Computing Unsatisfiable k-SAT Instances with Few Occurrences per Variable

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Abstract

(k, s)-SAT is the propositional satisfiability problem restricted to instances where each clause has exactly k distinct literals and every variable occurs at most s times. It is known that there exists an exponential function f such that for $s \leq f(k)$ all (k, s)-SAT instances are satisfiable, but $(k, f(k) + 1)$-SAT is already NP-complete ($k \geq 3$). Exact values of f are only known for $k = 3$ and $k = 4$, and it is open whether f is computable. We introduce a computable function f_1 which bounds f from above and determine the values of f_1 by means of a calculus of integer sequences. This new approach enables us to improve the best known upper bounds for $f(k)$, generalizing the known constructions for unsatisfiable (k, s)-SAT instances for small k.

Keywords: (k, s)-SAT, minimal unsatisfiable formulas, NP-completeness, integer sequences

1 Introduction

We consider CNF formulas represented as sets of clauses. Let k, s be fixed positive integers. We denote by (k, s)-CNF the set of formulas F where every clause of F has *exactly* k different literals and each variable occurs in *at most* s clauses of F. We denote the sets of satisfiable and unsatisfiable formulas by SAT and UNSAT, respectively.

It was observed by Tovey [12] that all formulas in $(3, 3)$-CNF are satisfiable, and the satisfiability problem restricted to $(3, 4)$-CNF is already NP-complete. This was generalized in Kratochvíl, et al. [7] where it is shown that for every $k \geq 3$ there is some integer $s = f(k)$ such that

1. all formulas in (k, s)-CNF are satisfiable, and
2. $(k, s + 1)$-SAT, the SAT problem restricted to $(k, s + 1)$-CNF, is already NP-complete.

The function f can be defined for positive integers k by the equation

$$f(k) := \max\{\, s : (k, s)\text{-CNF} \cap \text{UNSAT} = \emptyset \,\}.$$

From [12] it follows that $f(3) = 3$ and $f(k) \geq k$ for $k > 3$.

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Asymptotic upper and lower bounds for $f(k)$ have been obtained in [7, 9, 5]. Since typical formulas arising in practice have clauses of small width, it is interesting to know the exact values of $f(k)$ for small k. However, it is not known whether f is computable.

Dubois [4] constructs unsatisfiable formulas in $(4, 6)$-CNF and $(5, 11)$-CNF, respectively, which implies $4 \leq f(4) \leq 5$ and $5 \leq f(5) \leq 10$. As reported in [9], Stříbrná shows in her M.Sc. thesis [10] that $(4, 5)$-CNF contains unsatisfiable formulas, hence $f(4) = 4$. More recently, Berman, et al. [2] construct unsatisfiable formulas belonging to the classes $(3, 4)$-CNF, $(4, 6)$-CNF, $(5, 9)$-CNF, improving Dubois' upper bound for $f(5)$ to 8.

The quoted constructions are quite involved. We present a new and simple technique for generating unsatisfiable (k, s)-CNF formulas. By this new technique we can improve on best known upper bounds for $f(k)$; Table 1 gives an overview of upper bounds for $f(k)$.

By means of a construction due to Kratochvíl, et al. [7], one can construct from any unsatisfiable (k, s)-CNF formula an unsatisfiable $(k + 1, 2s)$-CNF formula; thus

$$f(k + 1) \leq 2f(k) + 1. \tag{1}$$

By generalization of a theorem by Savický and Sgall [9] one can derive the inequality $f(3k) \leq 3 \cdot 4^{k-1} f(k)$, yielding an asymptotic improvement over (1). The best known asymptotic upper bound is $f(k) \leq 2^k \cdot 8 \log_e k/k$ for $k \geq 2$, Hoory and Szeider [5]. However, for small k, (1) in conjunction with genuinely constructed formulas is preferable.

		Tov[12]	Dub[4]	Stř[10]	BKS[2]	this paper
3	$\leq f(3) \leq$	3	3	3	3	3
4	$\leq f(4) \leq$	7*	5	4	5	4
5	$\leq f(5) \leq$	15*	10	9*	8	7
7	$\leq f(6) \leq$	31*	21*	19*	17*	11
13	$\leq f(7) \leq$	63*	43*	39*	35*	17
24	$\leq f(8) \leq$	127*	87*	79*	71*	29
41	$\leq f(9) \leq$	255*	175*	159*	143*	51

Table 1: Best known lower and upper bounds of $f(k)$ for small k. Entries labeled by an asterisk are obtained via equation (1) from the preceding value of the respective paper. The lower bounds are taken from [2].

Our approach is to focus on a certain class MU(1) of unsatisfiable formulas. Formulas in MU(1) have a simple structure and can be constructed in a recursive way (see the next section). Therefore it is easier to search for unsatisfiable formulas in (k, s)-CNF $\cap$ MU(1) than in (k, s)-CNF.

For $k \geq 3$ let $f_1(k)$ denote the largest integer such that (k, s)-CNF$\cap$MU(1) $= \emptyset$. Since all formulas in MU(1) are unsatisfiable, always $f(k) \leq f_1(k)$ holds. Our examples below show that $f(k) = f_1(k)$ for $k = 3, 4$. It is interesting to know whether $f(k) = f_1(k)$ holds for $k \geq 5$.

We show that the existence of an unsatisfiable (k, s)-CNF formula in MU(1) is equivalent to a search problem on ordered integer sequences. This formulation lead to a saturation algorithm that calculates $f_1(k)$ exactly in time $O(4^{k^2})$. The next theorem summarizes the results we have obtained so far by running a C++ implementation of the saturation algorithm.

Theorem 1.
The following classes contain unsatisfiable formulas: $(3,4)$-CNF, $(4,5)$-CNF, $(5,8)$-CNF, $(6,12)$-CNF, $(7,18)$-CNF, $(8,30)$-CNF. $(9,52)$-CNF. *Hence, the satisfiability problem restricted to any of these classes is* NP-*complete.*

The existence of unsatisfiable formulas in $(5,8)$-CNF and $(6,12)$-CNF is certified by the derivations given in Fig. 3 and the appendix, respectively. For the other classes mentioned in Theorem 1, computer-generated certificates can be found in a file archive, available at the authors' homepages. The values of $f_1(k)$ are 3, 4, 7, 11, 17, 29, 51 for k = 3, 4, 5, 6, 7, 8, 9, respectively. The concise certificates we present prove the upper bound on $f_1(k)$. Proving that these bounds on $f_1(k)$ are exact requires re-running our program.

