From order lifting to social ranking: retrieving individual rankings from partial lifted preferences

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Abstract

Several methods have been introduced in the literature to extend preferences over items from a population to preferences over the groups they may form - a problem known as the Order Lifting problem. The converse matter of deducing preferences over items from expressed preferences over the coalitions that may be formed within a population - a problem known as the Social Ranking Problem - has also been studied more recently.

In this paper, we investigate the links between these two problems: after an examination of the general case, we consider the impact of missing information, by studying which social ranking methods allow for the most accurate recovery of initial preferences over items, depending on the amount of missing information about coalitions. Finally, we consider the specific case in which preferences are only expressed over coalitions of same size, and examine the accuracy of social ranking methods when facing partial information about preferences over k-sized coalitions.

1 Introduction

Decision-makers may express preferences over items in various forms. A moviegoer, for example, might rank films from most to least preferred, but could also express preferences over bundles of films, by evaluating different streaming platforms, each described as a set of films available to users. It seems reasonable to assume that preferences over sets of items are related to the preferences over single items.

The order lifting problem (also known as “Ranking sets of objects") consists in deducing preferences over sets of items from a given ranking of the items [11, 17]. It finds applications in many settings, such as in computational social choice, when trying to deduce voters' preferences over committees or bundles from their individual preferences, as it proves less demanding to require preferences over items than over all possible sets. In their overview of various approaches to the order lifting problem, [2, 13] describe various ways of extending preferences over individual items into a ranking over all the coalitions these items may form. Some of these methods consider that a decision maker will base their evaluation on the performance of extreme elements in a set. The idea that one would prefer a set containing the best item, or that one would dislike a set containing lowly-ranked items seem to intuitively describe respectively optimistic and pessimistic ways to evaluate groups of items.

On the other hand, [15, 16] introduce the social ranking problem, to infer a ranking over items from expressed preferences over the powerset of the population. The goal of this problem is to measure items' influence within the population, by measuring their impact on group performance: an item's influence is supposed linked to the positive or negative impact of its addition to a group to the latter's evaluation (see also [4, 10]).

In certain settings, however, it may happen that some information about a decision maker's preferences be missing, and that only a partial ranking is made available. For instance, the moviegoer may only be able to rank existing streaming platforms, and their preferences over other combinations of films remains unknown. Therefore, in this paper we are more interested in problems where preferences can only be observed on a given collection of subsets. As far as we know, in the area of lifting preferences from individual items to sets, this problem has been studied in the article [5], where only sets of a fixed size are ranked, and in the article [14] where a ranking over a family of subsets is built using alternative versions of the properties of dominance and independence. For a social ranking problem based on a given collection of subsets, we mention the article [21], where some lexicographic solutions are axiomatically characterized using alternative domains of subsets of items. In other articles, a similar problem has been studied in connection with specific applications of social rankings: to measure the alignments of norms with moral values taking into account some compatibility constraints [20]; to rank arguments based on their strength in an argumentation framework, and considering as input and extension-ranking semantics providing a partial ordering over subsets of arguments [3]; and to sort individual elements in a greedy heuristic aimed at solving a combinatorial optimization problem on the basis of the performances of a limited set of groups of elements [22].

Our study explores whether social ranking solutions allow for the recovery of initial preferences over items, when considering rankings over coalitions supposed to have been determined using a specific lifting method. The interest in addressing this question is twofold. First, inferring preferences over individual items based on a partial observation of the preferences over subsets can be useful to assess the impact of each single element in the overall evaluation of groups of items, for instance, to rank individual items according to their relevance for a user. Second, performing this inference through the lens of a predefined lifting rule makes it possible to reverse the process and extend the observed preferences over a limited family of subsets to all possible subsets in a consistent manner. This allows, for instance, to select new most-preferred bundles of items for a user.

The first section of this paper formally introduces the order lifting and social ranking problems, as well as some classic solutions proposed in the literature. We then study the links between the problems, namely by wondering whether these are inverse problems, *i.e.* if giving the output data of problem A as input to problem B will always lead to problem B's output being problem A's initial input. In a third section, we study the robustness of these results when only partial preferences are available. Finally, we focus on the particular scenario wherein (partial) preferences are given over coalitions of same size k.

2 Introduction to the problems

A *binary relation* $\succsim$ over a finite set X is a subset of the Cartesian product $X \times X$. It is called *a partial preorder* (or simply a *preorder*) if it is *reflexive* ($\forall x \in X, x \succsim x$) and *transitive* ($\forall x, y, z \in X, (x \succsim y) \wedge (y \succsim z) \Rightarrow x \succsim z$). A binary relation $\succ$ is *an order* if it is *irreflexive* ($\forall x \in X, x \not\succ x$), *asymmetric* ($\forall x, y \in X, x \succ y \Rightarrow \neg(y \succ x)$) and *transitive*. A given (pre)order is said to be *total* if $\forall x, y \in X, (x \succ y) \vee (y \succ x)$. Given a finite set $A \subseteq X$ and a total order $\succ_X$ over X, we denote by $\min(A)$ *(resp.* $\max(A)$*)* the worst *(resp. the best)* element in A, *i.e.* such that $\forall x \in X, x \succeq \min(A)$ *(resp.* $\max(A) \succeq x$*).*

Considering a population X of n items, and given a ranking over respectively items and the powerset of the population $\mathcal{P}(X) = \{C \subseteq X\}$, several solutions to the order lifting and social ranking problems have been introduced in the literature.

2.1 Order lifting

Barberà et al. [2] introduce several order lifting methods, of which we consider four, that we deem best suited to our ordinal framework: two minmax-based and two lexicographical order-based methods.

Minmax-based orderings compare two coalitions using only their best and worst components: *minmax* considers a coalition C_1 is preferred to another C_2 if $\min(C_1)$ is better than $\min(C_2)$ or if they are of similar worst element but its best element $\max(C_1)$ is preferred to $\max(C_2)$; *maxmin* uses the same process, but starts by comparing $\max(C_1)$ and $\max(C_2)$, then using the worst elements as tie-breakers.

Definition (Minmax ordering)**.** The *minmax ordering* $\succsim_{mnx}$ is such that, given a total order $>_X$ over

$X, \forall A, B \subseteq X$,

$$A \succsim_{mnx} B \Leftrightarrow \begin{cases} \min(A) >_X \min(B), \text{ or} \\ \min(A) = \min(B) \text{ and } \max(A) >_X \max(B) \end{cases}$$

Definition (Maxmin ordering)**.** The *maxmin ordering* $\succsim_{mxn}$ is such that, given a total order $>_X$ over $X, \forall A, B \subseteq X$,

$$A \succsim_{mxn} B \Leftrightarrow \begin{cases} \max(A) >_X \max(B), \text{ or} \\ \max(A) = \max(B) \text{ and } \min(A) >_X \min(B) \end{cases}$$

In a more detailed study of pairs of coalitions, lexicographical order-based orderings compare every element in each coalition, considering them either from worst to best (leximin) or from best to worst (leximax). Note that the leximin ordering is also known in the literature as the *anti-lexcel* ordering [20].

Definition (Leximin ordering)**.** The *leximin ordering* $\succsim^L_{min}$ is such that, given a total order $>_X$ over X, $\forall A, B \subseteq X$, $A = \{a_1, \ldots, a_{|A|}\}$ and $B = \{b_1, \ldots, b_{|B|}\}$,

$$A \succsim^L_{min} B \Leftrightarrow \begin{cases} A = B, \text{ or} \\ |A| > |B| \text{ and } a_{\sigma_A(i)} = b_{\sigma_B(i)}, \forall i \in \{1, \ldots, |B|\}, \text{ or} \\ \exists i \in \{1, \ldots, \min\{|A|, |B|\}\} \text{ such that } \forall j < i, a_{\sigma_A(j)} = b_{\sigma_B(j)} \text{ and } a_{\sigma_A(i)} >_X b_{\sigma_B(i)}, \end{cases}$$

with $\sigma_X : \{1, \ldots, |X|\} \to \{1, \ldots, |X|\}$ a permutation such that $a_{\sigma_X(i+1)} >_X a_{\sigma_X(i)}$.

Definition (Leximax ordering)**.** The *leximax ordering* $\succsim^L_{max}$ is such that, given a total order $>_X$ over $X, \forall A, B \subseteq X$, $A = \{a_1, \ldots, a_{|A|}\}$ and $B = \{b_1, \ldots, b_{|B|}\}$,

$$A \succsim^L_{max} B \Leftrightarrow \begin{cases} A = B, \text{ or} \\ |A| < |B| \text{ and } a_{\rho_A(i)} = b_{\rho_B(i)}, \forall i \in \{1, \ldots, |A|\}, \text{ or} \\ \exists i \in \{1, \ldots, \min\{|A|, |B|\}\} \text{ such that } \forall j < i, a_{\rho_A(j)} = b_{\rho_B(j)} \text{ and } a_{\rho_A(i)} >_X b_{\rho_B(i)} \end{cases}$$

with $\rho_X : \{1, \ldots, |X|\} \to \{1, \ldots, |X|\}$ a permutation such that $a_{\rho_X(i)} >_X a_{\rho_X(i+1)}$.

Additionally, Darmann and Klamler [7] introduce an order lifting method using the Borda scores traditionally employed in voting theory to associate a value to every coalition. Given a total order $>_X$ over X, and let $pos(x)$ denote the rank of item x in $>_X, \forall x \in X$. Let b be a n-sized vector of (decreasing) weights, such that b_i is the weight associated to the element ranked at the i-th position, the Borda value of a set $A \subseteq X$ is defined using vector b such that

$$w_b(A) = \sum_{x \in A} b_{pos(x)}.$$

The Borda-sum ordering then ranks items based on their Borda value.

Definition (Borda-sum ordering)**.** The *Borda-sum ordering* $\succsim_b$ is such that, $\forall A, B \subseteq X$,

$$A \succsim_b B \iff w_b(A) \geq w_b(B).$$

While these five order lifting methods are introduced in the literature using a total order as input, they can be extended to total preorders in a similar manner.

2.2 Social ranking

The purpose of the social ranking problem is to determine a ranking over items in X based on expressed preferences $\succsim$ over $\mathcal{P}(X)$, taking into account their interactions with other items from the population.

Let a, b be two same-sized vectors, we denote by $\geq^L$ the *lexicographical relation* such that $a \geq^L b \Leftrightarrow (a = b \text{ or } \exists t \text{ s.t. } a_t \succ b_t \text{ and } a_r \sim b_r, \forall r \in \{1, \ldots, t-1\})$. An *equivalence class*, denoted by Σ, in a preorder $\succsim$ is a set of elements deemed equivalent by $\succsim$. The *quotient order* of $\succsim$ is denoted by $\Sigma_1 \succ \cdots \succ \Sigma_l$, with each equivalence class Σ_k, $k \in \{1, \ldots, l\}$, generated by the symmetric part of $\succsim$.

Considering that the most influent item is the most present in top coalitions, [1, 4] introduce the *lexicographical excellence* method, defined as follows:

Definition. Let $\succsim$ be a ranking over the powerset of a given set X. The *lexicographical excellence (lexcel)* preference relation is a binary relation $\geq_{lex} \subseteq X \times X$ such that, $\forall i, j \in X$,

$$i \geq_{lex} j \iff \theta^{\succsim}(i) \geq^L \theta^{\succsim}(j),$$

with $\theta^{\succsim}(i)$ the occurrence vector of i, such that $\theta^{\succsim}(i)_k = |\{C \in \Sigma_k \mid i \in C\}|, \forall k \in \{1, \ldots, l\}$.

Example 1. Let $X = \{1, 2, 3\}$, and $\succsim$ be such that

$$23 \succ 1 \sim 123 \succ 2 \succ 13 \succ 3 \sim 12,$$

we compute the vectors $\theta^{\succsim}(1) = (0, 2, 0, 1, 1)$, $\theta^{\succsim}(2) = (1, 1, 1, 0, 1)$ and $\theta^{\succsim}(3) = (1, 1, 0, 1, 1)$, which, compared using the lexicographical relation, lead to the preference relation $2 >_{lex} 3 >_{lex} 1$.

In another interpretation of an item's impact on coalitions, [10] determine pairwise preference relations based on items' impact *ceteris paribus*, *i.e.* all things considered equal. An element $x \in X$ is therefore preferred to another $y \in X$ if a majority of coalitions $C \in X \setminus \{x, y\}$ verifies that $C \cup \{x\} \succ C \cup \{y\}$.

Definition. Let $\succsim$ be a ranking over the powerset of a given set X. The *CP-majority* preference relation is a binary relation $\geq_{CP} \subseteq X \times X$ such that, $\forall i, j \in X$,

$$i \geq_{CP} j \iff d_{ij}(\succsim) \geq d_{ji}(\succsim),$$

with $d_{ij}(\succsim) = |\{S \subseteq X \setminus \{i, j\} \mid S \cup \{i\} \succsim S \cup \{j\}\}|$.

Example 2. Let $X = \{1, 2, 3\}$, and $\succsim$ be such that

$$13 \succ 1 \sim 123 \succ 3 \succ 23 \succ 2 \sim 12.$$

To compare 1 and 2, we compare 1 to 2 and 13 to 23, and observe that $1 \succ 2$ and $13 \succ 23$, which leads us to conclude that $1 \geq_{CP} 2$; to compare 2 and 3, we compare 2 to 3 and 12 to 13, and observe that $3 \succ 2$ and $13 \succ 12$, which leads us to conclude that $3 \geq_{CP} 2$; finally, to compare 1 and 3, we compare 1 to 3 and 12 to 23, and observe that $1 \succ 3$ and $23 \succ 12$, which leads us to conclude that $1 =_{CP} 3$.

Remark. The CP-majority relation may return non-transitive pairwise comparisons.

Given a ranking $\succsim$ over the powerset of a given population X, and let $i \in X$, we denote by $u_i^{+,\succsim} = |\{S \subseteq X \mid S \cup \{i\} \succ S\}|$ and $u_i^{-,\succsim} = |\{S \subseteq X \mid S \succ S \cup \{i\}\}|$ the number of coalitions for which we observe a respective increase and decrease when i is added. [12] adapts the Banzhaf index to an ordinal framework, and evaluates the influence of an item $x \in X$ by computing its marginal contribution to all coalitions, *i.e.* counting positively the number of coalitions $C \subseteq X \setminus \{x\}$ such that $C \cup \{x\} \succ C$, and negatively those verifying that $C \succ C \cup \{x\}$. Items are then ranked based on their marginal contribution score.

Definition. Let $\succsim$ be a ranking over the powerset of a given set X. The *ordinal Banzhaf* preference relation is a binary relation $\geq_{Banz} \subseteq X \times X$ such that, $\forall i, j \in X$,

$$i \geq_{Banz} j \iff Score^{\succsim}_{Banz}(i) \geq Score^{\succsim}_{Banz}(j),$$

with $Score^{\succsim}_{Banz}(i) = u_i^{+,\succsim} - u_i^{-,\succsim}$.

Example 3. Let $X = \{1, 2, 3\}$, and $\succsim$ be such that

$$23 \succ 1 \sim 123 \succ 2 \succ \emptyset \succ 13 \succ 3 \sim 12,$$

we compute $Score^{\succsim}_{Banz}(1) = |\{\emptyset, 3\}| - |\{2, 23\}| = 0$, $Score^{\succsim}_{Banz}(2) = |\{\emptyset, 3, 13\}| - |\{1\}| = 2$ and $Score^{\succsim}_{Banz}(3) = |\{2, 12\}| - |\{\emptyset, 1\}| = 0$, which leads to the preference relation $2 \geq_{Banz} 1 =_{Banz} 3$.

