

APPROXIMATE CORE OF PARTICIPATORY BUDGETING VIA LINDAHL EQUILIBRIUM

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ABSTRACT. Participatory budgeting is a democratic process in which citizens decide how to allocate public funds among proposed projects. In practice, participants typically submit ordinal preferences—such as rankings or approvals—rather than numerical utilities. A central fairness concept in this setting is the proportional core, which ensures that no group of agents can reallocate their proportional share of the budget to strictly improve their outcomes. However, the core may be empty under general ordinal preferences, motivating the study of approximate core solutions that relax this requirement while preserving its fairness spirit.

We improve the best-known approximation bound from 32 (as established by [Jiang et al. \(2020\)](#)) to 6.24 under general monotone ordinal preferences. In structured domains such as committee selection with approval ballots or ranking preferences, we further refine our approach to achieve stronger guarantees of 5.15 and 5.10, respectively. Our main innovation is the introduction of the Lindahl Equilibrium with Ordinal preference (LEO), a novel continuous relaxation inspired by the classical Lindahl equilibrium. This framework bridges traditional economic insights with the discrete, ordinal setting of social choice and participatory budgeting, and has potential applicability in a broad range of settings.

1. INTRODUCTION

Since its inception in Porto Alegre, Brazil, in 1989, participatory budgeting has spread worldwide, allowing citizens to allocate public funds to community projects democratically (Wamper et al. 2021). This has led to extensive research on the design of voting systems for participatory budgeting (Aziz and Shah 2020). In the general participatory budgeting model, each project has a cost, and voters have *ordinal* preferences over *combinations* of projects. The goal is to select a *feasible* combination—one that stays within budget while fairly reflecting the voter’s preferences. A key special case, known as committee selection, arises when all projects have equal cost (normalized to 1) and the budget corresponds to the predetermined size of the committee.

A defining feature of participatory budgeting (PB) is the selection of multiple projects, rather than a single winner. While one could, in principle, treat each feasible combination of projects as an alternative and apply single-winner voting methods, such approaches fail to capture the core challenge of PB: ensuring proportional fairness. Specifically, no group of voters should be able to afford and strictly prefer an alternative set of projects whose cost is proportional to their size. This concept, known as the *proportional core* (core for short), is derived from the Lindahl equilibrium and the core of economies with divisible public goods (Foley 1970). Ever since its first introduction in the PB setting by (Fain et al. 2016), the concept has gained significant traction as it ensures fairness and representation in *all* subgroups.

The proportional core is a demanding solution concept that may not always exist (Fain et al. 2018). To address this, Jiang et al. (2020) introduced the γ -approximate core, where a project combination is stable if no alternative is preferred by more than γ times its cost-to-budget ratio of the total number of voters. They elegantly showed that there is a 32-approximate core solution for general monotone preferences and a 16-approximate solution for committee selection with approval ballots and ranking preferences. The lower bound is known to be 2 (Cheng et al. (2019)), which already holds for ranking preferences. Little progress in reducing the approximation factor has been made since Jiang et al. (2020).

Given this challenge, the literature has explored alternative approximate core stability using cardinal preferences, notably the (α, β) -core, where blocking coalitions must secure an α -fold utility gain and exceed the standard core constraint by a factor of β . Although this approach has advanced many problems, it has two drawbacks. Eliciting cardinal valuations can be impractical, as voters cannot easily quantify trade-offs between public goods. Moreover, it weakens individual rationality: the $(1 + \epsilon, \beta)$ -core permits blocking coalitions in which *all* voters strictly prefer an alternative, even when its cost is negligible.

Our main contribution is to study the original concept of the approximate core, assuming ordinal preferences and preserving individual rationality. We construct a 6.24-approximate core solution for participatory budgeting with monotone ordinal preferences. For special cases of committee selection with approval ballots and ranking preferences, we achieve improved approximation factors of 5.15 and 5.10, respectively.

To achieve this, we introduce a new class of continuous relaxations for participatory budgeting, inspired by the Lindahl equilibrium in public economics, which we term the *Lindahl Equilibrium with Ordinal Preferences* (LEO). LEO brings classical economic insights—particularly the use of individual prices or taxes—into discrete social choice settings. In this framework, given an agent’s income level, individual prices guide her to “consume” an optimal social outcome. These prices serve to coordinate all agents toward selecting the same outcome. To ensure this outcome lies in the core, it must also coincide with the choice of a centralized agent who maximizes expected revenue. However, unlike the traditional Lindahl equilibrium, LEO relies only on ordinal preferences. To address the discontinuities inherent in discrete social choice, we depart from the standard assumption of fixed incomes and instead allow agents’ incomes to be drawn from a common continuous distribution. This randomization, together with a modified consistency condition linking agents’

demands to the producer’s allocation, enables the application of standard fixed-point arguments to establish the existence of a LEO.

The paper is organized as follows. After reviewing related work, Section 2 introduces the model and fundamental solutions concepts. Section 3 covers the Lindahl equilibrium and extends it to ordinal preferences with the introduction of LEO. Section 4 present methods for constructing an approximate deterministic core solution using LEO. Section 5 focuses on committee selection with ranking and approval preferences, providing improved approximation.

Related work. The intersection of social choice and computation is a central topic in both economics and computer science. While the literature on participatory budgeting is relatively recent, it is growing rapidly. Two excellent surveys—Aziz and Shah (2020) and Rey and Maly (2023)—provide comprehensive overviews. Here, we focus only on works most closely related to our results and approach.

The approaches in the literature studying participatory budgeting problems differ according to the assumptions about preferences and the solutions used. The most general form involves ordinal preferences over combinations of projects, as considered in our paper. More specialized preference classes, such as approval, ranking, or additive preferences over individual projects, are also studied due to their succinct descriptions, which make it easier for voters to express their preferences (Rey and Maly 2023). Although a large number of solution concepts have been proposed, a significant portion of them fall into the category of *justified representation* (JR). JR requires that no *cohesive* subgroup, defined as a group of voters with sufficiently *similar* preference profiles, has the incentive to deviate. Numerous variations of JR concepts have been proposed: EJ (Aziz et al. 2017), EJ1 (Peters and Pierczynski (2021)), FJR (Peters and Pierczynski 2021), to name a few.

The proportional core, in contrast, seeks to prevent *any* of the exponentially many subgroups, cohesive or not, from deviating to another outcome. Since exact core solutions may not exist, approximate cores have been studied. Jiang et al. (2020) were the first to provide a constant-factor approximation, though a significant gap remains between known upper and lower bounds. Their 32-approximation for general monotone preferences motivates our work. To improve this bound, we develop new methods that extend the classical economic framework of Lindahl equilibrium.

Lindahl equilibrium is a classical concept for public goods and market design with externalities, originating from early economic theory (Foley 1966, 1970, Rader 1973) and recently developed further in economics (Gul and Pesendorfer 2020) and computer science (Fain et al. 2016, Kroer and Peters 2025). However, existing approaches rely on preferences having a cardinal representation. This reliance stems from the fact that establishing the existence of a Lindahl equilibrium typically requires continuity, which in turn depends on cardinal utility. In settings with ordinal preferences and a discrete outcome space, the standard workaround has been to relax agents’ choices to lotteries by extending their preferences over deterministic outcomes to preferences over lotteries—an approach that implicitly requires cardinal information. We depart from this by not using lottery extensions. Instead, we relax agents’ incomes from fixed values to continuous random variables. This technique, originally introduced in the context of resource allocation by Nguyen et al. (2025), allows us to establish Lindahl equilibria under purely ordinal preferences in public good settings.

The approximate core has also been studied in specialized settings of committee selection, where additional structure or assumptions are imposed on preferences. Our 6.24-approximation result already improves upon existing guarantees in the literature, and in this paper, we further refine these bounds for both ranking and approval preferences.

For ranking preferences, the first approximate core guarantee is established by Cheng et al. (2019) and Jiang et al. (2020), who show the existence of a 16-approximate core solution. This was later improved to a 9.8217-approximation by Charikar et al. (2025). A lower bound of 2 is also shown by Cheng et al. (2019), implying that exact core solutions cannot always be guaranteed under ranking preferences.

For approval preferences, the existence of an exact core solution remains an important open question. The best known approximation factor is 16, again due to [Cheng et al. \(2019\)](#) and [Jiang et al. \(2020\)](#). There are, however, positive results in restricted or related settings. For example, when the committee size is small (fewer than 8), [Peters \(2025\)](#) show that an exact core solution exists. In a relaxed setting where multiple copies of a candidate may be included in the committee, [Brill et al. \(2024\)](#) prove that the Proportional Approval Voting (PAV) rule yields a core solution. However, this argument does not extend to the standard setting, where each candidate can appear only once in the committee.

Recent works have also relaxed the notion of the approximate core by allowing weaker individual rationality conditions, leading to the concept of the (α, β) -core with $\alpha > 1$. Several studies have explored this direction ([Fain et al. 2018](#), [Mavrov et al. 2023](#), [Munagala et al. 2022](#), [Peters and Skowron 2020](#)). However, these works rely on cardinal measures of voter preferences and their techniques do not directly apply to our framework.

2. NOTATIONS AND PRELIMINARIES

Budgeted Social Choice. For full generality and ease of notation, we use a more abstract model. There is a finite set of social outcomes $\mathcal{O}$, with each outcome $o \in \mathcal{O}$ having a cost $c(o) \geq 0$. We assume that $\mathcal{O}$ contains the outcome $\emptyset$ with $c(\emptyset) = 0$, corresponding to the outside option. We also assume $\emptyset$ is the only outcome with 0 cost.

Given a budget B , a feasible outcome is an outcome in $\mathcal{O}$ with a cost of at most B .

A lottery over $\mathcal{O}$ is a discrete random variable that takes values in $\mathcal{O}$. It can be represented as a vector in $[0, 1]^{|\mathcal{O}|}$ whose coordinates sum to 1. The set of all such lotteries is denoted by $\Delta(\mathcal{O})$. We use a tilde over a capital letter, such as $\tilde{O}$, to denote an element of $\Delta(\mathcal{O})$. We also refer to a lottery over $\mathcal{O}$ as a probabilistic or randomized outcome.

Let the set of agents (voters) be N and the number of agents $|N| = n$. Each agent i has a preference order $\succeq_i$ over $\mathcal{O}$. We assume that the order is strict.

The tuple $(\mathcal{O}, c(\cdot), \{\succeq_i\}_{i=1}^n, B)$ is called an instance of *budgeted social choice*.

We make the following assumption about the outcome space and the preference profiles of voters.

ASSUMPTION 1. *There exists a merging operation $\oplus$ s.t. for any two outcomes $o, o' \in \mathcal{O}$, $o \oplus o' \in \mathcal{O}$, and it satisfies that $c(o \oplus o') \leq c(o) + c(o')$ and $o \oplus o' \succeq_i o, o'$ for every voter i .*

It is straightforward to see that the problem of participatory budgeting with monotone ordinal preferences (and, by extension, committee selection) fits within our framework. In these settings, there is a finite set of possible projects, and an outcome is a subset of these projects. In committee selection, similarly, the goal is to select a subset of candidates. Monotonicity implies that each agent weakly prefers any set to its strict subsets; the cost of an outcome is simply the cardinality of the selected set and the merging operation corresponds to taking the union of two outcome sets.¹

Our abstract model goes beyond traditional settings where outcomes are simply subsets of projects or candidates. For example, two separate projects—a Health Clinic and a Childcare Center—can be merged into a Family Wellness Hub that offers both services in a single facility. Even if the total cost of building the two projects separately is within budget, other constraints—such as land availability or location limitations—may prevent implementing them individually. In such cases, the two separate projects are infeasible, but the combined facility offers a viable alternative.

Our main solution concepts are the core and the approximate core, defined as follows.

DEFINITION 2.1. *Given an instance of budgeted social choice $(\mathcal{O}, c(\cdot), \{\succeq_i\}_{i=1}^n, B)$, a feasible outcome o^* , i.e., $c(o^*) \leq B$, lies in the core if, for every outcome $o \in \mathcal{O}$, the number of agents who prefer o to o^* is less than $\frac{c(o)}{B}n$. That is, $|\{i \in N : o^* \prec_i o\}| \leq \frac{c(o)}{B}n$.*

¹The same argument applies when multiple copies of each project are allowed in the outcome.

Given that the core can be empty, we will adopt the following notion of approximate core.

DEFINITION 2.2. *Given an instance of budgeted social choice $(\mathcal{O}, c(\cdot), \{\succeq_i\}_{i=1}^n, B)$, a feasible outcome o^* is in γ -approximate core if for every outcome $o \in \mathcal{O}$, $|\{i \in N : o^* \prec_i o\}| \leq \gamma \cdot \frac{c(o)}{B} \cdot n$.*

3. LINDAHL EQUILIBRIUM WITH ORDINAL PREFERENCES

This section introduces our main technical tool. We begin with the traditional Lindahl equilibrium and its existence proof, along with its relation to the core. We then adapt this framework to the setting with ordinal preferences to obtain a fractional solution analogous to the classical case. Finally, we show how to convert this fractional solution into a probabilistic one using dependent rounding, while preserving the desirable properties of the fractional core. This probabilistic outcome enables the development of a constant-factor core approximation in the next section.

3.1. Classical Lindahl equilibrium. To better understand our new notion of Lindahl equilibrium with Ordinal preferences, and how it helps us to construct an approximate core solution, we first revisit the classical version in a convex economy with a continuous set of alternatives. While there are several equivalent formulations, we adopt the following for simplicity: every $\mathbf{x} \in \mathbb{R}_+^m$ is assumed to be a feasible alternative. Each agent i has a strictly increasing and concave utility function $u_i(\mathbf{x})$ and is endowed with a fixed token budget (income), normalized to 1. Given a personalized price vector $\mathbf{p}_i \in \mathbb{R}_+^m$, the agent chooses a bundle from their demand set: $D_i(\mathbf{p}_i) = \arg \max_{\mathbf{x} \in \mathbb{R}_+^m, (\mathbf{p}_i)^\top \mathbf{x} \leq 1} u_i(\mathbf{x})$.