2 The Class MU(1)

A CNF formula is *minimal unsatisfiable* if it is unsatisfiable and removing any of its clauses makes it satisfiable. We denote the class of minimal unsatisfiable CNF formulas by MU. Since every unsatisfiable formula F has a minimal unsatisfiable subset F', and since $F \in (k,s)$-CNF implies $F' \in (k,s)$-CNF, we can restrict ourselves to the class MU. In other words,

$$f(k) = \max\{\, s : (k,s)\text{-CNF} \cap \text{MU} = \emptyset \,\}.$$

The *deficiency* $\delta(F)$ of a formula with n variables and m clauses is defined as $\delta(F) = m - n$. It is known that formulas in MU have always positive deficiency [1]; therefore it is natural to parameterize MU by deficiency and to consider the classes $\text{MU}(d) := \{\, F \in \text{MU} : \delta(F) = d \,\}$ for $d \geq 1$.

Let us consider the function

$$f_1(k) = \max\{\, s : (k,s)\text{-CNF} \cap \text{MU}(1) = \emptyset \,\}. \qquad (2)$$

Evidently, we have $f_1(k) \geq f(k)$, and so any upper bound for $f_1(k)$ is also an upper bound for $f(k)$. In the sequel we will show that f_1 is computable, and that for small k we can actually compute the exact value of $f_1(k)$.

Formulas in MU(1) have been widely studied (see, e.g., [1, 3, 8, 6, 11]). In particular, the following result of Davydov, et al. [3] (a proof is implicitly present in [1]), shows that formulas in MU(1) can be recursively decomposed ($\mathsf{var}(F)$ denotes the set of variables which occur (positively or negatively) in the formula F).

Lemma 1 (Davydov, et al. [3]). $F \in \text{MU}(1)$ *if and only if either* $F = \{\emptyset\}$ *or* F *is the disjoint union of formulas* F_1', F_2' *such that for a variable* x *we have*

- $\mathsf{var}(F_1') \cap \mathsf{var}(F_2') = \{x\}$ *and* $\{x, \overline{x}\} \subseteq \bigcup_{C \in F} C$;
- $F_1 := \{\, C \setminus \{x\} : C \in F_1' \,\} \in \text{MU}(1)$;
- $F_2 := \{\, C \setminus \{\overline{x}\} : C \in F_2' \,\} \in \text{MU}(1)$.

If F has a variable x with the properties stated in the above lemma, then following [6] we call the pair (F_1, F_2) a *disjunctive splitting of F in x*. Note that $x \notin var(F_1) \cup var(F_2)$ since the minimal unsatisfiable formulas F_1 and F_2 contain no pure literals. Furthermore we call the number of clauses of F in which x occurs the *degree* of the splitting (F_1, F_2).

For example, the formula $F = \{\{x, z\}, \{\overline{x}, y\}, \{\overline{y}, z\}, \{\overline{z}, w\}, \{\overline{z}, \overline{w}\}\}$ belongs to MU(1) since it can be decomposed by disjunctive splittings as displayed in Fig. 1. Note that $F \in (2, 4)$-CNF since all clauses have size 2 and every variable occurs at most 4 times. In general, if we decompose a formula F by splittings of degree $\leq s$, then evidently every variable of F occurs in at most s clauses. Hence we have the following lemma.

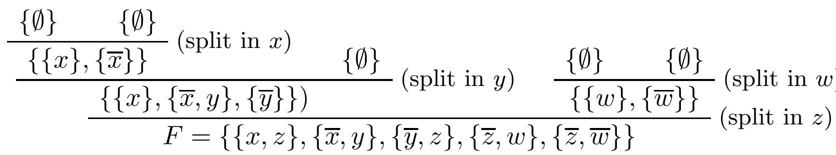

Figure 1: Decomposition of a formula $F \in$ MU(1) by disjunctive splittings.

Lemma 2. *If all clauses of a nonempty formula F have size k, then $F \in (k, s)$-CNF $\cap$ MU(1) if and only if F can be decomposed by disjunctive splittings of degree $\leq s$.*

3 A Calculus of Integer Sequences

Let $\sigma = (a_1, \ldots, a_n)$ be a *finite nonincreasing sequence of positive integers* (a *stairway*, for short). That is, $a_1 \geq \cdots \geq a_n \geq 1$. We call a_i an *entry* of σ, n the *length* of σ, and denote the *empty sequence* by ε. For a finite sequence of nonnegative integers σ let σ^{ord} denote the stairway obtained from σ by removing 0's and by ordering the entries nonincreasingly.

For a fixed integer $s \geq 2$ we consider the (nondeterministic) binary rule $N(s)$ that allows to infer a stairway σ from stairways σ_1, σ_2 as follows: For $i = 1, 2$ obtain σ_i' from σ_i by decrementing $s_i \geq 1$ entries by one, $s_1 + s_2 \leq s$, and put $\sigma := (\sigma_1' \sigma_2')^{\mathrm{ord}}$.

For dealing formally with the rule $N(s)$ in the proofs below, the following concept is convenient. Consider stairways $\sigma_1 = (a_1, \ldots, a_j)$ and $\sigma_2 = (a_{j+1}, \ldots, a_m)$. The definition of $N(s)$ says that a stairway σ can be inferred from σ_1, σ_2 if and only if there is a set $I \subseteq \{1, \ldots, m\}$ with $I \cap \{1, \ldots, j\} \neq \emptyset$, $I \cap \{j+1, \ldots, m\} \neq \emptyset$, and $|I| \leq s$ such that $\sigma = (a_1', \ldots, a_m')^{\mathrm{ord}}$ where

$$a_i' = \begin{cases} a_i - 1 & \text{if } i \in I; \\ a_i & \text{otherwise.} \end{cases}$$

We call the set I an *index set* associated with the inference. Note that the index set I is not necessarily unique.

An $N(s)$-*derivation* is a finite binary rooted tree T whose vertices are labeled by stairways such that if a vertex v labeled by σ has parents v_1, v_2 labeled by

σ_1, σ_2, respectively, then σ can be *inferred* from σ_1, σ_2 by the rule $N(s)$. For a set of stairways Γ and a stairway σ we write $\Gamma \vdash_{N(s)} \sigma$ if there is an $N(s)$-derivation T whose root is labeled by σ and whose leaves are labeled by sequences from Γ. In particular, we have $\Gamma \vdash_{N(s)} \sigma$ if $\sigma \in \Gamma$. If Γ is a singleton $\{\sigma'\}$ we simply write $\sigma' \vdash_{N(s)} \sigma$.