3 From Social Ranking to Order Lifting, and back

As each problem's input fits the other's output (and conversely), we investigate whether giving problem A's output as input to problem B returns the initial input to problem A, *i.e.* if these are inverse problems.

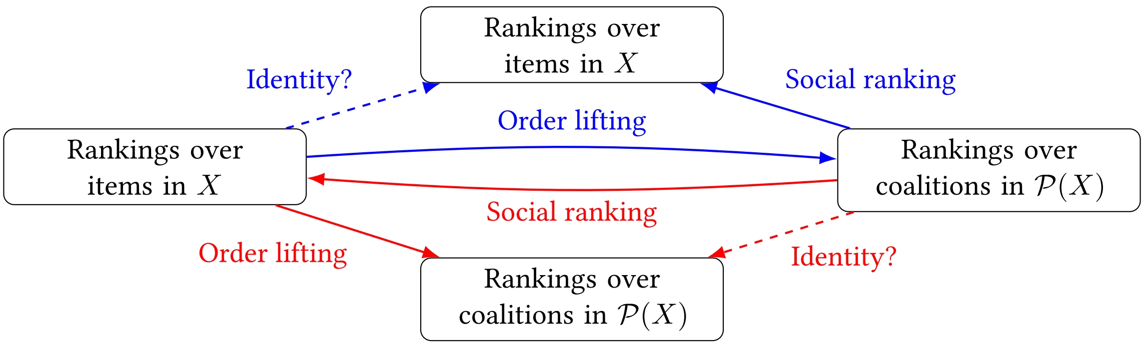


Figure 1: Representation of the studied links between the problems

3.1 Order Lifting applied to Social Ranking

We start by considering the application of order lifting methods to social ranking results, as represented by the red arrows in Figure 1. Thus, from a ranking $\geq_m$ over X, with $m \in \{lex, CP, Banz\}$ indicating that the social ranking rule used is respectively Lexcel, CP-majority or ordinal Banzhaf, we investigate whether the order lifting solutions introduced in subsection 2.1 allow for the recovery of the initial ranking $\succsim$ over coalitions from which $\geq_m$ has been obtained.

Proposition 1. [1] *Given a population of elements X, and let $\geq_m$ be the preferences over X obtained using a social ranking method $m \in \{lex, CP, Banz\}$ on given preferences $\succsim$ over $\mathcal{P}(X)$. No order lifting rule may systematically return the correct preferences on $\mathcal{P}(X)$ from $\geq_m$.*

In particular, for each of the introduced order lifting solutions, we find that there exists an example for which it returns an incorrect ranking over coalitions. This result is as expected for several reasons, but notably because, as detailed in subsection 2.2, the social ranking problem is set on detecting interactions between elements, while all introduced order lifting solutions consider extensions of preferences over items to be the sole indicator of set evaluation, without accounting for possible reaction to other items' presence. It would therefore be too optimistic to expect order lifting rules to account for interactions over which they have no knowledge.

[1] Proofs for every proposition in this paper are available in the Appendix

However, social ranking methods may still respond well to preferences which do not model any interaction between items. In the following subsection, we investigate social ranking solutions' ability to recover the correct ranking over items based on their lifted preferences, as represented by the blue arrows in Figure 1.

3.2 Social Ranking applied to Order Lifting

Let $\succsim$ be a preorder over $\mathcal{P}(X)$ given as input to the social ranking problem, obtained by applying one of the order lifting rules introduced in subsection 2.1 to a total order $>_X$ over X, we look into each social ranking rule's *(as introduced in subsection 2.2)* ability to recover $>_X$.

Proposition 2. *Given a total order $>_X$ over elements in a set X, and let $\succsim$ be the ranking obtained over $\mathcal{P}(X)$ by applying an order lifting solution on $>_X$, i.e. $\succsim \in \{\succsim_{mnx}, \succsim_{mxn}, \succsim^L_{min}, \succsim^L_{max}\}$. Lexcel, CP-majority and ordinal Banzhaf always return the correct order $>_X$ over X from $\succsim$.*

Using the "classic" Borda weight vector $b = (n-1, \ldots, 1, 0)$, one notices that the Borda-sum ordering is not monotonic, because of the *Irrelevance of Worst Object* axiom introduced in [7]: being of weight 0, the worst element in X can be removed from any coalition without changing the coalition's evaluation. Therefore, it is not always true that adding any element to a coalition strictly improves its performance - this only holds if the added element is not the element placed last by $\succsim_b$.

Proposition 3. *Given a total order $>_X$ over items in X, and let $\succsim_b$ be the preorder obtained over $\mathcal{P}(X)$ by applying the Borda-sum extension rule on $>_X$ using the weight vector $b = (n-1, \ldots, 1, 0)$. Lexcel and CP-majority always return the correct order over X from $\succsim_b$.*

Proposition 4. *Given a total order $>_X$ over items in X, and let $\succsim_b$ be the preorder obtained over $\mathcal{P}(X)$ by applying the Borda-sum extension rule on $>_X$ using the weight vector $b = (n-1, \ldots, 1, 0)$. Ordinal Banzhaf will never return the correct order over X from $\succsim_b$.*

The Order Lifting Problem and the Social Ranking Problem are therefore not inverse problems, as, from the order over items determined by a social ranking method, no order lifting method systematically manages to recover the initial order over coalitions. However, aside from ordinal Banzhaf's negative reaction to Borda-sum orderings, our social ranking rules always recover the correct initial order over items given the order over coalitions obtained using any of the introduced order lifting methods.

4 Recovering initial preferences from partial extensions: Social ranking applied to partially-lifted orders

Note that when all singletons are present in the expressed preferences over coalitions, and knowing that these preferences are the product of an order lifting method, the problem of determining the correct order over items is trivial: it is sufficient to remove all coalitions of size greater than 1 from the preferences. This problem becomes harder when at least a couple of the singletons are missing from the ranking, *i.e.* when a partial preorder is taken as input. Furthermore, preferences over the entirety of a population's powerset are in practice rare and rather impractical to obtain: it is much more common to only have access to preferences over some coalitions, with an idea of the underlying reasoning behind this ranking. There has been work dealing with incomplete preference such as [19], which is focused on completing partial preorders over coalitions to uncover individual preferences using elicitation. However, we choose to consider that no additional information over the preorder is made available, and study the impact of this lack of information on social ranking methods' efficiency in recovering the initial order over items.

We now consider a variant of the social ranking problem, in which, given a partial preorder over the powerset of a population, and with the knowledge that these preferences have been obtained through an order lifting method, we try and recover the initial order over items. Note that, from this point on, we only consider partial preorders in which at least two singletons are missing from the ranking, so as to ensure we do not consider trivial cases.

4.1 Theoretical results

We start by studying expected results from the three social ranking methods, and try to determine if one specific method may (in theory) fare better than the others under partial information. In the rest of this paper, we will say that a preorder $\succsim$ is *compatible* with another preorder $\succsim^+$ iff $\succsim \subseteq \succsim^+$, *i.e.* $\forall x, y \in X, x \succsim y \Rightarrow x \succsim^+ y$.

We define three levels of precision, which must hold for any preorder given as input, with level 0 representing the lowest threshold for precision, and level 2 the highest before exactness:

- **Level 0:** *(Impossibility of reversal)* the social ranking rule may not return a preorder that is the reverse of the initial one, *i.e.* $>_X \neq \emptyset$ and $\exists x, y \in X$ s.t. $x >_X y$ and the rule m returns that $x \geq_m y$.
- **Level 1:** *(Preservation of the extremes)* the social ranking rule may return incorrect pairwise preferences, but in a worst-case scenario where it would return the worst element as a top element *(i.e. $w \in X$ such that $\forall x \in X \setminus \{w\}$, $x \geq_X w$ is also such that $w \geq_m x$)*, it will also return the best element as top element. Similarly, in a worst-case scenario where it would return the best element as a bottom element *(i.e. $b \in X$ such that $\forall x \in X \setminus \{w\}$, $b \geq_X x$ is also such that $x \geq_m b$)*, it will also return the worst element as bottom element.
- **Level 2:** *(Preservation of pairwise preferences)* let $x, y \in X$ be such that $x >_X y$, the social ranking rule m will never determine that $y >_m x$ *(note that it may however return that $x =_m y$)*.

We then study the level of precision achieved by each social ranking method. Note that if a given rule does not verify level i of precision, it will not verify any level j such that $j > i$.

Proposition 5. *Given a ranking $>_X$ over X from which we determine $\succsim$ using one of the introduced order lifting rules, let $\succsim'$ be a partial preorder over $\mathcal{P}(X)$ compatible with $\succsim$. When applied to $\succsim'$, lexcel does not satisfy level 0 of precision, $\forall \succsim \in \{\succsim_{mnx}, \succsim_{mxn}, \succsim^L_{min}, \succsim^L_{max}, \succsim_b\}$.*

Proposition 6. *Given a ranking $>_X$ over X from which we determine $\succsim$ using one of the minmax-based orderings, let $\succsim'$ be a partial preorder over $\mathcal{P}(X)$ compatible with $\succsim$. When applied to $\succsim'$, ordinal Banzhaf satisfies level 1 of precision but not level 2, $\forall \succsim \in \{\succsim_{mnx}, \succsim_{mxn}\}$.*

Proposition 7. *Given a ranking $>_X$ over X from which we determine $\succsim$ using one of the lexicographical order-based orderings. Let b be the best item according to $>_X$, and let $\succsim'$ be a partial preorder over $\mathcal{P}(X)$ compatible with $\succsim$ such that $\exists C \subseteq X \setminus \{b\}$ verifying that $C, C \cup \{b\} \in \succsim'$. When applied to $\succsim'$, ordinal Banzhaf satisfies level 1 of precision but not level 2, $\forall \succsim \in \{\succsim^L_{min}, \succsim^L_{max}\}$.*

Remark. If there is no coalition $C \subseteq X \setminus \{b\}$ verifying that $C, C \cup \{b\} \in \succsim'$, ordinal Banzhaf may not satisfy level 1 of precision from lexicographical order-based orderings.

Proposition 8. *Given a ranking $>_X$ over X from which we determine $\succsim$ using one of the introduced order lifting rules, let $\succsim'$ be a partial preorder over $\mathcal{P}(X)$ compatible with $\succsim$. When applied to $\succsim'$, CP-majority satisfies level 2 of precision, $\forall \succsim \in \{\succsim_{mnx}, \succsim_{mxn}, \succsim^L_{min}, \succsim^L_{max}, \succsim_b\}$.*

For all five order lifting solutions, Lexcel does not verify even level 0 of precision, while CP-majority satisfies level 2, which indicates that CP-majority should be the recommended method in every scenario. We note that ordinal Banzhaf satisfies level 1 of precision, although, for the lexicographical order-based rules, with a hypothesis on the presence of at least one coalition. Note that this hypothesis is not very limiting, however, and ordinal Banzhaf thus remains a better option than Lexcel.

4.2 Experimental results

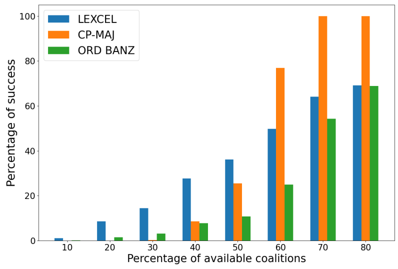
We proceed to several experiments over synthetic data to test each method's performance in practice. For each order lifting method, we randomly generate a ranking over items, from which we infer a ranking over coalitions. We remove a varying number of coalitions from the latter ranking, so that only a given percentage of the coalitions (among the 2^n) remains available: this constitutes the partial ranking given as input to each social ranking method. We then observe the output of each social ranking method and compare it to the correct initial ranking over items.

Recovering the initial order We start by studying the impact of missing coalitions on each social ranking method's ability to recover the correct initial order. In this section, we count as a success each occurrence in which a social ranking method has managed to recover the initial ranking over items such that errors *(i.e.* $a >_X b$*, but it is determined that* $b >_m a$*, with* m *the social ranking method)* or imprecision *(i.e. there exists a strict preference between* a *and* b*, but the social ranking method* m *finds that* $a =_m b$*)* are avoided.

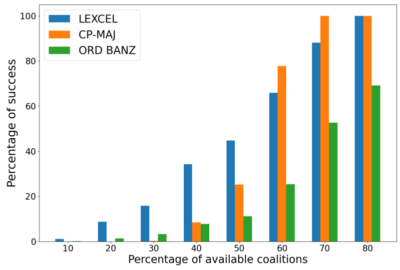
Figure 2 displays experimental results for a population of size 4; Figure 3 for a population of size 8. We already notice changes in conclusions: lexcel performs particularly well over a small population when a relatively small percentage of coalitions is available, while CP-majority appears to be systematically more efficient over a larger population. This is explained by the fact that, the smaller the population, the smaller the number of coalitions studied when considering a given percentage p of $\mathcal{P}(X)$. As we will observe in the next figures, CP-majority will often be unable to distinguish between elements when faced with fewer coalitions, while lexcel has a better chance to have access to informative top coalitions when $\mathcal{P}(X)$ is smaller, regardless of p. Naturally, as the total number of coalitions in the powerset increases, we observe that it is harder for all social ranking methods to determine the correct order based on only a small percentage of all the coalitions. This result differs when it comes to the Borda-sum ordering *(cf. Figure 3e)*, which succeeds in retrieving the correct initial order for over half of the runs with as little as 20% of all the coalitions. This result is due to the weakly additive nature of Borda-sum, to which CP-majority reacts rather well. Additionally, these experimental results are also coherent with our theoretical findings, displaying ordinal Banzhaf's poor reaction to preferences extended using Borda-sum *(cf. Figures 2e and 3e)*. What's more, it appears that ordinal Banzhaf is always suboptimal, regardless of the type of lifting rule considered.

Finally, we observe that, over a small population, the performance of each social ranking method (excepted ordinal Banzhaf for Borda-sum) remains somewhat similar over each lifting rule. That is not the case, however, when the size of the population increases, as Figure 3a and 3b highlight an impossibility for any social ranking method to find the correct initial order with fewer than 60% of all coalitions if the preferences have been obtained via minmax-based orderings.

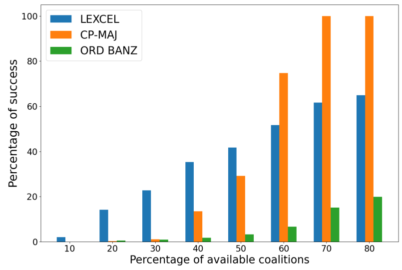
To get a better sense of the difference in performance between all three social ranking methods, we then focus on what we will refer to as the *weak Kendall-Tau distance* between the ranking uncovered by a social ranking method and the correct initial order. It is defined over two rankings $\geq_1$ and $\geq_2$ as the number of pairs $(a, b) \in X$ such that $(a, b) \in \geq_1$ but $(a, b) \notin \geq_2$. This weakened version of the classic Kendall-Tau distance will allow us to highlight a key characteristic of the results of CP-majority.



