August 2001

Polynomial–Time Recognition of Minimal Unsatisfiable Formulas with Fixed Clause–Variable Difference

Herbert Fleischner [a] Oliver Kullmann [b,1] Stefan Szeider [a,2]

[a] *Institute of Discrete Mathematics, Austrian Academy of Sciences, Sonnenfelsgasse 19, A–1010 Vienna, Austria*

[b] *Department of Computer Science, University of Toronto, Toronto, Ontario M5S 3G4*

Abstract

A formula (in conjunctive normal form) is said to be *minimal unsatisfiable* if it is unsatisfiable and deleting any clause makes it satisfiable. The deficiency of a formula is the difference of the number of clauses and the number of variables. It is known that every minimal unsatisfiable formula has positive deficiency. Until recently, polynomial–time algorithms were known to recognize minimal unsatisfiable formulas with deficiency 1 and 2. We state an algorithm which recognizes minimal unsatisfiable formulas with any fixed deficiency in polynomial time.

Key words: minimal unsatisfiable, SAT, deficiency, matching, autarky, polynomial time algorithm

1 Introduction

A formula F (in conjunctive normal form, CNF for short) is minimal unsatisfiable, if F is unsatisfiable, but omitting any clause yields a satisfiable formula.

[1] Supported by the Natural Sciences and Engineering Research Council of Canada and by the Communications and Information Technology Ontario.
[2] Supported by the Austrian Science Fund, P13417-MAT

Papadimitriou and Wolfe ([18]) showed that recognizing minimal unsatisfiable formulas is D^p–complete. D^p is the class of problems which can be considered as the difference of two NP–problems (D^p corresponds to the second level of the boolean hierarchy; see e.g., [10]).

For a formula F let $\delta(F)$ be the difference between the number of clauses of F and the number of variables occurring in F. Tarsi's Lemma ([1]) states that $\delta(F) \geq 1$ for every minimal unsatisfiable formula. Kleine Büning ([12]) showed that, if k is a fixed integer, then the recognition of minimal unsatisfiable formulas F with $\delta(F) \leq k$ is in NP.

Moreover, Kleine Büning conjectured the following ([12], see also [11]).

Conjecture 1 *For fixed integer k, it can be decided in polynomial time whether a formula F with $\delta(F) \leq k$ is minimal unsatisfiable.*

The main result of this paper is a proof of this conjecture [3]; we state an algorithm with running time $\mathcal{O}(\ell \cdot n^{k+1/2})$ where ℓ is the length and n the number of variables of the input formula.

So far, polynomial–time algorithms were only known for cases $\delta(F) = 1$ and $\delta(F) = 2$, with running time $\mathcal{O}(\ell^2)$ and $\mathcal{O}(n^3)$, respectively ([12,4]). Whence, in the cases $k = 1, 2$, the time complexity of our general algorithm is similar to the complexities of the quoted algorithms. (Note that $n = \mathcal{O}(\ell)$ and $\ell = \mathcal{O}(n^2)$.)

Zhao and Ding [19] considered formulas F with $\delta(F) = 3$ and $\delta(F) = 4$ satisfying a strong additional condition and obtained decision algorithms with running time $\mathcal{O}(n^5)$ and $\mathcal{O}(n^9)$, respectively.

[3] A preliminary version of this proof can be found in [6]; independently, in [13] Conjecture 1 has also been proven. The attempt in the present paper (and in [6]) can be seen as searching for a *satisfying truth assignment*, while [13] is based on searching for a *resolution refutation*.

2 Basic notations and results

2.1 Formulas

Let var be an infinite alphabet of *variables*; we will think of the elements of var as boolean variables. We define the *literals* to be elements of the form a or $\overline{a}$, where $a \in \mathsf{var}$. Literals which are variables are called positive; the others are called negative.

A *clause* is a finite set of literals not containing literals a and $\overline{a}$ at the same time, i.e., a clause is "non-tautological." A *formula* is a finite set of clauses. Thus, clauses do not contain "multiple occurrences" of literals, and formulas do not contain "multiple occurrences" of clauses. For a clause C we let $\mathsf{var}(C)$ be the set of variables a such that a or $\overline{a}$ is in C. For a formula F we put $\mathsf{var}(F) := \bigcup_{C \in F} \mathsf{var}(C)$.

The *length* of a formula F is given by $\sum_{C \in F} |C|$. Following [7] we call $\delta(F) := |F| - |\mathsf{var}(F)|$ the *deficiency of* F.

A *truth assignment* to a formula F is a map $f : \mathsf{var}(F) \to \{0, 1\}$. We define $f(\overline{a}) := 1 - f(a)$. Further, for $C \in F$ we define $f(C) := 1$ if $f(x) = 1$ for at least one literal $x \in C$; otherwise $f(C) := 0$. Furthermore, we put $f(F) := \min_{C \in F} f(C)$. (Sometimes we will also consider *partial truth assignments* to F, which are maps $f : S \to \{0, 1\}$ defined on a subset $S \subseteq \mathsf{var}(F)$.)

A formula F is *satisfied* by a truth assignment f if $f(F) = 1$. A formula F is called *satisfiable* if there exists a truth assignment which satisfies F; otherwise F is called *unsatisfiable*. Finally, a formula F is *minimal unsatisfiable*, if it is unsatisfiable but $F \setminus \{C\}$ is satisfiable for every $C \in F$.

