Mixed Voting Rules for Participatory Budgeting

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Abstract

Designing and analyzing voting rules for Participatory Budgeting (PB) elections is an active research area in computational social choice. Many PB voting rules aim to optimize a specific objective. For instance, the ubiquitous *Greedy rule* attempts to maximize utilitarian welfare, while the *Method of Equal Shares (MES)* aims to achieve proportional representation of voter preferences. However, it is often desirable to achieve good outcomes on multiple objectives rather than a close-to-perfect outcome for one. Inspired by mixed-member systems that are often used for parliamentary elections, we introduce *mixed voting rules* for PB. These are composed of a sequence of two or more rules that can each spend some fraction of the overall budget in order to add projects to the set selected by earlier rules. We develop a theoretical framework for formulating and analyzing mixed PB voting rules, and explore how existing rules can be adapted to this framework. We particularly focus on MES and its potential to address imbalances in representation created by earlier rules. We propose different ways to adjust MES voter budgets based on how satisfied voters are with previously chosen projects, and examine how well the resulting rules approximate well-known proportionality axioms such as EJR+. We complement our theoretical results with an empirical analysis of real-world PB elections, investigating how mixed rules perform compared to their constituent rules.

1 Introduction

Participatory Budgeting (PB) is a democratic innovation that lets citizens vote on how public money is spent [10, 28]. Typically, community members suggest projects, each with a specific cost. Voters are then asked to express preferences over these projects, based on which a voting rule selects a subset of the projects to fund, while making sure that their total cost stays within a given budget. Designing and analyzing PB voting rules is a very active research area in computational social choice [2, 25].

PB voting rules have different strengths and weaknesses. For example, the *Greedy rule* simply ranks projects by the number of votes they receive and funds them in that order until the budget runs out. While this method is straightforward to implement and explain, it may fail to represent minority interests. In contrast, the *Method of Equal Shares (MES)* [24] is guaranteed to provide proportional representation, but computing it requires more complex calculations that are harder to explain. Instead of choosing between different rules (and the different objectives that these rules aim for), we propose a framework that combines multiple rules. Ideally, this approach preserves the advantages of the constituent rules while mitigating their downsides. In particular, we introduce the framework of *mixed voting rules* for PB. A mixed voting rule is defined as a sequence of rules, and each rule in that sequence is allocated a specific portion of the overall budget. The process can be visualized as an assembly line (see Figure 1): the first rule selects projects using its assigned budget share, then passes both its selection and any unused budget to the next rule. Subsequent rules cannot alter the projects already chosen by earlier rules but can only add new projects using their budget allocation. For example, we might let the Greedy rule spend 60% of the budget first and then allow MES to spend the remaining 40%.

This idea is inspired by *mixed-member electoral systems*, which are used for parliamentary elections around the world [26]. Prominent examples include Germany's *Mixed-Member Proportional Representation* system and Scotland's *Additional Member System*. In these electoral systems, parliamentary seats are allocated according to two distinct selection methods: Some representatives are elected directly (usually through first-past-the-post voting in local districts), while additional seats are allocated to ensure — or at least approximate — the proportional representation of political parties in the parliament as a whole. Thus, these systems effectively divide the total resource (i.e., the parliamentary seats) between two complementary selection methods, with the second method specifically designed to enhance proportionality. Similarly, our mixed voting rules for PB allocate portions of the total budget to different selection methods, and later rules may enhance the proportionality of the outcome.[1]

To be able to apply existing PB voting rules in our mixed framework, we must adapt them to handle scenarios where some projects have already been selected. For the Greedy rule, this adaptation is straightforward: We can simply restrict attention to the remaining unselected projects and iteratively choose those with the highest vote counts. Adapting MES, on the other hand, is more subtle. Since the rule is defined via individual voter budgets, we need to decide how to divide the available budget among the voters. One trivial way of doing that is to split the budget equally, essentially ignoring the set of previously selected projects. In order to enhance proportionality, however, we need more sophisticated methods that explicitly take the previously selected projects into account. In this paper, we propose several approaches to adapting MES to this context, and we analyze — theoretically and empirically — how these methods perform in terms of proportional representation, among other metrics. All of these approaches are based on the intuition that voters who are already well-represented by the previously selected projects should be allocated less budget to spend during the execution of MES.

Beyond their role in our mixed framework, these adaptations might be of independent interest, as they allow PB voting rules to be applied to situations where certain projects must be included (maybe due to administrative or legal constraints). For example, a municipality might have ongoing projects that need to continue or legally required initiatives that must be funded. Another special case that is covered by our framework is the class of PB "completion methods," i.e., rules that are designed to extend an outcome of another rule and make it exhaustive, in the sense that no more projects can be afforded with the unused budget. Completion methods are often studied in the context of MES, because the rule often spends only a relatively small fraction of the total budget [24].

Our Contribution In this paper we extend PB voting rules to work with a set of *pre-selected projects*. We then use these extended rules to define *mixed PB voting rules*, composed of a selection of voting rules that are executed in sequence (Section 3). We adapt MES to the mixed rules framework by defining several "pre-allocation" methods to account for pre-selected projects (Section 4) and establish proportionality guarantees for these methods using parametrized variants of EJR+ (Section 5). We complement these theoretical guarantees with an empirical analysis of mixed rules on real-world PB instances (Section 6). Omitted proofs can be found in Appendix A.

Related Work Recent years have witnessed a lot of work from the (computational) social choice community on multiwinner elections [13, 20] and PB [2, 25]. Contrastingly, mixed-member electoral systems are mostly studied within the political science literature [26]. Proportionality in the PB setting has been a key research direction [4, 21, 8]. Prime examples of proportionality notions include *Extended Justified Representation* [3, 24], its "up to one" variant [24], and its strengthening, *EJR+* [6]. Several proportional rules have been proposed, most prominently the *Method of Equal Shares* [23]. Recently, there has been some work on analyzing the performance of (non-mixed) multiwinner voting rules according to competing objectives [12, 7] and on best-of-both-worlds approaches [22]. Extending an already selected set of projects has been studied in the context of PB completion methods [24]. More generally, the issue of extending partially specified solutions also features (at least implicitly) in inter-temporal fairness notions where repeated decisions need to be made [15, 18, 16].

[1]The analogy has limitations, as PB elections often (but not always) lack the concept of geographic districts that is central to most mixed-member electoral systems. Moreover, voters often (but not always) submit two ballots in a mixed-member system.

2 Preliminaries

Let P be set of projects and $N = [n] = \{1, ..., n\}$ a set of voters with $n = |N|$. We assume approval preferences and let $A_i \subseteq P$ denote the set of projects approved by voter $i \in N$. For a project $p \in P$, we let $N_p = \{i \in N : p \in A_i\}$ denote the *supporters of* p, i.e., the set of voters approving p.

An (approval-based PB) *instance* $I = (B, P, A, c)$ consists of (i) a budget limit $B \in \mathbb{R}_{>0}$; (ii) a finite set of projects P; (iii) an approval profile $A = (A_1, \ldots, A_n)$; and (iv) a cost function $c : P \to \mathbb{R}_{>0}$. We assume that voters have *cost satisfaction* functions [8], i.e., voter i's satisfaction (or utility) from project p is $\mu_i(p) = 0$ if $p \notin A_i$ and $\mu_i(p) = c(p)$ if $p \in A_i$. For a subset of projects $P' \subseteq P$, we write $c(P') = \sum_{p \in P'} c(p)$ and $\mu_i(P') = \sum_{p \in P'} \mu_i(p) = c(P' \cap A_i)$.

Any subset of projects is called an *outcome*. An outcome $P' \subseteq P$ is *feasible* for instance $I = (B, P, A, c)$ if $c(P') \leq B$. For an outcome P' and an instance I we say that a project $p \in P \setminus P'$ is *affordable* if $c(P') + c(p) \leq B$. An outcome is *exhaustive* if there are no unchosen affordable projects.

Voting rules map each PB instance to a feasible outcome. In order to facilitate the definition of mixed voting rules in Section 3, we define voting rules to take two additional inputs: a budget B_R (that is upper bounded by the instance budget B but can be strictly smaller) and a set P_0 of pre-selected projects that are required to be in the output of the rule.

Definition 1. *A* voting rule R *takes as an input (i) a PB instance $I = (B, P, A, c)$, (ii) a rule budget $B_R \in \mathbb{R}_{>0}$ with $B_R \leq B$, and (iii) a set $P_0 \subseteq P$ of pre-selected projects with $c(P_0) \leq B_R$; it outputs an outcome $R(I, B_R, P_0) = P^*$ with $P_0 \subseteq P^*$ and $c(P^*) \leq B_R$.*

Whenever $P_0 = \emptyset$ and $B_R = B$, this definition reduces to the standard definition of voting rules in the PB literature. When the instance I is clear from the context, we often write $P^* = R(B_R, P_0)$. We say that a voting rule is *exhaustive* if it always produces an exhaustive outcome (with respect to B_R).

We introduce two voting rules that are widely used in the PB literature and real-world elections, the Greedy rule and the Method of Equal Shares. We start by describing these rules in the standard setting.

Greedy Rule. Given a budget B, $\textsc{Greedy}(B, \emptyset)$ iteratively selects an affordable project p with the largest number of supporters $|N_p|$, with arbitrary tie-breaking, until no projects are affordable.

This rule greedily maximizes the utilitarian welfare $\sum_{i \in N} \mu_i(P^*)$. Note that $\textsc{Greedy}(B_{\mathrm{G}}, P_0)$, with $P_0 \neq \emptyset$ and $B_{\mathrm{G}} < B$, can be defined in the exact same way, greedily maximizing total voter satisfaction, given that P_0 must be included in its outcome, and checking project affordability with respect to B_{G}.

Method of Equal Shares [24]. MES$(B, \emptyset)$ assigns each voter $i \in N$ an initial budget of $b_i = \frac{B}{n}$ and iteratively selects projects as follows. Let $P^{(k-1)}$ be the set of projects chosen after step $k - 1$ of MES. During step k, for each affordable project $p \in P \setminus P^{(k-1)}$, we try to find $\rho(p)$ such that $\sum_{i \in N_p} \min(b_i, \rho(p)c(p)) = c(p)$. We select $p_k = \arg\min\{\rho(p) \mid p \in P \setminus P^{(k-1)}\}$, with ties broken arbitrarily. This is the project that can be bought by its supporters while minimizing the maximum payment per unit satisfaction. We add p_k to our selection ($P^{(k)} = P^{(k-1)} \cup \{p_k\}$) and update voter budgets to $b_i - \min(b_i, \rho(p_k)c(p_k))$ after every round. The algorithm terminates when no more projects can be afforded by their supporters, i.e., when $c(p) > \sum_{i \in N_p} b_i$ for all $p \in P \setminus P^{(k-1)}$.

By giving the voters equal budgets, and allowing them to spend these on projects they approve of, MES aims to make its outcome as proportional as possible. This has been formalized in the following proportionality notion, which the outcome of MES$(B, \emptyset)$ always satisfies [6].

Definition 2 ([6])**.** *An outcome $P^* \subseteq P$ satisfies* EJR+ up to any project *if for every group of voters $N' \subseteq N$ and every project $p \in \bigcap_{i \in N'} A_i \setminus P^*$, there is a voter $i \in N'$ with $c(A_i \cap P^*) + c(p) > \frac{|N'|B}{n}$.*

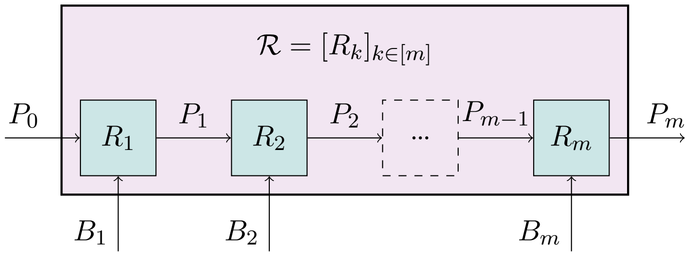


Figure 1: Illustration of a mixed voting rule $\mathcal{R}$ as a sequence of rules with inputs and outputs.

3 Mixed Voting Rules

In this section, we formally introduce a general framework for combining voting rules sequentially.