A centralized producer, indexed by 0, has a strictly increasing linear cost function $c(\mathbf{x})$ and chooses an allocation $\mathbf{x} \in \mathbb{R}_+^m$ to maximize total revenue subject to a budget constraint $B > 0$. The producer's choice comes from the set: $D_0(\mathbf{p}_1, \dots, \mathbf{p}_n) := \arg \max_{\mathbf{x} \geq 0} (\sum_{i=1}^n \mathbf{p}_i)^\top \mathbf{x} \quad \text{s.t.} \quad c(\mathbf{x}) = B$.

In a Lindahl equilibrium, all agents and the producer agree on a common outcome $\mathbf{x}^*$, with personalized prices ensuring individual optimality and feasibility.

DEFINITION 3.1 (Lindahl Equilibrium). *A Lindahl equilibrium consists of an allocation $\mathbf{x}^* \in \mathbb{R}_+^m$ and personalized price vectors $(\mathbf{p}_1, \dots, \mathbf{p}_n) \in (\mathbb{R}_+^m)^n$ such that:*

- (1) **Individual optimality:** *For each agent $i \in \{1, \dots, n\}$, $\mathbf{x}^* \in D_i(\mathbf{p}_i)$.*
- (2) **Producer optimality:** *The producer chooses $\mathbf{x}^* \in D_0(\mathbf{p}^1, \dots, \mathbf{p}^n)$.*

THEOREM 3.1 ((Foley 1970)). *A Lindahl equilibrium exists, and the equilibrium allocation $\mathbf{x}^*$ lies in the core. That is, for any alternative $\mathbf{x}$: $|\{i : u_i(\mathbf{x}) > u_i(\mathbf{x}^*)\}| < c(\mathbf{x}) \cdot \frac{n}{B}$.*

Sketch of the proof. The existence of equilibrium is established using a standard fixed-point argument, which follows from the continuity of the demand correspondence $D_i(p)$ with respect to prices p , along with the concavity of utilities and the linearity of the cost function.

To show that the equilibrium is in the core, let $\mathbf{x}^*$ be a Lindahl equilibrium. Since utilities are strictly increasing, each agent exhausts their budget, $\mathbf{p}_i \cdot \mathbf{x}^* = 1$. Thus, we have $\sum_{i=1}^n \mathbf{p}_i \cdot \mathbf{x}^* = n$.

Because of the cost is linear, the producer maximizes the profit-to-cost ratio, so for any alternative $\mathbf{x}$:

$$\frac{\sum_{i=1}^n \mathbf{p}_i \cdot \mathbf{x}}{c(\mathbf{x})} \leq \frac{\sum_{i=1}^n \mathbf{p}_i \cdot \mathbf{x}^*}{c(\mathbf{x}^*)} = \frac{n}{B} \quad \Rightarrow \quad \sum_{i=1}^n \mathbf{p}_i \cdot \mathbf{x} \leq c(\mathbf{x}) \cdot \frac{n}{B}.$$

Now suppose some agents prefer $\mathbf{x}$ to $\mathbf{x}^*$, i.e., $u_i(\mathbf{x}) > u_i(\mathbf{x}^*)$. Then, by revealed preference, $\mathbf{p}_i \cdot \mathbf{x} > 1$. Let $I = \{i : u_i(\mathbf{x}) > u_i(\mathbf{x}^*)\}$, then: $|I| < \sum_{i \in I} \mathbf{p}_i \cdot \mathbf{x} \leq c(\mathbf{x}) \cdot \frac{n}{B}$.

□

3.2. Lindahl equilibrium with Ordinal Preference (LEO). Adapting the classical Lindahl equilibrium concept to our setting—with ordinal preferences and a discrete set of outcomes—requires expanding the outcome space from discrete to continuous. A natural starting point is to consider lotteries over outcomes, represented by $\Delta(\mathcal{O})$. However, as we will show below, this is insufficient

for adapting the proof above. Instead, we work with outcomes represented by vectors $\mathbf{y} \in \mathbb{R}_+^{|\mathcal{O}|}$, which can be interpreted as fractional allocations over the discrete alternatives.

Even with this relaxation of the outcome space, significant challenges remain in formulating a Lindahl-type equilibrium relying only on ordinal preferences. We address each challenge as follows:

Lack of continuity: In classical settings, continuous demand enables fixed-point arguments. In our discrete setting with ordinal preferences, this continuity fails. To address this, we adopt the *random income* method from [Nguyen et al. \(2025\)](#), assuming incomes are drawn from a continuous distribution $\mathcal{I}$ on $[0, 1]$.² For each income realization, agents choose their most preferred affordable outcome, with $\emptyset$ (priced at zero) ensuring nonempty demand. This induces a random demand that varies continuously with prices, restoring the needed continuity.

Saturated utilities: In classical models, strictly increasing utilities ensure that all agents fully exhaust their budgets and consume the same allocation, which also coincides with the producer's output. This fails in our setting because utilities saturate over a finite outcome set. Instead, we adopt a condition akin to competitive equilibrium: each agent's randomized demand $\mathbf{x}_i \in \mathbb{R}_+^{|\mathcal{O}|}$ must not exceed the producer's allocation $\mathbf{y}$ coordinate-wise. If $x_{i,o} < y_o$ for some outcome o , then $p_{i,o} = 0$. This weaker condition allows for heterogeneous demands across agents, which is necessary to support a fixed-point argument, and still suffices to ensure core-like guarantees.

Revenue maximization and the core: Classical core arguments rely on the producer maximizing the revenue-to-cost ratio. But restricting outcomes to lotteries over $\mathcal{O}$ (i.e., $\sum_o y_o = 1$) blocks this approach. To recover it, we relax the outcome space to $\mathbf{y} \in \mathbb{R}_+^{|\mathcal{O}|}$ and let the producer maximize revenue subject to $\sum_{o \in \mathcal{O} \setminus \{\emptyset\}} y_o c(o) = B$. This retains the revenue-to-cost ratio logic but yields a fractional outcome, which must be rounded to a valid lottery. This is where we incur a constant-factor approximation loss.

These modifications naturally lead to a new solution concept, which we call *Lindahl equilibrium with Ordinal Preferences (LEO)* and formally define as follows.

Each agent i is endowed with a random income $\mathcal{I}$ supported on $[0, 1]$ (common for all voters) and has a *personalized* price vector $\mathbf{p}_i \in \mathbb{R}_+^{|\mathcal{O}|}$. We use $p_{i,o}$ to denote the personalized price of $o \in \mathcal{O}$ for an agent i with $p_{i,\emptyset} = 0$.

Given a fixed deterministic income, a voter selects her most preferred affordable outcome. Under a random income distribution $\mathcal{I}$, voter i 's *random demand* is the distribution over such choices across income realizations:

$$\mathcal{D}_i(\mathbf{p}_i, \mathcal{I}) := \left\{ \max_{\succ_i} \{o \in \mathcal{O} : p_{i,o} \leq b\} \mid b \sim \mathcal{I} \right\}.$$

Under the assumption that $\mathcal{I}$ is supported on $[0, 1]$ and $p_{i,\emptyset} = 0$, $\emptyset$ is affordable for every possible realization of the random income and therefore $\mathcal{D}_i$ is guaranteed to be a valid lottery over $\mathcal{O}$.

Similar to the standard Lindahl case, the producer aims to produce a fractional central allocation of outcomes maximizing the total profits given individual price vectors. In particular, given a total budget constraint B , the demand of the producer is

$$\mathcal{D}_0(\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n) = \arg \max_{\mathbf{z} \in \mathbb{R}_+^{|\mathcal{O} \setminus \{\emptyset\}|}} \sum_{o \in \mathcal{O} \setminus \{\emptyset\}} \left(\sum_{i=1}^n p_{i,o} \right) \cdot z_o \quad \text{s.t.} \quad \sum_{o \in \mathcal{O} \setminus \{\emptyset\}} c(o) z_o = B.$$

Note that in this formulation, the producer only produces outcomes other than $\emptyset$. This is purely without loss of generality: allowing production of $\emptyset$ does not affect the producer's decision, since $p_{i,\emptyset} = 0$ and $c(\emptyset) = 0$.

Now we introduce the following definition of Lindahl equilibrium with ordinal preference (LEO).

DEFINITION 3.2. *Given an instance of budgeted social choice $(\mathcal{O}, c(\cdot), \{\succeq_i\}_{i=1}^n, B)$, and a random income $\mathcal{I}$ supported on $[0, 1]$, the Lindahl equilibrium with ordinal preference (LEO) consists of*

²Our approach works for any bounded, continuous distribution; we use $[0, 1]$ for convenience.

individual consumptions $\mathbf{x}_{i=1}^n \in [0, 1]^{|\mathcal{O}|}$, a common allocation $\mathbf{y} \in [0, B]^{|\mathcal{O} \setminus \{\emptyset\}|}$ and personalized prices $\mathbf{p}_{i=1}^n \in [0, 1]^{|\mathcal{O}|}$ with $p_{i,\emptyset} = 0$ such that

- (1) for any $i \in N$ and $o \in \mathcal{O}$, $x_{i,o} = \Pr(\mathcal{D}_i(\mathbf{p}_i, \mathcal{I}) = o)$,
- (2) for any $i \in N$ and $o \in \mathcal{O} \setminus \{\emptyset\}$, $y_o \geq x_{i,o}$ with strict inequality only when $p_{i,o} = 0$,
- (3) $\mathbf{y} \in \mathcal{D}_0(\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n)$.

We first establish the existence of a LEO; the proof is provided in Appendix A.1.

THEOREM 3.2. *Given an instance of budgeted social choice $(\mathcal{O}, c(\cdot), \{\succeq_i\}_{i=1}^n, B)$, if the cumulative distribution function $F_{\mathcal{I}} : [0, 1] \rightarrow [0, 1]$ of $\mathcal{I}$ is continuous, then a LEO exists.*

Next, we introduce an additional parameter $\alpha > 0$ into the LEO framework. The only modification is in the second condition of LEO. The existence of an α -LEO follows directly from Theorem 3.2 (see Appendix A.2). This parameter will later be selected to optimize the approximation ratio in our final construction.

DEFINITION 3.3. *Given an instance of budgeted social choice $(\mathcal{O}, c(\cdot), \{\succeq_i\}_{i=1}^n, B)$, a random income $\mathcal{I}$ supported on $[0, 1]$ and $\alpha > 0$, an α -LEO consists of individual consumptions $\mathbf{x}_{i=1}^n \in [0, 1]^{|\mathcal{O}|}$, a common allocation $\mathbf{y} \in [0, B]_+^{|\mathcal{O} \setminus \{\emptyset\}|}$, and prices $\mathbf{p}_{i=1}^n \in [0, 1]^{|\mathcal{O}|}$ with $p_{i,\emptyset} = 0$, such that*

- (1) for any $i \in N$ and $o \in \mathcal{O}$, $x_{i,o} = \Pr(\mathcal{D}_i(\mathbf{p}_i, \mathcal{I}) = o)$,
- (2) for any $i \in N$ and $o \in \mathcal{O} \setminus \{\emptyset\}$, $y_o \geq \alpha \cdot x_{i,o}$ with strict inequality only when $p_{i,o} = 0$;
- (3) $\mathbf{y} \in \mathcal{D}_0(\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n)$.

THEOREM 3.3. *Given an instance of budgeted social choice $(\mathcal{O}, c(\cdot), \{\succeq_i\}_{i=1}^n, B)$ and $\alpha > 0$, if the cumulative distribution function $F_{\mathcal{I}} : [0, 1] \rightarrow [0, 1]$ of $\mathcal{I}$ is continuous, then an α -LEO exists.*

The following two properties of an α -LEO are important for our construction. We apply the α -LEO to a specific income distribution, $\mathcal{I} = \mathbf{U}[1 - \epsilon, 1]$ (i.e. uniform distribution over $[1 - \epsilon, 1]$), which is used to approximate the constant income of \$1 in the classical Lindahl equilibrium.

The first property is analogous to the classical setting and provides a bound on the total revenue of the common allocation. It follows from the definition of agents' consumption and the market-clearing condition.

PROPOSITION 3.1. *Given an α -LEO $(\mathbf{x}_{i=1}^n, \mathbf{y}, \mathbf{p}_{i=1}^n)$ with random income $\mathcal{I} = \mathbf{U}[1 - \epsilon, 1]$, the total revenue of the common allocation satisfies $\sum_{o \in \mathcal{O}} \sum_{i=1}^n p_{i,o} y_o < \alpha n$.*

The second property bounds the total price of each outcome, mirroring the revenue-to-cost ratio condition in the classical case. This property arises from the producer's revenue-maximization behavior.

PROPOSITION 3.2. *Let $(\mathbf{x}_{i=1}^n, \mathbf{y}, \mathbf{p}_{i=1}^n)$ be an α -LEO with a total budget B and a random income $\mathcal{I} = \mathbf{U}[1 - \epsilon, 1]$. Then, for any $o \in \mathcal{O}$, $\sum_{i=1}^n p_{i,o} \leq \alpha \cdot \frac{c(o)}{B} \cdot n$.*

3.3. Dependent rounding and Lindahl Lottery. In most cases, the fractional common allocation $\mathbf{y}$ may not be a lottery over $\mathcal{O}$. So, we adopt the technique of negatively dependent rounding from (Byrka et al. 2017) to turn $\mathbf{y}$ into a lottery.

PROPOSITION 3.3. *Byrka et al. (2017) Given $\mathbf{y} \in [0, 1]^m$ s.t. $\sum_{k=1}^m a_k y_k \leq B$, there exists a distribution $\tilde{Y}$ over $\{0, 1\}^m$ satisfying the following properties:*

- *Preservation of marginals:* $E[\tilde{Y}_k] \geq y_k$ for all $k \in [m]$,
- *Preservation of weights up to one:* $\sum_{k=1}^m \tilde{Y}_k \leq B + \max_{k: y_k > 0} a_k$ with probability 1,
- *Negative dependence between entries:* for $S \subseteq [m]$, $\Pr(\bigwedge_{k \in S} \tilde{Y}_k = 0) \leq \prod_{k \in S} (1 - y_k)$.

When $a_k = 1$ for all k , we additionally guarantee that $\sum_{k=1}^m \tilde{Y}_k \leq \lceil B \rceil$ with probability 1.

Given this dependent rounding scheme, we now have the tool to construct a lottery over social outcomes from a fractional solution. Note that dependent rounding runs in linear time, and thus the algorithm below also operates in linear time.

Note that we allow the common allocation $\mathbf{y}$ of a LEO to have entries exceeding 1. When generating a lottery from $\mathbf{y}$ with dependent rounding, we truncate its entries at 1.