2.2 Graphs and signed graphs

For graph theoretic terminology not defined here, the reader is referred to [5].

All *graphs* considered are finite and simple. For a graph G, the sets of *vertices* and *edges* are denoted by $V(G)$ and $E(G)$, respectively. $E_v(G)$ denotes the edges of G which are incident with a vertex $v \in G$. For $X, Y \subseteq V(G)$ we write

$E_G(X, Y)$ for the set of edges $e = xy \in E(G)$ with $x \in X$ and $y \in Y$. $N_G(v) := \{ w \in V(G) : vw \in E(G) \}$ is the set of *neighbors* of a vertex $v \in V(G)$; for $X \subseteq V(G)$ we put $N_G(X) := \left(\bigcup_{v \in X} N_G(v) \right) \setminus X$, and $\bar{N}_G(X) := N_G(X) \cup X$.

A graph G is *bipartite* if its vertices can be partitioned into two classes U and W such that no vertices of the same class are adjacent. We write $U(G)$ and $W(G)$ to denote a specific vertex–bipartition.

A *signing* Σ of a graph G is a map $\Sigma : E(G) \to \{+, -\}$ which assigns to each edge of G either $+$ or $-$. A graph G with a specified signing $\Sigma(G)$ is called a *signed graph*. We call an edge e of a signed graph *positive* (*negative*) if $\Sigma(e) = +$ ($\Sigma(e) = -$). The sets of positive and negative edges are denoted by $E^+(G)$ and $E^-(G)$, respectively. Similarly, for $\delta \in \{+, -\}$ we put $E_v^\delta(G) := E_v(G) \cap E^\delta(G)$ and $N_G^\delta(v) := \{ w \in V(G) : vw \in E^\delta(G) \}$. A vertex v of a signed graph G is a *sink* if $E_v^+(G) = \emptyset$; we put $W^-(G) := \{ w \in W(G) : w \text{ is a sink of } G \}$.

A set M of edges in a graph G is a *matching* if no two elements of M are adjacent. A vertex is *matched by M* if it is incident with an element of M. Let X be a set of vertices in G. A matching of G is X*–perfect* if all vertices in X are matched by M. The *matching number* of a graph G is defined by $\nu(G) := \max\{ |M| : M \text{ is a matching of } G \}$. A matching M of G is *maximum* if $|M| = \nu(G)$. A bipartite graph G has a $U(G)$–perfect ($W(G)$–perfect) matching if and only if $\nu(G) = |U(G)|$ ($\nu(G) = |W(G)|$).

A *cover* of a graph G is a set C of vertices such that every edge of G is incident with at least one vertex in C. The *covering number* of a graph G is defined by $\tau(G) := \min\{ |C| : C \text{ is cover of } G \}$. A cover C of G is *minimum* if $|C| = \tau(G)$. Note that if C is a cover of a bipartite graph G, then

$$E_G\Big(U(G) \setminus C, W(G) \setminus C\Big) = \emptyset.$$

3 Formula graphs

We use signed bipartite graphs to represent formulas.

Definition 1 Let F be a formula and G a signed bipartite graph. We call G the *formula graph* of F if there exist bijective maps $g : U(G) \to \mathsf{var}(F)$ and

$h : W(G) \to F$ such that

$$uw \in E^+(G) \text{ if and only if } g(u) \in h(w), \quad \text{and}$$
$$uw \in E^-(G) \text{ if and only if } \overline{g(u)} \in h(w).$$

Clearly, such formula graph of F always exists for given F; and since all formula graphs of a formula F are isomorphic, it is admissible to call G *the* formula graph of F. Moreover, formula graphs contain no loops or parallel edges. See Figure 1 for an example.

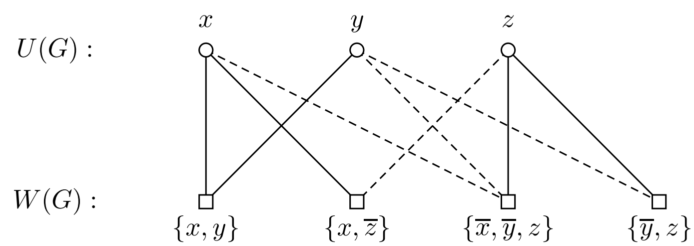

Fig. 1. Example of a formula graph of $F = \{\{x, y\}, \{x, \overline{z}\}, \{\overline{x}, \overline{y}, z\}, \{\overline{y}, z\}\}$. Positive edges are drawn by solid lines, negative edges by dashed lines.

In the following we summarize some observations which are easy to prove.

Lemma 1 *Let G be a signed bipartite graph.*

(1) G is the formula graph of some formula F if and only if $U(G)$ contains no isolates, and for $w, w' \in W(G)$, if $N_G^+(w) = N_G^+(w')$ and $N_G^-(w) = N_G^-(w')$, then $w = w'$.
(2) If G is the formula graph of a formula F, $W' \subseteq W(G)$, then the subgraph of G induced by $\bar{N}_G(W')$ is the formula graph of a subset of F.
(3) If G is the formula graph of a minimal unsatisfiable formula, then G is connected. (This follows from (2) and the definition of minimal unsatisfiability.)
(4) If G is the formula graph of a formula F, then $|E(G)|$ equals the length of F and $|W(G)| - |U(G)| = \delta(F)$.

