} \in \{v, w\}$ of the edge and consider the length-4 line subsegment $s(p, \{v, w\}; 2, 6)$ of $\rho_{\{v,w\}}$ starting at geodesic distance 2 from p and ending at geodesic distance 6 from p. Let us translate $s(p, \{v, w\}; 2, 6)$ by one unit in direction orthogonal to the extension direction of $s(p, \{v, w\}; 2, 6)$ itself. The now modified $\rho_{\{v,w\}}$ consists of three disconnected components, which we join by two unit-length line segments, yielding our definitive form of $\rho_{\{v,w\}}$. The length of the original SRP $\rho_{\{v,w\}}$ was hence artificially increased by two units by performing a local detour geometrically reminding of a rectangular U-turn. This guarantees that all present SRPs are now of length congruent 2 modulo 8. The absence of crossing edges is clearly maintained.

Step 2. Determine on $\rho_{\{v,w\}}$ the point at geodesic distance 5 from p (according to the geometry of the curve $\rho_{\{v,w\}}$ itself). Let us regard it as the embedded "splitting" vertex $x_{\{v,w\}}$ of G'. Then, append to the $x_{\{v,w\}}$-embedding a fresh length-2 straight path, meant to host the embeddings of $y_{\{v,w\}}$ and $z_{\{v,w\}}$, in a way ensuring that one coordinate-entry of $y_{\{v,w\}}$ is divisible by 8. Eventually, we have found an embedding with SRPs for all edges of G'.

Note that for each edge $e = \{v, w\}$ of G, the number of lattice points of $\mathbb{Z} \times \mathbb{Z}$ which are covered by the SRP of G' connecting $x_{\{v,w\}}$ and v is divisible

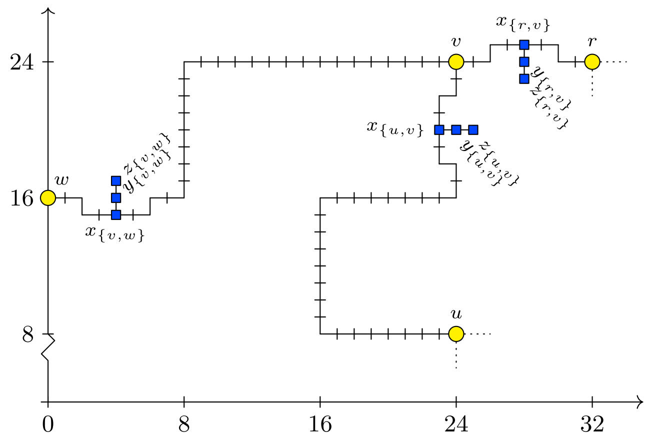

Fig. 2. Up to the detours (visible as lengthy U-turns) and the paths between the quadratic vertices, the drawing shows an initial SRP embedding already scaled by a factor of 8. The Euclidean distance between u and v, e.g., is assumed to have increased eightfold from initially 2 to eventually 16. The splitting vertices x_e, $e \in E$, lie—with respect to the associated SRP—at geodesic distance 5 from an endpoint.

by 4 when this number renounces counting the embedded endpoints $x_{\{v,w\}}$ and v. The same is true for the SRP connecting $x_{\{v,w\}}$ and w. On these $x_{\{v,w\}}$-incident SRPs let us consider these intermediate $\mathbb{Z} \times \mathbb{Z}$ points as embedded fresh subdivisors of the combinatorial edges of G; call this new graph $\tilde{G}$. In particular, as they lie at Manhattan distance 1, no combinatorial subdivision of the edges $\{x_e, y_e\}, \{y_e, z_e\}$ occurs. $\tilde{G}$ is a (subcubic planar) induced partial grid and has the form of the transformed graphs from Lemma 3 with

$$4 \leq \min\{L_{\{v,x_e\}}, L_{\{w,x_e\}}\} \equiv \max\{L_{\{v,x_e\}}, L_{\{w,x_e\}}\} \equiv 0 \pmod 4,$$
$$0 = L_{\{x_e,y_e\}} = L_{\{y_e,z_e\}} \equiv 0 \pmod 4,$$

for all $e = \{v, w\} \in E$. Thus, by Lemma 3, the existence of a feasible f for $\textsc{PosInflDom}(G, k)$ is equivalent to the existence of a feasible f' on the induced partial grid $\tilde{G}$ of weight at most $k + \sum_{e=\{v,w\}\in E} L_{\{v,x_e\}}/2 + L_{\{w,x_e\}}/2$. $\square$

The latter result has implications for the class of unit disk graphs on which domination-type problems are often considered [5].