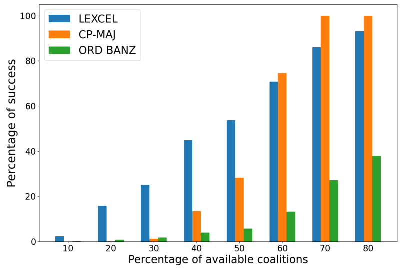
(a) Minmax ordering



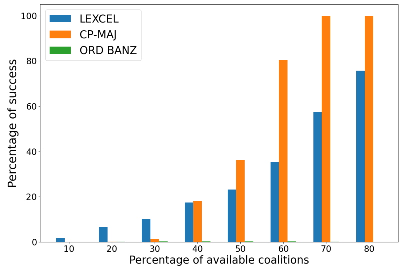
(b) Maxmin ordering



(c) Leximin ordering



(d) Leximax ordering



(e) Borda-sum ordering

Figure 2: Percentage of runs for which each social ranking method finds the correct initial order (over 10000 runs with $|X| = 4$) for each lifting rule

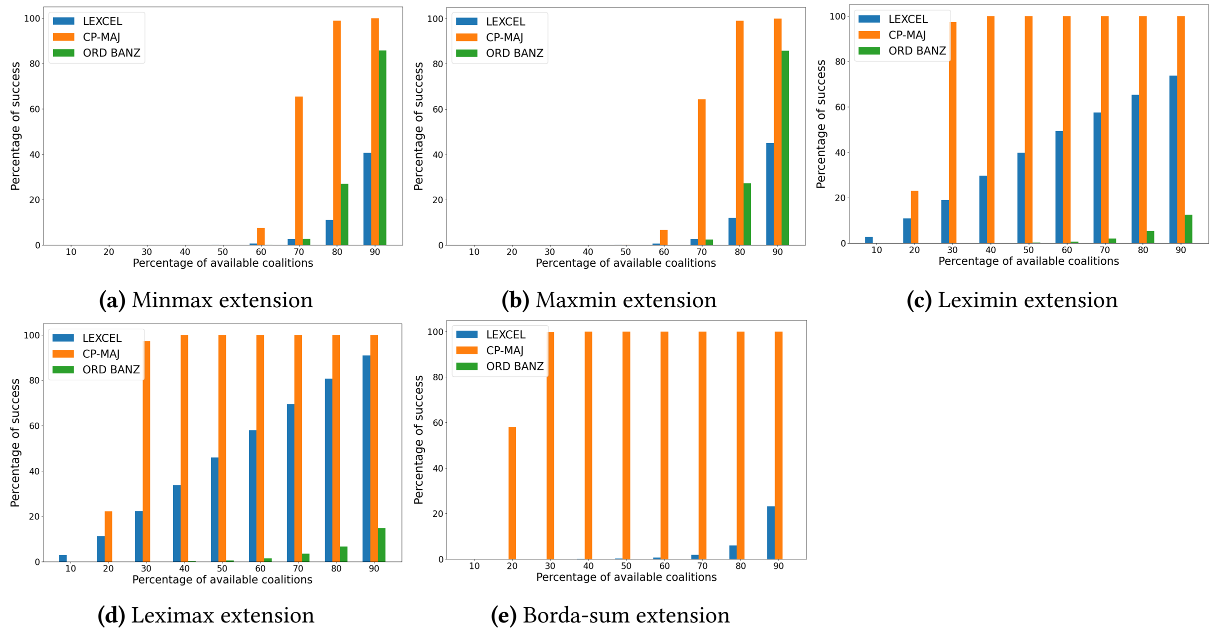
A study of weak Kendall-Tau distance to the correct ranking We now focus on measuring the distance between the output ranking over items and the initial ranking.

Figure 4 displays experimental results of each social ranking method for a population of size 8. A surprising observation relates to CP-majority, for which we have shown that no incorrect preference may be returned, *i.e.* if $a >_X b$, then CP-majority cannot return that $b >_{CP} a$. Yet, for every lifting rule, and every population size, we observe that CP-majority systematically displays the worst weak Kendall-Tau distance to the truth when only a few coalitions are available. This is explained by the fact that the weak Kendall-Tau distance considers equivalence to be a difference: if a is preferred to b in one order, but another order considers a to be equivalent to b, then the pair (a, b) is counted by the weak Kendall-Tau method. These figures therefore show that CP-majority returns many equivalences, particularly when only a small percentage of coalitions is available.

While this seems a reasonable phenomenon to observe, as the more information is lacking, the harder it will be to determine a strict preference with certainty, we note that lexcel maintains what appears to be a low weak Kendall-Tau distance, providing distinct preferences between the items.

This leads to a dual conclusion, relative to one's position regarding equivalence and exactness of results. Indeed, if one favours the robustness of the resulting ranking over its guaranteed expressiveness, then CP-majority appears to be the best method. If, however, one wishes to get clear preferences, with some tolerance for possible errors, then it would appear best to favour lexcel - at least when only a small percentage of all coalitions is available.

Combining social ranking methods It is worth noting that some pairs of items over which lexcel returns an incorrect strict preference may be such that their strict preference relation is returned correctly by CP-majority. This leads us to introduce another method altogether, which combines the guarantee of CP-majority's strict preference results and lexcel's low weak Kendall-Tau distance. We choose to combine the two rules, starting from the ranking determined using CP-majority, then using lexcel as a tie-breaker for any pair of items determined equivalent. We also study a version where ordinal Banzhaf is used as a tie-breaker, to compare the performance of both combinations.

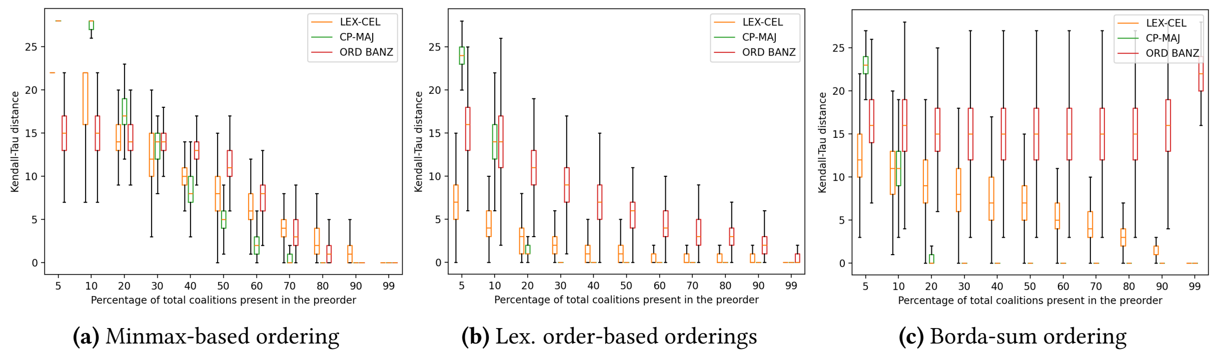


(a) Minmax extension **(b)** Maxmin extension **(c)** Leximin extension

(d) Leximax extension **(e)** Borda-sum extension

Figure 3: Number of times where each social ranking method finds the correct initial order (over 10000 runs with $|X| = 8$) for each extension

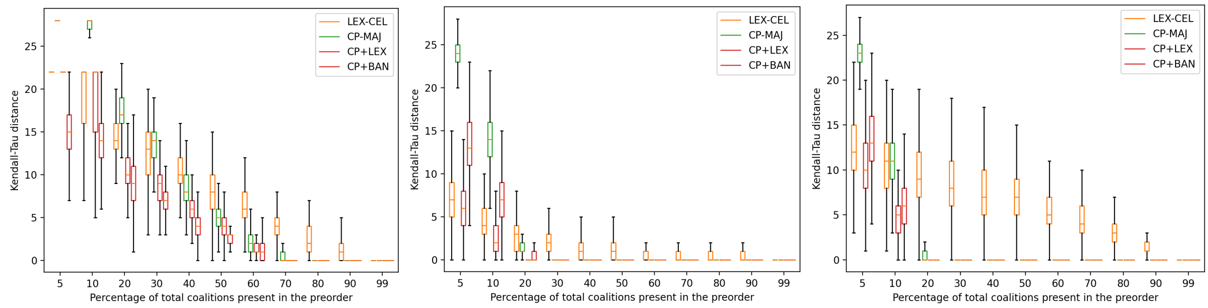
Figure 5 displays experimental results of each combination of social ranking methods for a population of size 8. Its subfigure 5a seems to indicate that the combination of CP-majority and Banzhaf is optimal under minmax-based orderings, while the combination of CP-majority and lexcel is the optimal method for all other lifting rules. We also observe that, in any case, both combinations yield results with a lower weak Kendall-Tau distance to the truth than any social ranking method by itself. This reinforces our assumption that the tie-breaking combination improves the overall result, and leads us to conclude that it is always better to opt for one of these combinations.



(a) Minmax-based ordering **(b)** Lex. order-based orderings **(c)** Borda-sum ordering

Figure 4: Weak Kendall-Tau distance to the initial order (over 10000 runs with $|X| = 8$) for each method and each family of lifting rules

All in all, while it appears that CP-majority yields the overall best results by itself, it seems to display a weakness in its propensity to return equivalence when faced with insufficient information. While this may be a reasonable weakness, as one may consider that, in situations of severe lack of information, it is reasonable to answer that strict preferences over items are not retrievable, it can also present as a drawback if one prefers to obtain strict preferences with a tolerance for some errors. In the latter case, we observe that using another social ranking method as a tie-breaker on CP-majority's results allow for a more decisive resulting ranking *(in that it will contain more strict preferences)* while maintaining the lowest weak Kendall-Tau distance to the truth, therefore guaranteeing the highest possible level of correctness of strict preferences among all other methods. Note, however, that adding other social



(a) Minmax-based orderings **(b)** Lex. order-based orderings **(c)** Borda-sum ordering

Figure 5: Weak Kendall-Tau distance to the initial order (over 10000 runs with $|X| = 8$) for each combination of methods and each family of lifting rules

ranking methods to CP-majority as tie-breakers weakens its preciseness level. Indeed, as lexcel and ordinal Banzhaf can return incorrect orders, *Preservation of pairwise preferences* may be violated when they are used to decide the preference relation between elements deemed equivalent by CP-majority.

Finally, we consider a second variation of the social ranking problem, in which the input (partial) ranking is only over coalitions of a same size k.

5 On the impact of structured preferences: the case of *k*-sized coalitions

In real-life applications, it is often the case that only groups of a certain size are studied and compared. Sports teams, for instance, are generally of a fixed size, and it is not particularly relevant or possible to consider teams of smaller or larger sizes. [6] namely studies the application of social ranking rules to a real-life dataset of NBA games between teams of 5 players, and highlights the challenges caused by the absence of coalitions of any other size, as well as missing comparisons between some pairs of teams.

In this section, we investigate the ability of social ranking rules to recover the correct order over items based on a (partial) preorder over coalitions of a given size k, when this preorder is supposed to have been obtained using one of the five introduced order lifting solutions. Given preferences $\succsim$ over $\mathcal{P}(X)$, we will denote by $X_k = \{C \subseteq X \mid |C| = k\}$ the set of coalitions of a given size k, and by $\succsim^k$ the preferences restricted to the set X_k. Note that $\succsim^k$ is total over X_k.

From this point on, we only consider (partial) preorders over k-sized coalitions in a population $X = \{1 \ldots, n\}$, for $k \in \{2, \ldots, n-1\}$, as the reader can easily convince themselves that considering $k = 1$ renders the problem trivial, and $k = n$ impossible. Finally, by construction, ordinal Banzhaf is unable to evaluate any item using only coalitions of a same size k, and will therefore always return as a result that all elements are equivalent (because all elements have a score 0), which makes it irrelevant in this particular setting, and we will therefore only consider lexcel and CP-majority going forward.

5.1 Theoretical results

We investigate each social ranking method's ability to recover the initial order over items, based on the ranking obtained using order lifting solutions only over the set X_k.

Proposition 9. *When $k \leq \lceil \frac{n}{2} \rceil$, lexcel recovers the initial order $>_X$ from the total preorder $\succsim$, $\forall \succsim \in \{\succsim^k_{mnx}, \succsim^k_{mxn}\}$.*

Proposition 10. *When $k \leq \lceil \frac{n}{2} \rceil$, CP-majority recovers the initial order $>_X$ from the total preorder $\succsim$, $\forall \succsim \in \{\succsim^k_{mnx}, \succsim^k_{mxn}\}$.*

Proposition 11. *Lexcel always recovers the initial order $>_X$ from the preorder $\succsim, \forall \succsim \in \{(\succsim_{min}^L)^k, (\succsim_{max}^L)^k), \succsim_b^k\}$.*

Proposition 12. *CP-majority always recovers the initial order $>_X$ from the preorder $\succsim, \forall \succsim \in \{(\succsim_{min}^L)^k, (\succsim_{max}^L)^k), \succsim_b^k\}$.*

We observe that Maxmin-based extensions are more restrictive, as they only allow for the recovery of the initial preferences when k is at most $\lceil \frac{n}{2} \rceil$, while other order lifting solutions allow for each social ranking method to recover the correct order over items, regardless of the value of $k \in \{2, \dots, n-2\}$. In the next subsection, we proceed to experiments to observe the impact of missing coalitions on the precision of the recovery of the initial order, when considering a ranking over X^k, for a given size k.

5.2 Experimental results

We proceed to experiments on the impact of missing information when considering only k-sized coalitions, for each applicable social ranking rule (as well as their combination), and for different population sizes[2]. Our results seem to indicate that, the closer k gets to $\frac{n}{2}$, the easier it gets for each method to determine the correct ranking over items, even from only partial preferences over the coalitions. That is particularly true for the minmax and maxmin extensions (respectively rules 1 and 2 in the tables). This may be explained by the fact that, as k gets closer to $\frac{n}{2}$, the number of coalitions of size k (*i.e.* $\binom{n}{k}$) increases. Therefore, as there are more coalitions to choose from, 10% of that set will represent a larger amount of useable coalitions. We also note that, while lexcel seems to be (slightly) more efficient with very little information, CP-majority rapidly outperforms it as the amount of accessible coalitions increases, but that the combination of CP and lexcel proves to be the most efficient approach.

It appears that the Borda-sum orderings lead to far better results than any other rule. On the other hand, we can observe that, while on small populations, the remaining four lifting rules seem to share similar results, the lexicographical order-based rules seem to lose efficiency as the size of the population grows, which is a different result than observed in section 4.2. Finally, we also observe that all lifting rules present poor results when $k = n - 1$, a scenario in which X^k is small, and missing information about coalitions is therefore particularly costly.

6 Conclusions and perspectives

We have determined in this work that, while the order lifting problem is not the inverse of the social ranking problem, the latter allows for the recovery of initial preferences over individuals, when they have been extended via preference lifting methods. Considering scenarios in which some information about the coalition ranking may be missing, we observed that CP-majority has some theoretical appealing properties, as well as encouraging experimental results, which leads us to believe that it is the best single method to recover individual preferences. However, we observe that combining methods increases efficiency: namely, using lexcel or ordinal Banzhaf as tie-breakers for the results yielded by CP-majority leads to better results than any social ranking method by itself.

While we have selected several methods for the order lifting and the social ranking problems, there exist many more: the order lifting and social ranking methods studied in this paper correspond to specific archetypes of decision makers, but others may allow for the description of different approaches to determining preferences over subsets. We therefore believe it to be of interest to see if our observations and results hold for other solutions. In particular, some approaches to preference modeling allow for the integration of possible interactions between items [8, 9, 18], which is a notion present in the social ranking problem, but not in order lifting. We believe that extending this work to consider such preference models to derive an order over coalitions from preferences over items could prove interesting.

[2]Tables detailing experimental results for varying population sizes are provided in the Appendix

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- Appendix -

3 From Social Ranking to Order Lifting, and back

3.1 Order Lifting applied to Social Ranking

Proposition 1. *Given a population of elements X, and let $\geq_m$ be the preferences over X obtained using a social ranking method $m \in \{lex, CP, Banz\}$ on given preferences $\succsim$ over $\mathcal{P}(X)$. No order lifting rule may systematically return the correct preferences on $\mathcal{P}(X)$ from $\geq_m$.*

Proof. A social ranking rule is a function taking as input the powerset of a given population X, and returning as output a ranking over the items in X. On the other hand, an order lifting rule is a function taking as input a ranking over X, and returning as output a ranking over its powerset $\mathcal{P}(X)$.