Definition 2 Let G be the formula graph of a formula F and let $X \subseteq U(G)$. We obtain a signed graph $r_X(G) = G'$ from G by letting $\Sigma_{G'}(e) \neq \Sigma_G(e)$ if e is incident with a vertex in X, and $\Sigma_{G'}(e) = \Sigma_G(e)$ otherwise. Note that $V(G') = V(G)$ and $E(G') = E(G)$. We call G' a *flipping* of G.

An example of a flipping is exhibited in Figure 2. The binary relation between

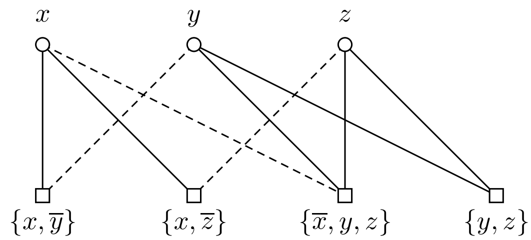

Fig. 2. A flipping (with $X = \{y\}$) of the formula graph in Figure 1.

formula graphs of being a flipping of each other is an equivalence relation. Flippings of formula graphs are closely related to renamings of formulas (c.f. [17]), where for a formula F and $A \subseteq \mathsf{var}(F)$ a formula $F' := r_A(F)$ is obtained by replacing in F every literal a by $\overline{a}$ and $\overline{a}$ by a whenever $a \in A$. Now, if G is the formula graph of F, then $r_X(G)$ is the formula graph of $r_A(F)$, where A is the set of variables which correspond to the vertices in X.

A formula F is satisfiable if and only if there is a renaming of F containing no negative clause (a clause is called negative if it contains no positive literal). The following lemma, which we shall use throughout this article, states this characterization in terms of formula graphs.

Lemma 2 *Let G be the formula graph of a formula F. Then F is satisfiable if and only if $W^-(G') = \emptyset$ for some flipping G' of G.*

PROOF. Let G be the formula graph of a formula F and $g : U(G) \to \mathsf{var}(F)$, $h : W(G) \to F$ bijections according to Definition 1. Assume that for F there is a truth assignment f to F which satisfies F. Let $X_f := \{\, u \in U(G) : f(g(u)) = 0 \,\}$ and consider the flipping $G' := r_{X_f}(G)$. Choose $w \in W(G) = W(G')$ arbitrarily, and put $C := h(w)$. Since $f(F) = 1$, there must be some literal $x \in C$ with $f(x) = 1$. If x is a positive literal (i.e., $x \in \mathsf{var}(F)$) let $u := g^{-1}(x) \in U(G)$. Consequently, $uw \in E^+(G)$ by Definition 1. Moreover, it follows that $u \notin X_f$; thus $uw \in E^+(G')$. On the other hand, if x is a negative literal, then $x = \overline{y} \in C$ for some $y \in \mathsf{var}(F)$. For $u := g^{-1}(y) \in U(G)$ it follows by Definition 1 that $uw \in E^-(G)$. However, $u \in X_f$ by the choice of x and because $x = \overline{y}$ by assumption; thus $uw \in E^+(G')$. We have therefore shown that every $w \in W(G') = W(G)$ is incident with some edge $uw \in E^+(G')$; whence $W^-(G') = \emptyset$.

Conversely, assume that for $X \subseteq U(G)$ and $G' := r_X(G)$ we have $W^-(G') = \emptyset$. We define a truth assignment f_X to F by setting $f_X(x) = 0$ if $g^{-1}(x) \in X$; otherwise $f_X(x) = 1$. Let $C \in F$ be an arbitrarily chosen clause and put $w := h^{-1}(C)$. Since $W^-(G') = \emptyset$, there is a vertex $u \in N^+_{G'}(w)$. If $u \notin X$, then $uw \in E^+_{G'}(u) = E^+_G(u)$. On the other hand, if $u \in X$, then $uw \in E^+_{G'}(u) = E^-_G(u)$. In the first case we have $x := g(u) \in C$; in the second case $\overline{x} := \overline{g(u)} \in C$. By definition of f_X and $f_X(C)$ it follows that $f_X(C) = 1$ in any case. Since $C \in F$ had been chosen arbitrarily, $f_X(F) = 1$ follows; i.e., F is satisfiable. $\square$

The following is an easy consequence of Lemma 2 and the definition of minimal unsatisfiability.

Lemma 3 *Let G be the formula graph of an unsatisfiable formula F. Then F is minimal unsatisfiable if and only if for every $w \in W(G)$ there is a flipping G' of G with $W^-(G') = \{w\}$.*

4 Matchings in signed graphs

Definition 3 Let G be a signed graph. A matching M of G is called *admissible* if $M \subseteq E^+(G)$.

Definition 4 Let M be a matching of a bipartite graph G. A path P of G is called *M–alternating* if the edges of M and $E(G) \setminus M$ alternate in P. An M–alternating path P is called *M–augmenting* if, say, it begins with an unmatched vertex in $U(G)$ and ends with an unmatched vertex in $W(G)$.

The following theorem is the main technical result of this paper; this result allows us to restrict our considerations (in testing for satisfiability) to truth assignments which correspond to matchings in the formula graph. For an application to the general SAT problem see Section 6 below.

