Epistemic EFX Allocations Exist for Monotone Valuations

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Abstract

We study the fundamental problem of *fairly* dividing a set of indivisible items among agents with (general) monotone valuations. The notion of *envy-freeness up to any item* (EFX) is considered to be one of the most fascinating fairness concepts in this line of work. Unfortunately, despite significant efforts, existence of EFX allocations is a major open problem in fair division, thereby making the study of approximations and relaxations of EFX a natural line of research. Recently, Caragiannis et al. [26] introduced a promising relaxation of EFX, called *epistemic* EFX (EEFX). An allocation is EEFX, if for every agent, it is possible to shuffle the items in the remaining bundles so that she becomes "EFX-satisfied". Caragiannis et al. [26] prove existence and polynomial-time computability of EEFX allocations for additive valuations. A natural question asks what happens when we consider valuations more general than additive?

We address this important open question and answer it affirmatively by establishing the *existence* of EEFX allocations for an arbitrary number of agents with general *monotone* valuations. To the best of our knowledge, besides EF1, EEFX is the only known relaxation of EFX to have such strong existential guarantees. Furthermore, we complement our existential result by proving computational and information-theoretic lower bounds. We prove that even for an arbitrary number of (more than one) agents with identical submodular valuations, it is PLS-hard to compute EEFX allocations and it requires exponentially-many value queries to do so.

1 Introduction

The theory of fair division addresses the fundamental problem of dividing a set of resources in a *fair* manner among individuals (often called agents) with varied preferences. This problem arises naturally in many real-world settings, such as division of inheritance, dissolution of business partnerships, divorce settlements, assigning computational resources in a cloud computing environment, course assignments, allocation of radio and television spectrum, air traffic management, course assignments, to name a few [33, 50, 63, 23, 53]. Although the roots of fair division can be found in antiquity, for instance, in ancient Greek mythology and the Bible, its first mathematical exposition dates back to the seminal work of Steinhaus, Banach, and Knaster [57]. Since then, the theory of fair division has received significant attention and a flourishing flow of research from areas across economics, social science, mathematics, and computer science; see [10, 19, 20, 55] for excellent expositions.

The development of fair division protocols plays a crucial role in ensuring equitable outcomes in the design of many social institutions. With the advent of internet, the necessity of having division rules that are both transparent and agreeable or, in other words, *fair* has become evident [51]. There are many examples to see how the principles of fair division are being applied in various technological platforms today.[3] [4]

Some of the central solution concepts and axiomatic characterizations in the fair division literature stem from the cake-cutting context [50] where the resource to be divided is considered to be a (divisible) cake

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$[0, 1]$. The quintessential notion of fairness—*envy-freeness*—was also mathematically formalized in this setup [36, 62]. We say an allocation is *envy-free* if every agent prefers their share in the division at least as much as any other agent's share. Strong existential guarantees of envy-free cake division that also establishes a connection with topology [58, 60], has made envy-freeness as the representative notion of fairness in resource-allocation settings. Unfortunately, an envy-free allocation is not guaranteed to exist when we need to fairly divide a set of indivisible items: consider two agents and a single valuable item. Only one agent can get the item, and the other agent will be envious. Furthermore, it is NP-hard to decide whether an envy-free allocation exists e.g., see [18]. Infeasibility along-with high computational complexity of envy-free allocations has led to study of its various relaxations for the discrete setting.

In this paper, we consider the setting where the resource is a set of discrete or indivisible items, each of which must be wholly allocated to a single agent. A fair division instance consists of a set $\mathcal{N} = \{1, 2 \ldots, n\}$ of n agents and a set $\mathcal{M}$ of items. Every agent i specifies her preferences via a valuation function $v_i \colon 2^{\mathcal{M}} \to \mathbb{R}$. We study general *monotone valuations* that pertains adding a good to a bundle cannot make it worse. The goal is to find a partition $X = (X_1, \ldots, X_n)$ of the items where every agent $i \in \mathcal{N}$ upon receiving bundle X_i considers X to be *fair*.