ALGORITHM 1: Dependent Rounding on Social Outcomes

Input: A fractional $\mathbf{y} \in \mathbb{R}_+^{|\mathcal{O}|}$ s.t. $\sum_{o \in \mathcal{O}} c(o)y_o \leq B$

Output: A lottery $\tilde{L}(\mathbf{y})$ over outcomes with the cost at most $B + \max_{o: y_o > 0} c(o)$

- 1 For each $o \in \mathcal{O}$ set $z_o := \min\{1, y_o\}$
 - 2 Generate a distribution $\tilde{Z}$ from z satisfying Proposition 3.3
 - 3 Sample $Z \in \{0, 1\}^{|\mathcal{O}|} \sim \tilde{Z}$
 - 4 Output $o = \bigoplus_{o': Z_{o'}=1} o'$ as a realization of $\tilde{L}(\mathbf{y})$, where $\bigoplus$ denotes the merging operation for $\mathcal{O}$.
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DEFINITION 3.4. *Given an α -LEO with common fractional allocation $\mathbf{y}$, the lottery $\tilde{L}(\mathbf{y})$ produced by Algorithm 1 is referred to as the Lindahl lottery.*

The main appeal of the Lindahl lottery is that it guarantees, for *every* agent, a constant probability of receiving a “good” outcome—one that is ranked above the agent’s *boundary outcome*, defined below. The term “boundary” reflects that this outcome lies at the edge of the agent’s random demand: an agent includes an outcome in her lottery only if it is at least as good as her boundary outcome. The boundary outcome plays a crucial role: any outcome strictly preferred to it must have a price of at least $1 - \epsilon$. This property is key to bounding the number of voters who can strictly prefer a different alternative.

DEFINITION 3.5. *Given an α -LEO $(\mathbf{p}, \mathbf{x}, \mathbf{y})$ with $\mathcal{I} = \mathbf{U}[1 - \epsilon, 1]$, the boundary outcome of agent i denoted by o_i is i ’s favorite outcome with price at most $1 - \epsilon$ i.e. $o_i = \max_{\succ_i} \{o \in \mathcal{O} : p_{i,o} \leq 1 - \epsilon\}$.*

We obtain the following result bounding the probability of a Lindahl lottery outputting an outcome worse than the boundary outcome. In particular, we show that the probability decreases exponentially with respect to α .

PROPOSITION 3.4. *Let $\tilde{L}(\mathbf{y})$ be a Lindahl lottery corresponding to an α -LEO $(\mathbf{p}, \mathbf{x}, \mathbf{y})$ with a budget B and income distribution $\mathbf{U}[1 - \epsilon, 1]$, then following properties hold:*

- *each realization of $\tilde{L}(\mathbf{y})$ has a cost of at most $B + \max_{o \in \mathcal{O}: y_o > 0} c(o)$;*
- *under this α -LEO, $\forall i \in N$, $\Pr_{o \sim \tilde{L}(\mathbf{y})} (o \prec_i o_i) \leq e^{-\alpha}$, where o_i is agent i ’s boundary outcome.*

4. MAIN RESULT: APPROXIMATE CORE

The main result of this section is the following.

THEOREM 4.1. *Given an instance of budgeted social choice $(\mathcal{O}, c(\cdot), \{\succeq_i\}_{i=1}^n, B)$, and under Assumption 1, there exists a 6.24-approximate core solution.*

To prove this result, we begin with a Lindahl lottery associated with an α -LEO, where α is a parameter to be chosen later. We will show that there exists a realization of this lottery, denoted o^* , that is approximately in the core. The intuition follows closely the classical Lindahl argument.

First, we show that o^* “covers” a large fraction of agents, in the sense that for each covered agent i , the following property holds: if some alternative outcome $o \succ_i o^*$, then the corresponding price satisfies $p_{i,o} \geq 1 - \epsilon$. By Proposition 3.2, which upper bounds the cumulative price of each

outcome, we can then conclude that the number of agents who are both covered by o^* and prefer another outcome o is small relative to its cost.

However, we cannot make any claims about agents not covered by o^* , who may prefer o to o^* . Still, the total number of uncovered agents is reduced by a constant factor relative to the total population. To obtain a constant-factor core approximation, we follow the iterative approach of Jiang et al. (2020), applying the same algorithm to the remaining uncovered agents using a scaled-down version of the Lindahl lottery. This process continues until no agents are left uncovered, and the outcomes obtained at different steps are merged together to produce the final output. By the convergence of a geometric sequence, this guarantees a constant-factor approximate core solution. The parameter α is chosen to optimize the trade-off between two factors: the convergence rate of the iterative process and the approximation guarantee obtained in each iteration.

We first define the notion of *covering*. Intuitively, an agent is said to be covered by an outcome if the outcome is at least as good as her boundary outcome. This, in turn, implies that any strictly better alternative must have a price of at least $1 - \epsilon$. This observation allows us to bound the number of such agents.

DEFINITION 4.1. *Let an α -LEO $(\mathbf{x}_{i=1}^n, \mathbf{y}, \mathbf{p}_{i=1}^n)$ be given under random incomes $\mathcal{I} \sim U[1 - \epsilon, 1]$. For each agent i , let $o_i \in \mathcal{O}$ be her boundary outcome as defined in Definition 3.5. Then, agent i is said to be covered by outcome o if $o \succeq_i o_i$.*

We now present Proposition 4.1, which uses the Lindahl lottery property from Proposition 3.4 and the linearity of expectation to establish the existence of a desirable outcome—one that not only covers a large portion of the population, but also ensures that only a small fraction of the covered agents can commonly find a better alternative.³

PROPOSITION 4.1. *Let V be a set of agents, and let $(\mathbf{x}_{i=1}^n, \mathbf{y}, \mathbf{p}_{i=1}^n)$ be an α -LEO with budget B and random budget $\mathcal{I} \sim U[1 - \epsilon, 1]$. Then there exists a realization $o^* \in \mathcal{O}$ from the Lindahl lottery satisfying:*

$$(1) \ c(o^*) \leq B + \max\{c(o) : y_o > 0\}$$

$$(2) \ \text{At most } e^{-\alpha} \cdot |V| \text{ voters are uncovered:}$$

$$|V \setminus S(o^*, V)| \leq e^{-\alpha} \cdot |V|, \quad \text{where } S(o^*, V) \text{ is the set of covered voters.}$$

$$(3) \ \text{For all } o \in \mathcal{O}, |\{i \in S(o^*, V) : o^* \prec_i o\}| \leq \frac{\alpha}{1-\epsilon} \cdot \frac{c(o)}{B} \cdot |V|.$$

The construction of an approximate core solution is presented in Algorithm 2. It ensures that all agents are covered while keeping the total cost within the budget B . At each iteration j , it restricts attention to outcomes with cost at most B_j , where B_j decays geometrically with rate γ . The Lindahl budget used in the α -LEO construction is set to $\gamma_0 B_j$.

By Proposition 4.1, each iteration yields an outcome o^j that covers a large fraction of the remaining agents; the uncovered agents proceed to the next round. The cost of o^j is at most $(\gamma_0 + 1)B_j$, accounting for rounding slack.⁴ The final outcome is the union of all o^j . Since B_j shrinks geometrically, the total cost remains bounded by B .

To prove the correctness of the algorithm, we first bound the number of agents preferring an alternative outcome at each stage.

³Proposition 4.1 is key to our improved approximation guarantee over Jiang et al. (2020). Their notion of partial coverage—termed “good” committees in Section 3.1 (which we refer to as “J-covering”)—shows that for any $\beta \leq 1$, there exists an outcome o^* that fails to J-cover at most a β -fraction of the population, and limits deviations among covered agents to at most $\frac{2}{\beta} \cdot \frac{c(o)}{B} \cdot |V|$. with the same number of “deviating” agents among covered agents, our formulation covers significantly more agents. This tighter coverage ensures faster convergence in our iterative procedure, yielding a stronger approximation ratio.

⁴The parameter γ_0 is introduced to better handle the additive slack arising from dependent rounding. In Jiang et al. (2020), γ_0 is implicitly set to 1, which can result in the size of each outcome being up to twice the Lindahl budget. Choosing a larger γ_0 helps reduce this rounding slack, leading to improved approximation guarantees.

ALGORITHM 2: Iterated Rounding with LEO

Input: a set V of n voters, a budget B , $\epsilon > 0$, parameters $\alpha > 0$ and $\gamma, \gamma_0 \geq 1$ s.t. $e^\alpha > \gamma$ and $\frac{\alpha}{\gamma_0} \geq \frac{e^\alpha}{e^\alpha - \gamma}$

Output: an outcome o^* with $c(o^*) \leq B$ s.t.

$$\left| \{i \in V : o^* \prec_i o\} \right| \leq \left\lfloor \frac{\alpha}{1-\epsilon} \cdot \frac{\gamma(\gamma_0+1)}{(\gamma-1)\gamma_0} \cdot \frac{e^\alpha}{e^\alpha - \gamma} \cdot \frac{c(o)}{B} \cdot n \right\rfloor \text{ for each } o \in \mathcal{O}$$

1 $j \leftarrow 0, o^* \leftarrow \emptyset, V_0 \leftarrow V, B_0 \leftarrow \frac{1}{\gamma_0+1} \cdot \frac{\gamma-1}{\gamma} \cdot B$

2 **while** $|V_j| \geq 1$ **do**

3 Given outcome space $\{o \in \mathcal{O} : c(o) \leq B_j\}$ and voters in V_j , generate an α -LEO $(\mathbf{x}_{i=1}^n, \mathbf{y}, \mathbf{p}_{i=1}^n)$ with a total budget of $\gamma_0 \cdot B_j$ and $\mathcal{I} = \mathbf{U}[1 - \epsilon, 1]$

4 Find an o^j with $c(o^j) \leq (\gamma_0 + 1) \cdot B_j$ satisfying Proposition 4.1

5 $o^* \leftarrow o^* \oplus o^j$

6 $B_{j+1} \leftarrow \frac{B_j}{\gamma}$

7 $V_{j+1} \leftarrow V_j \setminus S(o^j, V_j)$ i.e. V_{j+1} is the subset of V_j , who are not covered by o^j

8 $j++$

9 **end**

10 **return** o^*

PROPOSITION 4.2. *In round $j \in \mathbb{N}$ of Algorithm 2, let o be an outcome with $c(o) \leq B_j$. The number of agents in V_j who are covered by o^j and prefer o to o^j is at most:*

$$\left| \{i \in S(o^j, V_j) : o \succ_i o^j\} \right| \leq \left(\frac{\gamma}{e^\alpha} \right)^j \cdot \frac{\alpha}{\gamma_0(1-\epsilon)} \cdot \left(\frac{c(o)}{B_0} \cdot n \right).$$

With Proposition 4.2, we can establish the correctness of Algorithm 2. In particular, given an alternative outcome, we need to bound the total number of people deviating to the alternative accumulated during each stage. The key idea is that at the same time as B_j shrinks, $|V_j|$ decreases at a *faster* rate. As a result, for any alternative o , the number of covered agents who prefer o over o^j also decays geometrically. This prevents the cumulative preference deviation from growing unbounded across iterations.

We now state the formal result establishing the correctness of Algorithm 2.

PROPOSITION 4.3. *Let the input of Algorithm 2 be a set V of agents, a budget $B > 0$, $\epsilon > 0$, and parameters $\gamma_0, \alpha, \gamma \geq 1$ with $e^\alpha > \gamma$. Then the output o^* of the algorithm has a cost of at most B and lies in the $\frac{\alpha}{1-\epsilon} \cdot \frac{\gamma(\gamma_0+1)}{(\gamma-1)\gamma_0} \cdot \frac{e^\alpha}{e^\alpha - \gamma}$ approximate core.*

Proof of Theorem 4.1. The main result (Theorem 4.1) easily follows from Proposition 4.3 by letting ϵ approach 0 and picking optimal parameters ($\alpha = 2.88, \gamma = 4.6, \gamma_0 = 3.88$) for the formula in Proposition 4.3. The details can be found in Appendix C.1.

5. COMMITTEE SELECTION

In this section, we derive improved approximation results for the special case of *committee selection* under additional structures on voter preferences. Let M be the set of m candidates (also referred to as items), and let the set of feasible outcomes be $\mathcal{O} \subseteq 2^M$, i.e., $\mathcal{O}$ consists of certain subsets of M . We assume that $\emptyset \in \mathcal{O}$. The cost of selecting an outcome $S \subseteq M$ is its cardinality, $|S|$.⁵

⁵Note that we now shift from using the abstract outcome o (as in Section 4) to a committee $S \subseteq M$.

In the committee selection problem we consider, we are given a budget B , and feasible outcomes are all subsets of candidates whose size is at most B , i.e., $\mathcal{O} = \{S \subseteq M : |S| \leq B\}$.

As before, there are n voters, indexed by the set N , each with preferences over subsets in $\mathcal{O}$. We focus on the following two types of preferences.

DEFINITION 5.1. *An agent i is said to have a ranking preference $\succ_i$ over $\mathcal{O}$ if it is induced by an underlying preference ordering $\succ_i^*$ over the ground set in the following way. If the favorite item in S is considered better than the favorite item in T according to $\succ_i^*$, then $S \succ_i T$.*

DEFINITION 5.2. *An agent i is said to have an approval preference $\succ_i$ over $\mathcal{O}$ if there exists a subset of approved candidates $M_i \subseteq M$ such that any subset $S \in \mathcal{O}$ has a utility $u_i(S) = |S \cap M_i|$ and for any $S, T \in \mathcal{O}$, we $S \succ_i T$ if and only if $u_i(S) > u_i(T)$.*

Our main results are as follows.

THEOREM 5.1. *For any integer budget $B \in \mathbb{Z}_+$, there exists a committee in the 5.10-approximate core under ranking preferences. Similarly, for any integer budget $B \in \mathbb{Z}_+$, there exists a committee in the 5.15-approximate core under approval preferences.*

5.1. LEO with Item Pricing (LEOI). To take advantage of the special structure of the outcome space and these preferences, we introduce a new variant of the LEO tailored to this setting. Unlike the earlier version of LEO, which assigns individual prices to each outcome (i.e., subset of candidates or bundle), this version employs *item pricing*. That is, each agent faces a price for each candidate, and the price of any subset is simply the sum of its item prices. While this simplification introduces more structure, it comes with a drawback: consistency between individual demand and the common allocation is no longer enforced at the level of full outcomes (i.e., subsets), but only *in expectation* over each individual candidate.