As an example, the $N(4)$-derivation displayed in Fig. 2 shows that $(3) \vdash_{N(4)} (1,1,1,1,1)$.

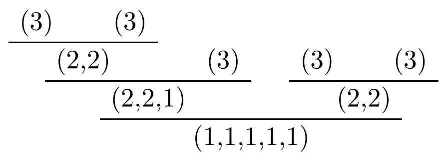

Figure 2: An $N(4)$-derivation.

Let $F = \{C_1, \ldots, C_m\} \neq \emptyset$ be a formula with $0 \leq |C_1| \leq \cdots \leq |C_m| \leq k$, and let n be the largest integer in $\{1, \ldots, m\}$ with $|C_n| < k$. We associate with F the stairway

$$\Sigma_k(F) := (k - |C_1|, \ldots, k - |C_n|).$$

Thus, $\Sigma_k(F)$ is the empty sequence if all clauses of F have size k.

The next lemma, which can be shown by induction, asserts that $N(s)$-derivations and formulas in $\mathrm{MU}(1) \cap (k, s)$-CNF are closely related.

Lemma 3. *For every stairway σ the following holds true. $(k) \vdash_{N(s)} \sigma$ if and only if there is a formula $F \in \mathrm{MU}(1)$ such that (i) $\Sigma_k(F) = \sigma$, (ii) all clauses of F have size at most k, and (iii) F can be decomposed by disjunctive splittings of degree $\leq s$.*

Proof. ($\Rightarrow$) Assume $(k) \vdash_{N(s)} \sigma$ and let T be an $N(s)$-derivation of σ from (k) with a minimal number n of inference steps (we count every non-leaf of T as an inference step). We proceed by induction on n. If $n = 0$ then σ is the axiom (k) and we put $F = \{\emptyset\}$. Clearly $\Sigma_k(F) = (k)$ and we are done. Now assume $n \geq 1$, and let σ_1, σ_2 be the stairways from which σ is inferred in T. Let $\sigma_1 = (a_1, \ldots, a_j)$, $\sigma_2 = (a_{j+1}, \ldots, a_m)$, and $\sigma = (c_1, \ldots, c_n)$. Let $I \subseteq \{1, \ldots, m\}$ be an index set associated with the inference of σ from σ_1, σ_2, so that we can write $\sigma = (a'_1, \ldots, a'_m)^{\mathrm{ord}}$.

By induction hypothesis (the subderivations of T ending in σ_1 and σ_2, respectively, have less than n steps), there are formulas $F_1, F_2 \in \mathrm{MU}(1)$ with $\Sigma_k(F_i) = \sigma_i$ such that F_i can be decomposed by disjunctive splittings of degree $\leq s$. We may assume that F_1 and F_2 do not share a variable (we can always rename variables). Let F'_i be the subset of F_i containing all clauses of size k, $i = 1, 2$. Since $\Sigma_k(F_i) = \sigma_i$, we can write $F_1 = \{C_1, \ldots, C_j\} \cup F'_1$ and $F_2 = \{C_{j+1}, \ldots, C_m\} \cup F'_2$ such that $a_i = k - |C_i|$ for $i = 1, \ldots, m$. We pick a new variable x and define $F := \{D_1, \ldots, D_m\} \cup F'_1 \cup F'_2$ where

$$D_i = \begin{cases} C_i \cup \{x\} & \text{if } i \in I \text{ and } i \leq j \\ C_i \cup \{\overline{x}\} & \text{if } i \in I \text{ and } i > j, \\ C_i & \text{otherwise.} \end{cases}$$

Consequently, (F_1, F_2) is a disjunctive splitting of F of degree $\le s$. Since $\Sigma_k(F) = \sigma$, the first part of the lemma is shown true.

($\Leftarrow$) Let $F \in \mathrm{MU}(1)$, $\Sigma_k(F) = \sigma$, be decomposable by disjunctive splittings of degree $\le s$. We show by induction on the number n of variables of F that $(k) \vdash_{N(s)} \sigma$. If $n = 0$ then $F = \{\emptyset\}$ and so $\sigma = (k)$; hence $(k) \vdash_{N(s)} \sigma$. Now assume $n > 0$. By assumption, F has a disjunctive splitting (F_1, F_2) of degree $\le s$. Let $\sigma_i := \Sigma_k(F_i)$, $i = 1, 2$. Since $|\mathsf{var}(F_i)| \le |\mathsf{var}(F)| - 1$, it follows by induction hypothesis that $(k) \vdash_{N(s)} \sigma_i$, $i = 1, 2$. It remains to show that σ can be inferred from σ_1, σ_2 by the rule $N(s)$.

By definition of a disjunctive splitting, F is the disjoint union of formulas F_1', F_2' such that for a variable x the conditions stated in Lemma 1 are satisfied. Consequently, for some nonempty subsets $G_i \subseteq F_i$, $i = 1, 2$, we have

$$\begin{aligned} F_1' &= \{\, C \cup \{x\} : C \in G_1 \,\} \cup (F_1 \setminus G_1), \\ F_2' &= \{\, C \cup \{\overline{x}\} : C \in G_2 \,\} \cup (F_2 \setminus G_2). \end{aligned}$$

Since the splitting is of degree $\le s$, $|G_1| + |G_2| \le s$ follows. Every clause in $G_1 \cup G_2$ corresponds bijectively to an entry a of σ_i which is decreased by one (thus either $a \ge 2$ and $a - 1$ is an entry of σ, or $a = 1$ and $a - 1$ is omitted in σ). The other clauses $C \in F_i \setminus G_i$ with $|C| < k$ correspond bijectively to entries $a = k - |C|$ of σ_i which give rise to entries of σ. Thus σ can indeed be inferred from σ_1, σ_2 by the rule $N(s)$ and so $(k) \vdash_{N(s)} \sigma$ follows. □

Note that in general there are many different formulas corresponding to one $N(s)$-derivation in the sense of Lemma 3.

For the example in Fig. 1, we have $F = \{\{x, z\}, \{\overline{x}, y\}, \{\overline{y}, z\}, \{\overline{z}, w\}, \{\overline{z}, \overline{w}\}\}$ and $\Sigma_3(F) = (1, 1, 1, 1, 1)$. The disjunctive splitting of degree ≤ 4 depicted in Fig. 1 corresponds to the $N(4)$-derivation in Fig. 2 by means of Lemma 3.

An immediate consequence of Lemma 3 is the following characterization of the function f_1 defined in (2). Recall that ε denotes the empty sequence.