Definition 3. *A* mixed voting rule $\mathcal{R} = [R_k]_{k\in[m]}$ *takes as an input (i) a PB instance $I = (B, P, A, c)$, (ii) a sequence of rule budgets $[B_k]_{k\in[m]}$, with each $B_k \in \mathbb{R}_{>0}$ and $0 < B_1 \leq B_2 \leq \ldots \leq B_m \leq B$, and (iii) a set $P_0 \subseteq P$ of pre-selected projects with $c(P_0) \leq B_1$; it outputs an outcome $P^* = \mathcal{R}(I, [B_k]_{k\in[m]}, P_0)$ with $P_0 \subseteq P^*$ and $c(P^*) \leq B_m$. The mixed rule $\mathcal{R}$ produces its outcome P^* using a series of intermediate outputs $(P_k)_{k\in[m]}$, created by the rules it contains, with $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_{m-1} \subseteq P_m = P^* \subseteq P$. The rules are resolved in sequence: Each set of output projects is iteratively defined as $P_k = R_k(I, B_k, P_{k-1})$ and is used as the next rule's set of input projects.*

This process is illustrated in Figure 1. When the instance I is clear from the context, we often write $P^* = \mathcal{R}([B_k], P_0)$. When $m = 1$, $R_1(B_1, P_0) = [R_1]([B_1], P_0)$.

We can think of a mixed rule $\mathcal{R}([B_k]_k, P_0) = [R_k]_k([B_k]_k, P_0)$ as splitting up the instance budget among the voting rules it contains, giving R_k a budget of at least $B_k - B_{k-1}$. However, we allow each rule to use any budget left unspent by the previous rules, giving R_k an *available budget* of $B_k - c(P_{k-1})$.

Definition 4. *Consider a mixed voting rule $\mathcal{R} = [R_k]_{k\in[m]}$ and let the outcome of R_{k-1} be P_{k-1}. We define the* available budget share *of R_k during the execution of $\mathcal{R}(I, [B_k]_{k\in[m]}, P_0)$ as*

$$\alpha_k = \frac{B_k - c(P_{k-1})}{B}.$$

The available budget share α_k represents the proportion of the overall budget available to R_k to spend on remaining projects. The available budget share of the first rule in a mixed rule is $\alpha_1 = \frac{B_1 - c(P_0)}{B}$. For $k > 1$, α_k depends on the set of projects chosen by earlier rules in the mix, and is unaffected by later rules. The available budget share of R_k is bounded from above by its "gross" budget share $\frac{B_k}{B} \geq \alpha_k$ (with equality if and only if $P_{k-1} = \emptyset$). We will refer to the available budget share as simply α when it is clear from the context which rule we are considering, writing "$R \in \mathcal{R}$ with available budget share α."

A subset[2] of "completion methods" [24] that are widely used in PB can be easily defined in the mixed voting rule framework. We call a rule $R_k \in \mathcal{R}$ a *completion rule* if $B_k = B_{k-1}$. That is, R_k is not allocated any extra budget, but can only spend budget that was left over from the previous rule. For instance, MES completed by Greedy can be defined as a mixed rule, Greedy$(B, \text{MES}(B, \emptyset))$.

Example 1. Consider $\mathcal{R} = [R_1, R_2, R_3]$ where $R_1 =$ MES, $R_2 =$ Greedy and $R_3 =$ Spend is a rule that picks the set of projects that maximizes total spending. Let the instance budget be $B = 100$, the rule budgets be $[B_1, B_2, B_3] = [50, 90, 100]$, and the project set $P = \{p_1, \ldots, p_5\}$. The cost function c and the approval profile A are given in Table 1.

[2]We only consider completion methods that add projects to the outcome of the rule they're completing, without modifying the already selected set. For instance, "completion by varying the budget" would not fall under this definition (and should perhaps not be called a completion method).

We compute $\mathcal{R}([50, 90, 100], \emptyset)$ in three steps:

(1) $\text{MES}(50, \emptyset)$ with available budget $B_1 - c(\emptyset) = 50$ and available budget share $\alpha_{\text{MES}} = 0.5$ selects $P_1 = \{p_1, p_2\}$ and terminates, as the remaining voter budgets ($b_1 = b_2 = b_3 = 3, b_4 = 0, b_5 = 1$) are not sufficient to afford any of the other projects. Note that $B_1 - c(P_1) = 10$ units of budget is left unspent.

Project	p_1	p_2	p_3	p_4	p_5	p_6
Cost	28	12	45	12	8	6
A_1	✓			✓		✓
A_2	✓		✓	✓		
A_3	✓		✓	✓	✓	
A_4	✓	✓	✓		✓	
A_5		✓	✓			
Selected by:	R_1	R_1	R_2	-	R_3	R_3

Table 1: Approval profile for Example 1.

(2) $\text{GREEDY}(90, P_1)$ with available budget $B_2 - c(P_1) = 50$ (note that this is the sum of $B_2 - B_1$ and $B_1 - c(P_1)$) and available budget share $\alpha_{\text{GREEDY}} = 0.5$ selects p_3 and terminates with outcome $P_2 = P_1 \cup \{p_3\}$, as no other projects can be afforded with GREEDY's remaining budget of $B_2 - c(P_2) = 5$.

(3) $\text{SPEND}(100, P_2)$ with available budget $B_3 - c(P_2) = 15$ and available budget share $\alpha_{\text{SPEND}} = \frac{B_3 - c(P_2)}{100} = 0.15$ selects $P_3 = P_2 \cup \{p_5, p_6\}$, which is the final outcome of our mixed rule $\mathcal{R}$. $\diamond$

We will analyze to what extent a mixed voting rule inherits the properties satisfied by its constituent rules. The following definition applies to a wide range of axiomatic properties.

Definition 5. *Let $\mathcal{I}_P$ be the set of all instances with project set P. A* monotone property *is a function $X : \mathcal{I}_P \times 2^P \to \{0, 1\}$ such that, for any $I \in \mathcal{I}_P$, $P' \subseteq P'' \subseteq P$ implies $X(I, P') \leq X(I, P'')$. We say that a set of projects $P^* \subseteq P$* satisfies *X for instance $I = (B, P, A, c)$ if $X(I, P^*) = 1$. Letting $\alpha \in \mathbb{R}_{>0}$, we say that P^* instead satisfies α-*budget *X for instance $I = (B, P, A, c)$ if $X((\alpha B, P, A, c), P^*) = 1$.*

Note that properties are defined for project sets rather than voting rules, and that project set P^* need not be a feasible outcome for the instance $(\alpha B, P, A, c)$. Every voting rule from the standard setting can be adapted to the mixed framework in a trivial way, by ignoring the pre-selected projects and setting $R(B, P_0) = P_0 \cup R(B, \emptyset)$. This "trivial adaptation" can be used to show the following statement.

Observation 1. *Suppose there exists a voting rule R whose outcome $P^* = R(B, \emptyset)$ always satisfies some monotone property X. Then, we can construct a voting rule R' such that if $R' \in \mathcal{R}$ with available budget share α, the outcome of $\mathcal{R}$ always satisfies α-budget X.*

Observation 1 helps establish a *theoretical baseline* for the axiomatic results of this paper. When we adapt a rule R whose outcome always satisfies property X to the mixed framework, it is desirable that, when $R \in \mathcal{R}$ with an available budget share of α, the outcome of $\mathcal{R}$ satisfies α'-budget X with $\alpha' \geq \alpha$. We will discuss a complementary empirical baseline in Section 6.

4 Adapting the Method of Equal Shares to the Mixed Rules Framework

In this section we generalize MES to account for a set of pre-selected projects P_0. We define several pre-allocation methods to account for the differences in voter satisfaction from P_0, each of which provides a different profile of initial voter budgets to MES.

4.1 Allocating Voter Budgets

Consider $\text{MES}(B_{\text{MES}}, P_0)$, with $P_0 \neq \emptyset$. Our goal is to determine how to initialize voter budgets b_i for each voter $i \in N$ to account for a pre-selected set of projects P_0, in order to maximize the proportionality achieved by subsequently running MES. We might no longer desire to equally split MES's available budget, αB, as the set of pre-selected projects P_0 need not be equally liked by all voters. Intuitively, voters that are more satisfied with P_0 should be provided with a smaller individual budget.

We formalize methods for determining voter budgets $(b_i)_{i\in N}$ (with $\sum_{i\in N} b_i = \alpha B$) as *pre-allocation methods*, and we write MES^M to refer to MES with pre-allocation method M. All our pre-allocation methods follow a two-step process.

Definition 6. *A* pre-allocation method *takes as input an instance I, a set $P_0 \subseteq P$ of pre-selected projects, and an available budget share α and proceeds in two stages:*

(1) It determines voter payments $(\pi_i)_{i\in N}$ *for projects from P_0, such that $\pi_i \geq 0$ for each voter $i \in N$ and $\sum_{i\in N} \pi_i \leq c(P_0)$.*

(2) It applies a rebalancing step *to determine* voter budgets $(b_i)_{i\in N}$ *in order to make voters' endowments $\{\pi_i + b_i\}_{i\in N}$ as equal as possible.*[3] *Formally, $(b_i)_{i\in N}$ are chosen to maximize $\min_{i\in N}(\pi_i + b_i)$ under the constraints $\sum_{i\in N} b_i = \alpha B$ and $b_i \geq 0$ for all $i \in N$.*

Note that we allow pre-allocation methods to have some voters pay more than their "fair share" of the MES budget, i.e., $\pi_i > \frac{B_{\text{MES}}}{n}$. It is also possible for the voter payments to only partially fund the projects in P_0, or not fund them at all. The rebalancing step (2) is the same for all pre-allocation methods. To motivate this, it can be shown that every "reasonable" voter budget profile $(b_i)_{i\in N}$ can be induced by picking appropriate voter payments in stage (1) — see Claim 1 in Appendix A for details.

4.2 Pre-allocation Methods

We now define four pre-allocation methods for $\text{MES}(B_{\text{MES}}, P_0)$ with available budget share α, each following the two-stage process in Definition 6. Alongside, we present a running example. All four methods can be computed in polynomial time.

Example 2. Consider an instance with $P = \{p_1, p_2, p_3\}$, budget $B = 32$, and approval profile and project costs as specified in Table 2. We assume that MES is given a budget $B_{\text{MES}} = 32$, and a pre-selected project set $P_0 = \{p_1, p_2\}$, resulting in an MES budget share of $\alpha = \frac{B_{\text{MES}} - c(P_0)}{B} = 0.25$.

The outcomes of the pre-allocation methods are illustrated in Figure 2.

Project	p_1	p_2	p_3
Cost	18	6	9
A_1	✓		✓
A_2	✓		
A_3	✓	✓	✓
A_4		✓	✓

Table 2: Approval profile for Example 2.

Method 1. <u>**Null**</u>: Set $\pi_i = 0$ for all voters, obtaining voter budgets $b_i = \frac{\alpha B}{n}$ using the rebalancing step. This pre-allocation method splits the remaining budget αB equally among all voters. ◇

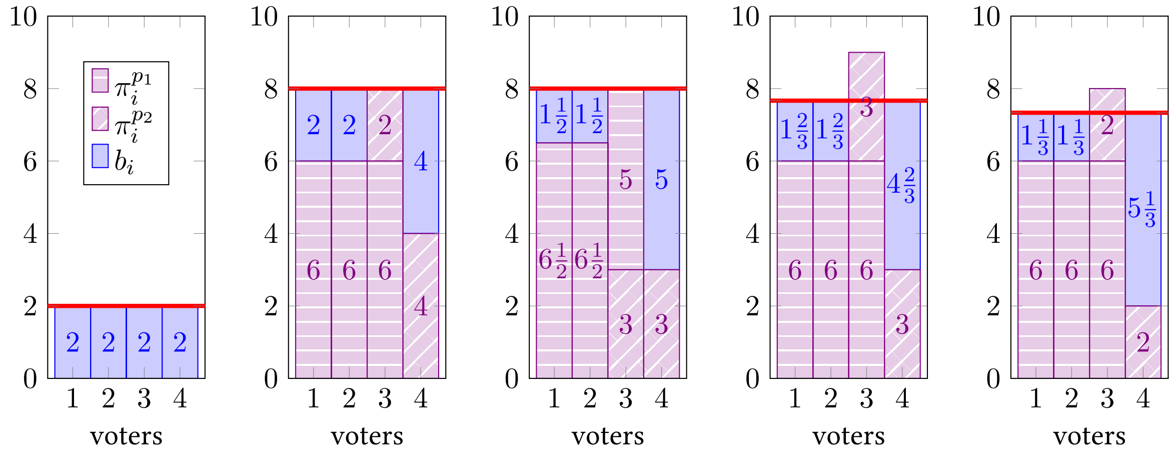
In Example 2, each voter $i \in N$ pays $\pi_i = 0$ for P_0 and gets an MES budget of $b_i = 2$.