Specifically, we define a new notion of Lindahl equilibrium in which prices are on items instead of bundles as in Definition 3.2. We call this equilibrium Lindahl Equilibrium with Ordinal Preferences and Item Pricing, which is abbreviated as LEOI.

Note that both ranking and approval preferences may allow ties. Given a preference $\succ_i$, we can extend it to a strict preference, $\succsim_i$, where all strict preferences are preserved, and ties are broken arbitrarily in a way that preserves set-inclusion monotonicity.⁶ As seen in the proofs, this does not affect the validity of our results, because the definition of approximate core only counts the number of agents *strictly* preferring another outcome to the current one in terms of the original preferences.

Now, given an item price vector $\mathbf{p}_{i=1}^m$ for a voter i , her random demand is

$$\mathcal{D}_i(\mathbf{p}_i, \mathcal{I}) := \left\{ \max_{\succsim_i} \{S : S \in \mathcal{O} \text{ and } \sum_{k \in S} p_{i,k} \leq b\} \mid b \sim \mathcal{I} \right\}.$$

Again, because $\sum_{k \in \emptyset} p_{i,k} = 0$, $\emptyset$ is affordable for every possible realization of the random income and therefore $\mathcal{D}_i$ is guaranteed to be a valid lottery over $\mathcal{O}$.

The demand of the producer, who aims to maximize the revenue subject to a budget of B , is:

$$\mathcal{D}_0(\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n) = \arg \max_{\mathbf{z} \in \mathbb{R}_+^{|M|}} \sum_{k \in M} \left(\sum_{i=1}^n p_{i,k} \right) \cdot z_k \quad \text{s.t.} \quad \sum_{k \in M} z_k = B.$$

DEFINITION 5.3. *Given a total budget B , a random income $\mathcal{I}$ supported on $[0, 1]$, and $\alpha > 0$, an α -LEOI consists of individual consumptions $\mathbf{x}_{i=1}^n \in [0, 1]^{|O|}$, a common allocation $\mathbf{y} \in [0, B]^m$, and personalized prices $\mathbf{p}_{i=1}^n \in [0, 1]^m$ such that*

- (1) *for any $i \in N$ and $S \in \mathcal{O}$, $x_{i,S} = \Pr(\mathcal{D}_i(\mathbf{p}_i, \mathcal{I}) = S)$,*

⁶We use $\succsim_i$ only in two definitions where tie-breaking is needed: first, in the definition of random demand; and second, in the definition of boundary committee (Definition 5.5).

(2) for any $i \in N$ and candidate $k \in M$,

$$\alpha \cdot \left(\sum_{S \in \mathcal{O}: k \in S} x_{i,S} \right) \leq y_k, \text{ with strict inequality only when } p_{i,k} = 0,$$

(3) $y \in \mathcal{D}_0(\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n)$

Similar to Theorem 3.2, we obtain the following result. The proof is provided in Appendix A.3.

THEOREM 5.2. *Given a set of feasible outcome $\emptyset \in \mathcal{O} \subset 2^M$, for any $\alpha > 0$ and $B > 0$, and for any random budget $\mathcal{I}$ with a continuous distribution, an α -LEOI exists.*

Now, similar to subsection 3.2, we have the following properties of equilibrium prices in a LEOI.

PROPOSITION 5.1. *Let $(\mathbf{x}_{i=1}^n, \mathbf{y}, \mathbf{p}_{i=1}^n)$ be an α -LEOI with total budget B and random income $\mathcal{I} \sim \mathbf{U}[1 - \epsilon, 1]$, for any $\epsilon \in (0, 1]$. Then:*

- (a) **(Total Revenue Bound)** $\sum_{k \in M} \sum_{i=1}^n p_{i,k} y_k < \alpha n$.
- (b) **(Item-wise Price Bound)** For any item $k \in M$, $\sum_{i=1}^n p_{i,k} \leq \alpha \cdot \frac{n}{B}$.
- (c) **(Set-wise Price Bound)** For any outcome $S \in \mathcal{O}$, $\sum_{i=1}^n \sum_{k \in S} p_{i,k} \leq \alpha \cdot \frac{|S|}{B} \cdot n$.

These properties allow us to bound the number of agents who prefer an alternative committee using the bound on individual equilibrium prices.

As in the previous section, we define a dependent rounding of the common fractional allocation $\mathbf{y}$ into a lottery over committees. We show that for each agent, the probability that a randomly drawn committee is strictly worse than their boundary candidate—that is, the committee fails to cover the agent—is at most $e^{-\alpha}$. This implies the existence of a committee that covers a large fraction of agents.

We then apply an iterative algorithm similar to Algorithm 2, which repeatedly selects subcommittees that cover the remaining uncovered agents. The output is the union of these subcommittees.

5.2. Dependent Rounding on LEOI. We apply dependent rounding with negative correlation, as in Proposition 3.3, for the special case where $a_k = 1$ for all $k \in M$.

DEFINITION 5.4. *Given an α -LEOI with common fractional allocation $\mathbf{y}$, we construct the corresponding Lindahl lottery $\tilde{L}(\mathbf{y})$ as follows:*

- Define $\mathbf{z} \in \mathbb{R}^m$ by setting $z_i = \min\{y_i, 1\}$ for each $i \in M$.
- Apply dependent rounding to $\mathbf{z}$ to obtain a lottery $\tilde{Z}$ over $\{0, 1\}^m$.
- Each realization of $\tilde{Z}$ corresponds to a subset of M , where item i is included if and only if the i -th coordinate is 1. This defines a realization of $\tilde{L}(\mathbf{y})$.

As in the previous section, we define the *boundary committee* and the notion of *covering*.

DEFINITION 5.5. *Let an α -LEOI $(\mathbf{x}_{i=1}^n, \mathbf{y}, \mathbf{p}_{i=1}^n)$ be given under random income $\mathcal{I} \sim \mathbf{U}[1 - \epsilon, 1]$. The boundary committee of voter i , denoted by S_i , is her most preferred bundle in terms of $\succsim_i$ under prices $\mathbf{p}_i$ subject to the lower bound on income: $S_i := \max_{\succsim_i} \{S \in \mathcal{O} : \sum_{k \in S} p_{i,k} \leq 1 - \epsilon\}$.*

DEFINITION 5.6. *A committee T is said to cover voter i if T is weakly preferred to the boundary committee S_i in terms of $\succsim_i$, that is, $T \succeq_i S_i$.*

The following proposition bound the probability that the Lindahl lottery fails to cover an agent with ranking preference.

PROPOSITION 5.2. *Suppose agent i has ranking preferences, and let $\tilde{L}(\mathbf{y})$ be the Lindahl lottery corresponding to an α -LEOI $(\mathbf{p}, \mathbf{x}, \mathbf{y})$ with budget B and income distribution $\mathbf{U}[1 - \epsilon, 1]$. Then:*

- Each realization of $\tilde{L}(\mathbf{y})$ has cost at most $\lceil B \rceil$;
- Let S_i be agent i 's boundary outcome under this α -LEOI. Then, $\Pr_{S \sim \tilde{L}(\mathbf{y})}(S \prec_i S_i) \leq e^{-\alpha}$.

Similarly, we can show that in the case of approval preference, the probability that the lottery returns a committee worse than the boundary committee is also small.

PROPOSITION 5.3. *Suppose an agent i has approval preference. Let $\tilde{L}(\mathbf{y})$ be a Lindahl lottery corresponding to an α -LEOI $(\mathbf{p}, \mathbf{x}, \mathbf{y})$ with a budget B and income distribution $\mathbf{U}[1 - \epsilon, 1]$. If $\alpha \geq 3$, the following properties hold:*

- *Each realization of $\tilde{L}(\mathbf{y})$ has a cost of at most $\lceil B \rceil$;*
- *Let S_i be agent i 's boundary outcome under this α -LEOI. Then, $\Pr_{S \sim \tilde{L}(\mathbf{y})}(S \prec_i S_i) \leq e^{-\alpha}$.*

While Proposition 5.3 is similar in form to Proposition 5.2, its proof relies on a different strategy. Specifically, it uses a novel concentration inequality—an adaptation of the Chernoff bound tailored to binary random vectors. This approach requires an additional condition: $\alpha \geq 3$. Full details are provided in Appendix D.

5.3. Iterative Rounding with LEOI. Now we can incorporate LEOI into the iterative framework in a way similar to Algorithm 2 and derive improved approximation results for the setting of committee selection under two important types of preferences. The improvement arises from the fact that, with the cost of each item being 1, the effect of extra additament of dependent rounding becomes minimal, which also simplifies the analysis.

PROPOSITION 5.4. *In the setting of committee selection with a size bound of $B \geq 2$, there exists an algorithm which, given inputs $\epsilon > 0$ and the $\alpha, \gamma > 1$ satisfying $e^\alpha > \gamma$, outputs a committee in the $\frac{\alpha}{1-\epsilon} \cdot \left(\frac{\gamma}{\gamma-1} + \frac{\log_\gamma(B)}{B} \right) \cdot \frac{e^\alpha}{e^\alpha - \gamma}$ approximate core for a set of agents with ranking preferences. If $\alpha \geq 3$, the algorithm can also output a committee in the $\frac{\alpha}{1-\epsilon} \cdot \left(\frac{\gamma}{\gamma-1} + \frac{\log_\gamma(B)}{B} \right) \cdot \frac{e^\alpha}{e^\alpha - \gamma}$ approximate core for a set of agents with approval preferences.*

The algorithm (Algorithm 3 in appendix) for constructing the desired approximate core solution in Proposition 5.4 uses a definition of covering similar to Definition 4.1, adapted to committee selection setting. At each step j , the algorithm rounds from an α -LEOI with budget B^j to cover a $(1 - \frac{1}{e^\alpha})$ -fraction of the population. The budget is reduced by a factor of γ at each step to maintain feasibility, and the process repeats on the uncovered population until every voter is covered.

This algorithm is simpler than in Algorithm 2: rounding is done item-wise, eliminating the need to restrict the outcome space at each step, and the parameter γ_0 is no longer required. The full algorithm, along with supporting propositions and proofs, is provided in Appendix D.

Proof of Theorem 5.1. Finally, to prove our main result, we consider two cases. In the case of $B > 60$, Theorem 5.1 follows from Proposition 5.4 by letting $\epsilon \rightarrow 0$ and optimizing over α and γ . The case of $B \leq 60$ is handled separately by applying Propositions D.3 and D.4 which are two propositions similar to Proposition 4.1 but adapted to the setting of LEOI. (See details in Appendix D.4).

6. CONCLUSION

We improve existing results for the approximate core of participatory budgeting and committee selection. Our key innovation is the introduction of Lindahl's Equilibrium for Ordinal Preferences (LEO), which extends classical economic insights — traditionally confined to continuous convex settings — to discrete economies with ordinal preferences.

There are several promising directions for future research. First, we believe that the LEO framework has broader applications in voting and market design settings with externalities, which we intend to explore further. Second, while our focus has been on existence results, computational aspects are also important. The computational direction goes beyond the main scope of this paper, which introduces the Lindahl Equilibrium with Ordinal Preferences as a new conceptual tool. We leave this avenue for future investigation.

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APPENDIX A. EXISTENCES OF LEO, LEOI

A.1. Proof of Theorem 3.2: Existence of LEO. We include the renowned Kakutani's fixed point theorem and Maximum theorem here for the sake of completeness.

THEOREM A.1 (Kakutani's fixed point theorem, [Kakutani \(1941\)](#)). *Let S be a non-empty, compact, and convex subset of some Euclidean space $\mathbb{R}^n$. Let $\psi : S \rightarrow 2^S \setminus \emptyset$ be a point-to-set function on S such that 1) ψ is upper-hemicontinuous and 2) $\psi(s)$ is non-empty and convex for all $s \in S$. Then there exists a fixed point $s \in S$ such that $s \in \psi(s)$.*

THEOREM A.2 (Maximum Theorem, [Brian \(1990\)](#)). *Given non-empty $\mathbf{X} \subset \mathbb{R}^L$ and $\Theta \subset \mathbb{R}^M$, let $f : \mathbf{X} \times \Theta \rightarrow \mathbb{R}$ be a continuous function on the product $\mathbf{X} \times \Theta$, and $\mathbf{C} : \Theta \rightrightarrows \mathbf{X}$ be a compact valued correspondence such that $\mathbf{C}(\theta) \neq \emptyset$ for all $\theta \in \Theta$. Define $f^*(\theta) = \sup\{f(x, \theta) : x \in \mathbf{C}(\theta)\}$ and the correspondence $\mathbf{C}^* : \Theta \rightrightarrows \mathbf{X}$ by $\mathbf{C}^*(\theta) = \{x \in \mathbf{C}(\theta) : f(x, \theta) = f^*(\theta)\}$. If $\mathbf{C}$ is continuous at θ , then f^* is continuous and $\mathbf{C}^*$ is upper hemi-continuous, non-empty, and compact valued. As a consequence, the sup can be replaced by max.*

Equipped with these theorems, we are ready to show Theorem 3.2.

