Two-Layer Planarization in Graph Drawing

Petra Mutzel and René Weiskircher

Max-Planck-Institut für Informatik, Im Stadtwald, D-66123 Saarbrücken
mutzel@mpi-sb.mpg.de, weiski@mpi-sb.mpg.de

Abstract. We study the two-layer planarization problems that have applications in Automatic Graph Drawing. We are searching for a two-layer planar subgraph of maximum weight in a given two-layer graph. Depending on the number of layers in which the vertices can be permuted freely, that is, zero, one or two, different versions of the problems arise. The latter problem was already investigated in [11] using polyhedral combinatorics. Here, we study the remaining two cases and the relationships between the associated polytopes.
In particular, we investigate the polytope $\mathcal{P}_1$ associated with the two-layer *planarization* problem with one fixed layer. We provide an overview on the relationships between $\mathcal{P}_1$ and the polytope $\mathcal{Q}_1$ associated with the two-layer *crossing minimization* problem with one fixed layer, the linear ordering polytope, the two-layer planarization problem with zero and two layers fixed. We will see that all facet-defining inequalities in $\mathcal{Q}_1$ are also facet-defining for $\mathcal{P}_1$. Furthermore, we give some new classes of facet-defining inequalities and show how the separation problems can be solved. First computational results are presented using a branch-and-cut algorithm. For the case when both layers are fixed, the two-layer planarization problem can be solved in polynomial time by a transformation to the heaviest increasing subsequence problem. Moreover, we give a complete description of the associated polytope $\mathcal{P}_2$, which is useful in our branch-and-cut algorithm for the one-layer fixed case.

1 Introduction

A *bipartite graph* is a graph $G = (A, B, E)$ with vertex sets A and B, called upper and lower layer, and an edge set E connecting a vertex in A with a vertex in B. There are no edges between two vertices in the same layer. A bipartite graph is *two-layer planar* $G = (A, B, E)$ if it can be drawn in such a way that all the vertices in A appear on a line (the upper line), the vertices in B appear on the lower line, and the edges are drawn as straight lines without crossing each other. The difference between a planar bipartite graph and a two-layer planar bipartite graph is obvious. For example, the graph shown in Fig. 1 is a planar bipartite graph, but not a two-layer planar graph.

Depending on the number of layers in which the permutation of the vertices is fixed, different problems arise:

– The permutations π_A and π_B of both layers A and B are fixed: Given a two-layer graph $G = (A, B, E, \pi_A, \pi_B)$ with weights $w_e > 0$ on the edges,

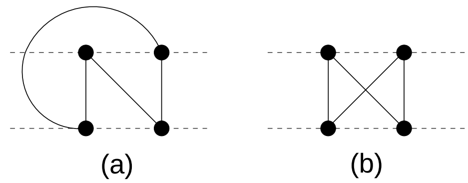

Fig. 1. (a) A planar bipartite graph that is (b) not 2-layer planar

the *two-layer planarization problem (2 layers fixed)* is to extract a subgraph $G' = (A, B, F, \pi_A, \pi_B)$, $F \subseteq E$, of maximum weight, i.e., the sum $\sum_{e \in F} w_e$ is maximum, which contains no crossings with respect to the given permutations π_A and π_B.

– The permutation π_A of one layer A is fixed: Given a two-layer graph $G = (A, B, E, \pi_A, \bullet)$ with weights $w_e > 0$ on the edges, the *two-layer planarization problem (1 layer fixed)* is to extract a subgraph $G' = (A, B, F, \pi_A, \bullet)$, $F \subseteq E$, of maximum weight, which contains no crossings with respect to the given permutation π_A of the upper layer.
– Both layers can be permuted: Given a two-layer graph $G = (A, B, E, \bullet, \bullet)$ with weights $w_e > 0$ on the edges, the *two-layer planarization problem (none layer fixed)* is to extract a two-layer planar subgraph $G' = (A, B, F, \bullet, \bullet)$, $F \subseteq E$, of maximum weight.