Method 2. <u>**MES-Style**</u>: In order to determine voter payments, choose any order for P_0 and initialize voter budgets to $b_i^0 = \frac{B_{\text{MES}}}{n}$. Iteratively fund the projects in P_0 as if they were sequentially selected by MES. If at any point the voters in N_p cannot fully fund some project $p \in P_0$, the remainder of its cost is discarded.[4] We do not fix the order for P_0. Our theoretical result (Proposition 5) holds regardless of the order. For our empirical results in Section 6, we order P_0 by the number of supporters. ◇

In Example 2, assume we order P_0 such that p_1 is funded first. Then the first 3 voters each pay 6 for it, obtaining intermediate budgets of $b_1' = b_2' = b_3' = 2$. When funding p_2, voter 3 can no longer pay their fair share, so they pay as much as they can, with the rest covered by 4. We obtain payments $(\pi_i)_{i\in[4]} = (6, 6, 8, 4)$ and budgets of $(b_i)_{i\in[4]} = (2, 2, 0, 4)$. If we instead fund p_2 first, we get $(\pi_i)_{i\in[4]} = (6.5, 6.5, 8, 3)$ and $(b_i)_{i\in[4]} = (1.5, 1.5, 0, 5)$.

[3] This can be thought of as pouring αB square units of water into a 2D bucket (see Figure 2), where the floor (hatched, violet) is a bar chart, where the height of each bar is π_i (and the width is 1), and the water (filled, blue) represents $(b_i)_{i\in N}$.

[4] Alternatively, we could fund such projects by overcharging their supporters; this would result in the same voter budgets.



(a) Null **(b)** MES-Style$[p_1, p_2]$ **(c)** MES-Style$[p_2, p_1]$ **(d)** Equal-Split **(e)** Value-Based

Figure 2: Pre-allocation outcomes for Example 2. Voter payments π_i are calculated separately by each pre-allocation method (and are split by project in the diagrams, with $\pi_i = \pi_i^{p_1} + \pi_i^{p_2}$), and the voter budgets b_i are obtained from the rebalancing step. The red line represents the minimum voter endowment $\min_{i \in N}(\pi_i + b_i)$.

Method 3. **Equal-Split**: Split the total cost of every project $p \in P_0$ equally among its supporters N_p, setting $\pi_i = \sum_{p \in A_i \cap P_0} \frac{c(p)}{|N_p|}$, with the voter budgets derived using the rebalancing step. Note that $\sum_{i \in N} \pi_i = c(P_0)$ and thus the voter payments fund the pre-selected projects completely. $\diamond$

In Example 2, supporters pay 6 each for p_1 and 3 each for p_2, with total payments $(\pi_i)_{i \in [4]} = (6, 6, 9, 3)$. Voter 3 has spent more than their fair share. From the rebalancing step, we get $(b_i)_{i \in [4]} = (\frac{5}{3}, \frac{5}{3}, 0, \frac{14}{3})$.

The MES-Style and and Equal-Split methods represent fairly reasonable ways to divide the cost of P_0 among voters. However, we will see in Section 5 that they do not achieve proportionality at a level dictated by our theoretical baseline, while the (trivial) Null method does. We introduce one more method that works similarly to Equal-Split, but achieves better proportionality guarantees. The idea behind this method is to partially fund the projects in P_0 from voter payments π_i in such a way that voters get good value for money whenever they contribute to a project.

Method 4. **Value-Based**: We define the utilitarian *value*[5] (for money) $v(p)$ of a project p to be $v(p) = |N_p|$. For a set of pre-selected projects P_0, we define the *threshold value* v^* as the value of the most valuable unselected project that is affordable under budget B_{MES} after every project from P_0 with greater value has been selected. Formally,

$$v^* = \max_{p \in P \setminus P_0} \{v(p) \mid p \text{ satisfies } c(U(p)) + c(p) \leq B_{\text{MES}}\},$$

where $U(p) = \{p' \in P_0 \mid v(p') \geq v(p)\}$ denotes the upper contour set of p in P_0, i.e., the set of pre-selected projects with value at least $v(p)$. If no project $p \in P \setminus P_0$ satisfies $c(U(p)) + c(p) \leq B_{\text{MES}}$, we let $v^* = 0$. In the case that all projects have distinct values, the threshold value v^* is the value of the first project from $P \setminus P_0$ that Greedy$(B_{\text{MES}}, \emptyset)$ would select.

When funding some project p^* with value v^*, voters in N_{p^*} would each pay $\frac{c(p^*)}{v^*}$, or equivalently $\frac{1}{v^*}$ per unit satisfaction they obtain from p^*. The idea of the Value-Based pre-allocation method is to allow voters to spend *at most* $\frac{1}{v^*}$ per unit satisfaction, defining voter payments as follows:[6]

$$\pi_i = \sum_{p \in A_i \cap P_0} \frac{c(p)}{\max\{v(p), v^*\}}.$$

[5]This definition captures the ratio of the *utilitarian welfare* of a project to its cost. For a satisfaction function $\mu(\cdot)$, the value of p is defined as $v(p) = \frac{\sum_{i \in N} \mu_i(p)}{c(p)}$. For the cost satisfaction function, this reduces to $v(p) = \frac{\sum_{i \in N_p} c(p)}{c(p)} = |N_p|$.

[6]We can easily extend the Value-Based method to other satisfaction functions by setting $\pi_i = \sum_{p \in A_i \cap P_0} \frac{\mu(p)}{\max\{v(p), v^*\}}$.

Thus, any project $p \in P_0$ with value $v(p) \geq v^*$ is thus funded identically to the Equal-Split method. $\diamond$

In Example 2, supporters pay 6 each for p_1 like they did under the Equal-Split method. The threshold value in this instance is $v^* = 3$, corresponding to the value of p_3. Thus, the Value-Based method allows voters 3 and 4 to partially fund p_2, with each paying $\frac{1}{3} \times 6 = 2$ for it, resulting in total payments of $(\pi_i)_{i \in [4]} = (6, 6, 8, 2)$. Using the rebalancing step, we get $(b_i)_{i \in [4]} = (\frac{4}{3}, \frac{4}{3}, 0, \frac{16}{3})$.

Arguably, the four methods outlined above choose voter payments π_i in a reasonable way, from the perspective of the voters. In particular, they never force a voter to pay for a project they do not approve.

5 Proportionality Guarantees for MES Variants

In this section, we study the proportionality of MES in the mixed framework, for each of the four pre-allocation methods defined in Section 4. We consider parametrized versions of the proportionality axiom *EJR+ up to any project* (Definition 2) and weakenings of it. The axioms we consider are parametrized using the *minimum voter budget share*, an important value that can be calculated from the output of a pre-allocation method. Notably, this creates a non-standard approach where the strength of our proportionality guarantees cannot be determined until the mixed rule is partially executed and we observe which projects were selected by earlier rules. We identify the Value-Based pre-allocation method as the sole method that improves on our theoretical baseline from Section 3.

Applying Definition 5 to *EJR+ up to any project* results in the following parameterized axiom.

Definition 7. *An outcome $P^* \subseteq P$ satisfies* α-budget EJR+ up to any project *if for every group of voters $N' \subseteq N$ and every project $p \in \bigcap_{i \in N'} A_i \setminus P^*$, there is a voter $i \in N'$ with $c(A_i \cap P^*) + c(p) > \alpha \frac{|N'|B}{n}$.*

Additionally, we will consider the following weakened version of this property.

Definition 8. *Let $k \in \mathbb{N}_{>0}$. An outcome $P^* \subseteq P$ satisfies* α-budget EJR+ up to any k projects *if for every group of voters $N' \subseteq N$ and every set of projects $P' \subseteq \bigcap_{i \in N'} A_i \setminus P^*$ with $|P'| = k$, there is a voter $i \in N'$ with $c(A_i \cap P^*) + c(P') > \alpha \frac{|N'|B}{n}$.*

This property reduces to Definition 7 for $k = 1$, and gets weaker for larger values of k. Similar notions have been defined in the fair division literature [1].

5.1 Minimum Voter Budget Share

The normative goal of the pre-allocation methods we defined in Section 4.2 is to select the profile of voter payments $(\pi_i)_{i \in N}$ in such a way that each voter gets good "value for money" whenever they contribute to a project in P_0. We can think of $\pi_i + b_i$ as the *total endowment* of the voter, which the rebalancing step tries to make as equal as possible among the set of voters. The following definition focuses on a voter with minimal total endowment and compares their total endowment to a voter's fair share of the instance budget, which is given by $\frac{B}{n}$ (dividing by $\frac{B}{n}$ is equivalent to multiplying with $\frac{n}{B}$).

Definition 9. *Consider an MES pre-allocation method M that is applied as part of a mixed rule. The* minimum voter budget share α^{M} *is defined as $\alpha^{\mathrm{M}} = \min_{i \in N}\{(\pi_i + b_i)\frac{n}{B}\}$.*

The minimum voter budget share represents how much the worst off voter gets to spend on (1) projects in P_0 and (2) during the execution of MES$^{\mathrm{M}}$, as a fraction of their fair share of the instance budget. A method M with a higher value of α^{M} is not necessarily more proportional as it is possible for M to spend voter budgets inefficiently on projects from P_0 (see the negative results in Table 3).

The rebalancing step (see Definition 6) places some constraints on the possible values of α^{M}.

Observation 2. *For any pre-allocation method M, the minimum voter budget share* α^M *is lower-bounded by MES's available budget share* $\alpha = \frac{B_{MES}-c(P_0)}{B}$ *and upper-bounded by MES's gross budget share* $\frac{B_{MES}}{B}$*. Furthermore,* $\frac{\alpha^M B}{n}$ *exceeds all voter budgets determined in the rebalancing step, i.e.,* $\frac{\alpha^M B}{n} \geq \max\{b_i\}_{i\in N}$.

Before considering proportionality guarantees for each of our pre-allocation methods, we derive the following relationship between their minimum voter budget shares.

Proposition 1. *Fix an instance* I *and a set of pre-selected projects* P_0*. The minimum voter budget shares for* $MES^M(B_{MES}, P_0)$ *with* $M \in \{\textsc{Null}, \textsc{MES-Style}, \textsc{Equal}, \textsc{Value-Based}\}$ *satisfy the following:*

$$0 \leq \alpha = \alpha^{\textsc{Null}} \leq \alpha^{\textsc{Value-Based}} \leq \alpha^{\textsc{Equal-Split}} \leq \alpha^{\textsc{MES-Style}} \leq \frac{B_{MES}}{B} \leq 1.$$

It is important to note that $\alpha^{\textsc{Null}} < \alpha^{\textsc{Value-Based}}$ whenever $P_0 \neq \emptyset$, and that the gap between those two values can be quite large in practice.[7]

5.2 Proportionality Guarantees

We will now consider which of our pre-allocation methods M guarantee that the outcome of voting rule MES^{M} satisfies a proportionality property of the form described in Definitions 7 and 8. These guarantees will be parameterized by the minimum voter budget share α^{M} corresponding to M.

Let $P^* = \text{MES}^{\text{M}}(B_{\text{MES}}, P_0)$ be the outcome of MES^{M} when it is provided with rule budget B_{MES} and set of pre-selected projects P_0. Any proportionality guarantee that we can prove for P^* directly translates to the same guarantee for the outcome of any mixed rule $\mathcal{R}$ with $R_k = \text{MES}^{\text{M}}$, provided that the input P_{k-1} for R_k equals P_0. This is because P^* will be contained in the output of $\mathcal{R}$.

Our results are summarized in Table 3. Interestingly, pre-allocation methods M with higher α^{M} (see Proposition 1) lead to worse proportionality guarantees: For instance, while $\alpha^{\textsc{Value-Based}} \leq \alpha^{\textsc{Equal-Split}}$, the outcome of $\text{MES}^{\textsc{Value-Based}}$ is guaranteed to satisfy $\alpha^{\textsc{Value-Based}}$-budget EJR+ up to any project, whereas the outcome of $\text{MES}^{\textsc{Equal-Split}}$ may violate the analogous property not only for $\alpha^{\textsc{Equal-Split}}$, but even for the (smaller) available budget share α.

Method M	Satisfied Properties	Violated Properties
Null	$\alpha^{\textsc{Null}}$-budget EJR+ up to any project (Theorem 1)	
Value-Based	$\alpha^{\textsc{Value-Based}}$-budget EJR+ up to any project (Theorem 2)	
Equal-Split†	$\alpha^{\textsc{Equal-Split}}$-budget EJR+ up to any two projects (Theorem 3)	α-budget EJR+ up to any project
Equal-Split	–	α-budget EJR+ up to any k projects
MES-Style	–	α-budget EJR+ up to any k projects

Table 3: Proportionality guarantees for the outcome $P^* = \text{MES}^{\text{M}}(B_{\text{MES}}, P_0)$. The marker "†" indicates that the corresponding results hold for the special case in which P_0 is selected by Greedy.