Since the cardinality of the set of rankings on items is lower than the cardinality of the set of rankings on coalitions, no mapping from the former to the latter may be surjective. This implies that there can be no bijection between the two and therefore no inverse function: no order lifting rule may systematically return the correct ranking over coalitions from the ranking over items. □

In particular, in the following counter-examples we present instances where each of the introduced order lifting rules returns an incorrect order over coalitions from preferences over items determined by each social ranking rule:

Example 1.
1. The rankings $\succsim_{mnx}$, $\succsim_{mxn}$, $\succsim^L_{min}$, $\succsim^L_{max}$ and $\succsim_b$ obtained from $\geq_{lex}$ may return preferences on $\mathcal{P}(X)$ different than $\succsim$.

Let $X = \{1, 2, 3\}$, and given the preorder $\succsim$ over $\mathcal{P}(X)$ such that

$$3 \succ 12 \succ 13 \succ 1 \succ 2 \succ 123 \succ 23.$$

From $\succsim$, lexcel will determine that $3 >_{lex} 1 >_{lex} 2$.
From $\geq_{lex}$, using the order lifting solutions, we obtain the following preorders over $\mathcal{P}(X)$:

- *Minmax:*

$$3 \succ_{mnx} 13 \succ_{mnx} 1 \succ_{mnx} 123 \sim_{mnx} 23 \succ_{mnx} 12 \succ_{mnx} 2,$$

- *Maxmin:*

$$3 \succ_{mxn} 13 \succ_{mxn} 23 \sim_{mxn} 123 \succ_{mxn} 1 \succ_{mxn} 12 \succ_{mxn} 2,$$

- *Leximin:*

$$3 \succ^L_{min} 13 \succ^L_{min} 1 \succ^L_{min} 23 \succ^L_{min} 123 \succ^L_{min} 12 \succ^L_{min} 2,$$

- *Leximax:*

$$3 \succ^L_{max} 13 \succ^L_{max} 123 \succ^L_{max} 23 \succ^L_{max} 1 \succ^L_{max} 12 \succ^L_{max} 2.$$

All of these rankings are different from $\succsim$.

2. The rankings $\succsim_{mnx}$, $\succsim_{mxn}$, $\succsim^L_{min}$, $\succsim^L_{max}$ and $\succsim_b$ obtained from $\geq_{CP}$ may return preferences on $\mathcal{P}(X)$ different than $\succsim$.

Let $X = \{1, 2, 3\}$, and given the preorder $\succsim$ over $\mathcal{P}(X)$ such that

$$23 \sim 13 \succ 123 \succ 3 \sim 1 \succ 12 \succ 2.$$

From $\succsim$, CP-majority will determine that $3 >_{CP} 1 >_{CP} 2$.

From $\geq_{CP}$, using the order lifting solutions, we obtain the same preorders than for 1., all of which are different from $\succsim$.

3. The rankings $\succsim_{mnx}$, $\succsim_{mxn}$, $\succsim^L_{min}$, $\succsim^L_{max}$ and $\succsim_b$ obtained from $\geq_{Banz}$ may return preferences on $\mathcal{P}(X)$ different than $\succsim$.

Let $X = \{1, 2, 3\}$, and given the preorder $\succsim$ over $\mathcal{P}(X)$ such that

$$12 \succ 3 \succ \emptyset \succ 13 \succ 123 \succ 23 \succ 2 \succ 1.$$

From $\succsim$, ordinal Banzhaf will determine that $3 >_{Banz} 1 >_{Banz} 2$.

From $\geq_{Banz}$, using the order lifting solutions, we obtain the same preorders than for 1., all of which are different from $\succsim$.

3.2 Social Ranking applied to Order Lifting

Proposition 2. *Given a total order $>_X$ over elements in a set X, and let $\succsim$ be the ranking obtained over $\mathcal{P}(X)$ by applying an order lifting solution on $>_X$, i.e. $\succsim \in \{\succsim_{mnx}, \succsim_{mxn}, \succsim^L_{min}, \succsim^L_{max}\}$. Lexcel, CP-majority and ordinal Banzhaf always return the correct order $>_X$ over X from $\succsim$.*

Proof. Let $x, y \in X$ be such that $x >_X y$.

1. Lexcel, CP-majority and Ordinal Banzhaf always return the correct order $>_X$ over X from $\succsim_{mnx}$.

- Lexcel: The comparison between x and y will be determined by the relation between the best non-equivalent coalitions $C_{x,-y}$ *(i.e. such that $x \in C_{x,-y}$ and $y \notin C_{x,-y}$)* and $C_{y,-x}$. By construction, any coalition $C_{x,-y}$ must be such that its minimal element is at best x; similarly, any coalition $C_{y,-x}$ must be of minimal element at best y. Since $x >_X y$, there will exist no coalition containing y and not x which will be preferred to a coalition containing x and not y, and lexcel will therefore determine that $x >_{lex} y$.

- CP-majority: In comparing x and y, CP-majority will consider every coalition $S \subseteq X \setminus \{x, y\}$ and compare $S \cup \{x\}$ to $S \cup \{y\}$. As coalitions are ranked according to their minimal element (and their maximal if there is a tie), any coalition S containing at least one element preferred to x and one to which y is preferred will lead to an indifference between $S \cup \{x\}$ and $S \cup \{y\}$ *(as they will have identical extreme elements)*, and will thus be irrelevant to the distinction between x and y. We therefore consider the two possible scenarios in which CP-majority will be able to observe a strict preference:

  1. if no element in S is preferred to x, then $\min(S \cup \{x\}) = \min(S)$ and $\min(S \cup \{y\}) = \min\{\min(S), y\}$. Since $x >_X y$, and, by hypothesis, $x >_X \min(S)$, it must hold that $S \cup \{x\} \succ_{mnx} S \cup \{y\}$.
  2. if no element in S is ranked lower than y, then $\min(S \cup \{x\}) = \min\{\min(S), x\}$ and $\min(S \cup \{y\}) = y$. Since $x >_X y$ and, by hypothesis, $\min(S) >_X y$, it must hold that $S \cup \{x\} \succ_{mnx} S \cup \{y\}$.

  In any case, any coalition $S \subseteq X \setminus \{x, y\}$ will lead to the observation that $S \cup \{x\} \succsim_{mnx} S \cup \{y\}$, with at least one coalition $S^* \subseteq X \setminus \{x, y\}$ verifying that $S^* \cup \{x\} \succ_{mnx} S^* \cup \{y\}$. CP-majority will therefore determine that $x >_{CP} y$.

- Ordinal Banzhaf: To compare x and y, ordinal Banzhaf will compute their scores based on their marginal contributions to coalitions. By construction, $S \cup \{x\} \succ_{mnx} S$ iff $x >_X \max(S)$, therefore any coalition S composed only of elements strictly worse than x will see its performance improve when x is added to it; similarly, $S \succ_{mnx} S \cup \{x\}$ iff $\min(S) >_X x$, therefore any coalition S composed only of elements strictly better than x will see its performance worsen when x is added to it. Even in the worst-case scenario, in which x and y are directly adjacent in $>_X$, we know that $u_x^{+,\succsim_{mnx}} > u_y^{+,\succsim_{mnx}}$, and $u_x^{-,\succsim_{mnx}} < u_y^{-,\succsim_{mnx}}$, therefore x will have a greater score than y, and ordinal Banzhaf will rank it before y.

2. Lexcel, CP-majority and Ordinal Banzhaf always return the correct order $>_X$ over X from $\succsim_{mxn}$.

- Lexcel: The comparison between x and y will be determined by the relation between the best non-equivalent coalitions $C_{x,-y}$ *(i.e. such that $x \in C_{x,-y}$ and $y \notin C_{x,-y}$)* and $C_{y,-x}$. By construction, any coalition $C_{x,-y}$ must contain the best element and x (or just x if it is the best element) - note that it may also contain any element $a \in X \setminus \{x, y\}$ such that $a >_X x$; similarly, any of coalition $C_{y,-x}$ must contain the best element (or second best if x is the best) and y - note that it may, too, contain any element $a \in X \setminus \{x, y\}$ such that $a >_X y$. Therefore $\max(C_{x,-y}) = \max(X)$ and $\max(C_{y,-x}) = \max(X \setminus \{x\})$. If $\max(X) = \max(X \setminus \{x\})$, then the minimal element will be used as a tie-breaker, with $\min(C_{x,-y}) = x$ and $\min(C_{y,-x}) = y$. From our hypothesis, it must then hold that $C_{x,-y} \succ C_{y,-x}$. Lexcel will therefore determine that $x >_{lex} y$.

- CP-majority: In comparing x and y, CP-majority will consider every coalition $S \subseteq X \setminus \{x, y\}$ and compare $S \cup \{x\}$ to $S \cup \{y\}$. As coalitions are ranked according to their maximal element (and their minimal if there is a tie), any coalition S containing at least one element worse than y and one preferred to x will lead to an indifference between $S \cup \{x\}$ and $S \cup \{y\}$ *(as they will have identical extreme elements)*, and will thus be irrelevant to the distinction between x and y. We therefore consider the two possible scenarios in which CP-majority will be able to observe a strict preference:

  1. if no element in S is preferred to x, then $\max(S \cup \{x\}) = x$ and $\max(S \cup \{y\}) = \max\{\max(S), y\}$. Since $x >_X y$, and, by hypothesis, $x >_X \max(S)$, it must hold that $S \cup \{x\} \succ_{mxn} S \cup \{y\}$.
  2. if no element in S is ranked lower than y, then if $\max(S \cup \{x\}) = \max(S \cup \{y\})$, we compare $\min(S \cup \{x\}) = \min\{\min(S), x\}$ to $\min(S \cup \{x\}) = y$ and, since $x >_X y$ and $\min(S) >_X y$, it must hold that $S \cup \{x\} \succ_{mxn} S \cup \{y\}$; if $\max(S \cup \{x\}) \neq \max(S \cup \{y\})$, then we must be in the previous scenario, and it must hold that $S \cup \{x\} \succ_{mxn} S \cup \{y\}$.

  In any case, any coalition $S \subseteq X \setminus \{x, y\}$ will lead to the observation that $S \cup \{x\} \succsim_{mxn} S \cup \{y\}$, with at least one coalition $S^* \subseteq X \setminus \{x, y\}$ verifying that $S^* \cup \{x\} \succ_{mxn} S^* \cup \{y\}$. CP-majority will therefore determine that $x >_{CP} y$.

- Ordinal Banzhaf: To compare x and y, ordinal Banzhaf will compute their scores based on their marginal contributions to coalitions. By construction, $S \cup \{x\} \succ_{mxn} S$ iff $x >_X \max(S)$, therefore any coalition S composed only of elements strictly worse than x will see its performance improve when x is added to it; similarly, $S \succ_{mxn} S \cup \{x\}$ iff $\min(S) >_X x$, therefore any coalition S composed only of elements strictly better than x will see its performance worsen when x is added to it. Even in the worst-case scenario, in which x and y are directly adjacent in $>_X$, we know that $u_x^{+,\succsim_{mxn}} > u_y^{+,\succsim_{mxn}}$, and $u_x^{-,\succsim_{mxn}} < u_y^{-,\succsim_{mxn}}$, therefore x will have a greater score than y, and ordinal Banzhaf will rank it before y.

3. Lexcel, CP-majority and Ordinal Banzhaf always return the correct order $>_X$ over X from $\succsim_{min}^L$.

- Lexcel:
  Lemma: *Let $x, y \in X$ be such that $x >_X y$, and let C be the most preferred coalition in $\succsim_{min}^L$ verifying that $y \in C$ and $x \notin C$. There exists a coalition C' verifying that $x \in C', y \notin C'$ and $C' \succ_{min}^L C$.*
  Proof: *Let $S = C \setminus \{y\}$, with $|S| = k$, and consider $C' = S \cup \{x\}$. Since C' and C differ by only one element, $\exists i \leq k+1$ st $\forall j < i, c'_{\sigma_{C'}(j)} = c_{\sigma_C(j)} = s_{\sigma_S(j)}$ but $c'_{\sigma_{C'}(i)} \neq c_{\sigma_C(i)}$. By construction of σ, it must then hold that $c_{\sigma_C(i)} = y$ and $c'_{\sigma_{C'}(i)} = \min\{x, s_{\sigma_S(i)}\}$. Since $x >_X y$ and $s_{\sigma_S(i)} >_X y$ (because y has been inserted before $s_{\sigma_S(i)}$), it must hold that $c'_{\sigma_{C'}(i)} >_X c_{\sigma_C(i)}$, and therefore $C' \succ_{min}^L C$.*
  Lexcel will consequently determine that $x >_{lex} y$.

- CP-majority: Let $S \subseteq X \setminus \{x, y\}$ such that $|S| = k$. When comparing $A = S \cup \{x\}$ to $B = S \cup \{y\}$, $\exists i \leq k+1$ st $\forall j < i, a_{\sigma_A(j)} = b_{\sigma_B(j)} = s_{\sigma_S(j)}$ but $a_{\sigma_A(i)} \neq b_{\sigma_B(i)}$. By construction of σ, it must then hold that $b_{\sigma_B(i)} = y <_X a_{\sigma_A(i)} = \min\{x, s_{\sigma_S(i)}\}$. Therefore $S \cup \{x\} \succ_{min}^L S \cup \{y\}$, which means that CP-majority will find that $x >_{CP} y$.

- Ordinal Banzhaf: Let $S \subseteq X \setminus \{x\}$, and denote by A the set $S \cup \{x\}$. If $x >_X \max(S)$, then by construction of σ, it must hold that $\forall i \in \{1, \ldots, |S|\}, s_{\sigma_S(i)} = a_{\sigma_A(i)}$ which, by definition, indicates that $A \succ_{min}^L S$ and will increase the score of x; if, however, $x <_X \max(S)$, then $\exists j \in \{1, \ldots, |S|\}$ such that $\forall i < j, s_{\sigma_S(i)} = a_{\sigma_A(i)}$ but $s_{\sigma_S(j)} \neq a_{\sigma_A(j)}$ and by construction, it must hold that $s_{\sigma_S(i)} >_X a_{\sigma_A(i)} = x$, therefore $S \succ_{min}^L A$. This means that coalitions containing only worse elements than x increase its

score; any other coalition will decrease it.
However, since $x >_X y$, and let x^- denote the number of elements to which x is preferred, we know that $x^- > y^-$, therefore $2^{x^-} > 2^{y^-}$ coalitions will increase the score of each compared element and $2^{n-1} - 2^{x^-} < 2^{n-1} - 2^{y^-}$ will decrease it, which means that $Score_{Banz}(x) > Score_{Banz}(y)$. In each case, ordinal Banzhaf will determine that $x >_{Banz} y$.

4. *Lexcel, CP-majority and Ordinal Banzhaf always return the correct order $>_X$ over X from $\succsim^L_{max}$.*

- Lexcel:
  Lemma: Let $x, y \in X$ be such that $x >_X y$, and let C be the most preferred coalition in $\succsim^L_{max}$ verifying that $y \in C$ and $x \notin C$. There exists a coalition C' verifying that $x \in C', y \notin C'$ and $C' \succ^L_{max} C$.
  Proof: Let $S = C \setminus \{y\}$, with $|S| = k$, and consider $C' = S \cup \{x\}$. Since C' and C differ by only one element, $\exists i \leq k+1$ st $\forall j < i, c'_{\rho_{C'}(j)} = c_{\rho_C(j)} = s_{\rho_S(j)}$ but $c'_{\rho_{C'}(i)} \neq c_{\rho_C(i)}$. By construction of ρ, it must then hold that $c'_{\rho_{C'}(i)} = x >_X c_{\rho_C(i)} = \max\{y, s_{\rho_S(i)}\}$. Since $x >_X y$ and $x >_X s_{\rho_S(i)}$ (because x has been inserted before $s_{\rho_S(i)}$), it must hold that $C' \succ^L_{max} C$.
  Therefore, lexcel will determine that $x >_{lex} y$.