To our knowledge, only the unweighted ($w_e = 1$ for all $e \in E$) two-layer planarization problems have been considered in the literature so far. Eades and Whitesides [4] showed NP-hardness for the latter two versions of the planarization problem and showed that the two layer fixed version can be solved by transforming it to a longest increasing subsequence problem. The none layer fixed version was first mentioned in [15]. The authors introduced the problem in the context of graph drawing. Recently, the weighted two-layer planarization problem has been attacked, in which the layers are allowed to be permuted freely [11]. The computational results are encouraging.

Directed graphs are widely used to represent structures in many fields such as economics, social sciences, mathematics and computer science. A good visualization of structural information allows the reader to focus on the information content of the diagram.

A common method for drawing directed graphs has been introduced by Sugiyama et al. [14] and Carpano [2]. In the first step, the vertices are partitioned into a set of k layers, and in the second step, the vertices within each layer are permuted in such a way that the number of crossings is small. In practice, this is done layerwise. Keep the permutation of one layer fix while permuting the other one, such that the number of crossings is reduced. We suggest an alternative approach for the second step.

Already for two-layer graphs the straight-line crossing minimization problem is NP-hard [6] even if one layer is fixed [5]. Exact algorithms based on branch and bound have been suggested by various authors (see, e.g., [9]). For $k \geq 2$, a vast amount of heuristics has been published in the literature (see, e.g., [14] and [3]). A new approach is to remove a minimal set of edges such that the remaining k-layer graph can be drawn without edge crossings. In the final drawing, the removed edges are reinserted. Since the insertion of each edge may produce many crossings, the final drawing may be far from an edge-crossing minimal drawing.

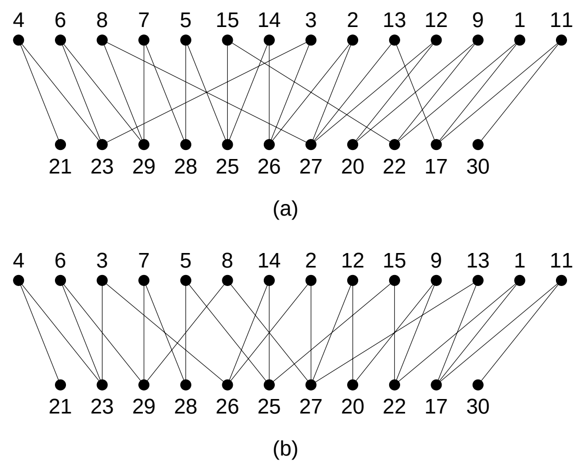

Fig. 2. A graph (a) drawn using k-planarization and (b) drawn with the minimal number of crossings computed by the algorithm in [9]

Figure 2(a) shows a drawing of a graph obtained by two-layer planarization, whereas Fig. 2(b) shows the same graph drawn with the minimal number of edge crossings (using the exact algorithm given in [9]). Although the drawing in Fig. 2(a) has 34 crossings, that is 41% more crossings than the drawing in Fig. 2(b) (24 crossings), the reader will not recognize this fact. This encourages us to study the k-layer planarization problem. We decided to first study the case $k = 2$ in order to learn for the general case $k \geq 3$.

In Sect. 2 we define the polytope $\mathcal{P}_1$ associated with the set of all possible two-layer planar subgraphs with respect to a given fixed permutation π_A. We then point out the relationships to related polytopes. This gives us hints about the structure of $\mathcal{P}_1$. In Sect. 3 we give a complete description of the polytope associated with all two-layer planar subgraphs when both permutations are fixed. This description is useful in the algorithm for solving the two-layer planarization problem (1 layer fixed case). Moreover, it provides a different polynomial time algorithm for solving the two-layer planarization problem (2 layers fixed). In Sect. 4, we investigate the structure of the polytope $\mathcal{P}_1$. We present an irredundant integer linear programming formulation and obtain additional classes of inequalities that tighten the associated LP-relaxation. In particular, besides some new classes of facets, we can show that all facet-defining inequalities of the linear ordering polytope transmit to the new polytope $\mathcal{P}_1$. In order to get practical use out of these inequalities, we have to solve the "separation problem". This question will be addressed in Sect. 5, where we also discuss a branch-and-cut algorithm based on those results. First computational results with a branch-and-cut algorithm are presented in Sect. 6. In this extended abstract we omit the proofs for most of the theorems.