We now state our proportionality guarantees formally, for mixed rules $\mathcal{R}$ containing MES^{M}. Note that the parameter α^{M} of the resulting guarantees cannot be directly inferred from the inputs to $\mathcal{R}$, but depends on the partial outcome provided to MES^{M} during the execution of $\mathcal{R}$. That is, we first need to partially run the rule $\mathcal{R}$ before determining how good a proportionality guarantee on its outcome we can give. While this might sound like a disadvantage, we remind the reader of our empirical observation that the values of α^{M} are often close to 1 in realistic scenarios (see Footnote 7).

We start with a guarantee for $\text{MES}^{\textsc{Null}}$, which is a straightforward extension of existing guarantees [6].

[7] For instance, when running [Greedy, MES]($[0.5B, B]$) on the real-world PB instances considered in Section 6, our pre-allocation methods obtain the following minimum voter budget shares on average: $\alpha^{\textsc{Null}} = 0.50$, $\alpha^{\textsc{Value-Based}} = 0.92$, $\alpha^{\textsc{Equal-Split}} = 0.94$, and $\alpha^{\textsc{MES-Style}} = 0.97$. See Appendix C.1 for a more detailed overview of these values in practice.

Theorem 1. *Consider a mixed voting rule $\mathcal{R}$ such that MES$^{\text{NULL}}$ $\in \mathcal{R}$ with available budget share $\alpha = \alpha^{\text{NULL}}$. Then, the outcome of $\mathcal{R}$ satisfies α-budget EJR+ up to any project.*

Thus, MES$^{\text{NULL}}$ meets — but does not exceed — our theoretical baseline from Section 3. (It does so directly, without following the "trivial adaptation" approach from Observation 1.) We can improve upon our baseline by employing the VALUE-BASED pre-allocation method.

Theorem 2. *Consider a mixed voting rule $\mathcal{R}$ such that MES$^{\text{VALUE-BASED}}$ $\in \mathcal{R}$ with available budget share α and minimum voter budget share $\alpha^{\text{VALUE-BASED}} \geq \alpha$. Then the outcome of $\mathcal{R}$ satisfies $\alpha^{\text{VALUE-BASED}}$-budget EJR+ up to any project.*

Theorem 2 is the main theoretical result of our paper. Its proof heavily relies on the fact that the voters spending per unit of utility is bounded during the VALUE-BASED pre-allocation.

Meanwhile, the EQUAL-SPLIT and MES-STYLE methods can subject voters to arbitrarily high payments per unit utility. As we show in Appendix B, the outcome of MES with either of those two pre-allocation methods may violate arbitrarily weak proportionality notions: For any fixed $k \in \mathbb{N}$, there are examples in which the outcome of MES$^{\text{EQUAL-SPLIT}}$ and MES$^{\text{MES-STYLE}}$ violate α-budget EJR+ up to any k projects.

Finally, we show that the EQUAL-SPLIT pre-allocation method performs reasonably well in the special case that the set of pre-selected projects was chosen by GREEDY.

Theorem 3. *Suppose $\mathcal{R} = [R_k]_{k\in[m]}$, where R_1 is GREEDY and R_2 is MES$^{\text{EQUAL-SPLIT}}$ with minimum voter budget share $\alpha^{\text{EQUAL-SPLIT}}$. Then, the outcome of $\mathcal{R}$ satisfies $\alpha^{\text{EQUAL-SPLIT}}$-budget EJR+ up to any two projects.*

6 Experimental Results

In this section, we empirically evaluate our framework by computing the results of mixed voting rules on a large dataset of real-world PB instances. In our experiments, we focus on mixed rules obtained by a combination of GREEDY and MES (with GREEDY completion), where MES is implemented with one of the pre-allocation methods introduced in Section 4. We compute the results for each of them while varying the fraction α_{G} of the budget provided to GREEDY from 0 to 1 in steps of 0.1. More formally, let $\mathcal{R}^{\text{M}} =$ [GREEDY, MES$^{\text{M}}$, GREEDY]. Then, for an instance $I = (B, P, A, c)$, we compute $\mathcal{R}^{\text{M}}([\alpha_{\text{G}}B, B, B]) =$ GREEDY$(B,$ MES$^{\text{M}}(B,$ GREEDY$(\alpha_{\text{G}}B, \emptyset)))$ for all M $\in$ {NULL, MES-STYLE, EQUAL, VALUE-BASED} and all $\alpha_{\text{G}} \in \{0, 0.1, 0.2, \ldots, 1\}$. We use the factor α_{G}, which we will refer to as the *GREEDY (budget) share*, to interpolate between the two rules, with $\alpha_{\text{G}} = 0$ corresponding to MES (with GREEDY completion) and $\alpha_{\text{G}} = 1$ corresponding to GREEDY.

Data The data for our experiments is obtained from PABULIB [14], a library of over 1300 PB instances. We compute the results of the mixed rules on all non-trivial real-world instances with at least 20 projects, which amounts to 313 instances.[8] We break ties lexicographically by project name.

Measures Since we are mixing a rule which aims to optimize the utilitarian welfare (GREEDY) with a rule which aims to provide proportional representation (MES), we evaluate the mixed rules on these two criteria. For a given instance $I = (B, P, A, c)$ and a set of selected projects $P^* \subseteq P$, we compute several numerical measures, averaging our results over all instances in our dataset. To measure welfare, we use the *utilitarian ratio* [19], defined as the utilitarian welfare of P^* as a fraction of the maximum achievable utilitarian welfare for the instance. For (proportional) representation, we consider three measures: (i) First, we check whether P^* satisfies EJR+ up to any project (henceforth abbreviated to

[8] PABULIB contains instances that are artificially generated and trivial instances, where all projects can be funded. The rationale for considering only instances with 20+ projects is discussed in Appendix C.3.

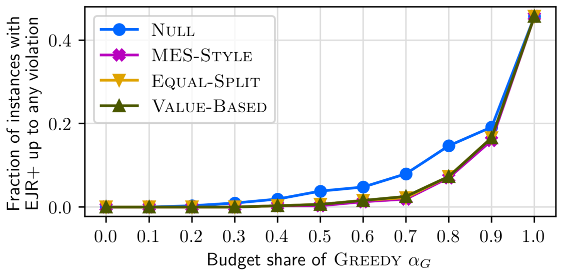


Figure 3: Fraction of instances for which EJR+ up to any project is violated.

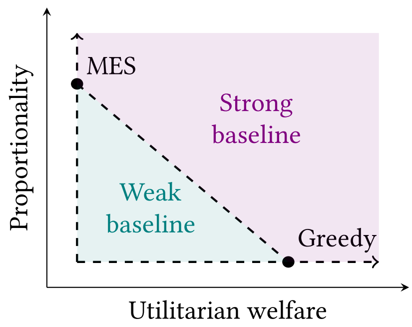
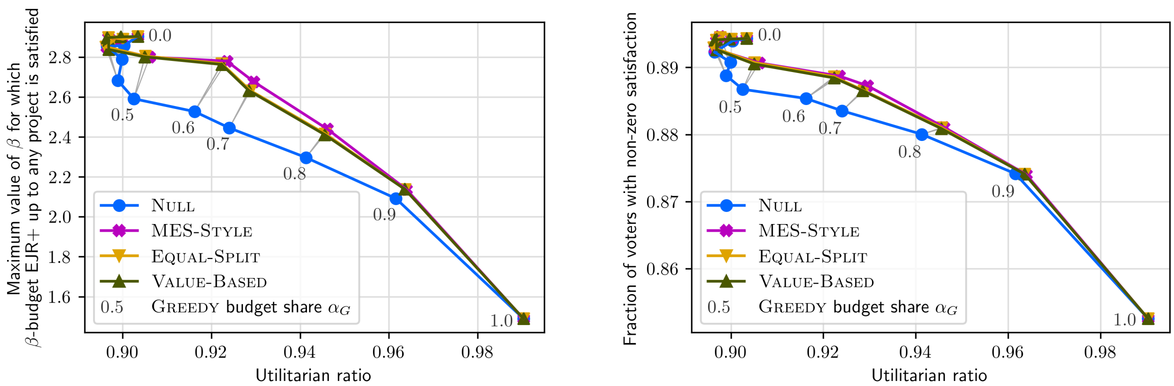


Figure 4: Empirical baselines for evaluating the performance of mixed rules.



(a) Maximum β for which β-budget EJR+ up to any project is satisfied over the utilitarian ratio.

(b) Fraction of represented voters over the utilitarian ratio.

Figure 5: Proportionality versus utilitarian welfare for the mixed rule $\mathcal{R}^M([\alpha_G B, B, B])$ for $M \in \{\text{NULL}, \text{MES-STYLE}, \text{EQUAL}, \text{VALUE-BASED}\}$ and $\alpha_G \in \{0, 0.1, 0.2, \ldots, 1.0\}$. Metrics are averaged over all instances with at least 20 projects.

EJR+X). (ii) To get a better idea of how "close" an outcome is to violating EJR+X, we compute the maximum value of β for which P^* satisfies β-budget EJR+X (Definition 7).[9] This yields a quantitative proportionality measure, as recently suggested by Bardal et al. [5]. (iii) Finally, we measure the fraction of represented voters (a.k.a. voters with non-zero satisfaction), i.e., $|\{i \in N \mid A_i \cap P^* \neq \emptyset\}|/n$.

Figures 3 and 5 show a summary of our results, averaged over all 313 instances. For all of these measures, the mixed rule containing MES$^{\text{NULL}}$ is mostly outperformed by the other three, which all perform similarly. In Figure 3 we can see that, for non-NULL pre-allocation methods, the number of proportionality violations only significantly increases once we give GREEDY a share of more than 70% of the budget, suggesting that even a comparatively small amount of budget allocated to MES is sufficient to achieve proportional outcomes. On the other hand, we can see that even GREEDY itself satisfies EJR+X on most of the instances, suggesting that it is relatively easy to satisfy in practice. This observation is further supported by Figure 5a, where we can see that the average values of β for which β-budget EJR+X is satisfied range between 1.4 and 3.0, much higher than the theoretical guarantees.

Figures 5a and 5b show the trade-offs between (proportional) representation and utilitarian welfare. When interpolating between GREEDY and MES, it is desirable to perform at least as well as the former with respect to representation and at least as well as the latter with respect to utilitarian welfare. In other words, we do not want the outcome of $\mathcal{R}^M$ to be "Pareto-dominated" by either constituent rule. We refer to this requirement as the *weak empirical baseline*, visualized in Figure 4. For GREEDY share

[9] While all theoretical guarantees in Section 5 have $\beta \leq 1$, actual values of β are often higher.

$\alpha_G \in \{0.1, 0.2, 0.3\}$, all mixed rules perform very similarly, with a slightly lower utilitarian welfare, but about the same proportionality as MES. Because of that, all rules $\mathcal{R}^M$ are dominated by MES for α_G up to 0.4. This decrease in utilitarian welfare when mixing 10 to 40 percent GREEDY into MES can potentially be explained by the GREEDY rule having to pick suboptimal projects due to its lower budget constraint. For $\alpha_G \geq 0.6$, the mixed rule $\mathcal{R}^M$ exceeds the weak empirical baseline with any of the four pre-allocation methods.

We can also compare our mixed rules to a hypothetical *randomized combination* of GREEDY and MES, where we run GREEDY with some probability $p \in [0, 1]$ and MES with probability $1 - p$. Ideally, we would want the outcome of our mixed rule $\mathcal{R}^M$ to not be Pareto-dominated by any of these randomized rules. We consider this to be the *strong empirical baseline*, also visualized in Figure 4. We can see that $\mathcal{R}^M$ with $M \neq$ NULL meets the strong empirical baseline for $\alpha_G \geq 0.6$. It is noteworthy that even for $\alpha_G = 0.9$, the proportionality of the mixed rule is around half-way between GREEDY and MES. However, this is mostly a side effect of splitting the budget into two parts, as can be seen by comparing to the mixed rule that combines GREEDY with GREEDY, essentially reducing the number of expensive projects that are chosen. For details, we refer to Appendix C.3.