Envy-freeness up to any item (EFX): One of the most compelling notions of fairness for discrete setting is *envy-freeness up to any item* (EFX). This notion was introduced by Caragiannis et al. [24]. An allocation is EFX if every agent prefers her own bundle to the bundle of any other agent, after removing *any* item from the latter. EFX is considered to be the "closest analogue of envy-freeness" for discrete setting [25]. Unfortunately, despite significant efforts over the past few years, existence of EFX allocations remains as the biggest and the most challenging open problem in fair division, even for instances with more than three agents with additive valuations [54, 28, 17, 4]. See Section 1.3 for a list of related results about EFX.

Epistemic envy-freeness up to any item (EEFX): A recent work of Caragiannis et al. [26] introduced a promising relaxation of EFX, called *epistemic* EFX (which adapts the concepts of *epistemic envy-freeness* defined by Aziz et al. [12]). We call an allocation X EEFX if for every agent $i \in [n]$, there exists an allocation Y such that $Y_i = X_i$ and for every bundle $Y_j \in Y$, we have $v_i(X_i) \geq v_i(Y_j \setminus \{g\})$ for every $g \in Y_j$. That is, an allocation is EEFX if, for every agent, it is possible to shuffle the items in the remaining bundles so that she becomes "EFX-satisfied". See Example 1.1 for a better intuition.

Example 1.1. Consider a fair division instance consisting of 7 items and 3 agents with additive valuations as described in Table 1. Now consider the allocation X where $X_1 = \{g_1, g_2, g_4\}, X_2 = \{g_3, g_5, g_6\}$, and $X_3 = \{g_7\}$. Note that X is envy-free, and hence, EFX and EEFX. Now assume that agent 1 and 2 exchange the items g_3 and g_4. Formally, let $Y = (\{g_1, g_2, g_3\}, \{g_4, g_5, g_6\}, \{g_7\})$. For $i \in \{1, 2\}$, have $v_i(Y_i) = 300 > 201 = v_i(X_i)$, and $v_3(Y_3) = v_3(X_3)$. Therefore, intuitively it seems that Y is a better allocation compared to X since agents 1 and 2 are strictly better off and agent 3 is as happy as before (i.e., Y Pareto dominates X). However, note that while allocation Y is still EEFX, it is not EFX. Namely, for agent 3 we have: $v_3(Y_1 \setminus \{g_1\}) = 100 > 55 = v_3(Y_3)$.

Caragiannis et al. [26] establish existence and polynomial-time computability of EEFX allocations for an arbitrary number of agents with a restricted class of *additive* valuations. Thus, the following question naturally arises:

Do EEFX allocations always exist for an arbitrary number of agents with general *monotone* valuations?

Table 1: The additive valuation functions of 3 agents for 7 goods.

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
v_1	100	100	100	1	1	1	1
v_2	1	1	1	100	100	100	1
v_3	1	50	50	1	1	1	55

1.1 Our Results

We answer the above question affirmatively and establish computational hardness and information-theoretic lower bounds for finding EEFX allocations:

1. EEFX allocations are guaranteed to *exist* for any fair division instance with an arbitrary number of agents having general *monotone* valuations; see Theorem 3.5.

2. Exponentially (in the number of goods) many valuation queries is required by any deterministic algorithm to compute an EEFX allocation for fair division instances with an arbitrary number of agents with identical submodular valuations; see Theorem 4.7.

3. The problem of computing EEFX allocations for fair division instances with an arbitrary number of agents having identical submodular valuations is PLS-hard; see Theorem 4.8.

It is relevant to note that, with the above results, the notion of *epsitemic*-EFX becomes the *second* known relaxation of EFX (besides EF1), that admits such strong existential guarantees. Along-with its hardness results, the notion of EEFX for discrete settings seems to enjoy results of *similar* flavor as that of envy-freeness for cake division [58, 59, 31].

Similar computational hardness and information-theoretic lower bounds are known for computing an EFX allocation between two agents with identical submodular valuations; see [52] and [39]. Observe that, the set of EEFX and EFX allocations are identical in instances with two agents. Hence, the computational hardness and information-theoretic lower bounds known for computing EFX allocations between two agents carry forward to EEFX allocations as well, but *only* for two agents. At first sight, it might seem trivial that finding an EEFX allocation can only get harder when the number of agents grows. However, note that when the number of agents grows, more bundles become "EEFX-feasible" for each agent, and hence, finding an EEFX allocation may be done faster. Nevertheless, in this work, we prove similar lower bounds for EEFX by reducing the problem of computing an EEFX allocation among an arbitrary number of agents with identical submodular valuations from the problem of computing an EFX allocation among two agents with identical submodular valuations. See Section 4.1 for further discussion on the PLS class [44].