2 The Polytope $\mathcal{P}_1$ and Its Related Polyhedra

Let us consider the two-layer planarization problem with one fixed layer more precisely: Given a two-layer planar graph $G = (A, B, E, \pi_A, \bullet)$ with a fixed permutation π_A of the vertices in A, we are looking for a permutation π_B of the vertices in B, and a subset F of edges in E such that the subgraph $G' = (A, B, F, \pi_A, \pi_B)$ is two-layer planar under the given permutations π_A and π_B of the sets A and B, respectively.

We introduce variables y_{uv} for $1 \le u < v \le |B|$ representing the permutation π_B of the vertices in B. That is, $y_{uv} = 1$ iff vertex u is before vertex v in π_B and $y_{uv} = 0$ otherwise. We denote the (row) vector $\bar{y} = (y_{1,2}, y_{1,3}, \ldots, y_L)$ with $L = \binom{B}{2}$. (Vectors are row vectors throughout the paper.) Moreover, we introduce variables x_e for $1 \le e \le |E|$ representing the subgraph induced by F. Variable x_e takes value 1 iff $e \in F$ and value 0 otherwise. For any tuple (π_B, F), where π_B is a permutation and $F \subseteq E$, we define an incidence vector $\chi^{(\pi_B,F)} \in \mathrm{R}^{L+|E|}$ with the i-th component $\chi^{(\pi_B,F)}(e_i)$ getting value 1 iff $e_i \in F$ and 0 if $e_i \notin F$ for $i > L$, and the j-th component $\chi^{(\pi_B,F)}(y_{uv})$ getting value 1 if vertex u is before vertex v in π_B and 0 otherwise for $j \le L$.

Now, we can define the two-layer planar subgraph polytope

$$\mathcal{P}_1 = \mathcal{P}_1(A, B, E, \pi_A, \bullet) = \mathrm{conv}\Big\{\chi^{(\pi_B,F)} \mid \pi_B \text{ is a linear ordering and } G' = (A, B, F, \pi_A, \pi_B) \text{ is a two-layer planar subgraph of } G\Big\}$$

as the convex hull of all incidence vectors $\chi^{(\pi_B,F)}$ that represent a two-layer planar subgraph $G' = (A, B, F, \pi_A, \pi_B)$ with respect to the valid orderings π_A and π_B.

For solving the two-layer crossing minimization problem with one fixed layer, we consider the polytope $\mathcal{Q}_1$ (see [9]). Again, we introduce variables $y_{ij} \in \{0, 1\}$ representing the permutation of the vertices in B. The incidence vector $\chi^{\pi_B} \in \mathrm{R}^L$ has the j-th component $\chi^{\pi_B}(y_{uv})$ value 1 if vertex u is before v and 0 otherwise. The polytope

$$\mathcal{Q}_1 = \mathcal{Q}_1(A, B, E, \pi_A, \bullet) = \mathrm{conv}\Big\{\chi^{\pi_B} \mid \pi_B \text{ is a permutation of the vertices in} B\Big\}$$

is identical to the linear ordering polytope that has been studied in [7]. If we denote the points in $\mathcal{P}_1$ by $(\bar{y}, \bar{x})$, where $\bar{y} \in \mathrm{R}^L$, $\bar{x} \in \mathrm{R}^{|E|}$, we have the following relationship between the two polytopes $\mathcal{P}_1$ and $\mathcal{Q}_1$: $\mathcal{Q}_1 \cong P_1 \cap \{\bar{x} = 0\}$.

This fact will lead us to investigate the hereditary property of the facet-defining inequalities in $\mathcal{Q}_1$ for $\mathcal{P}_1$.