Overall, the experiments show that mixing GREEDY with MES^M works best on large instances when we allow GREEDY to spend 60 to 90 percent of the budget. We achieve significantly better results when choosing the MES-STYLE, EQUAL-SPLIT or VALUE-BASED pre-allocation method over the NULL-method, with MES-STYLE consistently performing slightly better than the other two. However, as MES-STYLE fails to give *any* theoretical proportionality guarantees (see Table 3), the VALUE-BASED method might be preferable, as it gives strong theoretical guarantees with only a slight practical performance trade-off.

7 Conclusion

Taking inspiration from mixed-member electoral systems across the world, we have introduced mixed voting rules for participatory budgeting. Using combinations of GREEDY and MES as our primary examples, we have established a general framework for analyzing the performance of such mixed rules.

Our focus on MES stems from its strong proportionality guarantees and its potential to rebalance disproportional selections by earlier rules. Similar to how the proportional component in a mixed-member electoral system (like Scotland's *Additional Member System*) aims to correct the disproportionalities created by district voting, MES can enhance the proportionality of outcomes when incorporated as a later component of a mixed voting rule. This parallel has the potential to improve the explainability of mixed PB voting rules, particularly when voters are already familiar with mixed-member systems.

We proposed several methods to account for a set of already selected projects when setting initial voter budgets for MES. From these, the VALUE-BASED method was the only one to exceed our theoretical baseline in terms of proportionality guarantees. Our experiments suggest that mixing GREEDY and MES works best on large instances, when letting GREEDY spend at least $50\,\%$ of the budget. From the pre-allocation methods we defined, the VALUE-BASED method is among the best, performing significantly better than the naive approach of spreading the budget equally among voters.

In future work, we plan to consider the multiwinner (unit-cost) setting, where stronger proportionality guarantees are typically possible. We would also like to find a formal argument showing that the VALUE-BASED method achieves the best possible proportionality guarantee that meets our baseline. Additionally, we plan to complement our proportionality analysis by deriving *utilitarian* guarantees for mixed rules involving GREEDY and MES.

The mixed voting rules framework that we have introduced is applicable to a wide variety of rules and objectives. Some rules, like *Chamberlin–Courant* [11], might have a trivial adaptation, analogously to GREEDY. Others, like *Phragmén's sequential rule* [17, 9], might require processing the set of pre-selected projects, or an entirely different approach altogether.

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Appendix

A Omitted proofs

Observation 1. *Suppose there exists a voting rule R whose outcome $P^* = R(B, \emptyset)$ always satisfies some monotone property X. Then, we can construct a voting rule R' such that if $R' \in \mathcal{R}$ with available budget share α, the outcome of $\mathcal{R}$ always satisfies α-budget X.*

Proof. Fix $\alpha > 0$ and let $P^* = R(\alpha B, \emptyset)$. Define R' such that whenever it is provided with available budget share α and set of pre-selected projects P_0 it produces outcome $R'(\alpha B + c(P_0), P_0) = P_0 \cup P^*$. This outcome is feasible for R' as $c(P_0 \cup P^*) \leq c(P_0) + c(P^*) \leq c(P_0) + \alpha B$. Further, if P^* satisfies α-budget X then so does $P_0 \cup P^*$, completing the proof. Note that when $P_0 = \emptyset$, R' is identical to R. $\square$

A.1 Properties of the minimum voter budget share

Observation 2. *For any pre-allocation method M, the minimum voter budget share α^M is lower-bounded by MES's available budget share $\alpha = \frac{B_{MES} - c(P_0)}{B}$ and upper-bounded by MES's gross budget share $\frac{B_{MES}}{B}$. Furthermore, $\frac{\alpha^M B}{n}$ exceeds all voter budgets determined in the rebalancing step, i.e., $\frac{\alpha^M B}{n} \geq \max\{b_i\}_{i \in N}$.*

Proof. Pre-allocation methods defined using Definition 6 always perform weakly better (in terms of minimum voter budget share) than just giving each voter their fair share of MES's available budget $\frac{\alpha B}{n}$, which means that $\alpha^{\mathrm{M}} \geq \min_{i \in N}\{(\pi_i + \frac{\alpha B}{n})\frac{n}{B}\} \geq \frac{\alpha B}{n}\frac{n}{B} = \alpha$.

$\alpha^{\mathrm{M}} = \min\{(\pi_i + b_i)\frac{n}{B}\} \leq \sum_{i \in N}\{(\pi_i + b_i)\frac{n}{B}\}\frac{1}{n} \leq \frac{c(P_0) + \alpha B}{B} = \frac{B_{\mathrm{MES}}}{B}$.

As a consequence of the rebalancing step, for each voter $i \in N$ either $\frac{n}{B} b_i \leq \frac{n}{B}(\pi_i + b_i) = \alpha^{\mathrm{M}}$ or $b_i = 0$. $\square$

Claim 1. *Consider a target MES budget profile $(b_i)_{i \in N}$ with some minimum voter budget share α^M, satisfying the following constraints:*

- $0 \leq b_i \leq \frac{\alpha^M B}{n}$
- $\sum_{i \in N} b_i = \alpha B$
- $\alpha \leq \alpha^M \leq \frac{B_{MES}}{B}$

Then, there exists a choice of $(\pi_i)_{i \in N}$ (for some pre-allocation method M) with $\pi_i \geq 0$ and $\sum_{i \in N} \pi_i \leq c(P_0)$ such that the rebalancing step from Definition 6 outputs this profile.

Proof. Choose $\pi_i = \frac{\alpha^{\mathrm{M}} B}{n} - b_i$.

Then, $\pi_i \geq 0$ as $b_i \leq \frac{\alpha^{\mathrm{M}} B}{n}$ and $\sum_{i \in N} \pi_i = \alpha^{\mathrm{M}} B - \sum_{i \in N} b_i \leq \alpha^{\mathrm{M}} B - \alpha B \leq B_{\mathrm{MES}} - (B_{\mathrm{MES}} - c(P_0)) = c(P_0)$. Further, $\min_{i \in N}\{\pi_i + b_i\}\frac{n}{B} = \frac{\alpha^{\mathrm{M}} B}{n}\frac{n}{B} = \alpha^{\mathrm{M}}$ as required. $\square$

Proposition 1. *Fix an instance I and a set of pre-selected projects P_0. The minimum voter budget shares for MES$^{\mathrm{M}}(B_{MES}, P_0)$ with $M \in \{$Null, MES-Style, Equal, Value-Based$\}$ satisfy the following:*

$$0 \leq \alpha = \alpha^{\textsc{Null}} \leq \alpha^{\textsc{Value-Based}} \leq \alpha^{\textsc{Equal-Split}} \leq \alpha^{\textsc{MES-Style}} \leq \frac{B_{MES}}{B} \leq 1.$$

Proof. Fix an instance I, MES budget B_{MES} and set of pre-selected projects P_0. Suppose two different pre-allocation methods M and M' produce profiles $\{\pi_i^{\text{M}}\}_{i\in N}$ and $\{\pi_i^{\text{M'}}\}_{i\in N}$, which results in minimum voter budget shares of α^{M} and $\alpha^{\text{M'}}$ respectively computed during the rebalancing step. Using Definition 6 we can find that for all $i \in N$, $\pi_i^{\text{M}} \leq \pi_i^{\text{M'}}$, then $\alpha^{\text{M}} \leq \alpha^{\text{M'}}$. Using this, we can show the following:

- For all $i \in N$ we have $0 = \pi_i^{\textsc{Null}} \leq \pi_i^{\textsc{Value-Based}}$ so $\alpha^{\textsc{Null}} \leq \alpha^{\textsc{Value-Based}}$.
- For all $i \in N$ we again have $\pi_i^{\textsc{Value-Based}} \leq \pi_i^{\textsc{Equal-Split}}$ as voters pay weakly less for each project they support in the Value-Based method. Thus, $\alpha^{\textsc{Value-Based}} \leq \alpha^{\textsc{Equal-Split}}$.
- Define a new pre-allocation method, Equal-Split$'$, which selects voter payments $\pi_i^{\textsc{Equal-Split}'} = \min\{\pi_i^{\textsc{Equal-Split}}, \frac{b}{n}\}$. $\alpha^{\textsc{Equal-Split}'} = \alpha^{\textsc{Equal-Split}}$ as any voter with $\pi_i > \frac{b}{n}$ gets $b_i = 0$ in both cases and does not affect the objective of the rebalancing optimization.

  Now suppose there exists $i \in N$ for which $\pi_i^{\textsc{MES-Style}} < \pi_i^{\textsc{Equal-Split}'}$. In particular, this means that $\pi_i^{\textsc{MES-Style}} < \frac{b}{n}$ and thus i has never paid less than their fair share $\frac{c(p)}{|N_p|}$ for any project $p \in A_i \cap P_0$. However, under Equal-Split, voter i has always paid exactly $\frac{c(p)}{|N_p|}$ for every project $p \in A_i \cap P_0$. This leads to a contradiction, which means that $\pi_i^{\textsc{MES-Style}} \geq \pi_i^{\textsc{Equal-Split}'}$ for all $i \in N$, and therefore $\alpha^{\textsc{Equal-Split}} = \alpha^{\textsc{Equal-Split}'} \leq \alpha^{\textsc{MES-Style}}$.

□

A.2 Proportionality Theorems

Theorem 1. *Consider a mixed voting rule $\mathcal{R}$ such that MES$^{\textsc{Null}} \in \mathcal{R}$ with available budget share $\alpha = \alpha^{\textsc{Null}}$. Then, the outcome of $\mathcal{R}$ satisfies α-budget EJR+ up to any project.*

Proof. We adapt the proof of Brill and Peters [6] to the mixed voting rule setting. Let P^* be the outcome of MES$^{\textsc{Null}}(B_{\text{MES}}, P_0)$. It is sufficient for us to show that P^* satisfies α-budget EJR+ up to any project, as P^* will be contained in the outcome of $\mathcal{R}$. Suppose for a contradiction that P^* does not satisfy α-budget EJR+ up to any project. Then, there exists $p \notin P^*$ and voter set $N' \subseteq N_p$ with $c(A_i \cap P^*) + c(p) \leq \frac{|N'|}{n}\alpha B$ for all $i \in N'$.

Since $p \notin P^*$, we know that it was not affordable when MES$^{\textsc{Null}}$ terminated, and thus the remaining MES voter budgets b_i^r satisfy $\sum_{i\in N'} b_i^r < c(p)$. Therefore, we get that for projects from P_R:

$$\frac{\text{spending by voters in } N'}{\text{satisfaction of voters in } N'} = \frac{\sum_{i\in N'}(\frac{\alpha B}{n} - b_i^r)}{\sum_{i\in N'} c(A_i \cap P^*)} > \frac{|N'|\frac{\alpha B}{n} - c(p)}{|N'|(|N'|\frac{\alpha B}{n} - c(p))} = \frac{1}{|N'|}.$$

Hence, during the execution of MES$^{\textsc{Null}}$ at least one voter has to pay more than $\frac{1}{|N'|}$ per unit satisfaction they received, for a candidate from P^*. This means that at least one project p' with $\rho(p') > \frac{1}{|N'|}$ was selected by MES$^{\textsc{Null}}$. Just before the first such project was selected, each voter $i \in N'$ must have spent at most $\frac{c(A_i\cap P^*)}{|N'|} \leq \frac{\alpha B}{n} - \frac{c(p)}{|N'|}$ during the execution of R, and thus p must have been affordable with a $\rho(p) \leq \frac{1}{|N'|}$ at that point, so it should have been selected over p', which leads to a contradiction. □

Theorem 2. *Consider a mixed voting rule $\mathcal{R}$ such that MES$^{\textsc{Value-Based}} \in \mathcal{R}$ with available budget share α and minimum voter budget share $\alpha^{\textsc{Value-Based}} \geq \alpha$. Then the outcome of $\mathcal{R}$ satisfies $\alpha^{\textsc{Value-Based}}$-budget EJR+ up to any project.*

Proof. The proof builds on the proof of Theorem 1. Let $\alpha_v = \alpha^{\text{VALUE-BASED}}$. We again let the outcome of $\text{MES}^{\text{VALUE-BASED}}(B_{\text{MES}}, P_0)$ be P^* and assume for contradiction that there exist $p \notin P^*$ and voter set $N' \subseteq N_p$ with $c(A_i \cap P^*) + c(p) \leq \frac{|N'|}{n}\alpha_v B,\ \forall i \in N'$. Let v^* be the threshold value, from our definition of the VALUE-BASED pre-allocation method. We distinguish two cases.

