On Edge-Colored Graphs Covered by Properly Colored Cycles

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Abstract. We characterize edge-colored graphs in which every edge belongs to some properly colored cycle. We obtain our result by applying a characterization of 1-extendable graphs.

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1. Introduction, notation and statement of the main result

All graphs considered are finite and have no loops; multiple edges, however, are allowed. For a graph G and $X, Y \subseteq V(G)$ we denote the set of edges of G joining a vertex in X and a vertex in Y by $E(X,Y)$; for $v \in V(G)$ we put $E(v) := E(\{v\}, V(G))$. For $E' \subseteq E(G)$ we denote by $V(E')$ the set of all $v \in V(G)$ with $E(v) \cap E' \neq \emptyset$. Further graph theoretic terminology we refer to Diestel [2].

A graph G is called *edge-colored* if some positive integer $\chi(e)$ is assigned to every edge $e \in E(G)$; in this case we call $\chi(e)$ the *color* of e. In the sequel, G always denotes an edge-colored graph. For a vertex $v \in V(G)$ and a color c we write $E_c(v) := \{\ e \in E(v) \mid \chi(e) = c\ \}$, and we write $\chi(v) := \{\ c \mid E_c(v) \neq \emptyset\ \}$.

A cycle C in G is called *properly colored* if adjacent edges of C have different colors. We say that G is *covered by properly colored cycles* (*pcc covered*, for short) if every edge of G lies on some properly colored cycle. Note that if an edge-colored graph is connected and pcc covered, then it is "color connected" in the sense of Bang-Jensen and Gutin [1].

Edge-colored graphs which contain no properly colored cycles are well studied [3, 7, 8]; in the present paper we go to the other extreme and study edge-colored graphs in which every edge lies on some properly colored cycle. We characterize such graphs in terms of the newly introduced concept of "color restrictions." Ultimately, our characterization rests on Tutte's 1-Factor Theorem, applied in terms of a characterization of 1-extendable graphs due to Little, Grant, and Holton [4].

A survey on several results on edge-colored graphs can be found in Bang-Jensen and Gutin's book [1] where also applications to genetics are exhibited.

A *color restriction* of an edge-colored graph G is a map ρ which assigns to every vertex $v \in V(G)$ a set of colors $\rho(v) \subseteq \chi(v)$. We put $E_\rho(v) := \bigcup_{c \in \rho(v)} E_c(v)$, and $E_\rho := \bigcup_{v \in V(G)} E_\rho(v)$. We say that a color restriction ρ is *independent* if $E_\rho(v) \cap E_\rho(w) = \emptyset$ for every edge vw of G. Finally, for any subgraph G' of G (with given color restriction ρ) we put[1]

$$\Delta_\rho(G') := \begin{cases} \sum_{v\in V(G')} |\rho(v)| & \text{if } V(E_\rho)\cap V(G') \neq \emptyset; \\ 1 & \text{otherwise.} \end{cases}$$

Now we are in the position to state the main result.

Theorem 1 *An edge-colored graph G is pcc covered if and only if for every color restriction ρ of G exactly one of the following holds.*

(i) *$\Delta_\rho(G') > 1$ for some component G' of $G - E_\rho$;*
(ii) *ρ is independent and $\Delta_\rho(G') = 1$ for all components G' of $G - E_\rho$.*

The proof of this theorem is deferred to Section 3, where we will use the construction defined in the next section. This construction will also show that the question whether an edge-colored graph is pcc covered can be decided in polynomial time.

2. Transformation into 1-extendable graphs

A connected graph G is called 1-*extendable* (or *matching covered*) if G has a perfect matching, and every edge of G lies on some perfect matching (see Lovász and Plummer [5]). The following characterization of 1-extendable graphs (see Little et al. [4] for a proof and Yeo [9] for generalizations) can be shown easily by Tutte's 1-Factor Theorem, which states that a graph H has a perfect matching if and only if $c_o(H - S) \leq |S|$ for every $S \subseteq V(H)$; as usual, we denote by $c_o(H-S)$ the number of odd components (components with an odd number of vertices) of $H - S$.

Theorem 2 *A connected graph H having a perfect matching is* 1*-extendable if and only if every set $S \subseteq V(H)$ with $c_o(H - S) = |S|$ is independent.*

Note that for a 1-extendable graph H and $\emptyset \neq S \subseteq V(H)$, $H-S$ has no even components: otherwise, we choose an edge $uv \in E(H)$ such that u belongs to an even component of $H - S$ and v belongs to S; for $S' := S \cup \{u\}$ we have $|c_o(H - S')| = |S'|$ but S' is not independent, contradicting Theorem 2.