The polytope $\mathcal{P}_0$ associated with the two-layer planarization problem with two free layers (none fixed) has been introduced in [11]. Again, we denote an incidence vector $\chi^F \in R^E$ having the i-th component $\chi^F(x_e)$ value 1 if $x_e \in F$ and value 0 if $x_e \notin F$.

$$\begin{aligned}\mathcal{P}_0 = \mathcal{P}_0(A, B, E, \bullet, \bullet) = \operatorname{conv}\Big\{\chi^F \mid &\text{There exist orderings } \pi_A \text{ and } \pi_B \text{ such that} \\ &G' = (A, B, F, \pi_A, \pi_B) \text{ is a two-layer planar subgraph of } G\Big\} = \\ &= \operatorname{conv}\Big\{\chi^F \mid G' = (A, B, F, \bullet, \bullet) \text{ is a two-layer planar subgraph of } G\Big\}\end{aligned}$$

We can use our knowledge of the studied polyhedra $\mathcal{Q}_1$ and $\mathcal{P}_0$ for our investigation of $\mathcal{P}_1$. In particular, all facet-defining inequalities of $\mathcal{Q}_1$ and $\mathcal{P}_0$ are still valid inequalities for $\mathcal{P}_1$. Moreover, we will see that all facet-defining inequalities of $\mathcal{Q}_1$ are still facet-defining for $\mathcal{P}_1$.

Let us consider the two-layer planarization problem when the permutations of both layers are fixed. We define

$$\begin{aligned}\mathcal{P}_2 = \mathcal{P}_2(A, B, E, \pi_A, \pi_B) = \operatorname{conv}\{\chi^F \mid F \subseteq E &\text{ is a two-layer planar subgraph of} \\ G &\text{ with respect to the orderings } \pi_A \text{ and } \pi_B\}\end{aligned}$$

We have $\mathcal{P}_1 \cap \{\bar{y} = 0\} \supseteq \mathcal{P}_2$. In the following Section we will consider the structure of the polytope $\mathcal{P}_2$.

3 A Complete Description of the Polytope $\mathcal{P}_2$

In this Section we will consider the two-layer planarization problem when both layers are fixed. The set of all two-layer planar subgraph of $G = (A, B, E, \pi_A, \pi_B)$ defines an independence system $\mathcal{I}_P(G) = (E, \{F \mid F \subseteq E$ induces a two-layer planar graph$\})$ on E. Let us examine the circuits and the cliques of this independence system. *Circuits* are the minimal dependent sets in $(E, \mathcal{I})$ with respect to set inclusion. An independence set is called *k-regular* if each of its circuits is of size k. The set $F \subseteq E$ is a *clique* of $(E, \mathcal{I})$, if$|F| \geq k$ and all $\binom{|F|}{k}$ k-subsets of F are circuits of $(E, \mathcal{I})$. In [12] it is shown that a maximal clique $F \subseteq E$ in a k-regular independence system $(E, \mathcal{I})$ gives a facet-defining inequality, namely, the *clique inequality*

$$\sum_{e \in F} x_e \leq k - 1, \tag{1}$$

for $P_{\mathcal{I}}$, the polytope associated with $(E, \mathcal{I})$. The set of circuits in our system $\mathcal{I}(G)$ is

$$\mathcal{S} = \{\{(p, v), (q, u)\} \mid \pi_A(p) < \pi_A(q), \pi_B(u) < \pi_B(v), (p, v), (q, u) \in E\}.$$

Hence, $\mathcal{I}(G)$ is a 2-regular independence system. The maximal cliques in $\mathcal{I}(G)$ are the maximal sets of pairwise intersecting edges (with respect to set inclusion). We show, that the associated maximal clique inequalities and the trivial inequalities define the polytope $\mathcal{P}_2$.

Theorem 1. *The maximal clique inequalities of $\mathcal{I}(G)$ together with the inequalities $x_e \leq 1$ that are not contained in any clique and the trivial inequalities $0 \leq x_e$ for $e = 1, \ldots, |E|$, give a complete irredundant description of the two-layer planar subgraph polytope $\mathcal{P}_2$ (both layers fixed).*

Proof. (sketch) To proof the claim, we build a directed graph R with a single source s and a single sink t where every node apart of s and t corresponds to an edge in E. When every node has the capacity given by the corresponding component of a vector x in $[0, 1]^{|E|}$, than x belongs to polytope $\mathcal{P}_2$ if and only if there is no path in R from s to t where the sum of the capacities of the nodes is greater than 1. Every path from s to t corresponds to a maximal clique in $(E, \mathcal{I})$ and so a path where the sum of the capacities exceeds one corresponds to a violated clique inequality.