Theorem 2. $f_1(k) = \min\{\, s : (k) \vdash_{N(s)} \varepsilon \,\} - 1$.

Proof. Let $s \ge 2$ such that $(k) \vdash_{N(s)} \varepsilon$. By Lemma 3, there exists a formula $F \in \mathrm{MU}(1)$, $\Sigma_k(F) = \varepsilon$, which can be decomposed by splittings of degree $\le s$. Thus variables of F occur in at most s clauses. Moreover, $\Sigma_k(F) = \varepsilon$ implies that all clauses of F have size k, thus $F \in (k, s)$-CNF follows. Consequently $f_1(k) \le s - 1$.

Now assume $f_1(k) \ge s$; i.e., $(k, s)\text{-CNF} \cap \mathrm{MU}(1) = \emptyset$. Consequently, no $F \in \mathrm{MU}(1)$ with $\Sigma_k(F) = \varepsilon$ can be decomposed by splittings of degree $\le s$. By Lemma 3, it follows that $(k) \vdash_{N(s)} \varepsilon$ does not hold. Hence the theorem is shown true. □

4 Computing f_1

The results of the previous section suggest the following saturation algorithm for determining whether $f_1(k) \le s$ for given k, s:

- Start with the set $\mathcal{S}_0 = \{(k)\}$.
- For $i > 0$, obtain $\mathcal{S}_i$ as the union of $\mathcal{S}_{i-1}$ and the set of all sequences σ which can be inferred from $\sigma_1, \sigma_2 \in \mathcal{S}_{i-1}$ by the rule $N(s)$.

If we reach a set $\mathcal{S}_i$ which contains the empty sequence ε then we stop, as we then know that $f_1(k) < s$. Otherwise, if we reach a fixed-point i where $\mathcal{S}_i = \mathcal{S}_{i-1}$, then we know $f_1(k) \geq s$. We will show below that a refined saturation algorithm actually terminates, hence that a finite procedure for determining $f_1(k)$ exists.

When we run the saturation algorithm, it is desirable to avoid the derivation of sequences which are "worse" than other already derived sequences. For example, if we have already derived $(3, 2, 1)$, it is certainly superfluous to add the sequence $(3, 3, 1)$ or the sequence $(3, 2, 1, 1)$ to the cumulating set. We will see below that also, say, $(3, 3)$ can be ignored if we already have obtained $(3, 2, 1)$. Formally, we base the comparison of sequences on the following definition.

Let σ, σ' be stairways. We say that σ' is obtained from $\sigma = (a_1, \ldots, a_n)$ by *elementary flattening* if one of the following prevails:

1. For some $p \in \{1, \ldots, n\}$ we have $\sigma' = (a'_1, \ldots, a'_n)^{\mathrm{ord}}$ where
$$a'_i = \begin{cases} a_i - 1 & \text{if } i = p, \\ a_i & \text{otherwise.} \end{cases}$$
2. Consider σ to have an additional entry a_{n+1} with value 0. For some $p, q \in \{1, \ldots, n+1\}$ with $a_p > a_q$ we have $\sigma = (a'_1, \ldots, a'_{n+1})^{\mathrm{ord}}$ where
$$a'_i = \begin{cases} a_i - 1 & \text{if } i = p, \\ a_i + 1 & \text{if } i = q, \\ a_i & \text{otherwise.} \end{cases}$$
   We exclude the case $a_p = a_q + 1$ to ensure $\sigma \neq \sigma'$.

That is, σ' is obtained by decrementing some entry a_p and possibly incrementing some smaller entry a_q. We say that σ' *dominates* σ if either $\sigma' = \sigma$ or σ' can be obtained from σ by multiple applications of elementary flattening.

The next lemma states that if σ is dominated by σ', then σ is "worse" than σ' in the above sense.

Lemma 4. *If σ can be inferred from σ_1, σ_2 by rule $N(s)$, and if σ_i is dominated by $\sigma'_i \neq \varepsilon$, $i = 1, 2$, then σ is dominated by some σ' which can be inferred from σ'_1, σ'_2 by rule $N(s)$.*

Proof. Since σ_i is dominated by σ'_i, σ'_i can be obtained from σ_i by r_i applications of elementary flattening for some $r_i \geq 0$; in symbols, $\sigma_i \xrightarrow{r_i} \sigma'_i$. We proceed by induction on $r = r_1 + r_2$. If $r = 0$ then $\sigma_1 = \sigma'_1$, $\sigma_2 = \sigma'_2$, and we put $\sigma' = \sigma$.

Now assume $r > 0$. W.l.o.g., we may assume that $r_2 > 0$. Hence there is a stairway σ_2^* such that
$$\sigma_2 \xrightarrow{r_2 - 1} \sigma_2^* \xrightarrow{1} \sigma'_2.$$
The induction hypothesis yields that there is a stairway σ^* which dominates σ and can be obtained from σ'_1, σ_2^* by the rule $N(s)$. We have to show that there exists a stairway σ' which can be obtained from σ'_1, σ'_2 by rule $N(s)$ and which dominates σ^*; i.e., that the diagram
$$\begin{array}{ccc} \sigma'_1\sigma_2^* & \xrightarrow{\quad 1 \quad} & \sigma'_1\sigma'_2 \\ \big\downarrow{\scriptstyle N(s)} & & \big\downarrow{\scriptstyle N(s)} \\ \sigma^* & \xrightarrow{\quad \leq 1 \quad} & \sigma' \end{array}$$

commutes. Let $\sigma_1' = (a_1, \ldots, a_j)$, $\sigma_2^* = (a_{j+1}, \ldots, a_m)$, $\sigma^* = (a_1', \ldots, a_m')^{\mathrm{ord}}$, $a_{m+1} := 0$. Furthermore, let $b_1, \ldots, b_{m+1}$ be integers such that $\sigma_1'\sigma_2' = (b_1, \ldots, b_{m+1})^{\mathrm{ord}}$ where $a_i = b_i$ except $b_p = a_p - 1$ and possibly $b_q = a_q + 1$ for $a_p > a_q + 1$, $j \leq p < q \leq m+1$. We put $\sigma' = (b_1', \ldots, b_{m+1}')^{\mathrm{ord}}$ and define b_i' in the following case distinction.

First assume $b_p > 0$ or $a_p = a_p'$. We put $b_i' = b_i - a_i + a_i'$. It follows that σ' can be obtained from σ^* by one elementary flattening, thus σ' dominates σ^*.