At this junction we also state an elementary lemma which we will use below.

Lemma 1 *Let H be a graph, M a perfect matching of H, and $S \subseteq V(H)$. Then $|E(V(H'), S) \cap M| \equiv |V(H')| \pmod 2$ holds for every component H' of $H - S$.*

Next we present a construction which transforms a given edge-colored graph G into a graph H_G (with a certain fixed perfect matching M) such that G is pcc covered if and only if H_G is 1-extendable (this construction has already been used by Szeider [6,7]).

Consider $v \in V(G)$ with $\chi(v) = \{c_1, \ldots, c_k\}$. We split v into new vertices $v_{c_1}, \ldots, v_{c_k}$ such that edges in $E_{c_i}(v)$ become incident with v_{c_i}, $i = 1, \ldots, k$ (see Fig. 1 for an illustration). Next we add new vertices $v'_{c_1}, \ldots, v'_{c_k}$ and join v_{c_i} with v'_{c_i}, $i = 1, \ldots k$. Finally, we add new vertices $w_{1,v}$ and $w_{2,v}$, the edge $w_{1,v}w_{2,v}$, and join v'_{c_i} with $w_{j,v}$ for all $i = 1, \ldots, n$ and $j = 1, 2$. We put

$$V_{H_G}(v) := \{w_{1,v}, w_{2,v}\} \cup \{\ v_{c_i}, v'_{c_i} \mid 1 \leq i \leq k\ \}.$$

[1] The second case in this definition serves as a technical trick which allows us to formulate Theorem 1 for possibly disconnected graphs.

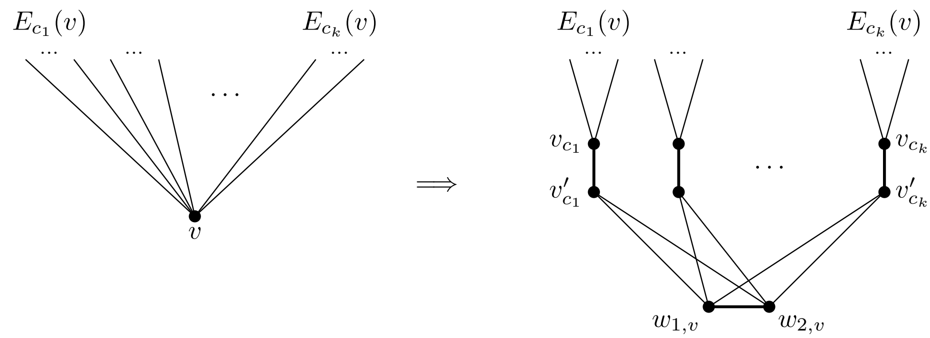

Fig. 1.

Applying this construction to all vertices of G, we obtain the graph H_G.

Let $V_1, V_2, V_3 \subseteq V(H_G)$ be the sets consisting of all vertices of the form v_{c_i}, v'_{c_i}, and $w_{j,v}$, respectively. Evidently, $V(H_G)$ is the disjoint union of V_1, V_2, and V_3, and the set

$$M := E(V_1, V_2) \cup E(V_3, V_3)$$

is a perfect matching of H_G. However, for brevity's sake we set $H := H_G$ in the following considerations.

Lemma 2 *Let G be a connected edge-colored graph. Then G is pcc covered if and only if H is* 1*-extendable.*

Proof. The lemma follows from the following observations. Properly colored cycles in G correspond in a natural way to M-alternating cycles in H (i.e., to cycles which alternate with respect to the perfect matching M). On the other hand, an edge $e \in E(H) \setminus M$ lies on some M-alternating cycle C if and only if e lies in a perfect matching M' of H (C is one of the cycles induced by the symmetric difference of M and M'). □

Since the construction of H can be carried out in polynomial time, and since we can decide whether H is 1-extendable by at most $|E(H)| - \frac{1}{2}|V(H)|$ applications of a matching algorithm, we have the following.

Theorem 3 *It can be decided in polynomial time whether an edge-colored graph is pcc covered.*

Note that Theorem 3 holds for disconnected graphs since it suffices to proceed by considering each of the components.