Theorem 1 *For every signed bipartite graph G there is some flipping G^* of G and an admissible matching M^* of G^* such that*

$$|M^*| = \nu(G^*) = \nu(G) \quad \textit{and} \quad W^-(G^*) \subseteq W^-(G).$$

PROOF. Let M be an admissible matching of G of maximum cardinality. We proceed be induction on $d = \nu(G) - |M|$. If $d = 0$ then the theorem holds trivially. Hence suppose $d \geq 1$. By Berge's Theorem ([3], see, e.g., [16, Theorem 1.2.1]) G has some M–augmenting path. Choose an M–augmenting path P such that $\ell^-(P)$ is minimal, where $\ell^-(P)$ is the number of negative edges in P. We obtain a matching M^* of G by setting

$$M^* := (M - E(P)) \cup (E(P) - M)\,;$$

observe that $|M^*| = |M| + 1$. However, M^* is not necessarily an admissible matching of G. Let X be the set of vertices in $U(G)$ which are incident with negative edges in M^*. It follows by definition of M^* that $X \subseteq V(P)$. Moreover, M^* is an admissible matching in the flipping $G^* := r_X(G)$. Since $|M^*| > |M|$ it remains to show that $W^-(G^*) \subseteq W^-(G)$.

Suppose to the contrary that some $s \in W^-(G^*) \setminus W^-(G)$ exists (this situation is illustrated in Figure 3). Since M^* is admissible, s cannot be matched by

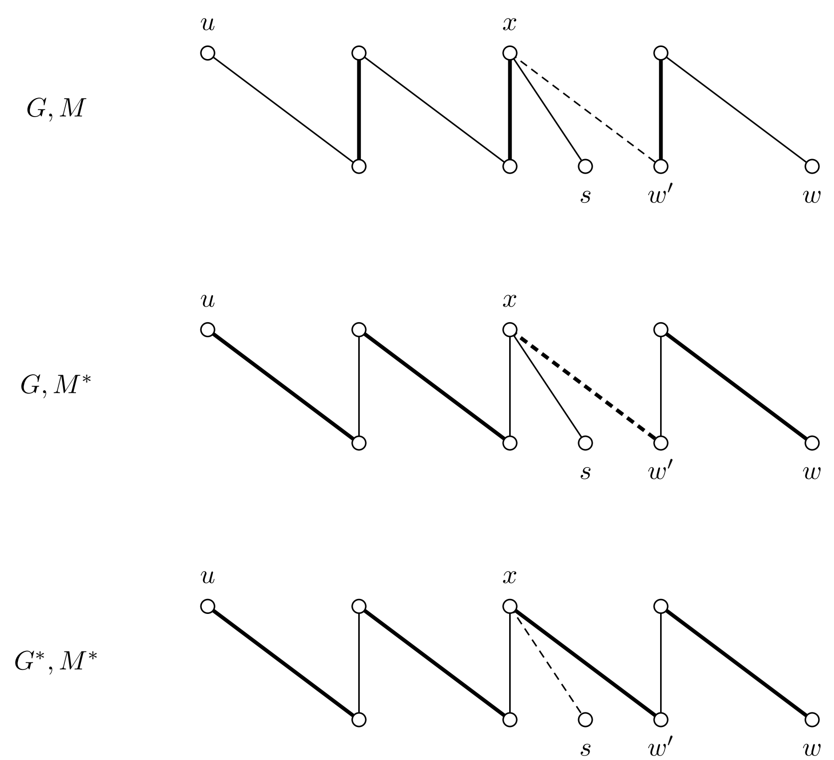

Fig. 3. Illustration for the (absurd) case that there is some $s \in W^-(G^*) \setminus W^-(G)$.

M^*. We observe that every $y \in W$ which is matched by M, is also matched by M^*; hence s is not matched by M as well. Since s became a sink through a flipping, we have $sx \in E^-(G^*)$ and $sx \in E^+(G)$ for some $x \in X \subseteq V(P)$.

Let $u \in U(G)$ and $w \in W(G)$ be the end–vertices of P. We split P into two paths $P_{u,x}$ and $P_{x,w}$ connecting u to x and x to w, respectively. Since $x \in X$, $P_{x,w}$ starts with an edge $xw' \in E^-(G)$, therefore $\ell^-(P_{x,w}) \geq 1$. Thus

$$\ell^-(P) = \ell^-(P_{u,x}) + \ell^-(P_{x,w}) \geq \ell^-(P_{u,x}) + 1. \tag{4.1}$$

Consider now the path P' from u to s obtained by juxtaposition of $P_{u,x}$ and the edge $xs = sx$. We observe that P' is an M–augmenting path with $\ell^-(P') = \ell^-(P_{u,x})$. By equation (4.1), $\ell^-(P') < \ell^-(P)$, a contradiction to the choice of P. Hence $s \in W^-(G^*) \setminus W^-(G)$ cannot exist; therefore, $W^-(G^*) \subseteq W^-(G)$ holds true. Since $\nu(G^*) - |M^*| < d$, the theorem follows now by induction. □

In this paper, we are faced several times with the problem of finding a matching of maximum cardinality in a bipartite graph G with $p = |V(G)|$ and $q = |E(G)|$. Therefore we can apply the well–known maximum cardinality matching algorithm of Hopcroft and Karp for bipartite graphs ([9]). Galil obtained the asymptotic bound $\mathcal{O}(q \cdot p^{1/2})$ for Hopcroft and Karp's algorithm, [8]. Hence we can state the following.