Proof. Define $\Gamma := \{\mathbf{z} \in \mathbb{R}_+^{|\mathcal{O} \setminus \{\emptyset\}|} \mid \sum_{o \in \mathcal{O} \setminus \{\emptyset\}} c_o z_o = B\}$. We construct the following correspondence:

$$\mathcal{L} : \prod_{i=1}^n [0, 1]^{|\mathcal{O}|} \times \Gamma \times \prod_{i=1}^n [0, 1]^{|\mathcal{O}|} \rightrightarrows \prod_{i=1}^n [0, 1]^{|\mathcal{O}|} \times \Gamma \times \prod_{i=1}^n [0, 1]^{|\mathcal{O}|}$$

with

$$\mathcal{L}((\{\mathbf{x}_i\}_{i \in N}, \mathbf{y}, \{\mathbf{p}_i\}_{i \in N})) = (\{\mathbf{x}'_i\}_{i \in N}, Y', \{\mathbf{p}'_i\}_{i \in N})$$

where

$$x'_{i,o} = \Pr[\mathcal{D}_i(\mathbf{p}_i, \mathcal{I}) = o] \quad \forall i \in N, o \in \mathcal{O}, \quad (1a)$$

$$Y' = \arg \max_{\mathbf{z} \in \Gamma} \left(\sum_{i \in N} \mathbf{p}_i \right)^T \mathbf{z}, \quad (1b)$$

$$p'_{i,o} = \begin{cases} \max \{ \min \{ 1, p_{i,o} + (x'_{v,o} - y_o) \}, 0 \} & \text{if } o \neq \emptyset \\ 0 & \text{if } o = \emptyset, \end{cases} \quad \forall i \in N, o \in \mathcal{O}. \quad (1c)$$

Our goal is to show that

- (i) By Theorem A.2, the correspondence $\prod_{i=1}^n [0, 1]^{|\mathcal{O}|} \rightrightarrows \Gamma$ (from $\mathbf{p}$ to Y') is upper hemi-continuous, non-empty, and compact valued. Furthermore, Y' is convex, bounded, and non-empty.
- (ii) By Theorem A.1, there exists a *fixed point* $(\{\mathbf{x}_i\}_{i \in N}, \mathbf{y}, \{\mathbf{p}_i\}_{i \in N})$ such that

$$(\{\mathbf{x}_i\}_{i \in N}, \mathbf{y}, \{\mathbf{p}_i\}_{i \in N}) \in \mathcal{L}(\{\mathbf{x}_i\}_{i \in N}, \mathbf{y}, \{\mathbf{p}_i\}_{i \in N}).$$

- (iii) The fixed point $(\{\mathbf{x}_i\}_{i \in N}, \mathbf{y}, \{\mathbf{p}_i\}_{i \in N})$ satisfies all the conditions of a LEO in Definition 3.2.

To show (i), we employ Theorem A.2 with $\mathbf{X} = \Gamma$ (the space of the producer's decision) and $\Theta = \prod_{i=1}^n [0, 1]^{|\mathcal{O}|}$ (the space of prices $\mathbf{p}$), both of which are non-empty. Let $f(\mathbf{z}, \mathbf{p}) = (\sum_{i \in N} \mathbf{p}_i)^T \mathbf{z}$ be the producer revenue given the producer decision $\mathbf{z} \in \Gamma$ and the prices $\mathbf{p} \in \Theta$. Given that for any prices $\mathbf{p} \in \Theta$, any producer decision in Γ is feasible, we set $\mathbf{C}(\mathbf{p}) = \Gamma$ for any $\mathbf{p} \in \Theta$. $\mathbf{C}$

is a compact valued correspondence since Γ is compact. f^* maps $\mathbf{p}$ to the maximum producer revenue, thus, given prices $\mathbf{p}$, we have $\mathbf{C}^*(\mathbf{p}) = \arg \max_{\mathbf{z} \in \Gamma} (\sum_{i \in N} \mathbf{p}_i)^T \mathbf{z}$. That is, $\mathbf{C}^*(\mathbf{p})$ is the set of the optimal producer decision. By Theorem A.2, because $\mathbf{C}$ is continuous at any $\mathbf{p} \in \Theta$, f^* is continuous and $\mathbf{C}^*$ is upper hemi-continuous, non-empty, and compact valued. Furthermore, $\mathbf{C}^*(\mathbf{p})$ is convex, non-empty, and bounded for any $\mathbf{p} \in \Theta$ since it is the set of the optimal solutions of a linear program with a bounded and non-empty feasible region.

To show (ii), we consider $S = \prod_{i=1}^n [0, 1]^{|\mathcal{O}|} \times \Gamma \times \prod_{i=1}^n [0, 1]^{|\mathcal{O}|}$ and let $\psi = \mathcal{L}$. The range of ψ is restricted to having $\mathbf{x}'$ and $\mathbf{p}'$ as a point and $Y' \subseteq \Gamma$ as a set. Clearly, S is non-empty, compact, and convex. For all $s \in S$, $\psi(s)$ is non-empty and convex since Y' is convex, non-empty, and bounded. The remaining is to show that $\mathcal{L}$ is upper hemi-continuous. To show this, we prove that (a) the mapping from $s \in S$ to $\mathbf{x}'$ is continuous, (b) the mapping from $s \in S$ to Y' is upper hemi-continuous, and (c) the mapping from $s \in S$ to $\mathbf{p}'$ is continuous. (b) directly holds from (i) and (c) holds once (a) holds since $p'_{i,o}$ defined in (1c) is a continuous function of $\mathbf{x}'$, $\mathbf{y}$, and $\mathbf{p}$. We focus on showing (a). Let $F_{\mathcal{I}}$ be the CDF for the budget given to an agent $i \in N$. We have that

$$\begin{aligned} x'_{i,o} &= \Pr[\mathcal{D}_i(\mathbf{p}_i, \mathcal{I}) = o] \\ &= \begin{cases} \min_{o' \in \mathcal{O}: o' \succ_v o} \left(F_{\mathcal{I}}(p_{i,o'}) - F_{\mathcal{I}}(p_{i,o}) \right)^+ & \text{if } o \in \mathcal{O} \text{ is not the top-ranked in } \succ_i, \\ 1 - F_{\mathcal{I}}(p_{i,o}) & \text{if } o \in \mathcal{O} \text{ is the top-ranked in } \succ_i. \end{cases} \end{aligned}$$

Intuitively, if o is the top-ranked in $\succ_i$, then i demands o as long as $p_{i,o}$ is at most the budget given to i . The probability of having such a budget is $1 - F_{\mathcal{I}}(p_{i,o})$. If o is the second-ranked and o' is the top-ranked in $\succ_i$, then i demands o if the budget is below $p_{i,o'}$ and at least $p_{i,o}$. This probability is captured by $(F_{\mathcal{I}}(p_{i,o'}) - F_{\mathcal{I}}(p_{i,o}))^+$. Note that if $p_{i,o'} < p_{i,o}$, then i never demands o since it is more expensive and less preferred. Applying analogous reasoning results in the closed form above for the probability that i demands o under a random budget $\mathcal{I}$. More specifically, consider $o' \succ_i o$, the probability that o is demanded by i under $\mathbf{p}_i$ is zero when there exists $p_{i,o'} \leq p_{i,o}$. Otherwise, it is the minimum difference between the CDF at $p_{i,o'}$ and $p_{i,o}$. Since $F_{\mathcal{I}}$ and the closed form above are continuous, (a) holds. By Theorem A.1, there exists $s \in S$ such that $s \in \mathcal{L}(s)$.

To show (iii), we prove that any fixed point $s = (\{\mathbf{x}_i\}_{i \in N}, \mathbf{y}, \{\mathbf{p}_i\}_{i \in N})$ such that $s \in \mathcal{L}(s)$ satisfies conditions (1), (2), and (3) in Definition 3.2. Conditions (1) and (3) follow directly by (1a) and (1b), respectively. For condition (2), we first show that $x_{i,o} \leq y_o$ for all $o \in \mathcal{O}$ and $i \in N$. Suppose for the sake of contradiction that $x_{i,o} > y_o$ for some $o \in \mathcal{O}$ and $i \in N$. Since $x_{i,o} - y_o > 0$ and s is a fixed point, from (1c), we must have $p_{i,o} = 1$. Consequently, i cannot afford a with a strictly positive probability unless $\Pr[\mathcal{I} = 1]$ is strictly positive. However, because $\mathcal{I}$ is a random variable with support on the unit interval $[0, 1]$, a strictly positive $\Pr[\mathcal{I} = 1]$ would imply that $F_{\mathcal{I}}$ is discontinuous, which is a contradiction. Therefore, $x_{i,o} \leq y_o$. Now suppose $x_{i,o} < y_o$, then from (1c), the fixed point s forces $p_{i,o} = 0$. Hence, $s = (\{\mathbf{x}_i\}_{i \in N}, \mathbf{y}, \{\mathbf{p}_i\}_{i \in N})$ satisfies all the conditions of Definition 3.2, so s is a LEO. $\square$

A.2. Proof of Theorem 3.3: Existence of α -LEO.

Proof. The theorem easily follows from Theorem 3.2. Let a budget B , a random income $\mathcal{I}$ supported on $[0, 1]$, and $\alpha > 0$ be given. Then let $(\mathbf{x}_{i=1}^n, \mathbf{y}, \mathbf{p}_{i=1}^n)$ be the LEO under budget $\frac{B}{\alpha}$ and random income $\mathcal{I}$. We claim $(\mathbf{x}_{i=1}^n, \alpha \cdot \mathbf{y}, \mathbf{p}_{i=1}^n)$ is the desired α -LEO. It is immediately clear that $(\mathbf{x}_{i=1}^n, \alpha \cdot \mathbf{y}, \mathbf{p}_{i=1}^n)$ satisfies part (1) and (2) of Definition 3.3 because $(\mathbf{x}_{i=1}^n, \mathbf{y}, \mathbf{p}_{i=1}^n)$ satisfies part (1) and (2) of Definition 3.2. Furthermore, since $\mathbf{y}$ maximizes revenue given prices $\mathbf{p}_{i=1}^n$ under budget $\frac{B}{\alpha}$, it follows that $\alpha \cdot \mathbf{y}$ maximizes revenue given prices $\mathbf{p}_{i=1}^n$ under budget B . Therefore part (3) of Definition 3.3 is satisfied and this completes the proof. $\square$

A.3. Proof of Theorem 5.2: Existence of LEOI.

Proof. The proof is analogous to that of Theorem 2. The difference is on the detail of the mapping.

In particular, we define $\Phi := \{\mathbf{z} \in \mathbb{R}_+^{|M|} \mid \sum_{k \in M} z_k = B\}$. We construct the following correspondence:

$$\mathcal{L} : \prod_{i=1}^n [0, 1]^{|M|} \times \Phi \times \prod_{i=1}^n [0, 1]^{|M|} \Rightarrow \prod_{i=1}^n [0, 1]^{|M|} \times \Phi \times \prod_{i=1}^n [0, 1]^{|M|}$$

with

$$\mathcal{L}((\{\mathbf{x}_i\}_{i \in N}, \mathbf{y}, \{\mathbf{p}_i\}_{i \in N})) = (\{\mathbf{x}'_i\}_{i \in N}, Y', \{\mathbf{p}'_i\}_{i \in N})$$

where

$$x'_{i,S} = \Pr[\mathcal{X}_i(\mathbf{p}_i, \mathcal{I}) = S] \quad \forall i \in N, S \in \mathcal{O}, \quad (2a)$$

$$Y' = \arg \max_{\mathbf{z} \in \Phi} \left(\sum_{i \in N} \mathbf{p}_i \right)^T \mathbf{z}, \quad (2b)$$

$$p'_{i,k} = \max \left\{ \min \left\{ 1, p_{i,k} + \left(\alpha \cdot \sum_{S: k \in S} x'_{i,S} - y_k \right) \right\}, 0 \right\} \quad \forall i \in N, k \in M. \quad (2c)$$

Thus, $\sum_{S: k \in S} x'_{i,S}$ represents agent i 's expected demand for item $k \in M$, Y' denotes the producer's response—an allocation that maximizes total revenue—and $p'_{i,k}$ captures the price adjustment.

The remainder of the proof follows the same structure as the proof of Theorem 3.2. Continuity of the mapping follows directly from that argument, which shows that agents' randomized demand is continuous in prices. Moreover, the price adjustment mechanism, together with the fact that the income distribution has support on the interval $[0, 1]$, ensures that any fixed point of the mapping corresponds to an α -LEOI. $\square$

APPENDIX B. MISSING PROOFS IN SECTION 3

Note that because of their fundamental importance, we have moved the proof of existence for LEO and LEOI to separate sections, A.1 and A.2 respectively, in the appendix.

B.1. Missing Proofs in Section 3.2.

Proof of Proposition 3.1. Given that the expected spending of an agent can never exceed the expected income, for any $i \in N$, we have

$$\sum_{o \in \mathcal{O}} p_{i,o} x_{i,o} \leq E[\mathcal{I}] < 1.$$

Then by condition (2) of Definition 3.3, given any $i \in N$, we can bound the revenue collected from agent i as

$$\sum_{o \in \mathcal{O}} p_{i,o} y_o = \sum_{o: \alpha x_{i,o} = y_o} p_{i,o} y_o \leq \sum_{o: \alpha x_{i,o} = y_o} p_{i,o} (\alpha x_{i,o}) \leq \alpha \sum_{o \in \mathcal{O}} p_{i,o} x_{i,o} < \alpha,$$

The first equality holds because of the second property of α -LEO, when $x_{i,o} < y_o$ the price $p_{i,o} = 0$. Now by summing across the revenue of individual agents, we can obtain

$$\sum_{o \in \mathcal{O}} \sum_{i=1}^n p_{i,o} y_o = \sum_{i=1}^n \left(\sum_{o \in \mathcal{O}} p_{i,o} y_o \right) < \sum_{i=1}^n \alpha = \alpha n.$$

$\square$

Proof of Proposition 3.2. We first suppose by contradiction that there exists an $\hat{o} \in \mathcal{O}$ with a cumulative price exceeding the threshold i.e.

$$\sum_{i=1}^n p_{i,\hat{o}} > \alpha \cdot \frac{c(\hat{o})}{B} \cdot n.$$

Then since $\mathbf{y}$ maximizes the total revenue under budget B , $\mathbf{y}$ can only pick outcomes with the highest revenue-to-cost ratio. In particular, this implies that for any o with $y_o > 0$, we can derive a *strict* lower bound on the revenue-to-cost ratio:

$$\frac{\sum_{i=1}^n p_{i,o}}{c(o)} \geq \frac{\sum_{i=1}^n p_{i,\hat{o}}}{c(\hat{o})} > \alpha \cdot \frac{n}{B}.$$

As a result, we can obtain a *strict* lower bound on the total revenue from $\mathbf{y}$ as:

$$\begin{aligned} \sum_{o \in \mathcal{O}} \sum_{i=1}^n p_{i,o} y_o &= \sum_{o: y_o > 0} \left(\frac{\sum_{i=1}^n p_{i,o}}{c(o)} \cdot c(o) y_o \right) \\ &> \sum_{o: y_o > 0} \left(\alpha \cdot \frac{n}{B} \cdot c(o) y_o \right) \\ &= \alpha \cdot \frac{n}{B} \cdot \sum_o c(o) y_o = \alpha n. \end{aligned}$$

This means that the total revenue from $\mathbf{y}$ is *strictly* greater than αn , which contradicts Proposition 3.1. Hence, for any $o \in \mathcal{O}$, $\sum_{i=1}^n p_{i,o} \leq \alpha \cdot \frac{c(o)}{B} \cdot n$, and completes the proof. $\square$

B.2. Missing Proofs in Section 3.3.