Every vector in $\mathcal{P}_2$ corresponds to a capacity function on the nodes of R such that there is no path from s to t where the sum of the capacities is greater than one. By shifting capacities in R, we can show that for every weighting of the edges in E and for every vector x in $\mathcal{P}_2$, there is another vector x' in $\mathcal{P}_2$ with the property that every component is either 0 or 1 and the sum of the weights of the edges whose nodes in R have capacity 1 is at least as large as the corresponding sum for x. Thus, we have a complete description of $\mathcal{P}_2$.

Since the separation problem for the clique inequalities can be solved in polynomial time (see Sect. 5), this yields a polynomial time algorithm for the two-layer planarization problem via the Ellipsoid method.

There is also a combinatorial algorithm for solving the problem. Eades and Whitesides [4] give a transformation of the unweighted two-layer planarization problem to the longest increasing subsequence problem. A similar transformation to the heaviest increasing subsequence problem works for the weighted version of the problem.

Lemma 1. *By transforming the two-layer planarization problem to an instance of the heaviest increasing subsequence problem, it can be solved in time* $\mathrm{O}(|E| \log |E|)$.

Both theorems are not surprising, since there are similar results for the trace polytope $\mathcal{T}_2$ on two sequences that has been introduced in [13] in the context of multiple sequence alignment. The set of circuits in the independence system $\mathcal{I}_T(G)$ is

$$\begin{aligned} \mathcal{S} &\cup \{\{(p,u),(p,v)\} \mid \pi_B(u) < \pi_B(v), (p,u) \in E, (p,v) \in E\} \\ &\cup \{\{(p,u),(q,u)\} \mid \pi_A(p) < \pi_A(q), (p,u) \in E, (q,u) \in E\}. \end{aligned}$$

In the following, we investigate the relations between the two-layer planar subgraph polytope $\mathcal{P}_2$ and the trace polytope $\mathcal{T}_2$.

Lemma 2. *Let $G = (V, E)$ be a graph. There exist transformations from G to $G' = (V', E')$ and $G'' = (V'', E'')$ with $E' \cong E \cong E''$, $|V'| = |V''| = 2|E|$, and*

$$\mathcal{P}_2(G) \cong \mathcal{P}_2(G') \cong \mathcal{T}_2(G') \text{ and } \mathcal{T}_2(G) \cong \mathcal{T}_2(G'') \cong \mathcal{P}_2(G'').$$

4 The Structure of the Polytope $\mathcal{P}_1$

First, we give an integer linear programming formulation for the two-layer planarization problem with one fixed layer. The notation is based on the previous Section. Let $G = (A, B, E, \pi_A, \bullet)$ be a two-layer graph and let $\bar{w} \in \mathrm{N}^{|E|}$ be the cost vector on the edges. Then, the two-layer planarization problem is to solve

$$\max\{\bar{w}\bar{x}^T \mid (\bar{y}, \bar{x}) \in \mathcal{P}_1, \bar{y} \in \mathrm{R}^L, \bar{x} \in \mathrm{R}^{|E|}\}.$$

We are interested in the integer points of $\mathcal{P}_1$.

Theorem 2. *The integer points of the two-layer planar subgraph polytope $\mathcal{P}_1 = \mathcal{P}_1(A, B, E, \pi_A, \bullet)$ are characterized by the following system of inequalities:*

$$\begin{aligned}
-y_{uv} - y_{vw} + y_{uw} &\le 0 && 1 \le u < v < w \le |B| && (2)\\
y_{uv} + y_{vw} - y_{uw} &\le 1 && 1 \le u < v < w \le |B| && (3)\\
0 \le y_{uv} &\le 1 && 1 \le u < v < w \le |B| && (4)\\
y_{uv} &\ \textit{integral} && 1 \le u < v < w \le |B| && (5)\\
y_{uv} + x_{(p,u)} + x_{(q,v)} &\le 2 && u < v, \pi_A(q) < \pi_A(p), (p,u), (q,v) \in E && (6)\\
-y_{uv} + x_{(p,u)} + x_{(q,v)} &\le 1 && u < v, \pi_A(p) < \pi_A(q), (p,u), (q,v) \in E && (7)\\
0 \le x_e &\le 1 && 1 \le e \le |E| && (8)\\
x_e &\ \textit{integral} && 1 \le e \le |E| && (9)
\end{aligned}$$