Theorem 2 *Let G be a bipartite graph with $n = |U(G)|$, and $\ell = |E(G)|$. If $k = |W| - |U|$ is fixed, then we can find a maximum matching of G in time $\mathcal{O}(\ell \cdot n^{1/2})$.*

Alt et al. ([2]) stated a matching algorithm with running time $\mathcal{O}(p^{3/2}\sqrt{q/\log p})$ which improves Hopcroft and Karp's algorithm for dense graphs. Consequently, applying the latter algorithm improves the running times of subsequently stated algorithms if formulas with dense formula graphs are considered.

5 Minimal unsatisfiability and the parameter k

The following is an unpublished result of Tarsi (see [1]). It is an easy consequence of Theorem 3 below.

Lemma 4 (Tarsi's Lemma) *If F is a minimal unsatisfiable formula, then $\delta(F) \geq 1$.*

For generalizations of Tarsi's Lemma see [14,15].

Theorem 3 ([1]) *Let G be the formula graph of a formula F. Then the following hold.*

(1) If G has a $W(G)$–perfect matching, then G is satisfiable.
(2) If F is minimal unsatisfiable, then G has a $U(G)$–perfect matching.

The preceding theorem holds also for infinite formulas, which is irrelevant, however, for the following considerations.

Next we state an algorithm by which satisfiability of a formula can be decided, provided that its formula graph G has a $U(G)$–perfect matching (in Section 6 we shall see how this algorithm can be applied to an arbitrary formula by first modifying the latter).

> **Algorithm** MATCHSAT
> **input:** a signed bipartite graph G with $\nu(G) = |U|$;
> $k := |W(G)| - |U(G)|$;
> **for all** $U_k \subseteq U(G)$ with $|U_k| = \min(k, |U(G)|)$ **do**
>   **for all** $X \subseteq U_k$ **do**
>   **let** $G' := r_X(G)$;
>   **let** $G'' := G' \setminus (U_k \cup N^+_{G'}(U_k))$;
>   **if** $\nu(G'') = |W(G'')|$ **return** 'yes';
>   **od**
> **od**
> **return** 'no';

Let a truth assignment f to a formula F be called a *matching truth assignment* if there exists an injective map $\phi : F \to \mathsf{var}(F)$ satisfying

$$\{\phi(C), \overline{\phi(C)}\} \cap C \neq \emptyset$$

and

$$f(\phi(C)) = \begin{cases} 1 & \text{if } \phi(C) \in C, \\ 0 & \text{otherwise,} \end{cases}$$

for all $C \in F$. Now algorithm MATCHSAT can be interpreted as running through all partial truth assignments f using at most k variables and checking whether after application of f (i.e., removing clauses which are satisfied by f and literals whose variable is in the domain of f) a formula is obtained which is satisfiable by a matching truth assignment.

Lemma 5 *Let G with $\nu(G) = |U(G)|$ be the formula graph of a formula F with $\delta(F) = k \geq 0$. Then* MATCHSAT(G) = 'yes' *if and only if F is satisfiable.*

PROOF. Suppose MATCHSAT(G) = 'yes'. There is a set $U_k \subseteq U(G)$ with $|U_k| = \min(k, |U(G)|)$ and $X \subseteq U_k$ such that for the flipping $G' := r_X(G)$ of G and for $Y := N^+_{G'}(U_k)$, the graph $G'' := G' \setminus (U_k \cup Y)$ has a matching M'' with

$$|M''| = |W(G'')| \,. \tag{5.2}$$

Let $X^* \subseteq U(G'')$ be the set of vertices in $U(G'')$ which are incident with negative edges in M''. We observe that $M'' \subseteq E^+(r_{X^*}(G'))$.

Consider the flipping $G^* := r_{X^*}(G')$. Note that $X \cap X^* = \emptyset$, whence G^* is the flipping of G w.r.t. $X \cup X^*$, i.e., $G^* = r_{X \cup X^*}(G)$. Since $X^* \cap U_k = \emptyset$ we have $N^+_{G^*}(U_k) = N^+_{G'}(U_k) = Y$. Thus

$$W^-(G^*) \cap Y = \emptyset. \tag{5.3}$$

On the other hand, by (5.2), every vertex in $W(G^*) \setminus Y = W(G'')$ is matched by M''; and since $X \cap U(G'') = \emptyset$, the matching M'' is an admissible matching in G^*. Whence

$$W^-(G^*) \setminus Y = \emptyset. \tag{5.4}$$

Combining (5.3) and (5.4) yields $W^-(G^*) = \emptyset$. Thus, since G^* is a flipping of G, it follows now by Lemma 2 that F is satisfiable.