Proof of Proposition 3.4. The first part of the proposition follows directly from the properties of dependent rounding. To see the second part, we fix an agent i and define $\hat{\mathcal{O}}_i = \{o \in \mathcal{O} : x_{i,o} > 0\}$, which is the set of all outcomes agent i includes in her random demand. We can observe that for each $\hat{o} \in \hat{\mathcal{O}}_i$, $o_i \preceq_i \hat{o}$, because by definition, $\hat{o}$ is agent i 's favorite outcome under some income realization $b \geq 1 - \epsilon$. Therefore, if agent i prefers a realization of $\tilde{L}(\mathbf{y})$ over o_i , it must be the case no outcome in $\hat{\mathcal{O}}_i$ has been merged into the realization.

Let $\mathbf{z}$ be the truncated vector appearing in line 1 of Algorithm 1 and $\tilde{Z}$ be the corresponding random vector. Given the reasoning in the last paragraph, we obtain the following inequality:

$$\Pr_{o \sim \tilde{L}(\mathbf{y})} (o \prec_i o_i) \leq \Pr_{\mathbf{z} \sim \tilde{Z}} (Z_{\hat{o}} = 0 \ \forall \hat{o} \in \hat{\mathcal{O}}_i). \quad (3)$$

Now we break into cases.

Case 1: If $o_i = \emptyset$, the proposition is trivially true because $\emptyset$ is assumed to be each agent's least favorite outcome and any realization of $\tilde{L}(\mathbf{y})$ is no worse. So we have $\Pr_{o \sim \tilde{L}(\mathbf{y})} (o \prec_i o_i) = 0$

and this completes the proof.

Case 2: $y_{\hat{o}} \geq 1$ for some $\hat{o} \in \hat{\mathcal{O}}_i$. In this case, $\hat{o}$ is merged into $\tilde{L}(\mathbf{y})$ with probability 1, and therefore $\Pr_{o \sim \tilde{L}(\mathbf{y})} (o \prec_i o_i) = 0$. This completes the proof.

Case 3: $o_i \neq \emptyset$ and $y_{\hat{o}} < 1 \ \forall \hat{o} \in \hat{\mathcal{O}}_i$. Then for each $\hat{o} \in \hat{\mathcal{O}}_i$, we have $z_{\hat{o}} = \min\{1, y_{\hat{o}}\} = y_{\hat{o}} \geq \alpha \cdot x_{i,\hat{o}}$.

Since $x_{i,\hat{o}}$ is a lottery over $\hat{\mathcal{O}}_i$, $\sum_{\hat{o} \in \hat{\mathcal{O}}_i} x_{i,\hat{o}} = 1$ and consequently

$$\sum_{\hat{o} \in \hat{\mathcal{O}}_i} z_{\hat{o}} \geq \alpha \cdot \sum_{\hat{o} \in \hat{\mathcal{O}}_i} x_{i,\hat{o}} = \alpha.$$

Now we apply the negative dependence property of dependent rounding to (3) and derive

$$\begin{aligned}
\Pr_{o \sim \tilde{L}(\mathbf{y})} (o \prec_i o_i) &\leq \Pr_{\mathbf{z} \sim \tilde{Z}} (Z_{\hat{o}} = 0 \ \forall \ \hat{o} \in \hat{\mathcal{O}}_i) \leq \prod_{\hat{o} \in \hat{\mathcal{O}}_i} (1 - z_{\hat{o}}) \\
&\leq \prod_{\hat{o} \in \hat{\mathcal{O}}_i} e^{-z_{\hat{o}}} = e^{-\sum_{\hat{o} \in \hat{\mathcal{O}}_i} z_{\hat{o}}} \leq e^{-\alpha}.
\end{aligned}$$

This completes the proof. $\square$

APPENDIX C. MISSING PROOF IN SECTION 4

Proof of Proposition 4.1. To find the desired outcome, we take the Lindahl lottery $\tilde{L}(\mathbf{y})$ corresponding to $\mathbf{y}$. By proposition 3.4, for each agent i , $\tilde{L}(\mathbf{y})$ fails to cover i with a probability of at most $e^{-\alpha}$. Then we can use linearity of expectation to obtain

$$E[|i \in N : \tilde{L}(\mathbf{y}) \text{ does not cover } i|] = \sum_{i \in V} \Pr(\tilde{L}(\mathbf{y}) \text{ does not cover } i) \leq |V| \cdot e^{-\alpha}.$$

This implies that there must exist a realization o^* of $\tilde{L}(\mathbf{y})$ with $|V \setminus S(o^*, V)| \leq e^{-\alpha} \cdot |V|$ as desired, thus satisfying part (2) of the proposition. Since all realization of $\tilde{L}(\mathbf{y})$ has a cost of at most $B + \max_{o \in \mathcal{O}: y_o > 0} c(o)$, so is o^* and thus part (1) is satisfied.

Now we are left to show o^* satisfies part (3). Let an arbitrary outcome $o \in \mathcal{O}$ be given. Suppose $o \succ_i o^*$ for a voter $i \in S(o^*, V)$, then by the transitivity of preference it must also be the case that $o \succ_i o_i$. However, since o_i is by definition i 's most preferred outcome with a price no greater than $1 - \epsilon$, this implies that $p_{i,o} \geq 1 - \epsilon$. And we know that the upper bound of the total prices for o is $\alpha \cdot \frac{c(o)}{B} \cdot |V|$ by Proposition 3.2. Thus we can bound the total number of covered agents preferring o as

$$|i \in V : o^* \prec_i o| \leq \frac{\sum_{i \in V} p_{i,o}}{1 - \epsilon} \leq \frac{\alpha}{1 - \epsilon} \cdot \frac{c(o)}{B} \cdot |V|$$

and this completes the proof. $\square$

Proof of Proposition 4.2. At step j , we have set the budget of the corresponding α -LEO to be

$$\hat{B} = \gamma_0 \cdot B_j = \gamma_0 \cdot \frac{B_0}{\gamma^j}.$$

By Part (2) of Proposition 4.1, $|V_{k+1}| \leq \frac{|V_k|}{e^\alpha}$ for each $k \geq 0$. Hence, $|V_j| \leq \frac{n}{e^{\alpha \cdot j}}$.

Now applying Part (3) of Proposition 4.1, we obtain

$$\left| \{i \in S(o^j, V_j) : o \succ_i o^j\} \right| \leq \frac{\alpha}{1 - \epsilon} \cdot \frac{c(o)}{\hat{B}} \cdot |V_j| \leq \left(\frac{\gamma}{e^\alpha}\right)^j \cdot \frac{\alpha}{\gamma_0(1 - \epsilon)} \cdot \left(\frac{c(o)}{B_0}\right) \cdot n$$

and this completes the proof. $\square$

Proof of Proposition 4.3. We first bound the cost of o^* . At step j , only outcomes with cost less than B_j are considered and the budget of the corresponding LEO is $\gamma_0 B_j$. Therefore, according to Proposition 4.1, the cost of o^j is at most $(\gamma_0 + 1)B_j$. Then by the merging assumption (Assumption 1), we can compute

$$c(o^*) \leq \sum_{|V_j| \geq 1} c(o_j) \leq \sum_{|V_j| \geq 1} \frac{(\gamma_0 + 1) \cdot B_0}{\gamma^j} \leq (\gamma_0 + 1)B_0 \cdot \sum_{j=0}^{\infty} \frac{1}{\gamma^j} \leq (\gamma_0 + 1)B_0 \cdot \frac{1}{1 - \frac{1}{\gamma}} = B,$$

which means that o^* is within the cost limit of B as desired.

Now given an outcome o , we want to bound the total number of agents preferring o to o^* . Let t be the first index for which $B_t < c(o)$. Then using Assumption 1 and Part (3) of Proposition 4.1

and assuming the worst-case scenario where all agents in V_t prefer o to o^* , we can obtain

$$\begin{aligned} |i \in N : o^* \prec_i o| &\leq \sum_{j=0}^{t-1} |i \in S(o^j, V_j) : o^* \prec_i o| + |V_t| \\ &\leq |V_t| + \left(\frac{\alpha}{\gamma_0(1-\epsilon)} \cdot \frac{c(o)}{B_0} \cdot n \right) \sum_{j=0}^{t-1} \left(\frac{\gamma}{e^\alpha} \right)^j. \end{aligned}$$

Then using the assumption on the value of α, γ, γ_0 and $c(o)$, we can bound $|V_t|$ as

$$\begin{aligned} |V_t| &\leq \frac{1}{e^{\alpha t}} \cdot n = \frac{B_0}{\gamma^t} \cdot \frac{n}{B_0} \cdot \left(\frac{\gamma}{e^\alpha} \right)^t = B_t \cdot \frac{n}{B_0} \cdot \left(\frac{\gamma}{e^\alpha} \right)^t \\ &\leq \frac{c(o)}{B_0} \cdot n \cdot \left(\frac{\gamma}{e^\alpha} \right)^t \leq \frac{\alpha/\gamma_0}{e^\alpha/(e^\alpha - \gamma)} \cdot \frac{c(o)}{B_0} \cdot n \cdot \left(\frac{\gamma}{e^\alpha} \right)^t \\ &= \frac{\alpha}{\gamma_0} \cdot \frac{c(o)}{B_0} \cdot n \cdot \frac{\left(\frac{\gamma}{e^\alpha} \right)^t}{1 - \frac{\gamma}{e^\alpha}} \leq \frac{\alpha}{\gamma_0(1-\epsilon)} \cdot \frac{c(o)}{B_0} \cdot n \cdot \sum_{j=t}^{\infty} \left(\frac{\gamma}{e^\alpha} \right)^j. \end{aligned}$$

Now combining the bound on $|V_t|$ and the previous inequality yields:

$$\begin{aligned} |i \in N : o^* \prec_i o| &\leq |V_t| + \left(\frac{\alpha}{\gamma_0(1-\epsilon)} \cdot \frac{c(o)}{B_0} \cdot n \right) \sum_{j=0}^{t-1} \left(\frac{\gamma}{e^\alpha} \right)^j \\ &\leq \left(\frac{\alpha}{\gamma_0(1-\epsilon)} \cdot \frac{c(o)}{B_0} \cdot n \right) \cdot \left(\sum_{j=0}^{t-1} \left(\frac{\gamma}{e^\alpha} \right)^j + \sum_{j=t}^{\infty} \left(\frac{\gamma}{e^\alpha} \right)^j \right) \\ &\leq \left(\frac{\alpha}{\gamma_0(1-\epsilon)} \cdot \frac{c(o)}{B_0} \cdot n \right) \cdot \sum_{j=0}^{\infty} \left(\frac{\gamma}{e^\alpha} \right)^j \\ &\leq \frac{\alpha}{\gamma_0(1-\epsilon)} \cdot \frac{c(o)}{B_0} \cdot n \cdot \frac{1}{1 - \frac{\gamma}{e^\alpha}} \\ &\leq \frac{\alpha}{1-\epsilon} \cdot \frac{\gamma(\gamma_0+1)}{(\gamma-1)\gamma_0} \cdot \frac{e^\alpha}{e^\alpha - \gamma} \cdot \frac{c(o)}{B} \cdot n. \end{aligned}$$

Thus o^* lies in the core with the desired approximation ratio and this completes the proof. $\square$

C.1. Proof of Theorem 4.1.

Proof. Fix α, γ, γ_0 satisfying the requirement for the inputs of Algorithm 2. For the ease of notation, we define $\omega(\alpha, \gamma, \gamma_0) := \alpha \cdot \frac{\gamma(\gamma_0+1)}{(\gamma-1)\gamma_0} \cdot \frac{e^\alpha}{e^\alpha - \gamma}$. Then by Proposition 4.3, for any $\epsilon > 0$, there exists a feasible outcome o_ϵ^* s.t. for any $o \in \mathcal{O}$,

$$|\{i \in N : o_\epsilon^* \prec_i o\}| \leq \frac{\omega(\alpha, \gamma, \gamma_0)}{1-\epsilon} \cdot \frac{c(o)}{B} \cdot n.$$

Since the left-hand side is integral, this implies that

$$|\{i \in N : o_\epsilon^* \prec_i o\}| \leq \lfloor \frac{\omega(\alpha, \gamma, \gamma_0)}{1-\epsilon} \cdot \frac{c(o)}{B} \cdot n \rfloor.$$

Because $c(o)$ is bounded from above and $\frac{\omega(\alpha, \gamma, \gamma_0)}{1-\epsilon} \rightarrow \omega(\alpha, \gamma, \gamma_0)$ as $\epsilon \rightarrow 0$, there must exist a small enough ϵ^* s.t. for any $o \in \mathcal{O}$,

$$\lfloor \frac{\omega(\alpha, \gamma, \gamma_0)}{1-\epsilon^*} \cdot \frac{c(o)}{B} \cdot n \rfloor = \lfloor \omega(\alpha, \gamma, \gamma_0) \cdot \frac{c(o)}{B} \cdot n \rfloor.$$

Let us denote the corresponding outcome as $o_{\epsilon^*}^*$. Then for any $o \in \mathcal{O}$,

$$|\{i \in N : o_{\epsilon^*}^* \prec_i o\}| \leq \lfloor \frac{\omega(\alpha, \gamma, \gamma_0)}{1 - \epsilon^*} \cdot \frac{c(o)}{B} \cdot n \rfloor \leq \omega(\alpha, \gamma, \gamma_0) \cdot \frac{c(o)}{B} \cdot n.$$

This means that for any α, γ, γ_0 satisfying the requirement for the inputs of Algorithm 2, there exists an outcome lying in the $\omega(\alpha, \gamma, \gamma_0)$ approximate core. Taking $\alpha = 2.88, \gamma = 4.6, \gamma_0 = 3.88$ gives the desired approximation factor of 6.24. $\square$

APPENDIX D. PROOFS IN SECTION 5

We want to point out that we have previously put the proof of Theorem 5.2, the existence of α -LEOI, in the separate sections of A.3 in the appendix due to its fundamental importance.

D.1. Omitted Proofs in Section 5.1.