Now assume that $0 = b_p = a_p - 1 = a_p'$. It follows that no entry a_q is incremented, since otherwise we would have $a_q < 0$. By assumption, σ_2' is not empty, hence we can pick some $t \in \{j+1, \ldots, m\} \setminus \{p\}$ with $b_t > 0$. If $a_t' = a_t - 1$, then we put $b_p' = b_p$ and $b_i' = b_i - a_i + a_i'$ for $i \neq p$; $\sigma' = \sigma^*$ follows (observe that $b_t' = b_t - 1$). Otherwise, if $a_t' = a_t$, then we put $b_p' = b_p$, $b_t' = b_t - 1$, and $b_i' = b_i - a_i + a_i'$ for $i \notin \{p, t\}$; in this case σ' arises from σ^* by an elementary flattening which decrements a_t'. It follows that σ' dominates σ^* in any case, hence in turn, σ' dominates σ as claimed. □

Repeated application of Lemma 4 yields the following result.

Corollary 1. *Let Γ and Γ' be sets of stairways such that every element of Γ is dominated by some element of Γ'. If $\Gamma \vdash_{N(s)} \sigma$ then σ is dominated by some σ' such that $\Gamma' \vdash_{N(s)} \sigma'$. In particular, $\Gamma \vdash_{N(s)} \varepsilon$ implies $\Gamma' \vdash_{N(s)} \varepsilon$.*

It would be interesting to know if there exists a more general notion of domination for which Corollary 1 holds.

Now it is easy to see that f_1 is computable: Assume that we want to decide whether $f_1(k) \leq s$. First decide whether $f_1(k-1) \leq s$ (we can inductively assume that this is possible); if $f_1(k-1) > s$ then clearly $f_1(k) > s$ and we are done. Otherwise, if $f_1(k-1) \leq s$, let T be an $N(s)$-derivation of ε from $(k-1)$, and let n denote the number of leaves of T. By changing all axioms of T from $(k-1)$ to (k), and by propagating this modification downward in T, we obtain an $N(s)$-derivation of the sequence 1^n, a sequence consisting of n 1s. Since every sequence of length at least n is dominated by 1^n, we can ignore all sequences of length greater than n in the saturation algorithm. On the other hand, all sequences containing an entry which is greater than k are dominated by (k); hence it follows that there is a finite number ($\leq (k+1)^n$) of sequences that have to be considered by the saturation algorithm. Hence it can be decided whether $f_1(k) \leq s$; thus f_1 is computable.

Theorem 3. *The function f_1 is computable.*

5 Restricting the Search Space

In this section we present further results which allow to speed up the computation of f_1.

5.1 A Deterministic Rule of Inference

Let $\sigma_1 = (a_1, \ldots, a_j)$, $\sigma_2 = (a_{j+1}, \ldots, a_n)$ be nonempty stairways, and let $(a_2, \ldots, a_j, a_{j+2}, \ldots, a_n)^{\mathrm{ord}} = (b_1, \ldots, b_{n-2})$. For given $s \geq 2$, we put $s' = \min(s, n) - 2$ and we define a stairway

$$\sigma_1 \oplus_s \sigma_2 := (a_1 - 1, a_j - 1, b_1 - 1, \ldots, b_{s'} - 1, b_{s'+1}, \ldots, b_{n-2})^{\mathrm{ord}}.$$

Thus, $\sigma_1 \oplus_s \sigma_2$ arises from $\sigma_1\sigma_2$ by decrementing the s largest entries of $\sigma_1\sigma_2$, ensuring that at least one entry of σ_1 and at least one entry of σ_2 is decremented.

Lemma 5. *Let σ_1, σ_2 be stairways. Then $\sigma_1 \oplus_s \sigma_2$ can be inferred from σ_1 and σ_2 by the rule $N(s)$; moreover, $\sigma_1 \oplus_s \sigma_2$ dominates all other sequences which can be inferred from σ_1 and σ_2 by the rule $N(s)$.*

Thus obtaining $\sigma_1 \oplus_s \sigma_2$ from σ_1, σ_2 is a special case of an inference by the rule $N(s)$. We denote the corresponding restricted form of the rule by $D(s)$.

Since every stairway is dominated by the empty sequence ε, Lemmas 4 and 5 immediately yield the following result.

Theorem 4. $f_1(k) = \min\{\, s : (k) \vdash_{D(s)} \varepsilon \,\} - 1.$

In Fig. 3 we give a $D(8)$-derivation of ε from (5), displayed as a sequence of inference steps. Since there is no $D(7)$-derivation of ε from (5), $f_1(5) = 7$ follows.

$$
\begin{aligned}
\sigma_0 &= (5) \\
\sigma_1 &= \sigma_0 \oplus_8 \sigma_0 = (4,4) \\
\sigma_2 &= \sigma_0 \oplus_8 \sigma_1 = (4,3,3) \\
\sigma_3 &= \sigma_0 \oplus_8 \sigma_2 = (4,3,2,2) \\
\sigma_4 &= \sigma_0 \oplus_8 \sigma_3 = (4,3,2,1,1) \\
\sigma_5 &= \sigma_0 \oplus_8 \sigma_4 = (4,3,2,1) \\
\sigma_6 &= \sigma_5 \oplus_8 \sigma_5 = (3,3,2,2,1,1) \\
\sigma_7 &= \sigma_5 \oplus_8 \sigma_6 = (3,2,2,2,1,1,1,1,1) \\
\sigma_8 &= \sigma_6 \oplus_8 \sigma_0 = (4,2,2,1,1) \\
\sigma_9 &= \sigma_7 \oplus_8 \sigma_0 = (4,2,1,1,1,1,1) \\
\sigma_{10} &= \sigma_8 \oplus_8 \sigma_0 = (4,3,1,1) \\
\sigma_{11} &= \sigma_8 \oplus_8 \sigma_{10} = (3,3,2,1,1,1) \\
\sigma_{12} &= \sigma_9 \oplus_8 \sigma_0 = (4,3,1) \\
\sigma_{13} &= \sigma_{11} \oplus_8 \sigma_0 = (4,2,2,1) \\
\sigma_{14} &= \sigma_{12} \oplus_8 \sigma_{13} = (3,3,2,1,1) \\
\sigma_{15} &= \sigma_{12} \oplus_8 \sigma_{14} = (3,2,2,2,1)
\end{aligned}
\qquad
\begin{aligned}
\sigma_{16} &= \sigma_{12} \oplus_8 \sigma_{15} = (3,2,2,1,1,1) \\
\sigma_{17} &= \sigma_{16} \oplus_8 \sigma_0 = (4,2,1,1) \\
\sigma_{18} &= \sigma_{17} \oplus_8 \sigma_{17} = (3,3,1,1) \\
\sigma_{19} &= \sigma_{17} \oplus_8 \sigma_{18} = (3,2,2,1) \\
\sigma_{20} &= \sigma_{17} \oplus_8 \sigma_{19} = (3,2,1,1,1) \\
\sigma_{21} &= \sigma_{20} \oplus_8 \sigma_0 = (4,2,1) \\
\sigma_{22} &= \sigma_{20} \oplus_8 \sigma_{21} = (3,2,1,1) \\
\sigma_{23} &= \sigma_{20} \oplus_8 \sigma_{22} = (2,2,1,1,1) \\
\sigma_{24} &= \sigma_{20} \oplus_8 \sigma_{23} = (2,1,1,1,1,1) \\
\sigma_{25} &= \sigma_{24} \oplus_8 \sigma_0 = (4,1) \\
\sigma_{26} &= \sigma_{24} \oplus_8 \sigma_{25} = (3,1) \\
\sigma_{27} &= \sigma_{24} \oplus_8 \sigma_{26} = (2,1) \\
\sigma_{28} &= \sigma_{24} \oplus_8 \sigma_{27} = (1,1) \\
\sigma_{29} &= \sigma_{24} \oplus_8 \sigma_{28} = (1) \\
\sigma_{30} &= \sigma_{29} \oplus_8 \sigma_{29} = \varepsilon
\end{aligned}
$$