Lemma 3 *Let G be a connected edge-colored graph and M the perfect matching of H as defined above. For a color restriction ρ of G with $E_\rho \neq \emptyset$, we define*

$$S_\rho := \{\ v_{c_i} \in V_1 \mid v \in V(G) \text{ and } c_i \in \rho(v)\ \}. \tag{1}$$

Then components of $G - E_\rho$ and components of $H - S_\rho$ are in a bijective correspondence such that

$$\Delta_\rho(G') = |E(V(H'), S_\rho) \cap M| \tag{2}$$

holds for all pairs G', H' of corresponding components of $G - E_\rho$ and $H - S_\rho$, respectively. Furthermore, ρ is an independent color restriction if and only if S_ρ is an independent set of vertices.

Proof. Note that for each component G' of $G - E_\rho$ the set $\bigcup_{v \in V(G')} V(v) \setminus S_\rho$ induces a component of $H - S_\rho$. Since $V(H - S_\rho) = \bigcup_{v \in V(G)} V_H(v) \setminus S_\rho$, the components of $G - E_\rho$ and $H - S_\rho$ are indeed in a bijective correspondence.

Let G' be a component of $G - E_\rho$ and let H' be the corresponding component of $H - S_\rho$. A vertex $v \in V(G')$ with $\rho(v) = \{c_1, \dots, c_r\}$ corresponds to edges $v_{c_1}v'_{c_1}, \dots, v_{c_r}v'_{c_r} \in M$ with $v_{c_i} \in S_\rho$ and $v'_{c_i} \in V(H')$ $(i = 1, \dots, r)$; thus (2) follows.

Moreover, $E(S_\rho, S_\rho) \neq \emptyset$ if and only if there are vertices $x_{c_i}, y_{c_j} \in V_1$ such that $x_{c_i}y_{c_i} \in E(S_\rho, S_\rho)$; i.e., $xy \in E(G)$ and by (1), $c_i \in \rho(x)$ and $c_j \in \rho(y)$. That is, $E(S_\rho, S_\rho) \neq \emptyset$ if and only if $E_\rho(x) \cap E_\rho(y) \neq \emptyset$ for some distinct vertices $x, y \in V_1$. □

3. Proof of the main result

Lemma 4 *Let G be a connected edge-colored graph and H the graph obtained from G by the above construction. H is 1-extendable if and only if for every color restriction ρ of G with $E_\rho \neq \emptyset$ either condition (i) or condition (ii) of Theorem 1 is satisfied.*

Proof. (⇒) Assume that H is 1-extendable and choose an arbitrary color restriction ρ of G with $E_\rho \neq \emptyset$.

In view of Lemma 3 we can write the components of $G - E_\rho$ and $H - S_\rho$ as $G_1, \dots, G_k$ and $H_1, \dots, H_k$, respectively, such that G_i and H_i correspond to each other satisfying

$$\Delta_\rho(G_i) = |E(V(H_i), S_\rho) \cap M| \text{ for } i = 1, \dots, k. \tag{3}$$

Now suppose that condition (i) of Theorem 1 does not hold; i.e., $\Delta_\rho(G_i) \leq 1$, $1 \leq i \leq k$. However, $E_\rho \neq \emptyset$ implies $S_\rho \neq \emptyset$ and therefore $H - S_\rho$ has no even components (see the remark following Theorem2). Thus $\Delta_\rho(G_i) = 1$, $1 \leq i \leq k$, follows of necessity. Consequently, Lemma 1 and (3) imply $c_o(H - S_\rho) = |S_\rho|$. Thus S_ρ is independent by Theorem 2 and so ρ is independent by Lemma 3. Whence condition (ii) of Theorem 1 is actually satisfied.

(⇐) Assume that for every color restriction ρ of G with $E_\rho \neq \emptyset$ either condition (i) or condition (ii) of Theorem 1 is satisfied. In order to apply Theorem 2, we choose arbitrarily $S \subseteq V(H)$ such that $c_o(H - S) = |S|$. In view of Lemma 1, it follows that

$$E(S, S) \cap M = \emptyset. \tag{4}$$

We will show that S is independent.

First we study the effect of removing certain vertices from S.

(a) Consider $v \in S \cap V_3$. By construction of H there is some $v' \in V_3$ with $vv' \in M$; by (4), $v' \notin S$. Thus, there is a component H' of $H - S$ with $v' \in V(H')$. H' must be an odd component because of $c_o(H - S) = |S|$. If a vertex $x \in V(H)$ is adjacent with v, then $v'x \in E(H)$ follows by the very structure of H, therefore $x \in V(H') \cup S$ follows. Hence, if we consider $S' := S \setminus \{v\}$, then H' turns into an even component H^* in $G - S'$ with $V(H^*) = V(H') \cup \{v\}$ and we have $c_o(H - S') = c_o(H - S) - 1$. Thus $c_o(H - S') = |S'|$, and $H - S'$ has some even component.