Proof. The inequalities (2)-(5) require the variables to represent a linear ordering π_B. Inequalities (6)-(9) are responsible for introducing no crossing with respect to the ordering π_B given by the vector $\bar{y}$. In particular, inequalities (6) and (7) link together the subgraph variables $\bar{x}$ and the linear ordering vertices $\bar{y}$. A crossing between two edges (p, u) and (q, v) occurs either if $\pi_A(q) < \pi_A(p)$ and u is before v in the ordering π_B given by $\bar{y}$, or if $\pi_A(p) < \pi_A(q)$ and v is before u in π_B.

Next, we address the question if the description given in Theorem 2 is tight.

Theorem 3. *The description given in Theorem 2 is an irredundant description of the two-layer planar subgraph polytope $\mathcal{P}_1 = \mathcal{P}_1(A, B, E, \pi_A, \bullet)$. In particular, the inequalities (2)-(4) and (6)-(8) are facet-defining for $\mathcal{P}_1$.*

In order to prove the facet-defining property of the inequalities, it is essential to know the dimension of the polytope.

Lemma 3. *The dimension of the two-layer planar subgraph polytope $\mathcal{P}_1 = \mathcal{P}_1(A, B, E, \pi_A, \bullet)$ is $L + |E|$, where $L = \binom{B}{2}$.*

Proof. We know from [7] that the linear ordering polytope is full dimensional. For every ordering of the nodes, a two-layer graph with only one edge is two-layer planar. So we can easily construct a set of $L + |E|$ affinely independent vectors that correspond to two-layer planar graphs.

In Sect. 2 we have seen that $\mathcal{P}_1$ is closely related to the linear ordering polytope $\mathcal{Q}_1$. The following theorem gives us the possibility to use the knowledge of the well-studied polytope $\mathcal{Q}_1$ for $\mathcal{P}_1$.

Theorem 4. *Let $\bar{c}\bar{y}^T \leq c_0$ be a facet-defining inequality of the linear ordering polytope $\mathcal{Q}_1 = \mathcal{Q}_1(A, B, E, \pi_A, \bullet)$. Then $\bar{c}\bar{y}^T \leq c_0$ is also facet-defining for the two-layer planar subgraph polytope $\mathcal{P}_1 = \mathcal{P}_1(A, B, E, \pi_A, \bullet)$.*

For the rest of this Section we will concentrate on new facet-defining inequalities for $\mathcal{P}_1$. Our practical experiments have supported the need for inequalities containing only $\bar{x}$-Variables.

We define a blocker $B = (u, l, r)$ to be a subgraph of $G = (A, B, E, \pi_A, \bullet)$ containing the edges (l, u) and (r, u) with $\pi_A(l) < \pi_A(r)$. We use the notation $x(B) = x_{(l,u)} + x_{(r,u)}$. A blocker forbids certain edges. An edge $e = (p, w)$ *crosses* blocker $B = (u, l, r)$ iff $\pi_A(l) < \pi_A(p) < \pi_A(r)$ and $w \neq u$. This fact leads to *e-blocker inequalities* which are valid and in some cases facet-defining for $\mathcal{P}_1$. Figure 3 shows some examples of configurations leading to these inequalities.

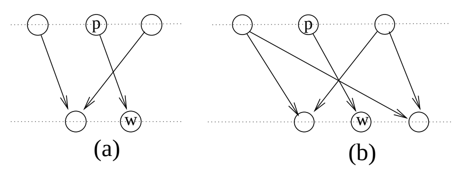

Fig. 3. Examples for support graphs of facet-defining e-blocker inequalities

Theorem 5. *Let $B_1, \ldots, B_k$ be a set of blockers $B_i = (u_i, l_i, r_i)$ with $u_i \neq u_j$ for $i, j = 1, \ldots, k$, $i \neq j$, and let $e = (p, w) \in E$ be an edge which crosses all blockers B_i. Then, the* e-blocker inequality

$$\sum_{i=1}^{k} x(B_i) + x_e \leq k + 1 \tag{10}$$

is valid for $\mathcal{P}_1(A, B, E, \pi_A, \bullet)$. If $l_i = l_j$ and $r_i = r_j$ for $i, j = 1, \ldots, k$, it is facet-defining for $\mathcal{P}_1(A, B, E, \pi_A, \bullet)$ (even for $k = 1$).