Case 1: $|N_p| \leq v^*$

From the definition of the VALUE-BASED method, we know that for each voter $i \in N$, $\pi_i + b_i \geq \frac{\alpha_v B}{N}$ Analogously to the proof of Theorem 1, we know that for projects from P^*:

$$\frac{\text{spending by voters in } N' \ (\pi_i \text{ and } b_i)}{\text{satisfaction of voters in } N'} \geq \frac{\sum_{i \in N'}(\frac{\alpha_v B}{n} - b_i^r)}{\sum_{i \in N'} c(A_i \cap P^*)} > \frac{1}{|N'|}$$

Hence, during either the pre-allocation of P_0 or the execution of MES at least one voter has to pay more than $\frac{1}{|N'|}$ per unit satisfaction they received, for a candidate from P^*. Further, this must be a candidate from $P^* \setminus P_0$ as whenever a voter funds projects from P_0 during the pre-allocation, they must spend at most $\frac{1}{v^*} \leq \frac{1}{|N_p|} \leq \frac{1}{|N'|}$ per unit satisfaction, from the definition of the VALUE-BASED pre-allocation method. Then, a contradiction can be obtained analogously to the proof of Theorem 1.

Case 2: $|N_p| > v^*$

From our definition of the VALUE-BASED method this must mean that there exists $P_0' \subset P_0$ with $c(P_0') + c(p) > B_{\text{MES}}$ and for each $p' \in P_0'$, $|N_{p'}| \geq |N_p|$, as p was not selected and $v(p) = |N_p| > v^*$.

Let the voter payments and budgets produced in the pre-allocation of $\text{MES}^{\text{VALUE-BASED}}(B_{\text{MES}}, P_0)$ be $(\pi_i)_{i \in N}$ and $(b_i)_{i \in N}$ respectively. In order to reach a contradiction, we will consider running $\text{MES}^{\text{VALUE-BASED}}(B_{\text{MES}}, P_0')$. We let the voter payments and budgets its pre-allocation produces be $(\pi_i')_{i \in N}$ and $(b_i')_{i \in N}$ respectively and let its available and minimum voter budget shares be α' and α_v' respectively. Clearly $\pi_i' \leq \pi_i$ for any voter i as $P_0' \subseteq P_0$.

We claim that the following inequalities hold:

(1) $\alpha_v' \geq \alpha_v$,

(2) $\frac{c(A_i \cap P_0')}{|N'|} + \frac{c(p)}{|N'|} \leq (\pi_i' + b_i')$ for every $i \in N'$,

(3) $\pi_i' \leq \frac{c(A_i \cap P_0')}{|N'|}$ for every $i \in N'$, and

(4) $b_i' < \frac{c(p)}{|N'|}$ for some $i \in N'$.

For (1), observe that one (perhaps not optimal) way to choose voter budgets $(b_i')_{i \in N}$ would be to give each voter $b_i' = b_i + (\pi_i - \pi_i') \geq b_i$. Thus, $\alpha_v' \geq \min_{i \in N}\{(\pi_i' + b_i')\frac{n}{B}\} \geq \min_{i \in N}\{(\pi_i + b_i)\frac{n}{B}\} = \alpha_v$.

For (2), recall that $P_0' \subset P^*$. Thus we know that for each voter $i \in N_p$ the following holds: $c(A_i \cap P_0') + c(p) \leq c(A_i \cap P^*) + c(p) \leq \frac{|N'|}{n}\alpha_v B \leq \frac{|N'|}{n}\alpha_v' B$ using (1). Further, from the definition of minimum voter budget share: $\alpha_v' \leq (\pi_i' + b_i')\frac{n}{b}$. Combining these, we obtain the statement above.

For (3), note that each project $p' \in P_0' \subseteq P_0$ has $|N_{p'}| \geq |N_p| \geq |N'|$ and was funded fully by the EQUAL-SPLIT method. This means that each voter $i \in N'$ paid at most $\frac{1}{|N'|}$ per unit satisfaction they obtained: $\frac{\pi_i'}{c(A_i \cap P_0')} \leq \frac{1}{|N'|}$.

For (4), observe that $c(p) > B_{\text{MES}} - c(P_0') = \alpha' B = \sum_{i \in N} b_i' \geq \sum_{i \in N'} b_i'$. Therefore, there exists $i \in N'$ such that $\frac{c(p)}{|N'|} > b_i'$.

Combining (2), (3), and (4), we obtain a contradiction. □

Theorem 3. *Suppose $\mathcal{R} = [R_k]_{k\in[m]}$, where R_1 is* Greedy *and R_2 is MES$^{\text{Equal-Split}}$ with minimum voter budget share $\alpha^{\text{Equal-Split}}$. Then, the outcome of $\mathcal{R}$ satisfies $\alpha^{\text{Equal-Split}}$-budget EJR+ up to any two projects.*

Proof. For this proof, we make some tweaks to the proof of the first case of Theorem 2 (the second case is not needed). Let $\alpha_v = \alpha^{\text{Equal-Split}}$. We again let the outcome of MES$^{\text{Equal-Split}}$ be P^*, but now assume that there exist $p_1, p_2 \notin P^*$ and voter set $N' \subseteq N_{p_1} \cap N_{p_2}$ with $c(A_i \cap P^*) + c(p_1) + c(p_2) \leq \frac{|N'|}{n}\alpha_v B$ for all $i \in N'$. Without loss of generality, assume $c(p_1) \leq c(p_2)$ and let $p = p_1$. Thus, we know that $c(A_i \cap P^*) + 2c(p) \leq \frac{|N'|}{n}\alpha_v B$ for all $i \in N'$.

Let the outcome of Greedy be P_0 (as it is the set of pre-selected projects for MES$^{\text{Equal-Split}}$). Greedy did not select p, which means that there exists $P_0' \subseteq P_0$ with $|N_{p'}| \geq |N_p|$ for all $p' \in P_0'$ and $c(P_0') + c(p) > B_1 \geq c(P_0)$

We now consider the spending of voters from N' for projects in $P_0' \cup (P^* \setminus P_0) \subseteq P^*$ (omitting $P_0 \setminus P_0'$ as that spending may have been inefficient), and the satisfaction they achieve from those projects:

$$\frac{\text{spending by voters in } N'}{\text{satisfaction of voters in } N'} \geq \frac{(\sum_{i\in N'} \frac{\alpha_v B)}{n} - b_i^r) - c(p)}{\sum_{i\in N'} c(A_i \cap P^*)} > \frac{|N'|\frac{\alpha_v B}{n} - 2c(p)}{|N'|(|N'|\frac{\alpha_v B}{n} - 2c(p))} = \frac{1}{|N'|}$$

Hence, during either the pre-allocation of $P_0' \subseteq P_0$ or the execution of MES (which additionally selected $P_R \setminus P_0$) at least one voter had to pay more than $\frac{1}{|N'|}$ per unit satisfaction they received, for a candidate from P_R. Further, this must be a candidate from $P_R \setminus P_0$ as whenever a voter funded projects from P_0' during the pre-allocation, they must have spent at most $\frac{1}{|N_p|} \leq \frac{1}{|N'|}$ per unit satisfaction. The rest of the argument follows from the proof of Theorem 1. ☐

B Proportionality Violations

In order to formulate the negative results in this section, we will use the proportionality notion of Extended Justified Representation (EJR), and its variants, from PB literature.

Definition 10. *Let $T \subseteq P$ and $N' \subseteq N$. We say that voter group N' is* T-cohesive *if and only if $T \subseteq \bigcap_{i\in N'} A_i$ and $c(T) \leq \frac{|N'|}{n}B$. We say that N' is α-budget T-cohesive if and only if $T \subseteq \bigcap_{i\in N'} A_i$ and $c(T) \leq \frac{|N'|}{n}\alpha B$*

Definition 11 (Extended to PB by Peters et al. [24])**.** *We say that an outcome $P^* \subseteq P$ satisfies* Extended Justified Representation *(EJR) if, for every T-cohesive group N', either $T \subseteq P^*$ or there exists a voter $i \in N'$ such that $c(A_i \cap P^*) \geq c(T)$. Following Definition 5, the outcome satisfies α-budget EJR if this is instead true for every α-budget T-cohesive group.*

A feasible outcome satisfying EJR always exists, but cannot be computed in polynomial time, unless P=NP [24], which motivated the following relaxation:

Definition 12. *We say that an outcome $P^* \subseteq P$ satisfies Extended Justified Representation up to one project (EJR1) if, for every T-cohesive group N', either $T \subseteq P^*$ or there exists a voter $i \in N'$ and a project $p \in A_i \cap (P \setminus P^*)$ such that $c(A_i \cap P^*) + c(p) > c(T)$. The outcome satisfies α-budget EJR1 if this is instead true for every α-budget T-cohesive group.*

The outcome of MES($B_{\text{MES}}, \emptyset$) always satisfies EJR1, and is computable in polynomial time. EJR+ up to any project implies EJR1 [6].[10]

We generalize the definition of EJR1 as follows.

[10]We are skipping over some intermediate notions between the two, such as EJR up to any project (EJRx), which are not needed for our results — see, e.g., Rey et al. [25] for an overview.

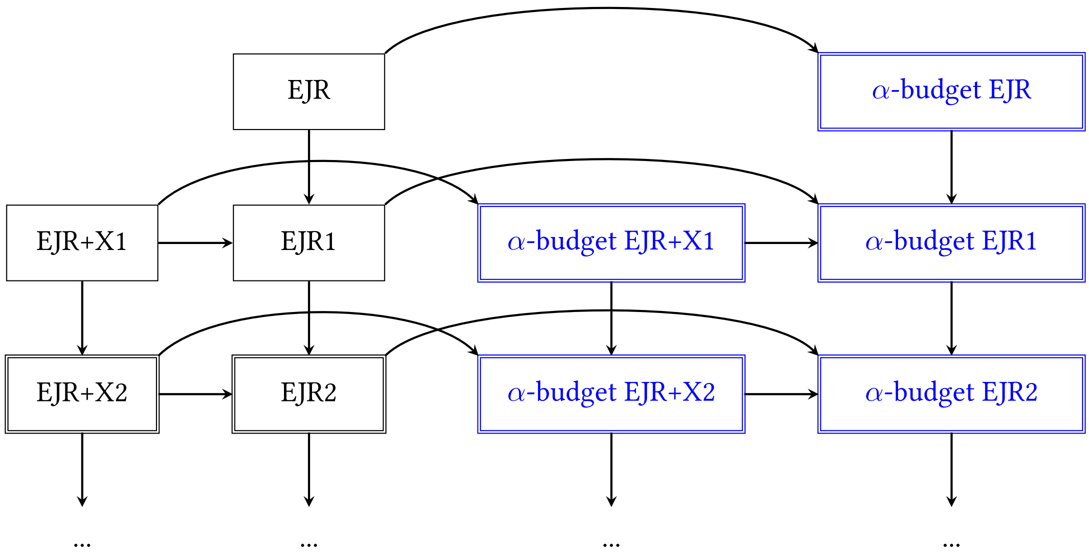


Figure 6: Relationships between PB proportionality notions, for $\alpha \leq 1$. We refer to "EJR+ up to any k projects" by the abbreviation *EJR+Xk* (so that EJR+X1 corresponds to EJR+X in Section 6). Nodes with a double border correspond to axioms that have been proposed in this paper.

Definition 13. *We say that an outcome $P^* \subseteq P$ satisfies EJR up to k projects (EJRk) if, for every T-cohesive group N', either $|T \cap P^*| > |T| - k$*[11] *or there exists a voter $i \in N'$ and a set of projects $P' \subseteq A_i \cap (P \setminus P^*)$ with $|P'| \leq k$ such that $c(A_I \cap P^*) + c(P') > c(T)$. P^* satisfies α-budget (EJRk) if the above instead holds for every α-budget T-cohesive group N'.*

The idea of EJRk is that either (i) N' is at most k-1 projects away from getting T, the set of projects they are cohesive over, or (ii) we can give some voter in N' k projects (possibly sourced from outside of T) to make them strictly better off than they would be from getting T. EJRk reduces to EJR1 when $k = 1$. An "up to k projects" style notion has not been considered for the PB setting, and we define it here analogously to the definition of envy-freeness up to k goods in fair division literature (see, e.g., Suksompong [27]).

We can show that $EJRk$ is a weaker axiom than EJR up to any k projects.