Conversely, assume that F is satisfiable. By Lemma 2 there is some flipping $G^* = r_{X^*}(G)$ such that $W^-(G^*) = \emptyset$; in view of Theorem 1 we may assume that G^* has a $U(G^*)$–perfect admissible matching M^* (note that $\nu(G) = |U(G)|$ by hypothesis). Let $W_k = \{w_1, \ldots, w_k\}$ be the set of vertices in $W(G^*)$ which are not matched by M^*. Observe that $N^+_{G^*}(w_i) \neq \emptyset$ for $1 \leq i \leq k$; hence we can choose some $u_i \in N^+_{G^*}(w_i)$ for $1 \leq i \leq k$ (possibly $u_i = u_j$ for $i \neq j$). Now consider any set $U_k \subseteq U(G)$ with $|U_k| = \min(k, |U(G)|)$ such that $u_i \in U_k$ $(1 \leq i \leq k)$. Put $X := X^* \cap U_k$ and let G' and G'' be the graphs as defined in Algorithm MATCHSAT w.r.t. X and U_k. Let $M'' := M^* \cap E(G'')$ be the (not necessarily admissible) matching in G''. It remains to show that M'' is $W(G'')$–perfect. Every $w \in W(G'') = W(G) \setminus N^+_{G'}(U_k) = W(G^*) \setminus N^+_{G^*}(U_k)$ is matched by some edge $e = uw \in M^*$. If $u \in U_k$ then $w \in N^-_{G'}(U_k) = N^-_{G^*}(U_k)$, and so $e \in E^-(G^*)$ which cannot be the case, since $M^* \subseteq E^+(G^*)$

by assumption. Thus $u \notin U_k$ and $e \in E(G'')$. It follows that M'' is in fact a $W(G'')$–perfect matching, which implies that $\nu(G'') = |W(G'')|$. Whence the lemma is shown true. $\square$

Lemma 6 *Let G be a signed bipartite graph with $\ell = |E(G)|$, $n = |U(G)|$, and fixed $k = |W(G)| - |U(G)|$. Then the Algorithm* MATCHSAT *runs with input G in time $\mathcal{O}(\ell \cdot n^{k+1/2})$.*

PROOF. Let $k' := \min(n, k)$. There are at most $\binom{n}{k'}$ different possibilities for choosing U_k; for each choice of U_k there are $2^{k'}$ possibilities for $X \subseteq U_k$. Hence, the instructions of the inner loop of the algorithm are performed at most $2^{k'}\binom{n}{k'} \leq 2^{k'}n^{k'}/k'! = \mathcal{O}(n^k)$ times. Thus, by Theorem 2, the claimed asymptotic bound follows. $\square$

For the following considerations let G^w be the subgraph of G induced by $\bar{N}_G(W(G) \setminus \{w\})$, i.e., $W(G^w) = W(G) \setminus \{w\}$ and $U(G^w) = N_G(W(G) \setminus \{w\})$.

Note that $U(G^w)$ contains no vertex u for which $N_G(u) = \{w\}$. Moreover, if G is the formula graph of a formula F and $w \in W(G)$, then G^w is the formula graph of $F \setminus \{C\}$ for some $C \in F$ (c.f. Lemma 1(2)).

The next algorithm makes use of MATCHSAT in deciding whether a given unsatisfiable formula is minimal unsatisfiable.

> **Algorithm MU**
> **input:** a formula graph G of an unsatisfiable formula with $\nu(G) = |U|$;
> **for all** $w \in W(G)$ **do**
>   **if** $\nu(G^w) < |U(G)|$ **then return** 'no' **fi**
>   **if** MATCHSAT(G^w) = 'no' **then return** 'no' **fi**
> **od**
> **return** 'yes'.

Lemma 7 *Let G with $\nu(G) = |U(G)|$ be the formula graph of an unsatisfiable formula F. Then* MU(G) = 'yes' *if and only if F is minimal unsatisfiable.*

PROOF. Let $h : W(G) \to F$ be a bijective map according to Definition 1. Assume MU(G) = 'yes', i.e., $\nu(G^w) = |U(G)| = |U(G^w)|$ and MATCHSAT(G^w) = 'yes' for all $w \in W(G)$. We show that $F' := F \setminus \{C\}$ is satisfiable for every clause $C \in F$: let $w \in W(G)$ such that $h(w) = C$. We observe that G^w is the formula graph of F'. Since $\nu(G^w) = |U(G^w)|$ it now follows by Lemma 5 (since MATCHSAT(G^w) = 'yes'), that F' is satisfiable. Because this holds for every $w \in W(G)$, therefore F is minimal unsatisfiable.

Conversely, assume that F is minimal unsatisfiable. Let $w \in W(G)$ be chosen arbitrarily and let $C \in F$ such that $h(w) = C$. By Lemma 3, there is a flipping G' of G such that $W^-(G') = \{w\}$. Moreover, by Theorem 1 there is a flipping G^* of G' and thus of G such that G^* has an admissible matching M^* with $|M^*| = \nu(G^*) = \nu(G)$ and $W^-(G^*) \subseteq W^-(G')$. Since F is unsatisfiable, $W^-(G^*) = \{w\}$ follows of necessity; and by the hypothesis $\nu(G) = |U(G)|$, it also follows that M^* is $U(G)$–perfect. Since $W^-(G^*) = \{w\}$, $M^* \subseteq E^+(G^*)$ does not match w. Therefore, $M^* \subseteq E(G^w)$ and thus $|M^*| = |U(G)| = |U(G^w)| = \nu(G^w)$. Applying again Lemma 5, we obtain that MATCHSAT(G^w) = 'yes' for all $w \in W(G)$. Whence MU(G) = 'yes'. $\Box$

Lemma 8 *Let G be a signed bipartite graph with $\ell = |E(G)|$, $n = |U(G)|$, and fixed positive $k = |W(G)| - |U(G)|$. Then the Algorithm* MU *runs with input G in time $\mathcal{O}(\ell \cdot n^{k+1/2})$.*