Corollary 2. *The problem* $\textsc{PosInflDom}(G, k)$ *remains NP-complete on unit disk graphs.*

5 Bounds on complete grids

This section derives lower and upper bounds on complete grids for γ_{PID} and $\gamma_{\times 2,\mathrm{t}}$. For this graph class the requirements of $\gamma_{\times 2,\mathrm{t}}$ are strictly more constrained than those of γ_{PID}. In fact, tighter lower bounds for $\gamma_{\times 2,\mathrm{t}}$ will be found.

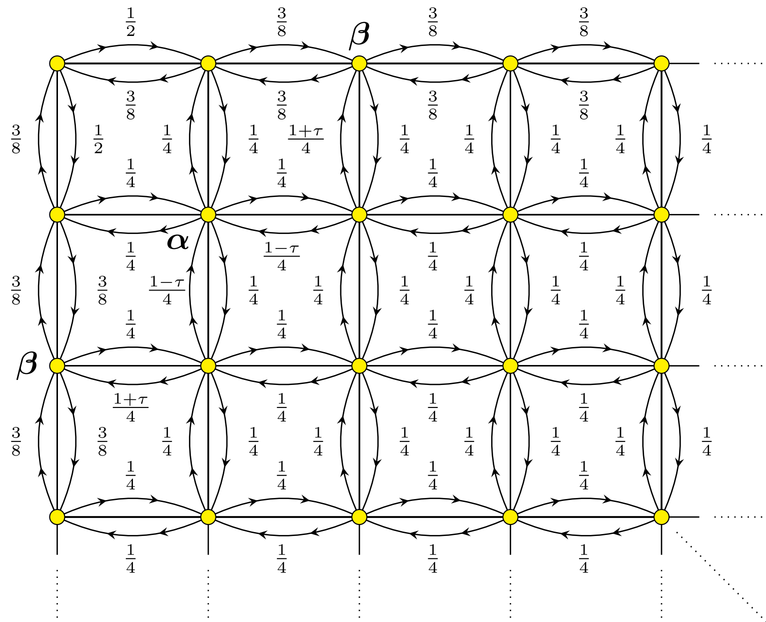

Fig. 3. The top-left area of an $m\times n$ grid subject to the discharging process. Coefficients attached to the arrows indicate the proportion of transferred charge. Vertices in the set A or B from the proof of Theorem 6 are marked by α or β, respectively. Exemplarily assuming that v is the upper β-marked vertex, the situation $f(v) = 1$ implies that the two neighbors of v on the grid's upper boundary receive additional charge of 3/8 whereas the interior neighbor just 1/4; if $f(v) = 0$, a zero-charge transfer to neighbors applies.

Theorem 5. *For* $m, n \geq 6$ *we have*

$$\frac{mn}{2} + \frac{m+n}{4} - \frac{3}{2} \leq \gamma_{\text{PID}}(P_m \Box P_n) \leq \frac{mn}{2} + \frac{m+n}{2} + \frac{3 - |\operatorname{mod}(n,4) - 1|}{2}. \quad (6)$$

Proof. As any double total dominating function on grids is automatically positive influence dominating, the upper bound of [15] for $\gamma_{\times 2,\text{t}}$ is valid for γ_{PID}, too; the latter is stated as the right-hand side of (6). For $\gamma_{\times 2,\text{t}}$ it constitutes the currently tightest upper bound with exception of the constellations $(\operatorname{mod}(m,4), \operatorname{mod}(n,4)) \in \{(0,0),(1,1)\}$, where it worsens the older bound [3] by 1.

Let us prove the lower bound by using the so-called *discharging method*, arguably best known due to the proof of Appel and Haken [2] of the Four Color Theorem. We transform a PID-feasible $f : [m] \times [n] \to \{0,1\}$ to a function $g : [m] \times [n] \to \mathbb{Q}$ in a weight-preserving manner, i.e., $\sum_{v \in [m]\times[n]} f(v) = \sum_{v \in [m]\times[n]} g(v)$ and then prove the bound for g: To obtain g—simultaneously for all vertices—we entirely redistribute the f-value of each vertex, according to a specific convex combination, among its neighbors. Adopting, conversely, a passive perspective, this means that the initial "charge" $f(v)$ of a vertex v is entirely replaced by the sum of the incoming charges from its neighborhood.