Figure 3: $D(8)$-derivation, certifying that $f(5) \le 7$.

5.2 Sequences of Length $s-1$ Suffice

In the above argument for showing that f_1 is computable (Theorem 3) we established an upper bound for the maximum length of sequences we have to consider for deciding whether $f_1(k) \le s$. This upper bound is very large and is not of practical help for actually determining $f_1(k)$ for small k. Next we present a construction which allows us to restrict the length of the sequences we have to consider to $s-1$.

Let $s \ge 1$ and let $\sigma = (a_1, \dots, a_n)$ be a stairway of length $n \ge s$. Consider the stairway

$$\sigma' = (a_1, \dots, a_{s-2}, a_{s-1}+1, a_s - 1, a_{s+1}, \dots, a_n)^{\mathrm{ord}};$$

we say that σ' is obtained from σ by *elementary s-sloping*. We can apply s-sloping to σ repeatedly, until we end up with a sequence of length $s-1$;

we denote this sequence by $\sigma|^s$, and for any stairway σ of length $< s$, we put $\sigma|^s = \sigma$.

The next result allows us for the saturation algorithm to apply s-sloping before we add a new sequence to the cumulating set.

Theorem 5. *Let Γ be a set of stairways and let $\Gamma' := \{\, \sigma|^s : \sigma \in \Gamma \,\}$. Then $\Gamma \vdash_{D(s)} \varepsilon$ if and only if $\Gamma' \vdash_{D(s)} \varepsilon$.*

Proof. ($\Leftarrow$) Since σ always dominates $\sigma|^t$, this direction of the theorem follows directly from Corollary 1.

($\Rightarrow$) Consider a $D(s)$-derivation T of ε from Γ. For every leaf v of T we count the number $k(v)$ of times we have to apply s-sloping to the sequence σ_v labeling v to obtain $\sigma_v|^s$. Let $k(T)$ denote the sum of $k(v)$ over all leaves of T. If $k(T) = 0$ then T is already a $D(s)$-derivation of ε from Γ', and we are done. Hence assume $k(T) > 0$. Below we describe a construction which modifies T in such a way that $k(T)$ is decreased; a repeated application of the construction yields to the case $k(T) = 0$.

We pick a leaf v_0 of T which is is labeled by $\sigma_0 = (a_1, \dots, a_n)$ for $n \geq s$.

Let $v_0, \dots, v_r$ be the sequence of vertices on the path P from v_0 to the root v_r of T. We introduce now a notion which will allow us to talk precisely about what happens to the entries of σ_0 on the path P.

Consider an entry a_j of σ_0. Following the path P from v_0 to v_r, we can track the entry a_j. At each step of inference, it is either decremented or it retains its value, until its value reaches 0 (we can always find its new position after sorting the sequence). We use this procedure to track $a_1, \dots, a_n$ so that at v_i their values are represented by the sequence $A_i := (a_1^{(i)}, \dots, a_n^{(i)})$, $i = 0, \dots, r$. Using the freedom in the choice of A_i, we can make sure that

$$a_1^{(i)} \geq \dots \geq a_{s-1}^{(i)} \quad \text{for } i = 0, \dots, r. \tag{3}$$

We call $\tau = (a_1^{(i)}, \dots, a_n^{(i)})_{i=0}^r$ a *trace* of v_0. Note that in general, v_0 has several possible traces. Since T is a $D(s)$-derivation, it follows that for any transition from A^i to A^{i+1}, if an entry of A^i is decremented, all strictly larger elements of A^i are decremented as well; we refer to this property of the trace as $>$-*preference*. For entries of A^i of equal value, we have some freedom in the choice of the trace. We assume that if an entry $a_t^{(i)}$ is decremented for $t \geq s$, then all entries $a_{t'}^{(i)} = a_t^{(i)}$ for $t' < s$ are decremented as well. We refer to this property of the trace as $=$-*preference*.

Let $i_0 \in \{1, \dots, r-1\}$ be the smallest index such that $a_s^{(i_0+1)} = a_s^{(i_0)} - 1$ (such i_0 exists, since the root v_r is labeled by the empty sequence, and so $A_r = (0, \dots, 0)$). At the transition from A_{i_0} to A_{i_0+1} at most $s-1$ entries are decremented; by the pigeon hole principle it follows that at least one $a_t^{(i_0)}$, $t < s$, is not decremented. $<$-preference implies $a_t^{(i_0)} \leq a_s^{(i_0)}$, and $=$-preference implies $a_t^{(i_0)} < a_s^{(i_0)}$. In view of (3), we may assume that $t = s-1$, therefore $a_{s-1}^{(i_0)} < a_s^{(i_0)}$.