Proposition 2. *Fix $P^* \subseteq P$ and $k \in \mathbb{N}^+$. If P^* satisfies EJR+ up to any k projects, then P^* satisfies EJRk.*

Proof. Suppose P^* satisfies EJR+ up to any k projects for some $k \geq 1$. Let $T \subseteq P$ and consider a T-cohesive group $N' \subseteq N$ with $|T \cap P^*| \leq |T| - k$. Then, consider a k-size subset of $T \setminus P^*$ and call it P'. We know from the definition of EJR+ up to any k projects that $c(A_i \cap P^*) + c(P') > \frac{|N'|[\alpha]B}{n}$ and we are done. □

The relationships between the proportionality notions restated and introduced in this paper are summarized in Figure 6. Our primary motivation for introducing weaker proportionality axioms is to show that some of our pre-allocation methods can perform arbitrarily badly, from a proportionality perspective.

We restate stronger versions of the results in Table 3 using the EJR-based proportionality notions defined above. Let $P^* = \text{MES}^{\text{M}}(B_{\text{MES}}, P_0)$ be the outcome of MES with available budget share $\alpha = \frac{B_{\text{MES}} - c(P_0)}{B}$. P^* does not necessarily satisfy the proportionality measure corresponding to that pre-allocation method in Table 4, with respect to the available budget share α. Note that each of the counterexamples we

[11] "$|T \cap P^*| > |T| - k$" represents a generalization of "$T \subseteq P^*$" from the definition of EJR1.

construct in this section produce an exhaustive outcome, and thus these violations are not a consequence of MES not spending enough of its available budget.

Method M	**Violated Properties**
NULL	α-budget EJR (Proposition 3), even if P_0 selected by GREEDY
VALUE-BASED	α-budget EJR (Proposition 3), even if P_0 selected by GREEDY
EQUAL-SPLIT	α-budget EJR1 (Proposition 6), even if P_0 selected by GREEDY
EQUAL-SPLIT	α-budget EJRk (Proposition 4), in general
MES-STYLE	α-budget EJRk (Proposition 5), even if P_0 selected by GREEDY

Table 4: Proportionality violations for the outcome $P^* = \text{MES}^{\text{M}}(B', P_0)$.

Proposition 3. *Suppose $R \in \mathcal{R}$ is MES with any pre-allocation method and available budget share α. Then the outcome of $\mathcal{R}$ does not necessarily satisfy α-budget EJR.*

This follows directly from the fact that the outcome of MES doesn't satisfy EJR [24], by considering $R(\alpha B, \emptyset)$.

Proposition 4. *Consider a mixed voting rule $\mathcal{R}$ such that MES$^{\text{EQUAL-SPLIT}} \in \mathcal{R}$ with available budget share α. Then, for any two arbitrary positive integers $l, k \in \mathbb{N}^+$, the outcome of $\mathcal{R}$ does not necessarily satisfy $\frac{\alpha}{l}$-budget EJR up to k projects.*

Proof. Let $l, k \in \mathbb{N}^+$ be two positive integers. We construct a PB instance I, which shows that the outcome of $\mathcal{R}$ does not satisfy $\frac{\alpha}{l}$-budget EJR up to k projects.

Consider the PB instance $I = (B, P, A, c)$ with an even number of voters $n = 8l$ and instance budget $B = n$. Let P contain the following projects:

- p_i for $1 \leq i \leq \frac{n}{2}$, with $c(p_i) = 1$ and $N_{p_i} = \{i\}$
- p'_j for $1 \leq j \leq 2k$ with $c(p'_j) = \frac{n}{8kl}$ and $N_{p'_j} = \{1, \ldots, \frac{n}{2}\}$
- p'' with $c(p'') = \frac{n}{2}$ and $N_{p''} = \{\frac{n}{2} + 1, \ldots, n\}$

Note, that the approval sets in A are implicitly defined through the approving voter sets N_p for all projects $p \in P$. We let $P_0 = \{p_1, \ldots, p_{\frac{n}{2}}\}$ and consider MES$^{\text{EQUAL-SPLIT}}(B, P_0)$ which has an available budget share $\alpha = 0.5$. The EQUAL-SPLIT method outputs the following voter payments and budgets:

- $\pi_i = 1$ and $b_i = 0$ for $1 \leq i \leq \frac{n}{2}$ and
- $\pi_i = 0$ and $b_i = 1$ for $\frac{n}{2} < i \leq n$.

MES$^{\text{EQUAL-SPLIT}}$ selects p'' and terminates with outcome $P^* = \{p_1, \ldots, p_{\frac{n}{2}}, p''\}$ with $c(P^*) = n$.

Now, we choose $T = \{p'_1, \ldots p'_{2k}\}$ and $N' = \{1, \ldots, \frac{n}{2}\}$ with $c(T) = \frac{n}{4l} = \frac{\frac{0.5}{l}\frac{n}{2}n}{n} = \frac{\frac{\alpha}{l}|N'|B}{n}$. Thus, N' is $\frac{\alpha}{l}$-budget T-cohesive (compare Definition 10). However, we have $|T \cap P^*| = 0 < k = |T| - k$ and for any voter $i \in N'$ and any set of projects $P' \subseteq A_i \cap (P \setminus P^*) = T$ with $|P'| \leq k$, we have

$$c(A_i \cap P^*) + c(P') \leq c(p_i) + k\frac{n}{8kl} = 1 + \frac{n}{8l} = 2 = \frac{n}{4l} = c(T).$$

Thus, the outcome of $\mathcal{R}$ violates $\frac{\alpha}{l}$-budget EJR up to k projects (compare Definition 13). ☐

Project	p_1	p_2	$\dots$	$p_{\frac{n}{2}}$	p'_1	p'_2	$\dots$	p'_{2k}	p''
Cost	1	1	$\dots$	1	$n/8kl$	$n/8kl$	$\dots$	$n/8kl$	$n/2$
Number of approvals	1	1	$\dots$	1	$n/2$	$n/2$	$\dots$	$n/2$	$n/2$
A_1	✓				✓	✓	$\cdots$	✓	
A_2		✓			✓	✓	$\cdots$	✓	
$\vdots$			$\ddots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$	
$A_{\frac{n}{2}}$				✓	✓	✓	$\cdots$	✓	
$A_{\frac{n}{2}+1}$									✓
$A_{\frac{n}{2}+2}$									✓
$\vdots$									$\vdots$
A_n									✓

Table 5: Example instance with a budget of n for the proof of Proposition 4.

Proposition 5. *Consider a mixed voting rule $\mathcal{R}$ such that MES$^{\text{MES-STYLE}} \in \mathcal{R}$ with available budget share α. Then, for any arbitrary positive integer $k \in \mathbb{N}^+$, the outcome of $\mathcal{R}$ does not necessarily satisfy α-budget EJR up to k projects. This holds even when MES is the second rule in $\mathcal{R}$ and the first rule in $\mathcal{R}$ is* Greedy *(with tie-breaking in favor of large projects).*

Proof. Let $k \in \mathbb{N}^+$ be an arbitrary positive integer. We construct a PB instance I, which shows that the outcome of $\mathcal{R}$ does not satisfy α-budget EJR up to k projects.

Let $I = (B, P, A, c)$ be a PB instance with $n = 100$ voters and a budget $B = 100$. Let parameter $\lambda \in \mathbb{N}$ be defined as $\lambda = \lceil \frac{k}{4} \rceil$. Let P contain the following subsets of projects:

- **Type 1**: For $1 \leq j \leq 50$ define p_j with cost $c(p_j) = 1$ and 40 approving voters $N_{p_j} = \{j\} \cup \{62, \dots, 100\}$.
- **Type 2**: For every unique 10-person subset of voters $N^* \subset \{1, \dots, 40\}$ define 5λ projects $p_j^{N^*}$ with $1 \leq j \leq 5\lambda$, with cost $c(p_j^{N^*}) = \frac{1}{\lambda} > 0$, approved by the 10 voters in N^*, i.e., $N_{p_j^{N^*}} = N^*$.
- **Type 3**: For every unique 40-person subset of voters $\hat{N} \subset \{1, \dots, 50\}$ define 40λ projects $p_j^{\hat{N}}$ with $1 \leq j \leq 40\lambda$ with cost $c(p_j^{\hat{N}}) = \frac{39}{40\lambda}$, approved by the 40 voters in $\hat{N}$, i.e., $N_{p_j^{\hat{N}}} = \hat{N}$.
- **Type 4**: Define one project p' with cost $c(p') = 11$, approved by 11 voters $N_{p'} = \{51, \dots, 61\}$.

Note, that the approval sets in A are implicitly defined through the approving voter sets N_p for all projects $p \in P$.

Let $P_0 = \{p_1, \dots, p_{50}\}$, which means that MES's available budget share is $\alpha = 0.5$. Note, that P_0 is exactly the set of projects that would be chosen by Greedy$(50, \emptyset)$ in this instance (with tie-breaking in favor of large projects). Consider the MES-Style pre-allocation method, with some arbitrary order of P_0. The method funds the first 40 projects in this ordering equally, with supporters paying $\frac{1}{40}$ per project. At this point, all voters in $\{62, \dots, 100\}$ run out of money and the remaining projects are funded solely by their supporters from $i \in \{1, \dots, 50\}$. Call the set of these 10 voters N^*.

The MES-Style pre-allocation method selects voter payments and induces voter budgets as follows:

- $\pi_i = 1; b_i = 0$ for $i \in N^* \cup \{62, \dots, 100\}$,

Project	p_1	p_2	$\dots$	p_{50}	Type 2	Type 3	p'
Cost	1	1	$\dots$	1	$1/\lambda$	$39/40\lambda$	11
Quantity					$\binom{40}{10} \cdot 5\lambda$	$\binom{50}{40} \cdot 40\lambda$	
Number of approvals	40	40	$\dots$	40	10	40	11
A_1	✓				(✓)	(✓)	
A_2		✓			(✓)	(✓)	
$\vdots$			$\ddots$		$\vdots$	$\vdots$	
A_{50}				✓	(✓)	(✓)	
A_{51}							✓
A_{52}							✓
$\vdots$							$\vdots$
A_{61}							✓
A_{62}	✓	✓	$\cdots$	✓			
A_{63}	✓	✓	$\cdots$	✓			
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$			
A_{100}	✓	✓	$\cdots$	✓			

Table 6: Example instance with a budget of $b = 100$ for the proof of Proposition 5. Each project of type 2 is approved by 10 voters in $\{1, \dots, 40\}$ and each project of type 3 is approved by 40 voters in $\{1, \dots, 50\}$.

- $\pi_i = \frac{1}{40}, b_i = \frac{39}{40}$ for $i \in \{1, \dots, 50\} \setminus N^*$, and
- $\pi_i = 0, b_i = 1$ for $i \in \{51, \dots, 61\}$.

This yields a minimum voter budget share of $\alpha_v = 1$. MES then selects the 40λ type 3 projects supported by $\{1, \dots, 50\} \setminus N^*$, and then the Type 4 project p', yielding an output P^* with $c(P^*) = 100$.

Each voter $i \in N^*$ obtains a (cost) satisfaction of $\mu_i(P^*) = c(A_i \cap P^*) = c(p_i) = 1$. However, there exist 5λ commonly approved unchosen projects $T = \{p_j^{N^*}\}_{1 \leq j \leq 5\lambda} \subseteq \bigcap_{i \in N^*} A_i$ with $c(T) = 5\lambda\frac{1}{\lambda} = 5 = \frac{\alpha|N^*|B}{n}$. Thus, N^* is α-budget T-cohesive (compare Definition 10).

However, we have $|T \cap P^*| = 0 < 5\frac{k}{4} - k \leq 5\lambda - k = |T| - k$ and for any voter $i \in N^*$ and any set of projects $P' \subseteq A_i \cap (P \setminus P^*)$ with $|P'| \leq k$, we have

$$c(A_i \cap P^*) + c(P') \leq 1 + k \cdot \frac{1}{\lambda} \leq 1 + k \cdot \frac{4}{k} = 5 = c(T).$$

Thus, the outcome of $\mathcal{R}$ violates α-budget EJR up to k projects (compare Definition 13). □

This example be tweaked to show that the $\frac{\alpha}{l}$-budget EJR up to k projects is not satisfied, for any arbitrarily arbitrarily high $l \in \mathbb{N}^+$ by choosing an appropriately large number of voters, similarly to the proof of Proposition 4.

Proposition 5 demonstrates that in order to achieve any kind of proportionality guarantees, it might be necessary to give a budget of less than their fair share of the MES budget, $\frac{B_{\text{MES}}}{n}$, to initially empty-handed voters (those who approve no projects in P_0), even if the cost of P_0 doesn't exceed the endowments of non-empty-handed voters. This might seem unfair, but it is in this case preferable to charging the cost of an entire project to one voter.