5 The Algorithm and Separation Routines

The *separation problem* is to decide for a given vector $\bar{x}$ and a polytope $\mathcal{P}$, whether $\bar{x} \in \mathcal{P}$, and, if $\bar{x} \notin \mathcal{P}$, find a vector $\bar{d}$ and a scalar d_0 such that the inequality $\bar{d}\bar{x}^T \leq d_0$ is valid with respect to $\mathcal{P}$ and $\bar{d}\bar{x}^T > d_0$.

Lemma 4. *The separation problems for the inequalities (2)-(4),(6)-(8), and (10) can be solved in polynomial time.*

For the two-layer planarization problem with one fixed layer, we implemented a branch-and-cut algorithm using the ABACUS-System [10]. Because of space limits, we cannot describe our branch-and-cut algorithm in more detail. We use separation routines for the inequalities given in Lemma 4 in order to get good upper bounds. Moreover, we try to use some information given to us by fractional solutions in order to get good lower bounds.

Our studies of the two-layer fix planarization problem is useful in two ways: For getting good lower bounds, we frequently use the combinatorial algorithm for the two-layer fix version given in Lemma 1. The upper bounds can be improved using the following strategy: In every branching step on variables in $\bar{y}$, we select a variable y_{uv} and set it to either 0 or 1. In the subproblems below this branching node, we have decided on a partial order for the vertices in B. For the partially ordered subsets, we can use the inequalities given by the complete description for $\mathcal{P}_2$ (see Sect. 3). Next, we will see that the separation problem for the inequalities (1) can be solved in polynomial time.

Theorem 6. *For the maximal clique inequalities, the separation problem can be solved in polynomial time by computing at most $|E|$ shortest path problems.*

According to earlier results (e.g., [8]), we can optimize a linear objective function over a polytope in polynomial time if and only if we can solve the separation problem in polynomial time. Hence, Theorem 6 gives us a polynomial time algorithm for solving the two-layer planarization problem when both layers are fixed.

6 Computational Results

To test the performance of our branch-and-cut algorithm for the two-layer planarization problem (1 layer fixed), we worked with the graphs from [1] that are called the *North DAGs*. These directed acyclic graphs have 10 to 100 nodes. We distributed them into sets G_i with i running form 1 to 9 such that the set G_i holds the graphs where the number of nodes is at least $10i$ and at most $10(i+1)-1$. We worked on 12 randomly chosen graphs out of each of the sets G_i. For each of the graphs, we distributed the nodes into pairwise disjoint sets L_j (called *layers*) such that for all edges the start-node is on a layer with smaller index than the end-node. This can be done using topological sorting.

After Inserting some dummy nodes, we get for each graph a number of bipartite graphs that consist of the nodes on two neighboring layers and the edges between these nodes. For a graph with k layers, we get $k-1$ bipartite graphs B_1 to B_{k-1} where B_i consists of the layers L_i and L_{i+1}. We used B_1 as input for our algorithm for solving the two-layer planarization problem (none layer fixed) resulting in a permutation for the layers L_1 and L_2. Then we applied the algorithm for the problem with one fixed layer to the rest of the problems beginning with B_2. Every optimization was stopped after 5 minutes if no optimum solution was found before. The following tabular shows for each set G_i the average optimization time (in seconds) for each layer and the maximum time used in any of the graphs of the set on a Sun Ultra Sparc 2/2x200. In all the 108 graphs we tested, there were only 4 graphs for which a planarization-problem could not be solved to optimality in 5 minutes computation time.

Nodes	10-19	20-29	30-39	40-49	50-59	60-69	70-79	80-89	90-99
Average	0.17	7.07	19.96	1.73	1.37	150.61	12.64	73.07	5.83
Maximum	1.1	17.0	78.53	14.1	27.04	300.16	116.8	300.14	50.16

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