- CP-majority: Let $S \subseteq X \setminus \{x, y\}$ such that $|S| = k$. When comparing $A = S \cup \{x\}$ to $B = S \cup \{y\}$, $\exists i \leq k+1$ st $\forall j < i, a_{\rho_A(j)} = b_{\rho_B(j)} = s_{\rho_S(j)}$ but $a_{\rho_A(i)} \neq b_{\rho_B(i)}$. By construction of ρ, it must then hold that $a_{\rho_A(i)} = x >_X b_{\rho_B(i)} = \max\{y, s_{\rho_S(i)}\}$. Therefore $S \cup \{x\} \succ^L_{max} S \cup \{y\}$, which means that CP-majority will find that $x >_{CP} y$.

- Ordinal Banzhaf: Let $S \subseteq X \setminus \{x\}$, and denote by A the set $S \cup \{x\}$. If $x >_X \min(S)$, then $\exists j \in \{1, \ldots, |S|\}$ st $\forall i < j, s_{\rho_S(i)} = a_{\rho_A(i)}$ but $s_{\rho_S(j)} \neq a_{\rho_A(j)}$ and, by construction of ρ, it must hold that $s_{\rho_S(j)} <_X a_{\rho_A(j)} = x$ because x was inserted before $s_{\rho_S(j)}$, therefore $A \succ^L_{max} S$; if, however, $x <_X \min(S)$, then by construction, $\forall i \in \{1, \ldots, |S|\}, s_{\rho_S(i)} = a_{\rho_A(i)}$, which indicates that $S \succ^L_{max} A$. This means that coalitions containing only better elements than x will decrease its score; any other coalition will increase it.
  However, since $x >_X y$, and let x^+ denote the number of elements preferred (or equivalent) to x, we know that $x^+ < y^+$, therefore $2^{x^+} < 2^{y^+}$ coalitions will decrease the score of each compared element, but $2^{n-1} - 2^{x^+} > 2^{n-1} - 2^{y^+}$ will increase it, which means that $Score_{Banz}(x) > Score_{Banz}(y)$. In each case, ordinal Banzhaf will determine that $x >_{Banz} y$.

$\square$

Proposition 3. *Given a total order $>_X$ over items in X, and let $\succsim_b$ be the preorder obtained over $\mathcal{P}(X)$ by applying the Borda-sum extension rule on $>_X$ using the weight vector $b = (n-1, \ldots, 1, 0)$. Lexcel and CP-majority always return the correct order over X from $\succsim_b$.*

Proof. Let $x, y \in X$ be such that $x >_X y$, and let w be the worst element in X according to $>_X$.

- Lexcel: The comparison between x and y will be determined by the relation between the best non-equivalent coalitions $C_{x,-y}$ *(i.e. such that $x \in C_{x,-y}$ and $y \notin C_{x,-y}$)* and $C_{y,-x}$. By construction of the Borda-sum ordering, $C_{x,-y}$ will be $X \setminus \{y\}$ *(equivalent to $X \setminus \{y, w\}$)*, and $C_{y,-x}$ will be $X \setminus \{x\}$ *(equivalent to $X \setminus \{x, w\}$)*. Without loss of generality, it thus holds that $w_b(C_{x,-y}) = w_b(X \setminus \{x, y\}) + b_{pos(x)}$ and $w_b(C_{y,-x}) = w_b(X \setminus \{x, y\}) + b_{pos(y)}$. Since, by hypothesis, $x >_X y$, then it must hold that $b_{pos(x)} > b_{pos(y)}$, hence $w_b(C_{x,-y}) > w_b(C_{y,-x})$. This means that $C_{x,-y} \succ_b C_{y,-x}$, and lexcel will therefore determine that $x >_{lex} y$.

- CP-majority: In comparing x and y, CP-majority will consider every coalition $S \subseteq X \setminus \{x, y\}$ and compare $S \cup \{x\}$ to $S \cup \{y\}$. By construction, $w_b(S \cup \{x\}) = w_b(S) + b_{pos(x)}$ and $w_b(S \cup \{y\}) = w_b(S) + b_{pos(y)}$. Since, by hypothesis, $x >_X y$, then it must hold that $b_{pos(x)} > b_{pos(y)}$, hence $w_b(S \cup \{x\}) > w_b(S \cup \{y\})$. This means that, $\forall S \subseteq X \setminus \{x, y\}$, $S \cup \{x\} \succ_b S \cup \{y\}$, and CP-majority will therefore determine that $x >_{CP} y$.

$\square$

Proposition 4. *Given a total order $>_X$ over items in X, and let $\succsim_b$ be the preorder obtained over $\mathcal{P}(X)$ by applying the Borda-sum extension rule on $>_X$ using the weight vector $b = (n-1, \ldots, 1, 0)$. Ordinal Banzhaf will never return the correct order over X from $\succsim_b$.*

Proof. Let $w \in X$ be the worst element in X according to $>_X$. Using the weight vector $b = (n-1, \dots, 1, 0)$, the addition of any element other than w to a coalition systematically improves its Borda-sum score, *i.e.* $\forall x \in X \setminus \{w\}, \forall S \subseteq X \setminus \{x\}, S \cup \{x\} \succ_b S$. Therefore, any element in $X \setminus \{w\}$ will get the same Banzhaf score of 2^{n-1}. As for w, since the weight associated to the worst element by b is 0, then it holds that $\forall S \subseteq X \setminus \{w\}, S \sim_b S \cup \{w\}$, and its Banzhaf score will always be of 0. Therefore, ordinal Banzhaf will always yield the following ranking:

$$\{x \mid x \in X \setminus \{w\}\} >_{Banz} w,$$

regardless of the correct initial ranking $>_X$. $\square$

4 Recovering initial preferences from partial extensions: Social ranking applied to partially-lifted orders

4.1 Theoretical results

- **Level 0:** *(Impossibility of reversal)* the social ranking rule may not return a preorder that is the reverse of the initial one, *i.e.* $>_X \neq \emptyset$ and $\exists x, y \in X$ s.t. $x >_X y$ and the rule m returns that $x \geq_m y$.
- **Level 1:** *(Preservation of the extremes)* the social ranking rule may return incorrect pairwise preferences, but in a worst-case scenario where it would return the worst element as a top element *(i.e. $w \in X$ such that $\forall x \in X \setminus \{w\}$, $x \geq_X w$ is also such that $w \geq_m x$)*, it will also return the best element as top element. Similarly, in a worst-case scenario where it would return the best element as a bottom element *(i.e. $b \in X$ such that $\forall x \in X \setminus \{w\}$, $b \geq_X x$ is also such that $x \geq_m b$)*, it will also return the worst element as bottom element.
- **Level 2:** *(Preservation of pairwise preferences)* let $x, y \in X$ be such that $x >_X y$, the social ranking rule m will never determine that $y >_m x$ *(note that it may however return that $x =_m y$)*.

Proposition 5. *Given a ranking $>_X$ over X from which we determine $\succsim$ using one of the introduced order lifting rules, let $\succsim'$ be a partial preorder over $\mathcal{P}(X)$ compatible with $\succsim$. When applied to $\succsim'$, lexcel does not satisfy level 0 of preciseness, $\forall \succsim \in \{\succsim_{mnx}, \succsim_{mxn}, \succsim^L_{min}, \succsim^L_{max}, \succsim_b\}$.*

Example 2. *Let $X = \{1, 2, 3, 4\}$, and let $1 >_X 2 >_X 3 >_X 4$.*

- *Minmax:* *The preference relation $\succsim_{mnx}$ obtained from $>_X$ is*

$$1 \succ_{mnx} 12 \succ_{mnx} 2 \succ_{mnx} 13 \sim_{mnx} 123 \succ_{mnx} 23 \succ_{mnx} 3 \succ_{mnx} 1234 \sim_{mnx} 134 \sim_{mnx} 124 \sim_{mnx} 14 \succ_{mnx} 234 \sim_{mnx} 24 \succ_{mnx} 34 \succ_{mnx} 4.$$

Consider the partial preorder $\succsim'_{mnx}$ compatible with $\succsim_{mnx}$ and such that $234 \succ'_{mnx} 34 \succ'_{mnx} 4$. Lexcel determines from $\succsim'_{mnx}$ that $4 >_{lex} 3 >_{lex} 2 >_{lex} 1$, which is the reverse order to $>_X$.

- *Maxmin:* *The preference relation $\succsim_{mxn}$ obtained from $>_X$ is*

$$1 \succ_{mxn} 12 \succ_{mxn} 13 \sim_{mxn} 123 \succ_{mxn} 14 \sim_{mxn} 124 \sim_{mxn} 134 \sim_{mxn} 1234 \succ_{mxn} 2 \succ_{mxn} 23 \succ_{mxn} 24 \sim_{mxn} 234 \succ_{mxn} 3 \succ_{mxn} 34 \succ_{mxn} 4$$

Consider the partial preorder $\succsim'_{mxn}$ compatible with $\succsim_{mxn}$ and such that $234 \succ'_{mxn} 34 \succ'_{mxn} 4$. Lexcel determines from $\succsim_{mxn}$ that $4 >_{lex} 3 >_{lex} 2 >_{lex} 1$, which is the reverse order to $>_X$.

- *Leximin:* *The preference relation $\succsim^L_{min}$ obtained from $>_X$ is*

$$1 \succ^L_{min} 12 \succ^L_{min} 2 \succ^L_{min} 13 \succ^L_{min} 123 \succ^L_{min} 23 \succ^L_{min} 3 \succ^L_{min} 14 \succ^L_{min} 124 \succ^L_{min} 24 \succ^L_{min} 134 \succ^L_{min} 1234 \succ^L_{min} 234 \succ^L_{min} 34 \succ^L_{min} 4$$

Consider the partial preorder $\succsim'_{min^L}$ compatible with $\succsim^L_{min}$ and such that $234 \succ'_{min^L} 34 \succ'_{min^L} 4$. Lexcel determines from $\succsim'_{min^L}$ that $4 >_{lex} 3 >_{lex} 2 >_{lex} 1$, which is the reverse order to $>_X$.

- Leximax: *The preference relation $\succsim^L_{max}$ obtained from $>_X$ is*

$$1 \succ^L_{max} 12 \succ^L_{max} 123 \succ^L_{max} 1234 \succ^L_{max} 124 \succ^L_{max} 13 \succ^L_{max} 134 \succ^L_{max} 14 \succ^L_{max} 2 \succ^L_{max} 23 \succ^L_{max} 234 \succ^L_{max} 24 \succ^L_{max} 3 \succ^L_{max} 34 \succ^L_{max} 4$$

Consider the partial preorder $\succsim'_{max^L}$ compatible with $\succsim^L_{max}$ and such that $234 \succ'_{max^L} 34 \succ'_{max^L} 4$. Lexcel determines from $\succsim'_{max^L}$ that $4 >_{lex} 3 >_{lex} 2 >_{lex} 1$, which is the reverse order to $>_X$.

- Borda-sum: *The preference relation $\succsim_b$ obtained from $>_X$ is*

$$1234 \sim_b 123 \succ_b 12 \sim_b 124 \succ_b 13 \sim_b 134 \succ_b 1 \sim_b 14 \sim_b 23 \sim_b 234 \succ_b 2 \sim_b 24 \succ_b 3 \sim_b 34 \succ_b 4 \sim_b \emptyset.$$

Consider the partial preorder $\succsim'_b$ compatible with $\succsim_b$ and such that $234 \succ'_b 34 \succ'_b 4$. Lexcel determines from $\succsim'_b$ that $4 >_{lex} 3 >_{lex} 2 >_{lex} 1$, which is the reverse order to $>_X$.

Proposition 6. *Given a ranking $>_X$ over X from which we determine $\succsim$ using one of the introduced order lifting rules, let $\succsim'$ be a partial preorder over $\mathcal{P}(X)$ compatible with $\succsim$. When applied to $\succsim'$, ordinal Banzhaf satisfies level 1 of preciseness but not level 2, $\forall \succsim \in \{\succsim_{mnx}, \succsim_{mxn}, \succsim^L_{min}\}$.*

Proof. Let $x, y \in X$ be such that $x >_X y$.

1. *When applied to $\succsim'$, ordinal Banzhaf satisfies level 1 of preciseness.*

Let $b \in X$ be the best element, and $w \in X$ be the worst element according to $>_X$, and suppose that w is returned by ordinal Banzhaf as a top element.

- Minmax: By construction of $\succsim_{mnx}$, an element x is improving when added to a coalition $C \subseteq \{a \in X \mid x >_X a\}$, because $\min(C) = \min(C \cup \{x\})$, yet $\max(C \cup \{x\}) >_X \max(C)$, which means that $C \cup \{x\} \succ_{mnx} C$; it is on the other hand detrimental to the performance of any coalition $C' \subseteq \{a \in X \mid a \succ x\}$, because $\min(C') >_X \min(C' \cup \{x\}) = x$, which means that $C' \succ_{mnx} C' \cup \{x\}$. Note that adding x to any other coalition will lead to an equivalence between the coalition and its augmented version.
  Therefore, let x^+ denote the number of elements in X that are preferred to x according to $>_X$; x^- the number of elements to which x is preferred. Any coalition containing exclusively elements of strictly higher rank will count negatively, while any coalition containing exclusively elements of strictly lower rank will count positively. This means that the Banzhaf score of w will always be negative or null *(because it will lower the performance of any coalition it is added to)*, while the Banzhaf score of b will always be positive or null.

  By hypothesis, w is a top element according to ordinal Banzhaf, its score must be at least as good as that of b. This can only happen if $Score_{Banz}(w) = Score_{Banz}(b) = 0$, which means that, $\forall C \subseteq X \setminus \{b\}, C$ or $C \cup \{b\}$ is absent from $\succsim'_{mnx}$ (*i.e.* no interaction is counted for b, for any interaction would have a positive impact). Similarly, this means that, $\forall C \subseteq X \setminus \{w\}, C$ or $C \cup \{w\}$ is absent from $\succsim'_{mnx}$ (*i.e.* no interaction is counted for w, for any interaction would have a negative impact).
  As w is a determined to be a top element by ordinal Banzhaf, then any element of similar score must also be a top element: this includes b.

  Similarly, under the other hypothesis that b is a bottom element according to Banzhaf, its score must be at least as low as that of w. This can only happen if $Score_{Banz}(w) = Score_{Banz}(b) = 0$.
  As b is a determined to be a bottom element by ordinal Banzhaf, then any element of similar score must also be a bottom element: this includes w.

- Maxmin: By construction of $\succsim_{mxn}$, an element x is improving when added to a coalition $C \subseteq \{a \in X \mid x >_X a\}$, because $\max(C) <_X \max(C \cup \{x\})$, which means that $C \cup \{x\} \succ_{mxn} C$; it is on the other hand detrimental to the performance of any coalition $C' \subseteq \{a \in X \mid a \succ x\}$, because $\max(C') = \max(C' \cup \{x\})$, yet $\min(C') >_X \min(C' \cup \{x\})$, which means that $C' \succ_{mxn} C' \cup \{x\}$. Note that adding x to any other coalition will lead to an equivalence between the coalition and its augmented version containing x.
  This puts us in the same setting as the proof for Minmax, therefore we reach the same conclusion: if w is determined to be a top element by ordinal Banzhaf, then b will also be determined a top element. Similarly, if b is determined to be a bottom element by ordinal Banzhaf, then w will also be determined as such.

2. When applied to $\succsim'$, ordinal Banzhaf does not satisfy level 2 of preciseness.
Let $X = \{1, 2, 3, 4\}$ and $>_X$ be such that $1 >_X 2 >_X 3 >_X 4$.