Although similar computational hardness and information-theoretic bounds hold true for finding EFX and EEFX allocations, our work has proved guaranteed existence of EEFX allocations for an arbitrary number of agents with monotone valuations, whereas existence of EFX allocations for more than three agents even with additive valuations remain a major open problem.

1.2 Our Techniques

Consider a fair division instance $\mathcal{I} = (\mathcal{N}, \mathcal{M}, \mathcal{V})$ which consists of a set $\mathcal{N}$ of n agents, a set $\mathcal{M}$ of m items and a set $\mathcal{V}$ consisting of agent-valuations over $\mathcal{M}$. and a *desirable property* $\mathcal{P}$ of a bundle $B \subseteq \mathcal{M}$

for an agent $i \in \mathcal{N}$. In this work, we consider the fairness property of whether B is n-*epistemic*-EFX for an agent i (see Definition 2.3). We say B is *desirable* to i when B satisfies the property $\mathcal{P}$ for agent i. The goal is to find an allocation $A = (A_1, \ldots, A_n)$ such that A_i is desirable to each agent $i \in \mathcal{N}$; we call such an allocation *desirable*.

For any partitioning of the items into n bundles $X_1, X_2, \ldots, X_n$, let us consider a bipartite graph $G(X)$ with one side representing the n agents and the other side representing the n bundles. There exists an edge (i, j) between (the node corresponding to) agent i and (the node corresponding to) bundle X_j, if and only if, bundle X_j is *desirable* to agent i. For any subset of the nodes $S \subseteq \mathcal{N}$, let us write $N(S)$ to denote the set of all neighbours of S in $G(X)$.

Note that, if $G(X)$ has a perfect matching, then this matching translates to a *desirable* allocation in $\mathcal{I}$. Therefore, let us assume that $G(X)$ does not admit a perfect matching and hence admits a Hall's violator set. That is, there exists a subset of agents $\{a_1, \ldots, a_{t+1}\}$ for which $|N(\{a_1, \ldots, a_{t+1}\})| \leq t$. But also, there exists a subset of bundles $\{X_{j_1}, \ldots, X_{j_{k+1}}\}$ for which $|N(\{X_{j_1}, \ldots, X_{j_{k+1}}\})| \leq k$. Let us assume that $\{X_{j_1}, \ldots, X_{j_{k+1}}\}$ is minimal. If $k \geq 1$, this means that we can find a non-empty matching of $((i_1, X_{j_1}), \ldots, (i_k, X_{j_k}))$ such that there exists no edge between agent $i \in \mathcal{N} \setminus \{i_1, \ldots, i_k\}$ and bundles $X_{j_1}, \ldots, X_{j_{k+1}}$. In other words, for all $\ell \in [k]$, X_{j_ℓ} is desirable to i_ℓ and is not desirable to any $i \notin \{i_1, \ldots, i_k\}$.

After finding such a matching, it is intuitive to allocate X_{j_ℓ} to i_ℓ for all $\ell \in [k]$ and then recursively find a desired allocation of the remaining goods to the remaining agents. In order to do so, we need to ensure two important conditions.

1. We can find a non-empty matching $((i_1, X_{j_1}), \ldots, (i_k, X_{j_k}))$ in each step.
2. After removing $X_{j_1} \cup \ldots \cup X_{j_k}$ from $\mathcal{M}$, we can still find desirable bundles (with respect to the original instance) for the remaining agents.

Whether ensuring these conditions is possible or not, depends on the property $\mathcal{P}$. In this work, we prove this approach works when the property $\mathcal{P}$ is n-epistemic-EFX, and thereby proving the existence of EEFX allocations for monotone valuations.

Although these two conditions might seem inconsequential, we prove that a stronger condition can simultaneously imply both of them. Namely, we only need to prove that at each step with n' remaining agents, for any remaining agent i, we can partition the remaining items into n' many bundles $X_1, \ldots, X_{n'}$ such that X_j is desirable to i for all $j \in [n']$. This way, at each step, we can ask one of the remaining agents to partition the remaining goods into n' many desirable bundles with respect to her own valuation. Then, we either find a perfect matching, or we find a non-empty matching and reduce the size of the instance.