Proposition 6. *Consider a mixed voting rule $\mathcal{R} = [R_j]_{1 \leq j \leq m}$ with $R_1 = $ Greedy and $R_2 = $ MES$^{\text{Equal-Split}}$ with available budget share α. Then the outcome of $\mathcal{R}$ does not necessarily satisfy α-budget EJR up to one project.*

Proof. We construct a PB instance I, which shows that the outcome of $\mathcal{R}$ does not satisfy α-budget EJR1.

Let $I = (B, P, A, c)$ be a PB instance with $n = 100$ voters and budget $B = 100$. Let P contain the following projects:

- p with cost $c(p) = 48.6$ and 11 approving voters $N_p = \{90, \ldots, 100\}$
- p_j for $1 \leq j \leq 10$ with cost $c(p_j) = 0.14$ and 1 approving voter $N_{p_j} = \{j\}$.
- p' with cost $c(p') = 2.85$, and 10 approving voters $N_{p'} = \{1, \ldots, 10\}$
- p''_j for $1 \leq j \leq 3$ with cost $c(p''_j) = 1.5$, and 9 approving voters $N_{p''_j} = \{1, \ldots, 9\}$
- $\hat{p}_j$ for $1 \leq j \leq 19$ with cost $c(\hat{p}_j) = 2.31$ and 4 approving voters $N_{\hat{p}_j} = \{4j + 7, \ldots, 4j + 10\}$
- $\hat{p}_{20}$ with cost $c(\hat{p}_{20}) = 1.88$ and 4 approving voters $N_{\hat{p}_{20}} = \{10, 87, 88, 89\}$

Note, that the approval sets in A are implicitly defined through the supporter sets N_p for all projects $p \in P$.

Consider the output of $\text{MES}^{\text{Equal-Split}}(100, \text{Greedy}(50, \emptyset))$. Greedy first chooses p, and then chooses $\{p_1, ..., p_{10}\}$ in some order, as it can no longer afford any other projects in P. $c(\{p, p_1, \ldots, p_{10}\}) = 50$, which provides MES with an available budget share of $\alpha = 0.5$.

The Equal-Split pre-allocation method assigns budgets as follows:

- $\pi_i = 0.14$ and $0.4375 \leq b_i \leq 0.4376$ for $1 \leq i \leq 10$,
- $\pi_i = 0$ and $0.5775 \leq b_i \leq 0.5776$ for $11 \leq i \leq 89$, and
- $\pi_i = \frac{48.6}{11}$ and $b_i = 0$ for $90 \leq i \leq 100$.

MES then selects p', which reduces the budgets of each voter $i \in \{1, \ldots, 10\}$ to $b_i \approx 0.1525$. Importantly, this means that the voters in $\{1, \ldots, 9\}$ can no longer afford any project from $\{p''_1, p''_2, p''_3\}$. MES then selects $\{\hat{p}_1, \ldots, \hat{p}_{20}\}$ and terminates with an exhaustive outcome $P^* = \{p, p_1, \ldots, p_{10}, p', \hat{p}_1, \ldots, \hat{p}_{20}\}$, having spent 48.62 units of its available budget.

Consider $T = \{p''_1, p''_2, p''_3\}$. Voter group $N' = \{1, \ldots, 9\}$ is 0.5-budget T-cohesive as $0.5\frac{|N'|}{n}B = 4.5 = c(T)$, and each voter $i \in N'$ has $c(A_i \cap P^*) = c(p') + c(p_i) = 2.99$, and thus for any project $p \in P \setminus P^* = \{p''_1, p''_2, p''_3\}$, $c(A_i \cap P^*) + c(p) = 2.99 + 1.5 < c(T)$.

Thus P^* does not satisfy EJR up to one project (compare Definition 12). □

C Further experimental results

In this section, we discuss some more detailed results from the experiments we conducted.

C.1 Minimum Voter Budget Shares in Practice

In Section 5.1 we introduce the concept of the *minimum voter budget share* α^M, based on which we show different proportionality guarantees for MES^M in Section 5.2. In particular, we show that the outcome of $\text{MES}^M(B', P_0)$ satisfies α^M-budget EJR up to any project for $M \in \{\text{Null}, \text{Value-Based}\}$ and α^M-budget EJR up to any two projects for $M = \text{Equal-Split}$ if P_0 was selected by Greedy. The

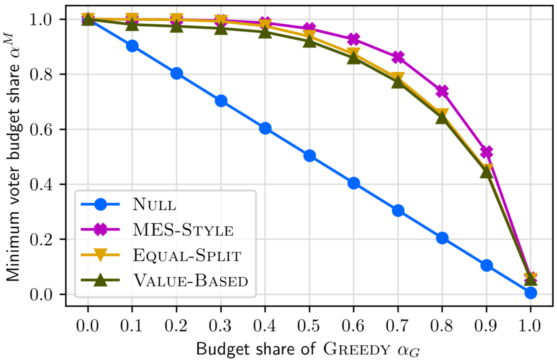


Figure 7: Minimum voter budget share α^{M} in practice for different pre-allocation methods and GREEDY budget shares α_{G}, averaged over all instances with at least 20 projects.

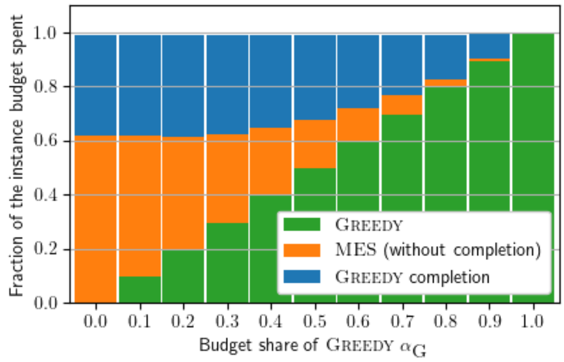
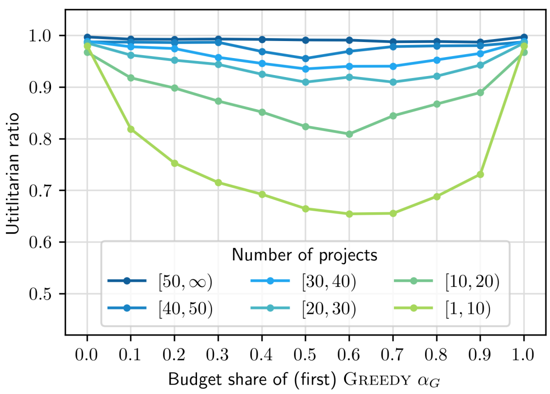
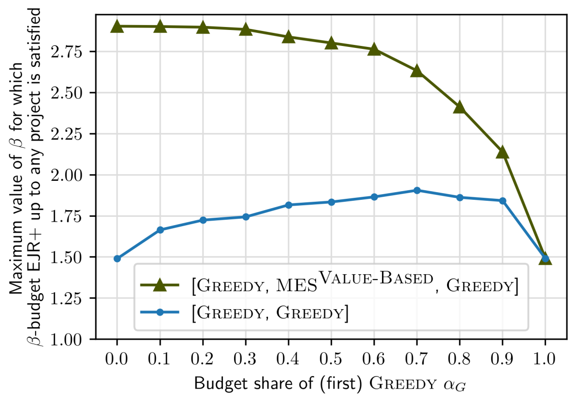


Figure 8: Budget spending of each individual rule in the mixed rule $\mathcal{R}^{\text{VALUE-BASED}}$, averaged over all instances with at least 20 projects.



(a) Loss of utilitarian welfare due to splitting the budget, averaged over instances of different sizes.



(b) Increase of proportionality due to splitting the budget, averaged over all instances with at least 20 projects. The values for the mixed rule $\mathcal{R}^{\text{VALUE-BASED}}$ are displayed for comparison.

Figure 9: Performance of mixing GREEDY with GREEDY for different (first) GREEDY budget shares α_{G} from 0 to 1.

relation between α^M for different pre-allocation methods M is stated in Observation 2. In particular, we know that $\alpha^{\text{Null}} \leq \alpha^{\text{Value-Based}} \leq \alpha^{\text{Equal-Split}}$, where α is the available budget share of MES^M. To get a feeling how far apart these values are in practice we computed them for the pre-allocation of $P_0 = \text{Greedy}(\alpha_G B, \emptyset)$ for $\alpha_G \in \{0, 0.1, 0.2, \ldots, 1\}$.

Figure 7 shows the average minimum voter budget share α^M for each pre-allocation method M, and for different Greedy shares of the budget, over all instances with at least 20 projects. We observe, that α^{Null} is significantly lower than the minimum voter budget share produced by the other three methods, suggesting that $\mathcal{R}^{\text{Value-Based}}$ provides a much better proportionality guarantee than $\mathcal{R}^{\text{Null}}$ in practice. The proportionality guarantees of $\mathcal{R}^{\text{Value-Based}}$ and $\mathcal{R}^{\text{Equal-Split}}$ are incomparable, as Equal-Split provides a weaker guarantee, but for a potentially larger fraction of the budget $\alpha^{\text{Equal-Split}}$. However Figure 7 shows that in practice $\alpha^{\text{Equal-Split}}$ is not much larger than $\alpha^{\text{Value-Based}}$.

C.2 Budget Spending of MES in a Mixed Rule

A well known problem of the Method of Equal Shares is that it is not exhaustive, i.e., after its execution, there can be unchosen affordable projects. This occurs because MES is constrained to only select projects that can be funded by their supporters. In practice this issue is often solved by using a completion method, as explained in Section 3.

Figure 8 shows how much of the budget is spent by each separate rule in $\mathcal{R}^{\text{M}} = [\text{Greedy}, \text{MES}^{\text{Value-Based}}, \text{Greedy}]$ as defined in Section 6, averaged over all instances with at least 20 projects. We can see, that while (the first) Greedy spends its available budget $\alpha_G B$ almost entirely, this is not true for MES. In fact, for large values of α_G, the spending of MES is close to zero. In these cases, $\mathcal{R}^M$ can almost be interpreted as mixing Greedy with Greedy, instead of Greedy with MES. We discuss mixing Greedy with itself in the subsection below.

A popular method to make MES use more of its available budget is "completion by varying the budget" [24], which involves iteratively increasing the starting budgets of all voters, until giving each voter one more unit of the budget would result in an outcome that exceeds the budget limit. In future work, it would be interesting to use this method in experiments, as it could solve the issue of MES only spending a comparatively small proportion of its available budget.

C.3 Effects of Splitting the Budget

Mixed Rules on Small Instances Since our experiments are limited to large instances (with at least 20 projects), this section addresses the challenges associated with running mixed rules on smaller instances.

Splitting the budget between multiple rules can sometimes have undesirable consequences — for example, a project costing more than $50\,\%$ of the budget is unlikely to ever be selected by a mixed rule that splits the budget into two equal sized parts. This problem will less likely occur in larger instances, since the average project cost tends to be lower for instances with a large number of projects. For example, the cost of the most expensive project in instances with less than 10 projects is $71\,\%$ of the budget limit on average, while this value is only $44\,\%$ over instances with at least 20 projects. We can empirically observe the effect of splitting the budget, which we explain below.

Mixing Greedy with Itself By comparing the outcomes of $\mathcal{R}_G = [\text{Greedy}, \text{Greedy}]$ to the outcome of Greedy, we can isolate the consequences of splitting the budget.

Figure 9a shows the utilitarian welfare for $\mathcal{R}_G([\alpha_G B, B])$, when varying α_G for different instance sizes (measured by the number of projects). Note, that for values 0 and 1 of α_G the mixed rule $\mathcal{R}_G$ is just the Greedy rule. The performance of $\mathcal{R}_G$ drops significantly when splitting the budget for instances with fewer than 20 projects, whereas the effect is much less pronounced for larger instances. For instances with at least 50 projects even splitting the budget in half seems to have almost no effect. This indicates that using mixed rules on small instances does not work well in practice.

Figure 9b shows the performance of $\mathcal{R}_{\mathrm{G}}$ in terms of proportionality, compared to that of $\mathcal{R}^{\mathrm{M}} = [\textsc{Greedy}, \mathrm{MES}^{\textsc{Value-Based}}, \textsc{Greedy}]$ as defined in Section 6. It seems surprising that the proportionality increases when splitting the budget of the Greedy rule into two parts, indicating that forcing Greedy to pick cheaper projects (without checking *who* approves these projects) increases proportionality. This could potentially be caused by correlations between the votes, or the fact that when larger projects are chosen, more projects remain unchosen, which could make an EJR+X violation more likely.