Let A be the set of vertices having precisely two degree-4 neighbors (lying at Manhattan distance 2 from a corner of the grid) and B be the set of vertices having only one degree-4 neighbor and being at distance 2 from a corner; for both see Fig. 3. In the following, we directly state the function g resulting from a tailored discharging process sketched in Fig. 3. Apart from fetching it from Fig. 3, more formally, the proportionality-scalar of the charge-transfer from a neighbor w to v can be read off the respective case for v in (7) from the coefficient of the unique summand associated to the index w. Denote by $D_t := \{v \in [m] \times [n] : \deg(v) = t\}$ specific preimages of the function $\deg(\cdot)$. For the parameter $\tau \in \{0, 1\}$ consider

$$g^\tau(v) := \begin{cases} \sum_{w \in N(v) \cap D_3} \frac{f(w)}{4} + \sum_{w \in N(v) \cap D_4} \frac{(1-\tau) f(w)}{4} & \text{if } v \in A, \\ \sum_{w \in N(v) \cap D_3} \frac{3f(w)}{8} + \sum_{w \in N(v) \cap D_4} \frac{(1+\tau) f(w)}{4} & \text{if } v \in B, \\ \sum_{w \in N(v) \cap D_2} \frac{f(w)}{2} + \sum_{w \in N(v) \cap D_3} \frac{3f(w)}{8} + & \\ \quad \sum_{w \in N(v) \cap D_4} \frac{f(w)}{4} & \text{if } v \in (D_2 \cup D_3) \setminus B, \\ \sum_{w \in N(v) \cap D_4} \frac{f(w)}{4} & \text{if } v \in D_4 \setminus A. \end{cases} \tag{7}$$

In the current proof, we are interested in the discharging process associated to $g := g^0$, i.e., $\tau = 0$ (the choice $\tau = 1$ concerns the proof of the later stated Theorem 6). As f is a PID-function, we have

$$\sum_{v \in [m] \times [n]} g(v) = \sum_{v \in D_4} g(v) + \sum_{v \in D_3} g(v) + \sum_{v \in D_2} g(v) \geq \frac{1}{2} |D_4| + \frac{5}{8} |D_3| + \frac{3}{8} |D_2|$$

$$= \frac{mn}{2} + \frac{m+n}{4} - \frac{3}{2}.$$

Here, the three sums have been estimated from below one by one with the following justification: By feasibility of f, every $v \in D_4$ has at least two neighbors with an f-value of 1; this simply allows to conclude that $g(v) \geq 1/4 + 1/4$, for $v \in D_4$. The estimate for each vertex in D_3 holds due to the fact, that again two neighbors with an f-value of 1 must exist leading in the lowest possible case to an inflow of charge $1/4 + 3/8 = 5/8$. The charge inflow for a corner must be at least $3/8$. □

Repeating the proof of Theorem 5—this time using $g := g^1$ defined as in (7)—we almost verbatim obtain the following result.

Theorem 6. *For $m, n \geq 6$ we have*

$$\frac{mn}{2} + \frac{m+n}{4} + 1 \leq \gamma_{\times 2,\mathrm{t}}(P_m \square P_n) \leq \frac{mn}{2} + \frac{m+n}{2} + \frac{3 - |\operatorname{mod}(n,4) - 1|}{2}. \tag{8}$$

Despite the simplicity of the discharging approach, the bound (8) on $\gamma_{\times 2,\mathrm{t}}$ slightly strengthens—the difference is precisely 1—the lower bound given in [15]. However, the more striking insight lies here in the possibility to give a considerably shorter proof for the lower bound than in the latter work.

6 Conclusion

We have shown that PosInflDom(G, k) and DblTotalDom(G, k) remain NP-complete problems even when restricted to planar subcubic graphs. The employed reductions in Sect. 3 lose certain structural properties raising two open questions: Is it possible to determine a small k—perhaps even $k = 3$—for which the hardness result of Theorem 2 still applies for k-subregular graphs in the setting of 2-*connected* planarity? Can we adapt the reductions to work for 3-regular instead of 3-subregular graphs?

Using a geometry-accented argument, we observed that NP-completeness for PID is inherited by the class of induced partial grids. Concerning the complete $m \times n$ grids we showed how to derive a lower bound for γ_{PID} by a short argument, which applied to the setting of $\gamma_{\times 2,\mathrm{t}}$ slightly tightens the known lower bound.

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A Appendix (supplementary proofs)

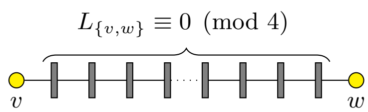

Fig. 4. Supplementary illustration of the edge subdivision in Lemma 3.

We restate Lemma 2 and provide its proof.