This technique works when the desirable property is, for instance, *proportionality* (moving-knife procedure [32]) or *maximin share* [57, 9, 45, 1, 42]. In this paper, we show that EEFX allocations under monotone valuations are also compatible with the above technique. Recently, Bu et al. [21] proved this technique also works for finding PROP1 allocations[5] among agents with additive valuations in a *comparison-based model*. In that model, two bundles are presented to an agent and she responds by telling which bundle she prefers.

1.3 Further Related Work

Plaut and Roughgarden [52] proved the existence of EFX for two agents with monotone valuations.

[5] PROP1 requires each agent's proportionality if one item is (hypothetically) added to that agent's bundle.

For three agents, a series of works proved the existence of EFX allocations when agents have additive valuations [28], *nice-cancelable* valuations [17], and finally when two agents have monotone valuations and one has an MMS-*feasible* valuation [4]. EFX allocations exist when agents have identical [52], binary [41], or bi-valued [9] valuations. Several approximations [29, 8, 27, 34] and relaxations [9, 25, 17, 49, 43, 16, 3] of EFX have become an important line of research in discrete fair division.

Another relaxation of envy-freeness proposed in discrete fair division literature is that of *envy-freeness up to some item* (EF1), introduced by Budish [22]. It requires that each agent prefers her own bundle to the bundle of any other agent, after removing some item from the latter. EF1 allocations always exist and can be computed efficiently [48].

Proportionality [32, 57] is another well-studied notion of fairness having its roots in cake division literature. An allocation is proportional if each agent gets a bundle of items for which her value exceeds her total value for all items divided by the number of agents. It is easy to see that proportional allocations do not necessarily exist for the setting of discrete items.

Among the relaxations of proportionality, the one that has received the lion's share of attention uses the so-called *maximin fair share* (MMS), i.e., the maximum value an agent can attain in any allocation where she is assigned her least preferred bundle, as threshold. Surprisingly, Kurokawa et al. [46] proved that MMS allocations may not always exist. Since then, research has focused on computing allocations that approximate MMS; e.g., see [7, 47, 38, 15, 37, 35, 5, 2] for additive, [15, 38, 61] for submodular, [38, 56, 6] for XOS, and [38, 56] for subadditive valuations.

Proportionality up to one good (PROP1) [30] is another relaxation of proportionality which can be guaranteed together with Pareto optimality [14]. Proportionality up to any good (PROPX) on the other hand, is not a guaranteed to exist in the goods setting [13].

An excellent recent survey by [10] discusses the above fairness concepts and many more. Another aspect of discrete fair division which has garnered an extensive research is when the items that needs to be divided are *chores*. We refer the readers to the survey by [40] for a comprehensive discussion.

1.4 Organization:

We begin by discussing the preliminaries in Section 2. We prove our key result of guaranteed existence of EEFX allocations for monotone valuations in Section 3. We conclude by proving information/theoretic lower bounds for computing an EEFX allocation in Section 4. Towards the end, we discuss a list of many interesting open problems motivated by this work in Section 5.

2 Definitions and Notation

For any positive integer k, we use $[k]$ to denote the set $\{1, 2, \ldots, k\}$. We denote a fair division instance by $\mathcal{I} = (\mathcal{N}, \mathcal{M}, \mathcal{V})$, where $\mathcal{N} = [n]$ is a set of n agents, $\mathcal{M}$ is a set of m items and $\mathcal{V} = (v_1, v_2, \ldots, v_n)$ is a vector of valuation functions. For any agent $i \in \mathcal{N}$, we write $v_i : 2^{\mathcal{M}} \to \mathbb{R}_{\geq 0}$ to denote her valuation function over the set of items. For all $i \in \mathcal{N}$, we assume v_i is normalized; i.e., $v_i(\emptyset) = 0$, and *monotone*; i.e., for all $i \in \mathcal{N}$, $g \in \mathcal{M}$ and $S \subset \mathcal{M}$, $v_i(S \cup \{g\}) \geq v_i(S)$. For simplicity, we sometimes use g instead of $\{g\}$ to denote a set with a single item. We use "items" and "goods" interchangeably.