Proof of Proposition 5.1, Part (a). The proof, which is based on the market clearing condition of LEOI, is similar to the proof of Proposition 3.1 bounding the total revenue in the case of LEO. For each agent $i \in N$, the expected spending cannot exceed her expected income. Therefore, we obtain the following bound:

$$\sum_{k \in M} \sum_{S \in \mathcal{O}: k \in S} p_{i,k} x_{i,S} = \sum_{S \in \mathcal{O}} \left(\sum_{k \in S} p_{i,k} \cdot x_{i,S} \right) \leq E[\mathcal{I}] < 1.$$

Then by condition (2) of Definition 5.3, given any $i \in N$, we can bound the revenue collected from agent i as

$$\sum_{k \in M} \sum_{S \in \mathcal{O}: k \in S} p_{i,k} y_k = \sum_{k: \alpha x_{i,k} = y_k} p_{i,k} y_k \leq \sum_{k: \alpha x_{i,k} = y_k} p_{i,k} (\alpha x_{i,k}) \leq \alpha \sum_{k \in M} \sum_{S \in \mathcal{O}: k \in S} p_{i,k} x_{i,S} < \alpha,$$

The first equality holds because of the second property of α -LEO, when $x_{i,k} < y_k$ the price $p_{i,k} = 0$. Now by summing across the revenue of individual agents, we can obtain

$$\sum_{k \in M} \sum_{i=1}^n p_{i,k} y_k = \sum_{i=1}^n \left(\sum_{k \in M} p_{i,k} y_k \right) < \sum_{i=1}^n \alpha = \alpha n.$$

$\square$

Proof of Proposition 5.1, Part (b): The proof is similar to the proof of Proposition 3.2 but simpler, which is based on the revenue maximization behavior of the producer. We first suppose by contradiction that there exists an item $\hat{k} \in M$ with a revenue exceeding the threshold i.e.

$$\sum_{i=1}^n p_{i,\hat{k}} > \alpha \cdot \frac{n}{B}.$$

Then since $\mathbf{y}$ maximizes the total revenue under budget B , $\mathbf{y}$ can only pick items with the highest revenue. This implies that for any k with $y_k > 0$:

$$\sum_{i=1}^n p_{i,k} \geq \sum_{i=1}^n p_{i,\hat{k}} > \alpha \cdot \frac{n}{B}.$$

As a result, we can obtain a *strict* lower bound on the total revenue from $\mathbf{y}$ as:

$$\begin{aligned}
\sum_{k \in M} \sum_{i=1}^n p_{i,k} y_k &> \sum_{k: y_k > 0} \left(\alpha \cdot \frac{n}{B} \cdot y_k \right) \\
&= \alpha \cdot \frac{n}{B} \cdot \sum_{k: y_k > 0} y_k = \alpha \cdot \frac{n}{B} \cdot B = \alpha n.
\end{aligned}$$

This means that the total revenue from $\mathbf{y}$ is *strictly* greater than αn , which contradicts Proposition 5.1, Part (a). Hence, for any $k \in M$, $\sum_{i=1}^n p_{i,k} \leq \alpha \cdot \frac{n}{B}$, and completes the proof. $\square$

Proof of Proposition 5.1, Part (c). This part is a straightforward corollary of Part (b). Fix a committee $S \in \mathcal{O}$. Then we can bound the total price of S as:

$$\sum_{i=1}^n \sum_{k \in S} p_{i,k} = \sum_{k \in S} \left(\sum_{i=1}^n p_{i,k} \right) \leq \sum_{k \in S} \alpha \cdot \frac{n}{B} \leq \alpha \cdot \frac{|S|}{B} \cdot n$$

and this completes the proof. $\square$

D.2. Omitted Proofs in Section 5.2.

Proof of Proposition 5.2. The first part of the proposition is a direct result of Proposition 3.3. Then it is left for us to prove the second part.

If $S_i = \emptyset$, the second part trivially follows. So we assume $S_i \neq \emptyset$. Let A be the set of items which appear as the best item of some realization of i 's random demand i.e.

$$A := \{k \in M : \exists S \in \mathcal{O} \text{ s.t. } x_{i,S} > 0 \text{ and } k = \max_{k' \in S} k'\},$$

which is a non-empty set because $S_i \neq \emptyset$.

Then we can see for any $k \in A$, $k \succeq_i S_i$. Otherwise, let $\mathcal{C}_k$ be the collection of committees in the random demand having k as the best item. Pick an arbitrary $C \in \mathcal{C}_k$. We will have $k \prec S_i$ and therefore $C \prec S_i$. Because $\succsim_i$ preserves strict preferences, $C \prec_i S_i$. Then C would not have been included in the random demand, which is a contradiction.

Given the discussion above, an agent i *strictly* prefers S_i to some realization of $\tilde{L}(\mathbf{y})$ in terms of $\succsim_i$ only if the realization does not contain any item from A . It then follows

$$\Pr_{S \sim \tilde{L}(\mathbf{y})} (S \prec_i S_i) \leq \Pr_{S \sim \tilde{L}(\mathbf{y})} (\forall k \in A : k \notin S).$$

Now we break into two cases.

- Case 1: If $y_k > 1$ for some $k \in A$, then $z_k = 1$. Consequently, item k is always included in $\tilde{L}(\mathbf{y})$ and the proof is complete.
- Case 2: $y_k \leq 1$ for all $k \in A$. Now let $\mathbf{z}$ be the vector created from $\mathbf{y}$ as in the formulation of Lindahl lottery and $\tilde{Z}$ be the corresponding random binary vector. Then by Part (2) of the definition of LEOI, we have

$$\sum_{k \in A} z_k = \sum_{k \in A} \min(y_k, 1) = \sum_{k \in A} y_k \geq \alpha \cdot \sum_{k \in A} \sum_{S \in \mathcal{O}: k \in S} x_{i,S} \geq \alpha \cdot \sum_{k \in A} \sum_{C \in \mathcal{C}_k} x_{i,C}. \quad (4)$$

And because we have already assumed $S_i \neq \emptyset$, we have

$$\{C \in \mathcal{O} : \exists k \in A \text{ s.t. } C \in \mathcal{C}_k\} = \{S \in \mathcal{O} : x_{i,S} > 0\}.$$

It then follows

$$\sum_{k \in A} \sum_{C \in \mathcal{C}_k} x_{i,C} = \sum_{S \in \mathcal{O}: x_{i,S} > 0} x_{i,S} = 1. \quad (5)$$

Combining inequalities (4) and (5) yields

$$\sum_{k \in A} z_k \geq \alpha \cdot \sum_{k \in A} \sum_{C \in \mathcal{C}_k} x_{i,C} = \alpha.$$

As in the proof of Proposition 3.4, the following inequality follows from negative dependence:

$$\Pr_{S \sim \tilde{L}(\mathbf{y})} (S \prec_i S_i) \leq \Pr_{S \sim \tilde{L}(\mathbf{y})} (\forall k \in A : k \notin S) \leq \Pr_{\mathbf{Z} \sim \tilde{\mathbf{Z}}} (Z_k = 0 \ \forall k \in A) \leq e^{-\sum_{k \in A} z_k} \leq e^{-\alpha},$$

and this completes the proof. $\square$

We have the following traditional Chernoff bound.

PROPOSITION D.1 (Chernoff bound). *Let $X_1, X_2, \dots, X_n$ be negatively correlated random variables in $[0, 1]$. Define $X = \sum_{i=1}^n X_i$ and let $\mu = E[X]$. Then for any $\delta > 1$,*

$$\Pr(X < (1 - \delta) \cdot \mu) \leq \left(\frac{e^{-\delta}}{1 - \delta^{1-\delta}} \right)^\mu,$$

and a weaker inequality is

$$\Pr(X < (1 - \delta) \cdot \mu) \leq e^{-\frac{\delta^2 \cdot \mu}{2}}.$$

Using the Chernoff bound above, we derive a similar result for discrete random variables.

PROPOSITION D.2. *Let $X_1, X_2, \dots, X_n$ be negative dependent random variables on $\{0, 1\}$ with mean $x_1, x_2, \dots, x_n$ and let $X = \sum_{i=1}^n X_i$. Let $\alpha, \Delta > 0$ be given and suppose $E[X] \geq \alpha \cdot \Delta$. Then if $\alpha \geq 3$ and $\Delta \geq 1$, the following inequality holds:*

$$\Pr(X < \Delta) \leq e^{-\alpha}.$$

Proof of Proposition D.2. Since X can only take integral values, it suffices to prove

$$\Pr(X \leq \max(\lfloor \Delta - 1 \rfloor, 0)) \leq e^{-\alpha}.$$

We break the analysis into 5 cases based on the value of Δ . For cases where $\Delta < 5$, we use the cumbersome but tighter form of Chernoff (Proposition D.1); when $\Delta \geq 5$, using the weaker form would suffice.

Case 1: First consider the case $1 \leq \Delta < 2$. In this case, we can see that $X \leq \max(\lfloor \Delta - 1 \rfloor, 0)$ only if $X_j = 0$ for any j . Then using the same deduction as in the proof Proposition 3.4, we can apply the negative dependence property to obtain:

$$\Pr(X_j = 0 \ \forall j) \leq e^{-\sum x_j} \leq e^{-\Delta \alpha} \leq e^{-\alpha}.$$

Case 2: Now consider the case $2 \leq \Delta < 3$. In this case, we take $\delta = \frac{5}{6}$ so that $1 - \Delta \geq \frac{1}{\alpha \cdot \Delta}$ and $(1 - \delta)^{(1-\delta)} \geq e^{-0.3}$. We can now compute

$$\begin{aligned} \Pr(X \leq 1) &\leq \Pr\left(\sum_j X_j \leq (1 - \delta) \cdot \alpha \cdot \Delta\right) \\ &\leq \left(\frac{e^{-\delta}}{(1 - \delta)^{(1-\delta)}} \right)^{\alpha \cdot \Delta} \\ &\leq \left(\frac{e^{-\frac{5}{6}}}{e^{-0.3}} \right)^{2\alpha} \leq e^{-\alpha}. \end{aligned}$$

Case 3: Now consider the case $3 \leq \Delta < 4$. In this case, we take $\delta = \frac{7}{9}$ so that $1 - \delta \geq \frac{2}{\alpha \cdot \Delta}$ and $(1 - \delta)^{(1-\delta)} \geq e^{-0.34}$. We can apply the same logic as in Case 2 to obtain

$$\Pr(X \leq 2) \leq \left(\frac{e^{-\frac{7}{9}}}{e^{-0.34}} \right)^{3\alpha} \leq e^{-\alpha}.$$

Case 4: Now consider the case $4 \leq \Delta < 5$. In this case, we can take $\delta = \frac{3}{4}$ so that $1 - \delta \geq \frac{3}{\alpha \cdot \Delta}$ and $(1 - \delta)^{(1-\delta)} \geq e^{-0.35}$. We can apply the same logic as in Case 4 to obtain

$$\Pr(X \leq 3) \leq \left(\frac{e^{-\frac{3}{4}}}{e^{-0.35}} \right)^{4\alpha} \leq e^{-\alpha}.$$

Case 5: Lastly, we consider the case $\Delta \geq 5$, where it would be sufficient to use the weaker form of Chernoff. In this case, we can take $\delta = \frac{2}{3}$ so that $1 - \delta \geq \frac{1}{\alpha} \geq \frac{\lfloor \Delta - 1 \rfloor}{\alpha \cdot \Delta}$. Then we have

$$\Pr(X \leq \lfloor \Delta - 1 \rfloor) \leq e^{-\frac{\delta^2 \cdot \alpha \cdot \Delta}{2}} \leq e^{-\alpha},$$

and this completes the proof. $\square$

Proof of Proposition 5.3. To make the presentation easier, we first introduce some simplified notations: given any agent i and item $k \in M$, we write

$$x_{i,k} := \sum_{S \in \mathcal{O}: k \in S} x_{i,S}.$$

Now we are ready to prove the proposition. Note that the first part of the proposition is a direct result of depending rounding with unit cost. It is left for us to prove the second part, which is equivalent to:

$$\Pr_{S \sim \tilde{L}(\mathbf{y})} \left(u_i(S) < u_i(S_i) \right) \leq e^{-\alpha}.$$

Now let us define $K := \{k \in M_i : y_k \geq 1\}$. Because all items in K are included in $\tilde{L}(\mathbf{y})$ with probability 1, it suffices to show

$$\Pr_{S \sim \tilde{L}(\mathbf{y})} \left(u_i(S \setminus K) < u_i(S_i) - |K| \right) \leq e^{-\alpha}.$$

Then let $\mathbf{z}$ be the vector truncating all entries of $\mathbf{y}$ to no more than 1 as in Definition 5.4 and $\tilde{Z}$ be the corresponding random vector on $\{0, 1\}^m$ obtained from dependent rounding. We define a random vector $Z_{M_i \setminus K} := \sum_{k \in M_i \setminus K} Z_k$. Equivalently, it is sufficient to prove

$$\Pr(Z_{M_i \setminus K} < u_i(S_i) - |K|) \leq e^{-\alpha}.$$

The next step is to verify that the prerequisite for applying Proposition D.2 has been satisfied. By definition, S_i is the worst committee with positive weight in i 's random demand in terms of $\succsim_i$. Then each realization of the random demand is also no worse than S_i according to $\succsim_i$ i.e. having a utility no lower than $u_i(S_i)$. Therefore, we can see that the expected utility of the random demand cannot be lower than $u_i(S_i)$ i.e.:

$$\sum_{S \in \mathcal{O}: x_{i,S} > 0} u_i(S) x_{i,S} \geq \sum_{S \in \mathcal{O}: x_{i,S} > 0} u_i(S_i) x_{i,S} = u_i(S_i) \cdot \sum_{S \in \mathcal{O}: x_{i,S} > 0} x_{i,S} = u_i(S_i).$$

We also know that the expected utility of the random demand equals the total weight of items in M_i . So, we have

$$\sum_{k \in M_i} x_{i,k} = \sum_{S \in \mathcal{O}: x_{i,S} > 0} u_i(S) x_{i,S} \geq u_i(S_i).$$

And since each realization of the random demand contains at most one unit of each item, it follows that $x_{i,k} \leq 1$ for each $k \in M$. This implies that the total weight of items in $M_i \setminus K$ is at least $u_i(S_i) - |K|$ i.e. :

$$\sum_{k \in M_i \setminus K} x_{i,k} = \sum_{k \in M} x_{i,k} - \sum_{k \in K} x_{i,k} \geq \sum_{k \in M} x_{i,k} - \sum_{k \in K} 1 \geq u_i(S_i) - |K|.$$

By our definition of K , $z_k = y_k$ for each $k \in M_i \setminus K$. Then by the market clearing condition, for each $k \in M_i \setminus K$, we have $z_k = y_k \geq \alpha \cdot x_{i,k}$, which allows us to lower bound the expected value of $Z_{M_i \setminus K}$ as:

$$E[Z_{M_i \setminus K}] = \sum_{k \in M_i \setminus K} z_k \geq \alpha \cdot \sum_{k \in M_i \setminus K} x_{i,k} = \alpha \cdot (u_i(S_i) - |K|).$$

Furthermore, we can assume that $u_i(S_i) - |K| \geq 1$; otherwise $\tilde{L}(\mathbf{y})$ would have produced utility no smaller than $u_i(S_i)$ with probability 1 and the proof would be complete. Now taking $X := Z_{M_i \setminus K}$ and $\Delta := u_i(S_i) - |K|$, we can see that the conditions for applying Proposition D.2 have been satisfied and our proof follows. $\square$

D.3. Omitted Algorithms, Propositions, and Proofs in Section 5.3. For both types of preferences, we can show the existence of a committee which covers a significant fraction of the population and ensures that only a small fraction of covered agents who all prefer an alternative committee.