- Minmax: The preorder $\succsim_{mnx}$ derived from $>_X$ is such that

$$1 \succ_{mnx} 12 \succ_{mnx} 2 \succ_{mnx} 13 \sim_{mnx} 123 \succ_{mnx} 23 \succ_{mnx} 3 \succ_{mnx} 14 \sim_{mnx} 124 \sim_{mnx} 134 \sim_{mnx} 1234 \succ_{mnx} 24 \sim_{mnx} 234 \succ_{mnx} 34 \succ_{mnx} 4.$$

  Consider the partial preorder $\succsim'$ compatible with $\succsim_{mnx}$ and such that

$$1 \succ' 12 \succ' 13 \sim' 123 \succ' 23 \succ' 14 \sim' 124 \sim' 134 \sim' 1234 \succ' 234 \succ' 4,$$

  $Score_{Banz}(2) = -1$ but $Score_{Banz}(3) = 0$, therefore ordinal Banzhaf will determine that $3 >_{Banz} 2$ even though $2 >_X 3$.

- Maxmin: The preorder $\succsim_{mxn}$ derived from $>_X$ is such that

$$1 \succ_{mxn} 12 \succ_{mxn} 123 \sim_{mxn} 13 \succ_{mxn} 14 \sim_{mxn} 124 \sim_{mxn} 134 \sim_{mxn} 1234 \succ_{mxn} 2 \succ_{mxn} 23 \succ_{mxn} 24 \sim_{mxn} 234 \succ_{mxn} 3 \succ_{mxn} 34 \succ_{mxn} 4.$$

  Consider the partial preorder $\succsim'$ compatible with $\succsim_{mxn}$ and such that

$$1 \succ' 12 \succ' 123 \succ' 14 \sim' 124 \sim' 134 \sim' 1234 \succ' 23 \succ' 3 \succ' 34 \succ' 4,$$

  $Score_{Banz}(2) = -1$ but $Score_{Banz}(3) = 0$, therefore ordinal Banzhaf will determine that $3 >_{Banz} 2$ even though $2 >_X 3$.

□

Proposition 7. *Given a ranking $>_X$ over X from which we determine $\succsim$ using one of the lexicographical order-based orderings. Let b be the best and w the worst item according to $>_X$, and let $\succsim'$ be a partial preorder over $\mathcal{P}(X)$ compatible with $\succsim$ such that $\exists C \subseteq X \setminus \{b\}$ verifying that $C, C \cup \{b\} \in \succsim'$. When applied to $\succsim'$, ordinal Banzhaf satisfies level 1 of preciseness but not level 2, $\forall \succsim \in \{\succsim^L_{min}, \succsim^L_{max}\}$.*

Proof.

1. When applied to $\succsim'$, ordinal Banzhaf satisfies level 1 of preciseness.
Let $b \in X$ be the best element, and $w \in X$ be the worst element according to $>_X$, and suppose that w is returned by ordinal Banzhaf as a top element.

- Leximin: By construction of $\succsim^L_{min}$, an element x is improving when added to a coalition $C \subseteq X \setminus \{x\}$ such that $x >_X \max(C)$; it is on the other hand detrimental when added to a coalition $C \subseteq X \setminus \{x\}$ such that $\max(C) >_X x$.
  By definition of a best element, there can be no coalition $C \subseteq X \setminus \{b\}$ such that $\max(C) >_X b$, which means that the Banzhaf score of b can never be strictly negative: in the worst case scenario, if it is impossible to find a coalition $C \subseteq X \setminus \{b\}$ such that both C and $C \cup \{b\}$ are in $\succsim'$, the Banzhaf score of b will be null. Similarly, by definition of a worst element, the only coalition $C \subseteq X \setminus \{w\}$ such that $w >_X \max(C)$ is the empty set, which means that the Banzhaf score of w can never be higher than 1: in the best case scenario, if the empty set is the only coalition $C \subseteq X \setminus \{w\}$ such that both C and $C \cup \{w\}$ are in $\succsim'$, the Banzhaf score of w will be 1.
  One of our hypotheses states that w is returned as a top element by ordinal Banzhaf, which may only occur if its Banzhaf score is the best it can possibly be *(in what we have called its best-case scenario)*. Our other hypothesis indicates that there exists at least one coalition $C \subseteq X \setminus \{b\}$ such that C and $C \cup \{b\}$ are present in $\succsim'$. In this case, the Banzhaf score of b is at least of 1, which is w's maximal Banzhaf score.

  Therefore, if w is determined as a top element by ordinal Banzhaf, then it must hold that b is also determined as a top element.

  Similarly, if b is determined to be a bottom element by ordinal Banzhaf, then it must be at its lowest possible score, which we have shown to be w's highest possible score. Therefore, if b is determined as a bottom element by ordinal Banzhaf, it must hold that w is also determined as such.

- Leximax: By construction of $\succsim^L_{max}$, an element x is improving when added to a coalition $C \subseteq X \setminus \{x\}$ such that $x >_X \min(C)$ *(note that the empty set is a special case, as it is preferred to any other coalition).*

  By definition of a best element, there can be no coalition $C \subseteq X \setminus \{b\}, C \neq \emptyset$ such that $\min(C) >_X b$, therefore b is improving to any coalition other than the empty set. Note that, as we are working under the hypothesis that there exists at least one coalition $C \subseteq X \setminus \{b\}$ such that C and $C \cup \{b\}$ are present in $\succsim'$, b's lowest possible Banzhaf score is 0. Similarly, there exists no coalition $C \subseteq X \setminus \{w\}$ such that $\min(C) <_X w$, therefore w's highest possible Banzhaf score is 0, in a best-case scenario where there exists no coalition $C \subseteq X \setminus \{w\}$ such that $C \in \succsim'$ and $C \cup \{w\} \in \succsim'$.

  Therefore, if w is determined as a top element by ordinal Banzhaf, then it must hold that b is also a determined as a top element.

  Similarly, if b is determined to be a bottom element by ordinal Banzhaf, then it must be at its lowest possible score, which we have shown to be w's highest possible score. Therefore, if b is determined as a bottom element by ordinal Banzhaf, it must hold that w is also determined as such.

2. *When applied to $\succsim'$, ordinal Banzhaf does not satisfy level 2 of preciseness.*

Let $X = \{1, 2, 3, 4\}$ and $>_X$ be such that $1 >_X 2 >_X 3 >_X 4$.

- Leximin: The preference relation $\succsim^L_{min}$ obtained from $>_X$ is

$$1 \succ^L_{min} 12 \succ^L_{min} 2 \succ^L_{min} 13 \succ^L_{min} 123 \succ^L_{min} 23 \succ^L_{min} 3 \succ^L_{min} 14 \succ^L_{min} 124 \succ^L_{min} 24 \succ^L_{min} 134$$
$$\succ^L_{min} 1234 \succ^L_{min} 234 \succ^L_{min} 34 \succ^L_{min} 4 \succ^L_{min} \emptyset$$

  Consider the partial preorder $\succsim'$ compatible with $\succsim^L_{min}$ and such that

$$1 \succ' 12 \succ' 13 \succ' 3 \succ' 14 \succ' 124 \succ' 134 \succ' 1234 \succ' 34 \succ' 4 \succ' \emptyset,$$

  $Score_{Banz}(2) = -4$ but $Score_{Banz}(3) = -1$, therefore ordinal Banzhaf will determine that $3 >_{Banz} 2$ when $2 >_X 3$.

- Leximax: The preference relation $\succsim^L_{max}$ obtained from $>_X$ is

$$\emptyset \succ^L_{max} 1 \succ^L_{max} 12 \succ^L_{max} 123 \succ^L_{max} 1234 \succ^L_{max} 124 \succ^L_{max} 13 \succ^L_{max} 134 \succ^L_{max} 14 \succ^L_{max} 2 \succ^L_{max} 23 \succ^L_{max} 234$$
$$\succ^L_{max} 24 \succ^L_{max} 3 \succ^L_{max} 34 \succ^L_{max} 4$$

  Consider the partial preorder $\succsim'$ compatible with $\succsim^L_{max}$ and such that

$$\emptyset \succ' 1 \succ' 12 \succ' 1234 \succ' 124 \succ' 134 \succ' 14 \succ' 2 \succ' 234 \succ' 24 \succ' 3 \succ' 34 \succ' 4,$$

  $Score_{Banz}(2) = 2$ but $Score_{Banz}(3) = 3$, therefore ordinal Banzhaf will determine that $3 >_{Banz} 2$ when $2 >_X 3$.

$\square$

.

Remark. If there is no coalition $C \subseteq X \setminus \{b\}$ verifying that $C, C \cup \{b\} \in \succsim'$, ordinal Banzhaf may not satisfy level 1 of precision, but it will still satisfy level 0.

Proof.

1. *When applied to $\succsim'$, ordinal Banzhaf satisfies level 0 of preciseness.*

Let $w \in X$ be the worst element and $b \in X$ be the best element according to $>_X$. In the proof of Proposition 7, we have determined that the highest possible Banzhaf score for w is 0, whereas the lowest possible Banzhaf score for b is 1. As Banzhaf scores always take integer values, and the only way $w >_{Banz} b$ is if both elements respectively take on their highest and lowest possible Banzhaf values, there can be no scenario in which an element $a \in X \setminus \{b, w\}$ is such that $Score_{Banz}(w) > Score_{Banz}(a) > Score_{Banz}(b)$, which would be necessary for ordinal Banzhaf to return the reverse of $>_X$.

2. *When applied to $\succsim'$, ordinal Banzhaf does not satisfy level 1 of preciseness.*

Let $X = \{1, 2, 3, 4\}$ st $1 >_X 2 >_X 3 >_X 4$.

- Leximin: The preference relation $\succsim^L_{min}$ derived from $>_X$ is

$$1 \succ^L_{min} 12 \succ^L_{min} 2 \succ^L_{min} 13 \succ^L_{min} 123 \succ^L_{min} 23 \succ^L_{min} 3 \succ^L_{min} 14 \succ^L_{min} 124 \succ^L_{min} 24 \succ^L_{min} 134 \succ^L_{min} 1234 \succ^L_{min} 234 \succ^L_{min} 34 \succ^L_{min} 4 \succ^L_{min} \emptyset.$$

  Consider the partial preorder $\succsim'$ compatible with $\succsim^L_{min}$ and such that $12 \succ' 13 \succ' 123 \succ' 234 \succ' 4 \succ' \emptyset$, then $Score_{Banz}(1) = Score_{Banz}(2) = 0$, $Score_{Banz}(3) = -1$ but $Score_{Banz}(4) = 1$, therefore ordinal Banzhaf would return $4 >_{Banz} 1 =_{Banz} 2 >_{Banz} 3$, determining that the worst element according to $>_X$ is the unique top element.

- Leximax: The preference relation $\succsim^L_{max}$ derived from $>_X$ is

$$\emptyset \succ^L_{max} 1 \succ^L_{max} 12 \succ^L_{max} 123 \succ^L_{max} 1234 \succ^L_{max} 124 \succ^L_{max} 13 \succ^L_{max} 134 \succ^L_{max} 14 \succ^L_{max} 2 \succ^L_{max} 23 \succ^L_{max} 234 \succ^L_{max} 24 \succ^L_{max} 3 \succ^L_{max} 34 \succ^L_{max} 4.$$

  Consider the partial preorder $\succsim'$ compatible with $\succsim^L_{max}$ and such that $\emptyset \succ' 1 \succ' 12 \succ' 13$, then $Score_{Banz}(1) = Score_{Banz}(2) = Score_{Banz}(3) = -1$ but $Score_{Banz}(4) = 0$, therefore ordinal Banzhaf would return $4 >_{Banz} 1 =_{Banz} 2 =_{Banz} 3$, determining that the worst element according to $>_X$ is the unique top element.

$\square$

Proposition 8. *Given a ranking $>_X$ over X from which we determine $\succsim$ using one of the introduced order lifting rules, let $\succsim'$ be a partial preorder over $\mathcal{P}(X)$ compatible with $\succsim$. When applied to $\succsim'$, CP-majority satisfies level 2 of preciseness, $\forall \succsim \in \{\succsim_{mnx}, \succsim_{mxn}, \succsim^L_{min}, \succsim^L_{max}, \succsim_b\}$.*

Proof. Let $x, y \in X$ be such that $x >_X y$.

- Minmax: If $y >_{CP} x$, there must be more coalitions $S \in \mathcal{P}(X \setminus \{x, y\})$ such that $S \cup \{x\} \succ_{mnx} S \cup \{y\}$ than coalitions $T \in \mathcal{P}(X \setminus \{x, y\})$ such that $T \cup \{y\} \succ_{mnx} T \cup \{x\}$. Let $C \in \mathcal{P}(X \setminus \{x, y\})$, we consider all the possible scenarios:
  - If $\min(C) <_X y$, then the comparison between $C \cup \{x\}$ and $C \cup \{y\}$ is based on their maximal elements.
    - if $\max(C) >_X x$, then $C \cup \{x\} \sim_{mnx} C \cup \{y\}$, and both coalitions will be equivalent - comparing them will advantage neither x not y.
    - if $\max(C) <_X x$, then $\max(C \cup \{x\}) = x$, and $\max(C \cup \{y\}) = \max(y, \max(C))$. As $x >_X y$, and since $\max(C) <_X x$, it must hold that $\max(C \cup \{x\}) \succ \max(C \cup \{y\})$, therefore comparing these coalition will count in favour of x.
  - if $\min(C) >_X y$, then $\min(C \cup \{y\}) = y$ and $\min(C \cup \{x\}) = \min(x, \min(C))$. As $x >_X y$ and since $\min(C) >_X y$, it must hold that $\min(C \cup \{x\}) >_X \min(C \cup \{y\})$, therefore comparing these coalitions will count in favour of x.

  None of the possible scenarios can be strictly in favour of y, therefore it is impossible for more coalitions to be evaluated in favour of y than of x using CP-majority. It is thus impossible to determine that $y >_{CP} x$.

- Maxmin: If $y >_{CP} x$, there must be more coalitions $S \in \mathcal{P}(X \setminus \{x, y\})$ such that $S \cup \{x\} \succ_{mxn} S \cup \{y\}$ than coalitions $T \in \mathcal{P}(X \setminus \{x, y\})$ such that $T \cup \{y\} \succ_{mxn} T \cup \{x\}$. Let $C \in \mathcal{P}(X \setminus \{x, y\})$, we consider all the possible scenarios:
  - If $\max(C) >_X x$, then the comparison between $C \cup \{x\}$ and $C \cup \{y\}$ is based on their minimal elements.
    - if $\min(C) <_X y$, then $C \cup \{x\} \sim_{mxn} C \cup \{y\}$, and both coalitions will be equivalent - comparing them will advantage neither x not y.
    - if $\min(C) >_X y$, then $\min(C \cup \{y\}) = y$, and $\min(C \cup \{x\}) = \min(x, \min(C))$. As $x >_X y$, and since $\min(C) >_X y$, it must hold that $\min(C \cup \{x\}) >_X \min(C \cup \{y\})$, therefore comparing these coalition will count in favour of x.

- if $\max(C) <_X x$, then $\max(C \cup \{x\}) = x$ and $\max(C \cup \{y\}) = \max(y, \max(C))$. As $x >_X y$ and since $x >_X \max(C)$, it must hold that $\max(C \cup \{x\}) >_X \max(C \cup \{y\})$, therefore comparing these coalitions will count in favour of x.

None of the possible scenarios can be strictly in favour of y, therefore it is impossible for more coalitions to be evaluated in favour of y than of x using CP-majority. It is thus impossible to determine that $y >_{CP} x$.