For a fair division instance $\mathcal{I} = (\mathcal{N}, \mathcal{M}, \mathcal{V})$ with monotone valuations, we consider the valuations to be accessed via an oracle. Note that, monotone valuations are the most general class of valuations when the set of items consists of only goods or only chores. A valuation function $v : 2^{\mathcal{M}} \to \mathbb{R}$ is submodular, if and only if for all subsets of items S and T, $v(S) + v(T) \geq v(S \cup T) + v(S \cap T)$.

An allocation $X = (X_1, X_2, \dots, X_n)$ of the items among agents is a partition of items into n bundles such that bundle X_i is allocated to agent i. That is, we have $X_i \cap X_j = \emptyset$ for all $i, j \in \mathcal{N}$ and $\cup_{i \in [n]} X_i = \mathcal{M}$.

Let us now define the concept of *strong envy* that characterizes one of the most compelling notions of fairness in the literature - *envy-freeness up to any item* (EFX).

Definition 2.1 (Strong Envy)**.** For a fair division instance, we say an agent i upon receiving a bundle $A \subseteq \mathcal{M}$ *strongly envies* a bundle $B \subseteq \mathcal{M}$, if there exists an item $g \in B$ such that $v_i(A) < v_i(B \setminus g)$. Under an allocation X, we say agent i *strongly envies* agent j, if upon receiving X_i, agent i strongly envies the bundle X_j.

Definition 2.2 (EFX)**.** For a fair division instance, an allocation $X = (X_1, X_2, \dots, X_n)$ is said to be "*envy-free up to any item*" or "EFX", if no agent strongly envies another agent. i.e., for all agents i and j, $v_i(X_i) \geq v_i(X_j \setminus g)$ for all $g \in X_j$.

Recently, Caragiannis et al. [26] introduced a promising new notion of fairness — *epistemic* EFX – by relaxing EFX, that we define next. They proved *epistemic* EFX allocations among an arbitrary number of agents with additive valuations can be computed in polynomial time.

Definition 2.3. For any integer k, agent $i \in [n]$ and subset of items $S \subseteq \mathcal{M}$, we say that a bundle $A \subseteq S$ is "k-*epistemic*-EFX" for i with respect to S, if there exists a partitioning of $S \setminus A$ into $k-1$ bundles $C_1, C_2, \dots, C_{k-1}$, such that for all $j \in [k-1]$, upon receiving A, i would not strongly envy C_j. We call $C = \{C_1, C_2, \dots, C_{k-1}\}$ a "k-*certificate*" of A for i under S. Also we define

$$\begin{aligned}\mathsf{EEFX}_i^k(S) :=& \{A \subseteq S \mid A \text{ is ``}k\text{-epistemic-EFX''} \\ & \text{for agent } i \text{ with respect to } S\}.\end{aligned}$$

Definition 2.4 (EEFX)**.** For a fair division instance, an allocation $X = (X_1, X_2, \dots, X_n)$ is said to be *epistemic* EFX or EEFX if for all agents i, $X_i \in \mathsf{EEFX}_i^n(\mathcal{M})$.

Note that the set of EFX and EEFX allocations coincide for the case of two agents. Next, we define a notion of EEFX-*graph* that plays a crucial role in proving the existence of EEFX allocations.

Definition 2.5. For a fair division instance, consider a partition of $\mathcal{M}$ into n bundles $Y_1, \dots, Y_n$. We define the EEFX-*graph* as an undirected bipartite graph $G = (V, E)$, where V has one part consisting of n nodes corresponding to the agents and another part with n nodes corresponding to the bundles $Y_1, \dots, Y_n$. There exists an edge (i, j) between (the node corresponding to) agent i and (the node corresponding to) bundle Y_j if and only if $Y_j \in \mathsf{EEFX}_i^n(\mathcal{M})$.

We abuse the notation and refer to the "nodes corresponding to agents" as "agents" and also refer to the "nodes corresponding to bundles" as "bundles". For any subsets V of nodes, $N(V)$ is the set of all neighbors of the nodes in V. For a matching M, $V(M)$ is the set of vertices of M.

3 Existence of Epistemic EFX Allocations

In this section, we prove our main result that establishes existence of EEFX allocations for any fair division instance with n agents having monotone valuations. We start by proving an important structural property (in Lemma 3.1) that enables us to reduce an instance to one with lower number of agents.