PROOF. For $w \in W(G)$, the matching number $\nu(G^w)$ can be computed in

$$\mathcal{O}(\ell \cdot n^{1/2}) \tag{5.5}$$

steps (see Theorem 2). If $\nu(G^w) = |U(G)|$, then $|W(G^w)| - |U(G^w)| = k - 1$. Consequently, since $|E(G^w)| < \ell$, it follows by Lemma 6 that MATCHSAT(G^w) requires at most

$$\mathcal{O}(\ell \cdot n^{(k-1)+1/2}) \tag{5.6}$$

steps. For $k \geq 0$, the estimate (5.6) absorbs (5.5). Since Algorithm MU considers at most $|W(G)| = \mathcal{O}(n)$ different choices for w, the claimed time complexity follows. $\Box$

Theorem 4 (Main Theorem) *Given a positive integer k, consider a formula F of length ℓ with n variables and such that $\delta(F) = k$. Then it can be decided in time $\mathcal{O}(\ell \cdot n^{k+1/2})$ whether F is minimal unsatisfiable.*

PROOF. We consider the formula graph G of F. Consequently, $\ell = |E(G)|$, $n = |U(G)|$, and $k = |W(G)| - |U(G)|$. Now we compute the matching number of G in time

$$\mathcal{O}(\ell \cdot n^{1/2}) \tag{5.7}$$

(c.f. Theorem 2). If $\nu(G) < n$ then F cannot be minimal unsatisfiable by Theorem 3. Hence assume $\nu(G) = n$. Now the hypotheses of Lemmas 5 and 6 are fulfilled, and we can test whether F is unsatisfiable in time

$$\mathcal{O}(\ell \cdot n^{k+1/2}). \tag{5.8}$$

If F is satisfiable, then F cannot be minimal unsatisfiable. Hence assume F is unsatisfiable. Now we can apply Lemmas 7 and 8, and test whether F is minimal unsatisfiable in time

$$\mathcal{O}(\ell \cdot n^{k+1/2}). \tag{5.9}$$

In view of the asymptotic estimates (5.7), (5.8), (5.9), the theorem follows. □

Thus Conjecture 1 is shown to be true.

6 Polynomial time SAT-decision based on bounded maximum deficiency

In this section we will indicate how Algorithm MATCHSAT can be made applicable for deciding satisfiability of an arbitrary formula.

Definition 5 The *maximum deficiency* of a bipartite graph G is defined by

$$\delta^*(G) := \max\{\, |Y| - |N_G(Y)| : Y \subseteq W(G)\,\}.$$

If G is the formula graph of a formula F, then we put $\delta^*(F) := \delta^*(G)$.

Note that the maximum deficiency of a bipartite graph is always non–negative, since for $Y = \emptyset$ we have $|Y| - |N_G(Y)| = 0$. Moreover, for a formula F we have

$$\delta^*(F) = \max\{\, \delta(F') : F' \subseteq F\,\}$$

(see [14,15] for a more detailed investigation of the maximum deficiency of formulas). By the following well–known result (see, e.g., [16, Theorem 1.3.1]), the maximum deficiency of a bipartite graph can be computed in polynomial time.

Lemma 9 *The maximum deficiency* $\delta^*(G)$ *of every bipartite graph* G *equals* $|W(G)| - \nu(G)$.

Lemma 10 *Every formula* F *can be transformed efficiently into a formula* F^* *such that*

- $\nu(G^*) = |U(G^*)|$ *for the formula graph* G^* *of* F^**;*
- $\delta(F^*) = \delta^*(F)$*;*
- F^* *is satisfiable if and only if* F *is satisfiable.*

PROOF. Let G be the formula graph of F and M a maximum matching of G. We obtain a set $C \subseteq V(G)$ by choosing for each edge $uw \in M$, ($u \in U(G)$, $w \in W(G)$) one of its end vertices as follows: if some M–alternating path P which starts in an unmatched vertex in $W(G)$ ends in u, then we choose u; otherwise we chose w (this implies that if P ends in u, then every $u' \in V(P) \cap U(G)$ is also in C). Thus $|C| = |M|$. Note that C can be obtained by breadth–first–search in linear time. It follows from the proof of Kőnig's Minimax Theorem ([5, Theorem 2.1.1]) that C is a minimum cover of G. Put $C_U := C \cap U(G)$, $C_W := C \cap W(G)$, and let G^* be the subgraph of G induced by $(W(G) \setminus C) \cup C_U$. Since C is a cover, we have

$$E_G(U(G) \setminus C, W(G) \setminus C) = \emptyset.$$

Hence $N_G(W(G) \setminus C) = C_U$, and so it follows that G^* is the formula graph of some $F^* \subseteq F$ (see Lemma 1(2)). By construction of C it follows that every vertex $u \in C_U = U(G^*)$ is incident with some edge $e = uw \in M$ with $w \notin C_W$; hence $e \in E(G^*)$. It follows that $M \cap E(G^*)$ is a $U(G^*)$–perfect matching in G^*, consequently $\nu(G^*) = |U(G^*)|$.