Lemma 2. *Let $k \in \mathbb{N} \cup \{0\}$. Further, let $G = (V, E)$ be a subcubic graph, and $L_e \in \mathbb{N} \cup \{0\}$ be divisible by 4, for $e \in E$. Consider $G' = (V', E')$ resulting from replacing each $e = \{v, w\} \in E$ by an undirected path $(v, p_e^1, \dots, p_e^{L_e}, w)$ of length $L_e + 1$, i.e., the former edge e is split by L_e fresh vertices. Let $U = \{(0,1,1,0), (0,0,1,0), (0,0,1,1), (0,1,1,1)\}$. If there exists a PID-feasible f' with $\sum_{v \in V'} f'(v) = K$, then there exists a PID-feasible $f'' : V' \to \{0,1\}$ satisfying $(f''(v), f''(p_e^1), f''(p_e^{L_e}), f''(w)) \notin U$, for $e = \{v, w\} \in E$, and $\sum_{v' \in V'} f''(v') \leq K$.*

Proof. An existing labeling f'' with the required properties is given by the output of Algorithm 1. In the subsequent let us prove its soundness.

PID-feasibility. The periodicity-4 patterns are designed to meet PID-feasibility certainly on the non-endpoint vertices of the inter-vertex path. On the two endpoints of the length-L_e pattern, PID-feasibility is ensured due to the increased label of v (and w).

Correctness and termination. The fact that constellations from U are successfully entirely removed and that the algorithm terminates is explained as follows: Initially $|M_{(0,1,1,0)}^{f''}|$ is a finite number. Each iteration of the loop removes a single occurrence of a type-$(0,1,1,0)$ constellation and does not cause a novel one of this type somewhere else in the graph to be handled in a later iteration—in fact, the overwriting process only increases labels for vertices in V. Similarly, the second while loop terminates: By the increased labels in the overwriting process in the iterations, no novel type-$(0,0,1,1)$ constellations are caused. Furthermore, no novel occurrences of constellations eliminated in Step 1 are caused. The same can be observed for the third while loop, where no novel type-$(0,0,1,1)$ constellations are caused. Additionally, none of the constellations eliminated in the prior two loops are caused. Finally, termination of the last loop is justified analogously. We observe that no constellation eliminated in previous loops is caused by the overwriting processes of the last loop.

No excess of weight. We show that in every loop, each iteration does not increase the weight.

Loop 1: Note that before overwriting, $(p_e^3, \dots, p_e^{L_e-2})$ is a path of length divisible by 4, hence, according to Lemma 1 its weight will be at least $(L_e -$

procedure *Canonize*$(V, E; (L_e)_{e \in E}; f')$
  $f'' \leftarrow f'$ // create a working copy of f'
  $M^{f''}_{(q_1,q_2,q_3,q_4)} := \{\{v, w\} \in E : (f''(v), f''(p_e^1), f''(p_e^{L_e}), f''(w)) = (q_1, q_2, q_3, q_4)\}$
  while $M^{f''}_{(0,1,1,0)} \neq \emptyset$ **do**
    Pick (v, w) from $M^{f''}_{(0,1,1,0)}$.
    Overwrite $F''_{(v,w)} \leftarrow (1, 0, 0, 1, \ldots, 1, 0, 0, 1)$.
    Overwrite $f''(v) \leftarrow 1$, $f''(w) \leftarrow 1$.
  while $M^{f''}_{(0,0,1,0)} \neq \emptyset$ **do**
    Pick (v, w) from $M^{f''}_{(0,0,1,0)}$.
    Overwrite $F''_{(v,w)} \leftarrow (0, 0, 1, 1, \ldots, 0, 0, 1, 1)$.
    Overwrite $f''(v) \leftarrow 1$.
  while $M^{f''}_{(0,0,1,1)} \neq \emptyset$ **do**
    Pick (v, w) from $M^{f''}_{(0,0,1,1)}$.
    Overwrite $F''_{(v,w)} \leftarrow (1, 0, 0, 1, \ldots, 1, 0, 0, 1)$.
    Overwrite $f''(v) \leftarrow 1$.
  while $M^{f''}_{(0,1,1,1)} \neq \emptyset$ **do**
    Pick (v, w) from $M^{f''}_{(0,1,1,1)}$.
    Overwrite $F''_{(v,w)} \leftarrow (1, 0, 0, 1, \ldots, 1, 0, 0, 1)$.
    Overwrite $f''(v) \leftarrow 1$.
  return f''

Algorithm 1: Constellations forbidden by the set U are eliminated from f'.