Lemma 3.1. *For any fair division instance, consider an agent $i \in \mathcal{N}$ and $A \subseteq \mathcal{M}$ such that $A \notin \mathsf{EEFX}_i^n(\mathcal{M})$. Then for all bundles $B \in \mathsf{EEFX}_i^{n-1}(\mathcal{M} \setminus A)$, we must have $B \in \mathsf{EEFX}_i^n(\mathcal{M})$.*

Algorithm 1 ALG = EEFX($\mathcal{I}$)

Input: A fair division instance $\mathcal{I} = (\mathcal{N}, \mathcal{M}, \mathcal{V})$ where agent $i \in \mathcal{N} = [n]$ has monotone valuation v_i over the set of items $\mathcal{M}$

Output: An allocation $X = (X_1, X_2, \ldots, X_n)$

if $\mathcal{N} = \emptyset$ **then**
    return $\emptyset$;
$n \leftarrow |\mathcal{N}|$
$(X_1, \ldots, X_n) \leftarrow$ an EFX allocation of $\mathcal{M}$ among n agents with valuation v_n;
$G \leftarrow$ EEFX-graph of $\{X_1, \ldots, X_n\}$;
Let $M = \{(k+1, X_{k+1}), \ldots, (n, X_n)\}$ be a matching of size at least 1 such that $N(\{X_{k+1}, \ldots, X_n\}) = \{k+1, \ldots, n\}$;
$\mathcal{N}' \leftarrow [k]$;
$\mathcal{M}' \leftarrow \mathcal{M} \setminus \bigcup_{\ell \in [n] \setminus [k]} X_\ell$;
$\mathcal{V}' \leftarrow (V_1, \ldots, V_k)$;
$(X_1, \ldots, X_k) \leftarrow$ EEFX($\mathcal{N}', \mathcal{M}', \mathcal{V}'$);
return $(X_1, X_2, \ldots, X_n)$;

Proof. For an agent $i \in \mathcal{N}$, let us assume that bundle $A \subseteq \mathcal{M}$ is such that $A \notin \mathsf{EEFX}_i^n(\mathcal{M})$. Now, consider any bundle $B \in \mathsf{EEFX}_i^{n-1}(\mathcal{M} \setminus A)$, i.e., there exists an $(n-1)$-certificate of B for i under $\mathcal{M} \setminus A$, we call it $C = \{C_1, \ldots, C_{n-2}\}$. By definition, we have that

$$v_i(B) \geq v_i(C_j \setminus g) \text{ for all } j \in [n-2] \text{ and } g \in C_j \tag{1}$$

If $v_i(A) \geq v_i(B)$, then combining it with equation (1), we obtain $\{B, C_1, \ldots, C_{n-2}\}$ is an n-certificate of A for i under $\mathcal{M}$ and $A \in \mathsf{EEFX}_i^n(\mathcal{M})$, leading to a contradiction. Hence, we must have

$$v_i(B) \geq v_i(A) \tag{2}$$

Finally, combining equations (1) and (2), we obtain $\{A, C_1, \ldots, C_{n-2}\}$ is an n-certificate of B for i under $\mathcal{M}$ and $B \in \mathsf{EEFX}_i^n(\mathcal{M})$. This completes our proof. $\square$

Lemma 3.1 implies that if an agent i finds a bundle A to be n-epistemic-EFX while no other agent finds A to be n-epistemic-EFX, we can safely allocate A to i, and remove i and A from the instance and find an EEFX allocation of $\mathcal{M} \setminus A$ to the remaining $n-1$ agents. Note that we can repeat this process iteratively and remove $t \geq 1$ agents and t bundles. The formal description is given in Corollary 3.2.

Corollary 3.2 (of Lemma 3.1)**.** *For a fair division instance, consider a partial allocation $(X_{k+1}, X_{k+2}, \ldots, X_n)$ to agents in the set $[n] \setminus [k]$. Let us assume that for all agents $i \in [n] \setminus [k]$ and all $j \in [k]$, we have $X_i \in \mathsf{EEFX}_i^n$, $X_i \notin \mathsf{EEFX}_j^n$. If $(X_1, \ldots, X_k)$ is an* EEFX *allocation of $\mathcal{M} \setminus \bigcup_{\ell \in [k]} X_\ell$ for agents in $[k]$, then $(X_1, X_2, \ldots, X_n)$ is an* EEFX *allocation for agents in $[n]$.*