- Leximin: Let $S \subseteq X \setminus \{x, y\}$. When comparing $A = S \cup \{x\}$ to $B = S \cup \{y\}$, $\exists i \leq k+1$ st $\forall j < i, a_{\rho_A(j)} = b_{\rho_B(j)} = s_{\sigma_S(j)}$ but $a_{\rho_A(i)} \neq b_{\rho_B(i)}$. By construction of ρ, it must then hold that $a_{\rho_A(i)} = \min\{x, s_{\rho_S(i)}\} >_X b_{\rho_B(i)} = y$. Therefore, $S \cup \{x\} \succ^L_{min} S \cup \{y\}$, $\forall S \subseteq X \setminus \{x, y\}$, and CP-majority will determine that $x >_{CP} y$. This means that if $\exists S \subseteq X \setminus \{x, y\}$ s.t. $S \cup \{x\}$ and $S \cup \{y\}$ are present in $\succsim'_{min^L}$, then CP-majority will find the correct relation. If, however, there exists no such coalition S, CP-majority will be unable to compare x and y, and will therefore consider them equivalent. In any case, CP-majority cannot find that $y >_{CP} x$.

- Leximax: Let $S \subseteq X \setminus \{x, y\}$. When comparing $A = S \cup \{x\}$ to $B = S \cup \{y\}$, $\exists i \leq k+1$ st $\forall j < i, a_{\sigma_A(j)} = b_{\sigma_B(j)} = s_{\sigma_S(j)}$ but $a_{\sigma_A(i)} \neq b_{\sigma_B(i)}$. By construction of σ, it must then hold that $a_{\sigma_A(i)} = x \succ b_{\sigma_B(i)} = \max\{y, s_{\sigma_S(i)}\}$. Therefore, $S \cup \{x\} \succ^L_{max} S \cup \{y\}$, $\forall S \subseteq X \setminus \{x, y\}$, and CP-majority will determine that $x >_{CP} y$. This means that if $\exists S \subseteq X \setminus \{x, y\}$ s.t. $S \cup \{x\}$ and $S \cup \{y\}$ are present in $\succsim'_{max^L}$, then CP-majority will find the correct relation. If, however, there exists no such coalition S, CP-majority will be unable to compare x and y, and will therefore consider them equivalent. In any case, CP-majority cannot find that $y >_{CP} x$.

- Borda-sum: Let $S \subseteq X \setminus \{x, y\}$. When comparing $S \cup \{x\}$ to $S \cup \{y\}$, the Borda-sum lifting rule compares their Borda-sum scores. By definition, $w_b(S \cup \{x\}) = w_b(S) + w_b(x)$, and $w_b(S \cup \{y\}) = w_b(S) + w_b(y)$. Since $x >_X y$, it must hold that $w_b(x) > w_b(y)$, meaning that $w_b(S \cup \{x\}) > w_b(S \cup \{y\})$. Therefore, there exists no coalition $C \subseteq X \setminus \{x, y\}$ such that $C \cup \{x\} \succ_b C \cup \{y\}$, and CP-majority will always determine that $x >_{CP} y$.

$\square$

5 On the impact of structured preferences: the case of k-sized coalitions

5.1 Theoretical results

Proposition 9. *When $k \leq \lceil \frac{n}{2} \rceil$, lexcel recovers the initial order $>_X$ from the total preorder $\succsim$, $\forall \succsim \in \{\succsim^k_{mnx}, \succsim^k_{mxn}\}$.*

Proof. Let $x, y \in X$ be such that $x >_X y$. The preference relation between x and y will be determined by the relation between the best non-equivalent coalitions $C^k_{x,-y}$ *(i.e. of size k and such that $x \in C^k_{x,-y}$ and $y \notin C^k_{x,-y}$)* and $C^k_{y,-x}$.

- Maxmin: By construction of the maxmin ordering, any coalition $C^k_{x,-y}$ must contain the best $k-1$ elements and x *(resp. the best k elements, if x is among the $k-1$ best.)*. Similarly, $C^k_{y,-x}$ must contain the best $k-1$ elements and y.
If y is not among the best $k-1$ elements, both coalitions are of identical best element, but since $x >_X y$, it must hold that $\min(C^k_{x,-y}) >_X \min(C^k_{y,-x})$, meaning that $C^k_{x,-y} \succ_{mxn} C^k_{y,-x}$.
Otherwise, if y is also among the $k-1$ best elements in the population, then no coalition of size k will allow us to distinguish x from y as long as they are of identical best and worst elements. Since lexcel considers the coalitions from best to worst, the best non-equivalent coalitions containing x and not y *(resp. y but not x)* will therefore be coalitions of best element x *(resp. y)*. Note that $C^k_{x,-y}$ must therefore contain x and the next $k-1$ best elements in $>_X$. We know this coalition exists because, for both x and y to be among the $k-1$ best elements, and since $k \leq \lceil \frac{n}{2} \rceil$, there must be $n-(k-2)$ elements ranked lower than x. In the worst-case scenario, *i.e.* if $k = \lceil \frac{n}{2} \rceil$, $n-(k-2) = n - \lceil \frac{n}{2} \rceil + 2 = \lceil \frac{n}{2} \rceil + 1$. As there are n elements in total, and x is among the $\lceil \frac{n}{2} \rceil - 2$ first elements, there will indeed remain $\lceil \frac{n}{2} \rceil - 1$ elements to add to x to form $C^k_{x,-y}$ as described. Similarly, there must be $n-(k-1)$ elements

ranked lower than y, which means $C^k_{y,-x}$ may also be formed as described.
Since $x >_X y$, it will hold that $\max(C^k_{x,-y}) >_X \max(C^k_{y,-x})$, meaning that $C^k_{x,-y} \succ_{mxn} C^k_{y,-x}$.

In either case, lexcel will therefore (correctly) determine that $x >_{lex} y$.

- Minmax: By construction of the minmax ordering, any coalition $C^k_{x,-y}$ must contain the best $k-1$ elements and x *(resp. the best k elements, if x is among the $k-1$ best.)*. Similarly, $C^k_{y,-x}$ must contain the best $k-1$ elements and y.
If y is not among the best $k-1$ elements, and since $x >_X y$, it must hold that $\min(C^k_{x,-y}) >_X \min(C^k_{y,-x})$, meaning that $C^k_{x,-y} \succ_{mnx} C^k_{y,-x}$.
Otherwise, if y is also among the $k-1$ best elements in the population, then no coalition of size k will allow us to distinguish x from y as long as they are of identical best and worst elements. Since lexcel considers the coalitions from best to worst, the best non-equivalent coalitions containing x and not y *(resp. y but not x)* will therefore be coalitions of best element x *(resp. y)*, *i.e.* coalitions containing x *(resp. y)* and its $k-1$ next best elements. As detailed in the proof for maxmin, we know that such coalitions will always exist. Since $x >_X y$, y will be among the $k-1$ elements added to x to form $C^k_{x,-y}$, meaning that $\min(C^k_{x,-y}) <_X \min(C^k_{y,-x})$, thus $C^k_{x,-y} \succ_{mnx} C^k_{y,-x}$.

  In either case, lexcel will therefore (correctly) determine that $x >_{lex} y$.

□

Proposition 10. *When $k \leq \lceil \frac{n}{2} \rceil$, CP-majority recovers the initial order $>_X$ from the total preorder $\succsim$, $\forall \succsim \in \{\succsim^k_{mnx}, \succsim^k_{mxn}\}$.*

Proof. Let $x, y \in X$ be such that $x >_X y$. To determine the preference relation between x and y, CP-majority will consider any coalition $C \subseteq X \setminus \{x, y\}$ of size $k-1$ and compare $C \cup \{x\}$ to $C \cup \{y\}$.

- Maxmin: By construction of the maxmin ordering, if C is such that $\max(C) >_X x$ and $\min(C) <_X y$, then it will always hold that $C \cup \{x\} \sim_{mxn} C \cup \{x\}$, and C will therefore not be useful in the comparison of x and y.

  In the worst-case scenario, x and y are directly adjacent and ranked in the middle positions of $>_X$. As $k \leq \lceil \frac{n}{2} \rceil$, however, there will always exist a coalition C of size $k-1$ such that $\max(C) <_X x$ or $\min(C) >_X y$.

  In the former case, $\max(C \cup \{x\}) = x$ and $\max(C \cup \{y\}) = \max\{\max(C), y\}$: since $x >_X y$ and $x >_X \max(C)$, it will hold that $\max(C \cup \{x\}) >_X \max(C \cup \{y\})$; in the latter case, $\max(C \cup \{x\}) = \max\{\max(C), x\}$ and $\max(C \cup \{y\}) = \max\{\max(C), y\}$: either $C \cup \{x\}$'s best element is preferred to $C \cup \{y\}$'s *(if* $\max(C \cup \{x\}) = x$*)*, or both are of similar best element, in which case we then consider the worst element and, knowing that $\min(C \cup \{x\}) = \min\{\min(C), x\}$ and $\min(C \cup \{y\}) = y$, it will hold that $\min(C \cup \{x\}) >_X \min(C \cup \{y\})$.

  In both cases, this will lead us to observe that $C \cup \{x\} \succ_{mxn} C \cup \{y\}$. CP-majority will therefore always observe that $C \cup \{x\} \succsim_{mxn} C \cup \{y\}$, with at least one coalition $C^* \subseteq X \setminus \{x, y\}$ of size k such that $C^* \cup \{x\} \succ_{mxn} C^* \cup \{y\}$, and determine that $x >_{CP} y$.

- Minmax: By construction of the minmax ordering, if C is such that $\max(C) >_X x$ and $\min(C) <_X y$, then it will always hold that $C \cup \{x\} \sim_{mnx} C \cup \{x\}$, and C will therefore not be useful in the comparison of x and y.

  In the worst-case scenario, x and y are directly adjacent and ranked in the middle positions of $>_X$. As $k \leq \lceil \frac{n}{2} \rceil$, however, there will always exist a coalition C of size $k-1$ such that $\min(C) >_X y$ or $\max(C) <_X x$.

  In the former case, $\min(C \cup \{x\}) = \min\{\min(C), x\}$ and $\min(C \cup \{y\}) = y$: since $x >_X y$ and $\min(C) >_X y$, it will hold that $\min(C \cup \{x\}) >_X \min(C \cup \{y\})$; in the latter case, $\min(C \cup \{x\}) = \min(C)$ and $\min(C \cup \{y\}) = \min\{\min(C), y\}$: either $C \cup \{x\}$'s worst element is preferred to $C \cup \{y\}$'s *(if* $\min(C \cup \{y\}) = y$*)*, or both are of similar worst element, in which case we then consider the best element, and, knowing that $\max(C \cup \{x\}) = x$ and $\max(C \cup \{y\}) = \max\{\max(C), y\}$, it will hold that $\max(C \cup \{x\}) >_X \max(C \cup \{y\})$.

In both cases, this will lead us to observe that $C \cup \{x\} \succ_{mnx} C \cup \{y\}$. CP-majority will therefore always observe that $C \cup \{x\} \succsim_{mnx} C \cup \{y\}$, with at least one coalition $C^* \subseteq X \setminus \{x, y\}$ of size k such that $C^* \cup \{x\} \succ_{mnx} C^* \cup \{y\}$, and determine that $x >_{CP} y$.

$\square$

Proposition 11. *Lexcel always recovers the initial order $>_X$ from the preorder $\succsim, \forall \succsim \in \{(\succsim^L_{min})^k, (\succsim^L_{max})^k), \succsim^k_b\}$.*

Proof. Let $x, y \in X$ be such that $x >_X y$. Let $C^k_{x,-y}$ *(resp.* $C^k_{y,-x}$*)* be the coalition of size k containing x and not y *(resp. y and not x)* ranked the highest in $\succsim$.

- Leximin: By construction of leximin, $C^k_{x,-y}$ must be composed of the $k-1$ best elements according to $>_X$ and of x *(or the k best elements if x is among them)*. Similarly, $C^k_{y,-x}$ must be composed of the $k-1$ best elements different from x, and y.

  If y is not among the $k-1$ best elements, then $\min(C^k_{x,-y}) >_X \min(C^k_{y,-x}) = y$, therefore leximin will determine that $C^k_{x,-y} \succ^L_{min} C^k_{y,-x}$.

  If, however, y is among the $k-1$ best elements, then so is x. Therefore $C^k_{x,-y}$ and $C^k_{y,-x}$ will differ from one element : x or y. By construction, there exists $a \in X \subseteq \{y\}$ such that a is ranked directly before y in $>_X$, *i.e.* such that $a >_X y$, then x will be inserted after a in the vector associated with $C^k_{x,-y}$ *(if it is different from a - otherwise it will be placed as a)*, while y will be inserted before a in the vector associated with $C^k_{y,-x}$. During the lexicographical comparison of the vectors, the first non-identical pair of elements to be compared will be a and y, which will lead to the conclusion that $C^k_{x,-y} \succ^L_{min} C^k_{y,-x}$. Lexcel will therefore determine that $x >_{lex} y$.

- Leximax: By construction of leximax, $C^k_{x,-y}$ must be composed of the $k-1$ best elements according to $>_X$ and of x *(or the k best elements if x is among them)*. Similarly, $C^k_{y,-x}$ must be composed of the $k-1$ best elements different from x, and y.

  If y is not among the $k-1$ best elements, then $\max(C^k_{x,-y}) = \max(C^k_{y,-x})$ but $\min(C^k_{x,-y}) >_X \min(C^k_{y,-x}) = y$, therefore leximax will determine that $C^k_{x,-y} \succ^L_{max} C^k_{y,-x}$.

  If, however, y is among the $k-1$ best elements, then so is x. Therefore $C^k_{x,-y}$ and $C^k_{y,-x}$ will differ from one element : x or y. If there exists $a \in X \subseteq \{x, y\}$ such that $x >_X a >_X y$, then x will be inserted before a in the vector associated with $C^k_{x,-y}$, while y will be inserted after a in the vector associated with $C^k_{y,-x}$. During the lexicographical comparison of the vectors, the first non-identical pair of elements to be compared will be x and a, which will lead to the conclusion that $C^k_{x,-y} \succ^L_{max} C^k_{y,-x}$. If, however, there exists no such a, then the first pair of non-identical elements to be compared will be x and y, which will also lead to the conclusion that $C^k_{x,-y} \succ^L_{max} C^k_{y,-x}$. In any case, lexcel will determine that $x >_{lex} y$.

- Borda-sum: Let T^X_{k-1} denote the $k-1$ best elements from the population X according to $>_X$, then

  $$C^k_{x,-y} = \begin{cases} T^X_{k-1} \cup \{x\} \text{ if } x \notin T^X_{k-1} \\ T^{X \setminus \{y\}}_k \text{ if } x \in T^X_{k-1} \end{cases}$$

  Similarly,

  $$C^k_{y,-x} = \begin{cases} T^{X \setminus \{x\}}_{k-1} \cup \{y\} \text{ if } y \notin T^{X \setminus \{y\}}_{k-1} \\ T^{X \setminus \{x\}}_k \text{ if } y \in T^{X \setminus \{x\}}_{k-1} \end{cases}$$

  Since $x >_X y$, we know that $w_b(x) > w_b(y)$.

  If $y \notin T^X_k$, then $w_b(C^k_{x,-y}) = w_b(T_{k-1}) + w_b(x)$, and $w_b(C^k_{y,-x}) = w_b(T_{k-1}) + w_b(y)$, therefore $w_b(C^k_{x,-y}) > w_b(C^k_{y,-x})$, meaning that $C^k_{x,-y} \succ_b C^k_{y,-x}$.

  If, however, $y \in T^X_k$, since we know that $w_b(T^{X \setminus \{y\}}_k) > w_b(T^{X \setminus \{x\}}_k)$, since the former contains x where the latter contains y, it must also hold that $w_b(C^k_{x,-y}) > w_b(C^k_{y,-x})$, meaning that $C^k_{x,-y} \succ_b C^k_{y,-x}$.