By construction of G^* we have $|W(G)| - \nu(G) = |W(G^*)| - |U(G^*)|$; i.e., $\delta(F^*) = \delta^*(F)$ by definition and Lemma 9, respectively. Hence it remains to show that F^* is satisfiable if and only if F is satisfiable. Clearly, if F is satisfiable, then so is $F^* \subseteq F$. Hence assume that F^* is satisfiable, i.e., there is a flipping $H^* = r_{Z^*}(G^*)$, $Z^* \subseteq U(G^*)$, such that $W^-(H^*) = \emptyset$. Let Z be the set of vertices in $U(G) \setminus C$ which are incident with some negative edge in M; it follows that $Z^* \cap Z = \emptyset$. We consider the flipping $H := r_{Z \cup Z^*}(G)$. Every $w \in W(H) \setminus C = W(H^*)$ is incident with a positive edge $e \in E^+(H^*) \subseteq E^+(H)$; and every $w \in W(H) \cap C = C_W$ is incident with some $e \in M \subseteq E^+(H)$. Thus $W^-(H) = \emptyset$, which implies that F is satisfiable. $\square$

Note that F^* is the normal form studied in [15] obtained by reduction with "matching autarkies."

Theorem 5 *For every fixed integer k, the satisfiability of a formula F with $\delta^*(F) \leq k$ can be decided in polynomial time.*

PROOF. Let F a formula with $\delta^*(F) \leq k$ be given. We first obtain in polynomial time a formula F^* in accordance with the proof of Lemma 10. Lemmas 5 and 6 apply to F^*, hence we can decide in polynomial time whether F^* is satisfiable. By the preceding lemma, F is satisfiable if and only if F^* is satisfiable. $\square$

7 Concluding remarks

We have presented polynomial–time algorithms

- for recognizing minimal unsatisfiable formulas with bounded deficiency, and
- for deciding the satisfiability of formulas with bounded maximum deficiency.

The key to our results is Theorem 1 which generalizes the concept of augmenting paths to signed graphs.

In both cases our algorithms use a "try all subsets of size k" strategy—is this an essential feature of the problem, or can we do better?

References

[1] R. Aharoni and N. Linial. Minimal non-two-colorable hypergraphs and minimal unsatisfiable formulas. *J. Combin. Theory Ser. A*, 43:196–204, 1986.

[2] H. Alt, N. Blum, K. Mehlhorn, and M. Paul. Computing a maximum cardinality matching in a bipartite graph in time $O(n^{1.5}\sqrt{m/\log n})$. *Information Processing Letters*, 37(4):237–240, 1991.

[3] C. Berge. Two theorems in graph theory. *Proc. Nat. Acad. Sci. U.S.A.*, 43:882–844, 1957.

[4] G. Davidov, I. Davydova, and H. Kleine Büning. An efficient algorithm for the minimal unsatisfiability problem for a subclass of CNF. *Annals of Mathematics and Artificial Intelligence*, 23:229–245, 1998.

[5] R. Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer Verlag, 2nd edition, 2000.

[6] H. Fleischner and S. Szeider. Polynomial-time recognition of minimal unsatisfiable formulas with fixed clause-variable difference. Technical Report TR00–049, Electronic Colloquium on Computational Complexity (ECCC), 2000.

[7] J. Franco and A. V. Gelder. A perspective on certain polynomial time solvable classes of satisfiability. To appear in Discr. Appl. Math., 2001.

[8] Z. Galil. Efficient algorithms for finding maximal matching in graphs. In *CAAP '83 (L'Aquila, 1983)*, volume 159, pages 90–113. Springer Verlag, 1983.

[9] J. E. Hopcroft and R. M. Karp. An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs. *J. Comput.*, 2:225–231, 1973.

[10] D. S. Johnson. A catalog of complexity classes. In J. van Leewen, editor, *Handbook of Theoretical Computer Science*, volume A, chapter 2, pages 67–161. Elsevier Science Publishers, North-Holland, 1990.

[11] H. Kleine Büning. An upper bound for minimal resolution refutations. In G. Gottlob, E. Grandjean, and K. Seyr, editors, *CSL'98*, volume 1584, pages 171–178. Springer Verlag, 1999.

[12] H. Kleine Büning. On subclasses of minimal unsatisfiable formulas. *Discr. Appl. Math.*, 107(1–3):83–98, 2000.

[13] O. Kullmann. An application of matroid theory to the SAT problem. In *Fifteenth Annual IEEE Conference of Computational Complexity*, pages 116–124, 2000. See also TR00-018, Electronic Colloquium on Computational Complexity (ECCC), March 2000.

[14] O. Kullmann. Investigations on autark assignments. *Discr. Appl. Math.*, 107(1-3):99–137, 2000.

[15] O. Kullmann. Lean clause-sets: Generalizations of minimally unsatisfiable clause-sets. Submitted to Discr. Appl. Math., June 2000.

[16] L. Lovász and M. D. Plummer. *Matching Theory*, volume 29 of *Annals of Discrete Mathematics*. Elsevier Science Publishers, North-Holland, 1986.

[17] B. Meltzer. Theorem-proving for computers: some results on resolution and renaming. *Comp. J.*, 8:341–343, 1966.

[18] C. H. Papadimitriou and D. Wolfe. The complexity of facets resolved. *J. of Computer and System Sciences*, 37(1):2–13, 1988.

[19] X. Zhao and D. Ding. Two tractable subclasses of minimal unsatisfiable formulas. *Science in China Ser. A*, 42(7):720–731, 1999.