Now we modify the labels of the vertices v_i, $i = 0, \dots, i_0$, as follows. We can replace in σ_{v_i} the entries $a_{s-1}^{(i)}$ and $a_s^{(i)}$ by $a_{s-1}^{(i)} + 1$ and $a_s^{(i)} - 1$, respectively (by assumption, $a_s^{(i)} = a_s$ for $i \leq i_0$). Let T' denote the new labeled tree. To show that T' is an $N(s)$-derivation, it suffices to justify the labels of $v_0, \dots, v_{i_0+1}$

by the rule $N(s)$. This is easy for $v_0, \dots, v_{i_0}$. By assumption, the inference that yields the label v_{i_0+1} involves decrementing $a_s^{(i_0)}$, ($a_s^{(i_0+1)} = a_s^{(i_0)} - 1$), but $a_{s-1}^{(i_0)}$ is not changed ($a_{s-1}^{(i_0+1)} = a_{s-1}^{(i_0)}$). In T', we simply swap the roles of these two entries, and obtain the original label of v_{i_0+1}. Hence T' is indeed an $N(s)$-derivation and, as we have applied elementary s-sloping to the label of v_0, $k(T') = k(T) - 1$.

In order to complete our inductive argument, we transform the $N(s)$-derivation T' into a $D(s)$-derivation T'' such that $k(T'') \leq k(T')$. We apply Lemmas 4 and 5 along the path P. That is, assume that vertex v_i, $1 \leq i \leq r$ is labeled by a sequence λ, and that its parents v_{i-1} and v'_{i-1} are labeled by λ_1 and λ_2, respectively. If we change λ_1 to some sequence λ'_1 which dominates λ_1, then, in view of Lemmas 4 and 5, we can change λ to $\lambda' := \lambda'_1 \oplus_s \lambda_2$ (λ' dominates λ). We apply this re-labeling to $v_1, v_2, \dots$ until we reach a vertex $v_{r'}$ which receives the label ε. The subtree T'' rooted in $v_{r'}$ is now a $D(s)$-derivation with $k(T'') \leq k(T') < k(T)$ as claimed. Hence, by iteration, we are finally left with a $D(s)$-derivation T^* with $k(T^*) = 0$, which is a $D(s)$-derivation of ε from Γ'. This completes the proof of the theorem. □

Corollary 2. *There exists an algorithm to calculate $f_1(k)$ with running time $O(4^{k^2})$.*

Proof. As suggested by previous discussion, consider the following saturation algorithm, that given k and s decides if ε is derivable from (k). Throughout, the algorithm maintains in its memory a database of known derivable sequences of length at most $s-1$. Initially the database consists of the sequence (k). As long as possible, the algorithm picks two derivable sequences σ_1, σ_2, calculates $(\sigma_1 \oplus \sigma_2)|^s$, and adds it to the database, provided it is not already there and that it is not dominated by (k). Finally, the algorithm checks if ε is in the database.

The maximal possible size of the database is bounded by the number of integer sequences $k \geq a_1 \geq a_2 \geq \cdots \geq a_{s-1} \geq 0$, which is $\binom{k+s-1}{k} \leq (k+s)^k$. Note that $k \geq a_1$ follows from the restriction to sequences not dominated by (k). To see this, consider the $\binom{k+s-1}{k}$ possible orderings of k white balls and $s-1$ black balls. Each such ordering is in one to one correspondence with the sequence $a_1, \dots, a_{s-1}$, where a_i is the number of white balls to the right of the i-th black ball.

Let M denote the maximal number of sequences in the database, and denote the time required to calculate $(\sigma_1 \oplus \sigma_2)|^s$ by T_{flat}. It can be easily verified that $T_{\text{flat}} = O(s+k)$. We calculate $f_1(k)$ by performing a binary search on s, to determine the maximal value of s such that ε is not derivable from (k). It is not difficult to verify that $f_1(k) \leq 2^{k-2}$ for a sufficiently large k, either by a direct proof, or by the results of Hoory and Szeider [5]. Therefore, $k+s$ may be bounded by 2^{k-1}, for large k. It follows that $f_1(k)$ can be calculated in time $k \cdot T_{\text{flat}} \cdot M^2 = O(k \cdot (k+s)^{2k+1}) = O(k \cdot 2^{(k-1)\cdot(2k+1)}) = O(4^{k^2})$.