PROPOSITION D.3. *Given a set of agents V with ranking preferences and $\alpha \geq 1$, let $(\mathbf{x}_{i=1}^n, \mathbf{y}, \mathbf{p}_{i=1}^n)$ be an α -LEOI with a budget B and a random budget $\mathcal{I} = U[1 - \epsilon, 1]$. Then there exists a realization $S^* \in \mathcal{O}$ of the Lindahl lottery satisfying the following properties:*

- (1) $|S^*| \leq \lceil B \rceil$
- (2) S^* leaves at most $e^{-\alpha} \cdot |V|$ voters in V uncovered; that is,

$$|V \setminus C(S^*, V)| \leq e^{-\alpha} \cdot |V|,$$

where $C(S^*, V)$ ⁷ is the set of voters in V covered by S^* .

- (3) for any $S \in \mathcal{O}$,

$$|\{i \in C(S^*, V) : S^* \prec_i S\}| \leq \frac{\alpha}{1 - \epsilon} \cdot \frac{|S|}{B} \cdot |V|.$$

PROPOSITION D.4. *Given a set of agents V with approval preferences and $\alpha \geq 3$, let $(\mathbf{x}_{i=1}^n, \mathbf{y}, \mathbf{p}_{i=1}^n)$ be an α -LEOI with a budget B and a random budget $\mathcal{I} = U[1 - \epsilon, 1]$. Then there exists a realization $S^* \in \mathcal{O}$ of the Lindahl lottery satisfying the following properties:*

- (1) $|S^*| \leq \lceil B \rceil$
- (2) S^* leaves at most $e^{-\alpha} \cdot |V|$ voters in V uncovered; that is,

$$|V \setminus C(S^*, V)| \leq e^{-\alpha} \cdot |V|,$$

where $C(S^*, V)$ is the set of voters in V covered by S^* .

- (3) for any $S \in \mathcal{O}$,

$$|\{i \in C(S^*, V) : S^* \prec_i S\}| \leq \frac{\alpha}{1 - \epsilon} \cdot \frac{|S|}{B} \cdot |V|.$$

Proof of Proposition D.3 and Proposition D.4. Due to the similarity of the two Propositions, we choose to prove them together. The main idea is very similar to the proof of Proposition 4.1.

⁷In the setting of committee selection, we change the notation from $S(V)$ to $C(V)$ because S has been used to denote committees

As in the proof of Proposition 4.1, we can apply Proposition 5.2 in the case of ranking preference (Proposition 5.3 in the case of approval preference) and linearity expectation to find a committee S^* which is a realization of the Lindahl lottery and satisfies Part (2) of both propositions. S^* also satisfies Part (1) because any realization of the Lindahl lottery has size at most $\lceil B \rceil$ by Proposition 3.3.

To show Part (3), notice that the total price for a committee S is at most $\alpha \cdot \frac{|S|}{B} \cdot |V|$ by Proposition 5.1, Part (c). If an agent is covered by S^* but prefers S to S^* , her personal price for S is at least $1 - \epsilon$. It then follows

$$|\{i \in C(S^*, V) : S^* \prec_i S\}| \leq \frac{\alpha}{1 - \epsilon} \cdot \frac{|S|}{B} \cdot |V|$$

and the proof is complete. $\square$

We are ready to give the iterative algorithm with LEOI.

ALGORITHM 3: Iterated Rounding with LEOI

Input: a set V of n voters with either ranking or approval preferences, a budget B , $\epsilon > 0$, parameters $\gamma \geq 1$, $\alpha \geq 1$ for ranking preferences ($\alpha \geq 3$ for approval preferences) satisfying $e^\alpha > \gamma$

Output: a committee S^* with $|S^*| \leq B$ s.t.

$$\left| \{i \in V : S^* \prec_i S\} \right| \leq \left\lfloor \frac{\alpha}{1 - \epsilon} \cdot \left(\frac{\gamma}{\gamma - 1} + \frac{\log_\gamma(B)}{B} \right) \cdot \frac{e^\alpha}{e^\alpha - \gamma} \cdot \frac{|S|}{B} \cdot n \right\rfloor \text{ for each } S \in \mathcal{O}$$

- 1 $j \leftarrow 0, S^* \leftarrow \emptyset, V_0 \leftarrow V, B_0 \leftarrow \frac{1}{\frac{\gamma}{\gamma - 1} + \frac{\log_\gamma(B)}{B}} \cdot B$
- 2 **while** $|V_j| \geq 1$ **and** $B_j \geq 1$ **do**
- 3 Given voters in V_j , we generate an α -LEO $(\mathbf{x}_{i=1}^n, \mathbf{y}, \mathbf{p}_{i=1}^n)$ with a total budget of B_j and $\mathcal{I} = \mathbf{U}[1 - \epsilon, 1]$
- 4 Find an S^j with $|S^j| \leq B_j$ satisfying Proposition D.3 in the case of ranking preference or satisfying D.4 in the case of approval preference
- 5 $S^* \leftarrow S^* \cup S^j$
- 6 $B_{j+1} \leftarrow \frac{B_j}{\gamma}$
- 7 $V_{j+1} \leftarrow V_j \setminus C(S^j, V_j)$ i.e. V_{j+1} is the subset of V_j , who are not covered by o^j
- 8 $j++$
- 9 **end**
- 10 **return** S^*

We next show the correctness of Algorithm 3.

PROPOSITION D.5. *In round j of Algorithm 3, for any $S \in \mathcal{O}$, the number of agents in V_j who are covered by o^j and prefer S to S^j is at most*

$$\left| \{i \in C(S^j, V_j) : S \succ_i S^j\} \right| \leq \left(\frac{\gamma}{e^\alpha} \right)^j \cdot \frac{\alpha}{\gamma(1 - \epsilon)} \cdot \left(\frac{|S|}{B_0} \cdot n \right).$$

Proof. According to Proposition D.4 and Proposition D.3, we have

$$\left| \{i \in C(S^j, V_j) : S \succ_i S^j\} \right| \leq \frac{\alpha}{1 - \epsilon} \cdot \frac{|S|}{B_j} \cdot |V_j|.$$

Since $|V_{j+1}| \leq \frac{|V_j|}{e^\alpha}$ and $B_{j+1} \leq \frac{B_j}{\gamma}$, it follows that $|V_j| \leq \frac{n}{e^{\alpha \cdot j}}$ and $B_j \leq \frac{B_0}{\gamma^j}$ for each j . Plugging these into the formula above gives the desired result. $\square$

Proof of Proposition 5.4. We first show that the output of Algorithm 3, S^* , is feasible i.e. has a size of less than B .

$$\begin{aligned}
|S^*| &\leq \sum_{j:B_j \geq 1} |S^j| \leq \sum_{j:B_j \geq 1} \lceil B_j \rceil \leq \sum_{j:B_j \geq 1} B_j + \sum_{j:B_j \geq 1} 1 \\
&\leq B_0 \sum_{j=0}^{\infty} \frac{1}{\gamma^j} + \log_{\gamma}(B) \leq B_0 \cdot \frac{\gamma}{\gamma-1} + \log_{\gamma}(B_0) \\
&\leq B_0 \cdot \left(\frac{\gamma}{\gamma-1} + \frac{\log_{\gamma}(B_0)}{B_0} \right) \leq B_0 \cdot \left(\frac{\gamma}{\gamma-1} + \frac{\log_{\gamma}(B)}{B} \right) \leq B,
\end{aligned}$$

where the last line uses the fact that $\frac{\log_{\gamma}(x)}{x}$ is a decreasing function for $x \geq 2$.

Now given $S \in \mathcal{O}$, we want to bound the total number of agents preferring S to S^* using Proposition D.5.

$$\begin{aligned}
|\{i \in V : S^* \prec_i S\}| &\leq \sum_{j=0}^{|V_j| \geq 1} |\{i \in C(S^j, V_j) : S \succ_i S^j\}| \leq \frac{\alpha}{\gamma(1-\epsilon)} \cdot \left(\frac{|S|}{B_0} \cdot n\right) \cdot \sum_{j=0}^{\infty} \left(\frac{\gamma}{e^{\alpha}}\right)^j \\
&\leq \frac{\alpha}{\gamma(1-\epsilon)} \cdot \left(\frac{|S|}{B_0} \cdot n\right) \cdot \frac{1}{1 - \frac{\gamma}{e^{\alpha}}} \leq \frac{\alpha}{1-\epsilon} \cdot \left(\frac{\gamma}{\gamma-1} + \frac{\log_{\gamma}(B)}{B}\right) \cdot \frac{e^{\alpha}}{e^{\alpha}-\gamma} \cdot \left(\frac{|S|}{B} \cdot n\right).
\end{aligned}$$

Therefore, S^* lies in the $\frac{\alpha}{1-\epsilon} \cdot \left(\frac{\gamma}{\gamma-1} + \frac{\log_{\gamma}(B)}{B}\right) \cdot \frac{e^{\alpha}}{e^{\alpha}-\gamma}$ -approximate core and this completes the proof. $\square$

D.4. Proof of Theorem 5.1.

Proof. As B is an integer, we divide the proof into two following cases.

Case 1: $B \leq 60$.

The reason we handle small B separately is because when B is small, the term $\frac{\log_{\gamma}(B)}{B}$ in Proposition 5.4 can be too big to derive the desired bound.

Now, consider the committee S^* as given in Proposition 5.2 and Proposition 5.3. Because B is an integer, S^* has a size of exactly B and is therefore feasible. Then, we claim that given $B \leq 60, \epsilon > 0, \alpha \geq 1$ ($\alpha \geq 3$ in the case of approval preferences), S^* is in the $(\frac{\alpha}{1-\epsilon} + 60e^{-\alpha})$ -approximate core. To see this, we can combine Part (2) and Part (3) of 5.2 (Proposition 5.3 in the approval case) to show that for any $S \in \mathcal{O}$,

$$\begin{aligned}
|i \in V : S^* \prec_i S| &\leq |i \in C(S^*, V) : S^* \prec_i S| + |i \notin C(S^*, V) : S^* \prec_i S| \\
&\leq |i \in C(S^*, V) : S^* \prec_i S| + |V \setminus C(S^*, V)| \\
&\leq \frac{\alpha}{1-\epsilon} \cdot \frac{|S|}{B} \cdot |V| + e^{-\alpha} \cdot |V| \leq \left(\frac{\alpha}{1-\epsilon} + \frac{e^{-\alpha}}{|S|/B}\right) \cdot \frac{|S|}{B} \cdot |V| \\
&\leq \left(\frac{\alpha}{1-\epsilon} + Be^{-\alpha}\right) \cdot \frac{|S|}{B} \cdot |V| \leq \left(\frac{\alpha}{1-\epsilon} + 60e^{-\alpha}\right) \cdot \frac{|S|}{B} \cdot |V|.
\end{aligned}$$

Now let $\epsilon \rightarrow 0$, we can obtain the existence of a committee in the $\alpha + 60e^{-\alpha}$ -approximate core. Optimizing over α gives the existence of the desired approximation ratio ($\alpha = 4.094, \alpha + 60e^{-\alpha} = 5.094$).

Case 2: $B \geq 61$.

Because this part of the proof is similar to the proof of Theorem 4.1, we choose to offer a sketch here. By Proposition 5.4, for every $\epsilon > 0$ and $\alpha, \gamma \geq 1$ satisfying the requirement for the inputs of Algorithm 3, we can find a committee in the $\frac{\alpha}{1-\epsilon} \cdot \left(\frac{\gamma}{\gamma-1} + \frac{\log_{\gamma}(B)}{B}\right) \cdot \frac{e^{\alpha}}{e^{\alpha}-\gamma}$ -approximate core. Because $\frac{\alpha}{1-\epsilon} \cdot \left(\frac{\gamma}{\gamma-1} + \frac{\log_{\gamma}(B)}{B}\right) \cdot \frac{e^{\alpha}}{e^{\alpha}-\gamma} \rightarrow \alpha \cdot \left(\frac{\gamma}{\gamma-1} + \frac{\log_{\gamma}(B)}{B}\right) \cdot \frac{e^{\alpha}}{e^{\alpha}-\gamma}$ as $\epsilon \rightarrow 0$ and the number of deviating agents must be an integer, there must exist a $\epsilon > 0$ for which the corresponding committee lies in the

$\alpha \cdot \left(\frac{\gamma}{\gamma-1} + \frac{\log_\gamma(61)}{61} \right) \cdot \frac{e^\alpha}{e^\alpha - \gamma}$ -approximate core, where we use the assumption of $B \geq 61$ and the fact that $\frac{\log(x)}{x}$ is monotone decreasing in $[2, \infty)$.

Then, for ranking preferences, taking $\alpha = 2.57, \gamma = 3.67$ yields an approximation factor of 5.10. For approval preferences, taking $\alpha = 3, \gamma = 4.55$ gives an approximation factor of 5.15. This completes the proof.

□