4)/2 regardless of its assigned values by f'. Necessarily $f'(p_e^2) = f'(p_e^{L_e-1}) = 1$ implying that the weight of f' on the entire length-L_e path is at least $L_e/2 + 2$, consequently, $f'(v) + f'(w) + \sum_{i=1}^{L_e} f'(p_e^i) \geq L_e/2 + 2$. On the other hand, by construction $f''(v) + f''(w) + \sum_{i=1}^{L_e} f''(p_e^i) = L_e/2 + 2$, i.e., the same or even a better weight is obtained while maintaining PID-feasibility.

Loop 2: Note that before overwriting, $(p_e^4, \ldots, p_e^{L_e-1})$ is a path of length divisible by 4, hence, according to Lemma 1 its weight will be at least $(L_e - 4)/2$ regardless of the values of f'. Necessarily $f'(p_e^2) = f'(p_e^3) = 1$ implying that the weight of f' on the entire length-L_e path is at least $L_e/2 + 1$, consequently, $f'(v) + f'(w) + \sum_{i=1}^{L_e} f'(p_e^i) \geq L_e/2 + 1$. Thus, the loop produces an equal-quality or even better update in terms of weight.

For the last two loops let us state the same argumentation compactly.

Loop 3: Recognize that the f'-weight of $(p_e^4, \ldots, p_e^{L_e-1})$ will be at least $(L_e - 4)/2$. Necessarily $f'(p_e^2) = f'(p_e^3) = 1$ implying that the weight of f' on the entire length-L_e path is at least $L_e/2 + 1$, consequently, $f'(v) + f'(w) + \sum_{i=1}^{L_e} f'(p_e^i) \geq L_e/2 + 1$ and the in-loop updates yield no worse alternative f''.

Loop 4: Recognize that the f'-weight of $(p_e^3, \ldots, p_e^{L_e-2})$ will be at least $(L_e - 4)/2$. Necessarily $f'(p_e^2) = 1$ implying that the weight of f' on the entire length-

L_e path is at least $L_e/2+1$, consequently, $f'(v)+f'(w)+\sum_{i=1}^{L_e} f'(p_e^i) \geq L_e/2+1$ and the in-loop updates yield no worse alternative f''. □

We restate Theorem 6 and provide its proof.

Theorem 6. *For* $m, n \geq 6$ *we have*

$$\frac{mn}{2} + \frac{m+n}{4} + 1 \leq \gamma_{\times 2,\mathrm{t}}(P_m \Box P_n) \leq \frac{mn}{2} + \frac{m+n}{2} + \frac{3 - |\operatorname{mod}(n,4) - 1|}{2}. \quad (9)$$

Proof. Again, as in Theorem 5 the right-hand side of (9) stems from [15].

For the lower bound, use the definitions of the proof of Theorem 5 and repeat its argumentation via the discharging method—this time using the premise $g := g^1$ with g^τ defined as in (7): As f is a valid double total dominating function we have

$$\begin{aligned}\sum_{v\in[m]\times[n]} g(v) &= \sum_{v\in D_4} g(v) + \sum_{v\in D_3\setminus B} g(v) + \sum_{v\in B} g(v) + \sum_{v\in D_2} g(v)\\ &\geq \frac{1}{2}|D_4| + \frac{5}{8}|D_3 \setminus B| + \frac{6}{8}|B| + \frac{6}{8}|D_2|\\ &= \frac{1}{2}(m-2)(n-2) + \frac{5}{8}(2m-8+2n-8) + \frac{6}{8}\cdot 8 + \frac{6}{8}\cdot 4\\ &= \frac{mn}{2} + \frac{m+n}{4} + 1.\end{aligned}$$

Here, the four sums have been estimated from below one by one with the following justification: By feasibility of f, every $v \in D_4$ has at least two neighbors with an f-value of 1; while for $v \in D_4 \setminus A$ this simply allows to conclude that $g(v) \geq 1/4 + 1/4$, for $v \in A \subseteq D_4$ we have to recognize that an inflowing charge of $1/4 + 1/4$ originates solely from the two degree-3 neighbors of v (due to the fact that both neighbors of a corner these are forced to attain an f-value of 1). The estimate for each degree-3 vertex not contained in B holds due to the fact, that again two neighbors with an f-value of 1 must exist leading in the lowest possible case to an inflow of charge $1/4+3/8 = 5/8$; concerning the estimate for the vertices in B this lowest case is slightly higher, namely $3/8+3/8 = 6/8$. By the aforementioned particularity of a corner, its charge inflow must be precisely $3/8 + 3/8 = 6/8$. □