□

Appendix: A $D(12)$-Derivation, Certifying that $f(6) \leq 11$

$\sigma_0 = (6)$
$\sigma_1 = \sigma_0 \oplus_{12} \sigma_0$
$\sigma_2 = \sigma_0 \oplus_{12} \sigma_1$
$\sigma_3 = \sigma_0 \oplus_{12} \sigma_2$
$\sigma_4 = \sigma_0 \oplus_{12} \sigma_3$
$\sigma_5 = \sigma_0 \oplus_{12} \sigma_4$
$\sigma_6 = \sigma_0 \oplus_{12} \sigma_5$
$\sigma_7 = \sigma_1 \oplus_{12} \sigma_1$
$\sigma_8 = \sigma_1 \oplus_{12} \sigma_6$
$\sigma_9 = \sigma_1 \oplus_{12} \sigma_8$
$\sigma_{10} = \sigma_1 \oplus_{12} \sigma_9$
$\sigma_{11} = \sigma_1 \oplus_{12} \sigma_{10}$
$\sigma_{12} = \sigma_1 \oplus_{12} \sigma_{11}$
$\sigma_{13} = \sigma_2 \oplus_{12} \sigma_{12}$
$\sigma_{14} = \sigma_6 \oplus_{12} \sigma_{12}$
$\sigma_{15} = \sigma_6 \oplus_{12} \sigma_{13}$
$\sigma_{16} = \sigma_7 \oplus_{12} \sigma_{12}$
$\sigma_{17} = \sigma_7 \oplus_{12} \sigma_{13}$
$\sigma_{18} = \sigma_{14} \oplus_{12} \sigma_0$
$\sigma_{19} = \sigma_{15} \oplus_{12} \sigma_0$
$\sigma_{20} = \sigma_{16} \oplus_{12} \sigma_0$
$\sigma_{21} = \sigma_{17} \oplus_{12} \sigma_0$
$\sigma_{22} = \sigma_{18} \oplus_{12} \sigma_0$
$\sigma_{23} = \sigma_{18} \oplus_{12} \sigma_1$
$\sigma_{24} = \sigma_{18} \oplus_{12} \sigma_{22}$
$\sigma_{25} = \sigma_{19} \oplus_{12} \sigma_0$
$\sigma_{26} = \sigma_{20} \oplus_{12} \sigma_1$
$\sigma_{27} = \sigma_{21} \oplus_{12} \sigma_0$
$\sigma_{28} = \sigma_{23} \oplus_{12} \sigma_{25}$
$\sigma_{29} = \sigma_{23} \oplus_{12} \sigma_{28}$
$\sigma_{30} = \sigma_{24} \oplus_{12} \sigma_0$
$\sigma_{31} = \sigma_{25} \oplus_{12} \sigma_{27}$
$\sigma_{32} = \sigma_{25} \oplus_{12} \sigma_{28}$
$\sigma_{33} = \sigma_{25} \oplus_{12} \sigma_{32}$
$\sigma_{34} = \sigma_{26} \oplus_{12} \sigma_{31}$
$\sigma_{35} = \sigma_{27} \oplus_{12} \sigma_{31}$
$\sigma_{36} = \sigma_{29} \oplus_{12} \sigma_0$
$\sigma_{37} = \sigma_{30} \oplus_{12} \sigma_{34}$
$\sigma_{38} = \sigma_{33} \oplus_{12} \sigma_0$
$\sigma_{39} = \sigma_{35} \oplus_{12} \sigma_{35}$
$\sigma_{40} = \sigma_{36} \oplus_{12} \sigma_0$
$\sigma_{41} = \sigma_{37} \oplus_{12} \sigma_0$
$\sigma_{42} = \sigma_{38} \oplus_{12} \sigma_{38}$
$\sigma_{43} = \sigma_{38} \oplus_{12} \sigma_{40}$
$\sigma_{44} = \sigma_{38} \oplus_{12} \sigma_{42}$
$\sigma_{45} = \sigma_{39} \oplus_{12} \sigma_0$
$\sigma_{46} = \sigma_{40} \oplus_{12} \sigma_{40}$
$\sigma_{47} = \sigma_{40} \oplus_{12} \sigma_{43}$
$\sigma_{48} = \sigma_{41} \oplus_{12} \sigma_0$
$\sigma_{49} = \sigma_{42} \oplus_{12} \sigma_{46}$
$\sigma_{50} = \sigma_{42} \oplus_{12} \sigma_{47}$
$\sigma_{51} = \sigma_{42} \oplus_{12} \sigma_{48}$
$\sigma_{52} = \sigma_{44} \oplus_{12} \sigma_0$
$\sigma_{53} = \sigma_{45} \oplus_{12} \sigma_0$
$\sigma_{54} = \sigma_{49} \oplus_{12} \sigma_{52}$
$\sigma_{55} = \sigma_{50} \oplus_{12} \sigma_0$
$\sigma_{56} = \sigma_{51} \oplus_{12} \sigma_{53}$
$\sigma_{57} = \sigma_{51} \oplus_{12} \sigma_{56}$
$\sigma_{58} = \sigma_{52} \oplus_{12} \sigma_{55}$
$\sigma_{59} = \sigma_{52} \oplus_{12} \sigma_{58}$
$\sigma_{60} = \sigma_{53} \oplus_{12} \sigma_{58}$
$\sigma_{61} = \sigma_{53} \oplus_{12} \sigma_{59}$
$\sigma_{62} = \sigma_{54} \oplus_{12} \sigma_0$
$\sigma_{63} = \sigma_{55} \oplus_{12} \sigma_{55}$
$\sigma_{64} = \sigma_{55} \oplus_{12} \sigma_{58}$
$\sigma_{65} = \sigma_{57} \oplus_{12} \sigma_0$
$\sigma_{66} = \sigma_{58} \oplus_{12} \sigma_{60}$
$\sigma_{67} = \sigma_{60} \oplus_{12} \sigma_{62}$
$\sigma_{68} = \sigma_{60} \oplus_{12} \sigma_{66}$
$\sigma_{69} = \sigma_{61} \oplus_{12} \sigma_{67}$
$\sigma_{70} = \sigma_{63} \oplus_{12} \sigma_0$
$\sigma_{71} = \sigma_{64} \oplus_{12} \sigma_0$
$\sigma_{72} = \sigma_{65} \oplus_{12} \sigma_{71}$
$\sigma_{73} = \sigma_{66} \oplus_{12} \sigma_{71}$
$\sigma_{74} = \sigma_{68} \oplus_{12} \sigma_0$
$\sigma_{75} = \sigma_{69} \oplus_{12} \sigma_0$
$\sigma_{76} = \sigma_{70} \oplus_{12} \sigma_{74}$
$\sigma_{77} = \sigma_{72} \oplus_{12} \sigma_{76}$
$\sigma_{78} = \sigma_{73} \oplus_{12} \sigma_0$
$\sigma_{79} = \sigma_{75} \oplus_{12} \sigma_{75}$
$\sigma_{80} = \sigma_{75} \oplus_{12} \sigma_{79}$
$\sigma_{81} = \sigma_{75} \oplus_{12} \sigma_{80}$
$\sigma_{82} = \sigma_{75} \oplus_{12} \sigma_{81}$
$\sigma_{83} = \sigma_{77} \oplus_{12} \sigma_{80}$
$\sigma_{84} = \sigma_{78} \oplus_{12} \sigma_{83}$
$\sigma_{85} = \sigma_{79} \oplus_{12} \sigma_{79}$
$\sigma_{86} = \sigma_{79} \oplus_{12} \sigma_{82}$
$\sigma_{87} = \sigma_{79} \oplus_{12} \sigma_{86}$
$\sigma_{88} = \sigma_{79} \oplus_{12} \sigma_{87}$
$\sigma_{89} = \sigma_{80} \oplus_{12} \sigma_{88}$
$\sigma_{90} = \sigma_{80} \oplus_{12} \sigma_{89}$
$\sigma_{91} = \sigma_{84} \oplus_{12} \sigma_0$
$\sigma_{92} = \sigma_{85} \oplus_{12} \sigma_{90}$
$\sigma_{93} = \sigma_{85} \oplus_{12} \sigma_{92}$
$\sigma_{94} = \sigma_{91} \oplus_{12} \sigma_{93}$
$\sigma_{95} = \sigma_{91} \oplus_{12} \sigma_{94}$
$\sigma_{96} = \sigma_{93} \oplus_{12} \sigma_{95}$
$\sigma_{97} = \sigma_{93} \oplus_{12} \sigma_{96}$
$\sigma_{98} = \sigma_{97} \oplus_{12} \sigma_0$
$\sigma_{99} = \sigma_{97} \oplus_{12} \sigma_{98}$
$\sigma_{100} = \sigma_{97} \oplus_{12} \sigma_{99}$
$\sigma_{101} = \sigma_{97} \oplus_{12} \sigma_{100}$
$\sigma_{102} = \sigma_{97} \oplus_{12} \sigma_{101}$
$\sigma_{103} = \sigma_{102} \oplus_{12} \sigma_{102} = \varepsilon$

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