  In any case, lexcel will determine that $x >_{lex} y$.

□

Proposition 12. *CP-majority always recovers the initial order* $>_X$ *from the total preorder* $\succsim, \forall \succsim \in \{(\succsim^L_{min})^k, (\succsim^L_{max})^k), \succsim^k_b\}$.

Proof. Let $x, y \in X$ be such that $x >_X y$, and let $S \subseteq X \setminus \{x, y\}$, $|S| = k - 1$.

- Leximin: The preference between $S \cup \{x\}$ and $S \cup \{y\}$ is determined by leximin via a comparison of its components from worst to best. As all elements in S remain unchanged, only the insertion of x and y will impact the lexicographical comparison. Since we know that $x >_X y$, then

  1. if $\exists a \in X \setminus \{x, y\}$ such that $x >_X a >_X y$ and $a \in S$, then y is placed before a in the vector associated to $S \cup \{y\}$, while x is placed after a in the vector associated to $S \cup \{x\}$. The element y will therefore be compared to a, and since $a >_X y$, it must hold that $S \cup \{x\} \succ^L_{min} S \cup \{y\}$
  2. otherwise, x and y will be compared to each other: since $x >_X y$, it will hold that $S \cup \{x\} \succ^L_{min} S \cup \{y\}$

  Therefore, CP-majority will determine that $x >_{CP} y$ from $(\succsim^L_{min})^k$, $\forall k$, $\forall k \in \{2, \dots, n-1\}$

- Leximax: The preference between $S \cup \{x\}$ and $S \cup \{y\}$ is determined by leximax via a comparison of its components from best to worst. As all elements in S remain unchanged, only the insertion of x and y will impact the lexicographical comparison. Since we know that $x >_X y$, then

  1. if $\exists a \in X \setminus \{x, y\}$ such that $x >_X a >_X y$ and $a \in S$, then x is placed before a in the vector associated to $S \cup \{x\}$, while y is placed after a in the vector associated to $S \cup \{x\}$. The element x will therefore be compared to a, and since $x >_X a$, it must hold that $S \cup \{x\} \succ^L_{max} S \cup \{y\}$
  2. otherwise, x and y will be compared to each other: since $x >_X y$, it will hold that $S \cup \{x\} \succ^L_{max} S \cup \{y\}$

  Therefore, CP-majority will determine that $x >_{CP} y$ from $(\succsim^L_{max})^k$, $\forall k$, $\forall k \in \{2, \dots, n-1\}$

- Borda-sum: The preference between $S \cup \{x\}$ and $S \cup \{y\}$ is determined by comparing their Borda-sum scores. By definition, $w_b(S \cup \{x\}) = w_b(S) + w_b(x)$ and $w_b(S \cup \{y\}) = w_b(S) + w_b(y)$. Since $x >_X y$, $w_b(x) > w_b(y)$, therefore it must hold that $w_b(S \cup \{x\}) > w_b(S \cup \{x\})$, *i.e.* that $S \cup \{x\} \succ_b S \cup \{y\}$.

  Therefore, CP-majority will determine that $x >_{CP} y$ from $\succsim^k_b, \forall k \in \{2, \dots, n-1\}$

□

5.2 Experimental results

This section contains the numerical results of experiments over the amount of missing information over coalitions of size k. For each lifting rule, serving as the structure for the preferences over X^k, we study the number of runs (out of 10 000) for which each social ranking rule (or combination of social ranking rules) manages to recover the exact initial order $>_X$ when only a given percentage of X^k is available. We consider different values for n, the population size, and consider each possible value of $k \in \{2, \ldots, n-1\}$, indicating how many coalitions of such a size exist in the population.

Note that, as we've shown that ordinal Banzhaf is not efficient in the k-sized setting, it is excluded from the results. Similarly, as we have shown that Minmax and Maxmin only allow for the recovery of $>_X$ if $k \leq \lceil \frac{n}{2} \rceil$, they are excluded from tests where k is higher.

		rule	SR	40% (3)	60% (4)	70% (5)
N=4	**k=2 (6 coals)**	1	lexcel	1923	3937	6633
			CP	0	0	10 000
			CP+lex	1923	5217	10 000
		2	lexcel	2031	3967	6630
			CP	0	0	10 000
			CP+lex	2031	5219	10 000
		3	lexcel	2022	3968	6614
			CP	0	0	10 000
			CP+lex	2022	5133	10 000
		4	lexcel	1915	4022	6696
			CP	0	0	10 000
			CP+lex	1915	5289	10 000
		5	lexcel	2991	6021	10 000
			CP	0	8001	10 000
			CP+lex	4463	10 000	10 000
	k=3 (4)	rule	SR	50% (2)	60% (3)	
		3	lexcel	0	2535	
			CP	0	0	
			CP+lex	0	2535	
		4	lexcel	0	2503	
			CP	0	0	
			CP+lex	0	2503	
		5	lexcel	0	2478	
			CP	0	0	
			CP+lex	0	2478	

Table 1: Number of times (out of 10 000 runs) each SR method finds the exact correct order over parts of the k-sized coalitions, depending on the lifting rule (1=minmax, 2=maxmin, 3=leximin, 4=leximax, 5=borda) for $n = 4$

N	k	rule	SR	20% (2)	30% (3)	40% (4)	50% (5)	60% (6)	70% (7)	80% (8)	90% (9)
N=5	**k=2 (10 coals)**	1	lexcel	0	260	930	1905	3154	4511	6254	8009
			CP	0	0	0	0	732	5818	10 000	10 000
			CP+lex	0	275	1101	2809	5531	8533	10 000	10 000
		2	lexcel	0	253	701	1356	2174	3306	4820	7033
			CP	0	0	0	0	684	5831	10 000	10 000
			CP+lex	0	283	936	2344	4951	8264	10 000	10 000
		3	lexcel	0	216	879	1890	3062	4555	6251	7986
			CP	0	0	0	0	729	5897	10 000	10 000
			CP+lex	0	228	1043	2816	5519	8596	10 000	10 000
		4	lexcel	0	227	723	1401	2140	3371	4946	6994
			CP	0	0	0	0	682	5758	10 000	10 000
			CP+lex	0	270	970	2395	4951	8182	10 000	10 000
		5	lexcel	0	319	1126	2859	5759	10 000	10 000	10 000
			CP	0	0	862	5642	10 000	10 000	10 000	10 000
			CP+lex	0	865	4189	8513	10 000	10 000	10 000	10 000
	k=3 (10)	rule	SR	20% (2)	30% (3)	40% (4)	50% (5)	60% (6)	70% (7)	80% (8)	90% (9)
		1	lexcel	0	515	2035	4966	10 000	10 000	10 000	10 000
			CP	0	0	0	3302	10 000	10 000	10 000	10 000
			CP+lex	0	515	2686	6596	10 000	10 000	10 000	10 000
		2	lexcel	0	497	2058	4942	10 000	10 000	10 000	10 000
			CP	0	0	0	3294	10 000	10 000	10 000	10 000
			CP+lex	0	497	2651	6663	10 000	10 000	10 000	10 000
		3	lexcel	0	275	675	1386	2207	3354	4883	7056
			CP	0	0	0	0	698	5818	10 000	10 000
			CP+lex	0	275	804	2311	5074	8319	10 000	10 000
		4	lexcel	0	234	958	1878	3168	4597	6270	7958
			CP	0	0	0	0	693	5787	10 000	10 000
			CP+lex	0	234	1054	2672	5419	8503	10 000	10 000
		5	lexcel	0	306	1189	2835	5666	10 000	10 000	10 000
			CP	0	0	889	5705	10 000	10 000	10 000	10 000
			CP+lex	0	858	4118	8571	10 000	10 000	10 000	10 000
	k=4 (5)	rule	SR	40% (2)			50% (3)		40% (4)		
		3	lexcel	0			0		1943		
			CP	0			0		0		
			CP	0			0		1943		
		4	lexcel	0			0		1972		
			CP	0			0		0		
			CP+lex	0			0		1972		
		5	lexcel	0			0		2038		
			CP	0			0		0		
			CP+lex	0			0		2038		

Table 2: Number of times (out of 10 000 runs) each SR method finds the exact correct order over parts of the k-sized coalitions, depending on the lifting rule (1=minmax, 2=maxmin, 3=leximin, 4=leximax, 5=borda) for $n = 5$

N	k	rule	SR	10% (3)	20% (6)	30% (9)	40% (12)	50% (14)	60% (17)	70% (20)	80% (23)	90% (26)
N=8	**k=2 (28 coals)**	1	lexcel	0	27	345	1157	1857	3354	4875	6625	8565
			CP	0	0	0	13	229	4013	9192	10 000	10 000
			CP+lex	0	31	679	3298	5952	8956	9891	10 000	10 000
		2	lexcel	0	4	34	102	202	476	1178	2755	6098
			CP	0	0	0	18	242	3985	9261	10 000	10 000
			CP+lex	0	6	203	1368	3523	7760	9782	10 000	10 000
		3	lexcel	0	22	328	1150	1944	3343	4892	6681	8597
			CP	0	0	0	10	220	4041	9158	10 000	10 000
			CP+lex	0	27	648	3391	5928	8986	9899	10 000	10 000
		4	lexcel	0	3	24	75	174	494	1234	2770	6149
			CP	0	0	0	10	224	4069	9183	10 000	10 000
			CP+lex	0	5	174	1305	3610	7755	9749	10 000	10 000
		5	lexcel	0	17	764	5326	10 000	10 000	10 000	10 000	10 000
			CP	0	278	7564	10 000	10 000	10 000	10 000	10 000	10 000
			CP+lex	36	3239	9702	10 000	10 000	10 000	10 000	10 000	10 000

N	k	rule	SR	5% (3)	10% (6)	20% (12)	30% (17)	40% (23)	50% (28)	60% (34)	70% (40)	80% (45)	90% (51)
N=8	**k=3 (56)**	1	lexcel	0	64	1808	5285	10 000	10 000	10 000	10 000	10 000	10 000
			CP	0	0	2429	9866	10 000	10 000	10 000	10 000	10 000	10 000
			CP+lex	0	114	6711	9956	10 000	10 000	10 000	10 000	10 000	10 000
		2	lexcel	0	26	641	3177	10 000	10 000	10 000	10 000	10 000	10 000
			CP	0	0	2375	9868	10 000	10 000	10 000	10 000	10 000	10 000
			CP+lex	0	62	5981	9940	10 000	10 000	10 000	10 000	10 000	10 000
		3	lexcel	0	18	195	527	1000	1609	2550	3945	5357	7591
			CP	0	0	0	17	3309	8848	9990	10 000	10 000	10 000
			CP+lex	0	23	891	4384	8775	9850	9999	10 000	10 000	10 000
		4	lexcel	0	5	37	88	247	511	988	1945	3392	6181
			CP	0	0	0	13	3350	8818	9982	10 000	10 000	10 000
			CP+lex	0	7	384	2954	8095	9754	9998	10 000	10 000	10 000
		5	lexcel	0	506	5241	10 000	10 000	10 000	10 000	10 000	10 000	10 000
			CP	0	1922	10 000	10 000	10 000	10 000	10 000	10 000	10 000	10 000
			CP+lex	364	6515	10 000	10 000	10 000	10 000	10 000	10 000	10 000	10 000

Table 3: Number of times (out of 10 000 runs) each SR method finds the exact correct order over parts of the k-sized coalitions, depending on the lifting rule (1=minmax, 2=maxmin, 3=leximin, 4=leximax, 5=borda) for $n = 8$

N	k	rule	SR	5% (4)	10% (7)	20% (14)	30% (21)	40% (28)	50% (35)	60% (42)	70% (49)	80% (56)	90% (63)
N=8	k=4 (70)	1	lexcel	10	263	7371	10 000	10 000	10 000	10 000	10 000	10 000	10 000
			CP	0	0	10 000	10 000	10 000	10 000	10 000	10 000	10 000	10 000
			CP+lex	10	637	10 000	10 000	10 000	10 000	10 000	10 000	10 000	10 000
		2	lexcel	7	261	7329	10 000	10 000	10 000	10 000	10 000	10 000	10 000
			CP	0	0	10 000	10 000	10 000	10 000	10 000	10 000	10 000	10 000
			CP+lex	0	0	10 000	10 000	10 000	10 000	10 000	10 000	10 000	10 000
		3	lexcel	1	16	82	236	429	884	1603	2645	4153	6543
			CP	0	0	0	248	6515	9831	10 000	10 000	10 000	10 000
			CP+lex	1	31	955	5677	9344	9979	10 000	10 000	10 000	10 000
		4	lexcel	1	12	83	230	466	934	1561	2620	4331	6543
			CP	0	0	0	251	6439	9818	10 000	10 000	10 000	10 000
			CP+lex	1	25	836	5688	9295	9968	10 000	10 000	10 000	10 000
		5	lexcel	545	2754	8227	10 000	10 000	10 000	10 000	10 000	10 000	10 000
			CP	0	5179	10 000	10 000	10 000	10 000	10 000	10 000	10 000	10 000
			CP+lex	2563	8960	10 000	10 000	10 000	10 000	10 000	10 000	10 000	10 000
	k=5 (56)	rule	SR	5% (3)	10% (6)	20% (12)	30% (17)	40% (23)	50% (28)	60% (34)	70% (40)	80% (45)	90% (51)
		3	lexcel	0	7	43	76	237	460	957	2011	3326	6233
			CP	0	0	0	20	3426	8828	9980	10 000	10 000	10 000
			CP+lex	0	11	421	3099	8192	9755	9996	10 000	10 000	10 000
		4	lexcel	0	18	163	498	986	1616	2579	3951	5398	7520
			CP	0	0	0	9	3311	8834	9976	10 000	10 000	10 000
			CP+lex	0	26	818	4275	8733	9855	9997	10 000	10 000	10 000
		5	lexcel	0	504	5268	10 000	10 000	10 000	10 000	10 000	10 000	10 000
			CP	0	1831	10 000	10 000	10 000	10 000	10 000	10 000	10 000	10 000
			CP+lex	313	6316	10 000	10 000	10 000	10 000	10 000	10 000	10 000	10 000
	k=6 (28)	rule	SR	10% (3)		20% (6)	30% (9)	40% (12)	50% (14)	60% (17)	70% (20)	80% (23)	90% (26)
		3	lexcel	0		4	28	86	178	494	1207	2801	6208
			CP	0		0	0	5	244	3931	9219	10 000	10 000
			CP+lex	0		5	181	1565	3782	7809	9769	10 000	10 000
		4	lexcel	0		25	351	1094	1864	3328	4949	6663	8559
			CP	0		0	0	11	229	4013	9169	10 000	10 000
			CP+lex	0		30	683	3280	5878	8883	9899	10 000	10 000
		5	lexcel	0		12	735	5440	10 000	10 000	10 000	10 000	10 000
			CP	0		288	7639	10 000	10 000	10 000	10 000	10 000	10 000
			CP+lex	36		3236	9680	10 000	10 000	10 000	10 000	10 000	10 000
	k=7 (8)	rule	SR	30% (3)				40% (4)	60% (5)		70% (6)	80% (7)	
		3	lexcel	0				0	0		0	1239	
			CP	0				0	0		0	0	
			CP+lex	0				0	0		0	1239	
		4	lexcel	0				0	0		0	1289	
			CP	0				0	0		0	0	
			CP+lex	0				0	0		0	1289	
		5	lexcel	0				0	0		0	1206	
			CP	0				0	0		0	0	
			CP+lex	0				0	0		0	1206	

Table 4: Number of times (out of 10 000 runs) each SR method finds the exact correct order over parts of the k-sized coalitions, depending on the lifting rule (3=leximin, 4=leximax, 5=borda) for $n = 8$ *(cont.)*