(b) Consider $v \in S \cap V_2$ and assume $S \cap V_3 = \emptyset$. By construction of H, there is some $v' \in V_1$ with $vv' \in M$. As above we conclude that there is a component H' of $H - S$ with $v' \in V(H')$. Since v is of degree 3 by construction of H, v is adjacent to $w, w' \in V_3$ with $ww' \in M$. Since $S \cap V_3 = \emptyset$, there is a component H'' of $H - S$ containing w and w'

(possibly $H' = H''$). We put $S' = S \setminus \{v\}$ and consider $H - S'$. Now H' and H'' turn into a component H^* of $H - S'$ with $V(H^*) = V(H') \cup V(H'') \cup \{v\}$. Note that H^* is odd if $H' \neq H''$ and H'' is odd, and even, otherwise. Thus $c_o(H - S) - 1 = c_o(H - S') = |S'|$ follows.

In order to show that S is independent, we assume to the contrary that S contains adjacent vertices x, y. Note that in view of (4), for any two adjacent vertices of S either both belong to V_1, or one belongs to V_2 and the other belongs to V_3. Hence we have to consider the following cases.

Case $x, y \in V_1$. By repeated application of steps (a) and (b) above, we obtain $S' := S \cap V_1$ with $c_o(H - S') = |S'| \geq 2$. We define a color restriction ρ of G by setting $\rho(v) := \{ c_i \in \chi(v) \mid v_{c_i} \in S' \}$ for all $v \in V(G)$. It can be verified that this definition is exactly the converse of (1) in Lemma 3; i.e., $S_\rho = S'$ holds. By Lemma 1, we conclude from $c_o(H - S') = |S'|$ that $|E(V(H'), S') \cap M| \leq 1$ for every component H' of $H - S'$; i.e., in view of Lemma 3, condition (i) of Theorem 1 does not hold. On the other hand, since $x, y \in S_\rho$ are adjacent, ρ is not independent by the last part of Lemma 3. Thus condition (ii) of Theorem 1 does not hold as well, and we have a contradiction.

Case $x \in V_2$, $y \in V_3$, *and* $S \cap V_1 \neq \emptyset$. As above, we apply steps (a) and (b), such that for $S' := S \cap V_1$ we have $c_o(H - S') = |S'| > 0$. However, $H - S'$ contains some even component, since $y \in S \setminus S'$ (see step (a) above). Exactly as in the preceding case we define a color restriction ρ such that $S_\rho = S'$, and we conclude—again by Lemma 3—that condition (i) of Theorem 1 does not hold. Therefore, by Lemma 3, $|E(V(H'), S_\rho) \cap M| = 1$ for all components H' of $H - S_\rho$. In view of Lemma 1 we conclude that all components of $H - S_\rho$ are odd, a contradiction.

Case $x \in V_2$, $y \in V_3$, *and* $S \cap V_1 = \emptyset$. We apply steps (a) and (b) to remove from S all vertices that belong to $V_3 \cup V_2 \setminus \{x\}$. Thus we end up with the singleton $S' := \{x\} \subseteq S$ such that $H - x$ has exactly one odd component H'. As in step (b) we conclude that apart from y there are exactly two more vertices adjacent with x, say $y' \in V_3$, $x' \in V_1$; thus $xx', yy' \in M$. Let H'' denote the component of $H - x$ which contains y and y'. Since y was removed from S by step (a), H'' is an even component; actually, this is the only even component of $H - x$, since H is connected and by construction of H, x has degree 3 and $yy' \in E(H)$.

Furthermore,

$$|E(H'', \{x\}) \cap M| = 0. \tag{5}$$

As in the preceding cases we define a color restriction ρ such that $S_\rho = S'$; by Lemma 3, H'' corresponds to a component G'' of $G - E_\rho$, with $\Delta_\rho(G'') = |E(H'', \{x\}) \cap M|$. However, $\Delta_\rho(G'') = 1$ by definition of ρ, a contradiction to (5).

Whence S must be independent in any case, and so H is 1-extendable by Theorem 2. □

If G is disconnected, and $G_1, \ldots, G_k$ are the components of G, then evidently G is pcc covered if and only if every G_i is pcc covered, $i = 1, \ldots, k$. Furthermore, a color restriction ρ of G decomposes into color restrictions ρ_i of G_i, $i = 1, \ldots, k$, and E_ρ is the disjoint union of all E_{ρ_i}, $i = 1, \ldots, k$. The proof of Theorem 1 hence reduces to applications of Lemmas 2 and 4 to the components of G.

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