We will now give a high-level overview of our constructive proof for establishing the existence of EEFX allocation among arbitrary number of agents with monotone valuations using ALG (see Algorithm 1). For a fair division instance $\mathcal{I} = (\mathcal{N}, \mathcal{M}, \mathcal{V})$, our algorithm ALG, starts by considering an EFX allocation $(X_1, \ldots, X_n)$ of $\mathcal{M}$ among n agents with valuation v_n. We know such an allocation exists by the work of Plaut and Roughgarden [52]. Next, we construct the EEFX-graph G between the bundles $X_1, \ldots, X_n$ and the agents. Lemma 3.3 proves that there will always exist a non-trivial matching[6]

[6]Without loss of generality, we can rename the bundles and agents in the matching M

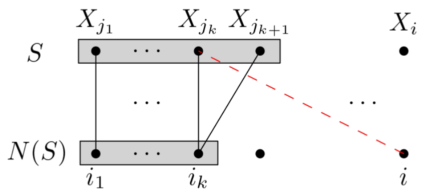

Figure 1: If $G(X)$ does not admit a perfect matching, then there exists a minimal subset $S = \{X_{j_1}, \dots, X_{j_{k+1}}\}$ of bundles such that $|N(S)| < k+1$. Then, for all agent $i \in N(S)$ and all $\ell \in [k+1]$, no edge between X_ℓ and i exists. In other words, no such red dashed edges can exist.

$M = \{(k+1, X_{k+1}), \dots, (n, X_n)\}$ such that $N(\{X_{k+1}, \dots, X_n\}) = \{k+1, \dots, n\}$. That is, for every $j \in [n] \setminus [k]$, bundle $X_j \in \mathsf{EEFX}^n_j(\mathcal{M})$.

Next, ALG reduces the instance by removing the agents $\{k+1, k+2, \dots, n\}$ from $\mathcal{N}$ with their bundles $X_{k+1}, X_{k+2}, \dots, X_n$ safely. Note that, no agent $i \in [k]$ has any edge in G to any bundle X_j for $j \in [n] \setminus [k]$. Finally, this also implies that finding an EEFX allocation $(X_1, X_2, \dots, X_k)$ in the reduced instance and combining it with $(X_{k+1}, X_{k+2}, \dots, X_n)$ leads to an overall EEFX allocation in the original instance. That is, our technique enables us to reduce our instance, find an EEFX allocation in the reduced instance, and combine it in such a way that we produce an EEFX allocation the the original instance.

We begin by proving Lemma 3.3.

Lemma 3.3. *For any fair division instance, consider an agent $i \in \mathcal{N}$, let $(X_1, \dots, X_n)$ be an* EFX *allocation for an instance consisting of n agents with identical valuations v_i. Let G be the* EEFX*-graph with n agents and n bundles $X_1, \dots, X_n$. Then there always exists a matching $M = \{(i_1, X_{j_1}), \dots, (i_k, X_{j_k})\}$ of size at least 1, such that $N(\{X_{j_1}, \dots, X_{j_k}\}) = \{i_1, \dots, i_k\}$.*

Proof. To begin with, if G has a perfect matching $M = \{(i_1, X_{j_1}), \dots, (i_n, X_{j_n})\}$, then the lemma trivially holds true.

Therefore, let us assume that no perfect matching exists in G. This implies that the Hall's condition is not satisfied, i.e., there exists a subset $S = \{X_{j_1}, \dots, X_{j_{k+1}}\}$ of bundles such that $|N(\{X_{j_1}, \dots, X_{j_{k+1}}\})| < k+1$. See Figure 1 for a better intuition. We assume that the subset $S = \{X_{j_1}, \dots, X_{j_{k+1}}\}$ is minimal. That is, for all $S' \subsetneq S$, we have $N(S') \geq |S'|$. Now consider $T = \{X_{j_1}, \dots, X_{j_k}\} \subsetneq S$. By minimality of S, we know that Hall's condition holds for T, i.e., there exists a perfect matching, say $M = \{(i_1, X_{j_1}), \dots, (i_k, X_{j_k})\}$ between the nodes in T and $N(T)$. Since $|N(S)| < k+1$ and $\{i_1, \dots, i_k\} \subseteq N(T) \subseteq N(S)$, it follows that $N(S) = N(T) = \{i_1, \dots, i_k\}$.

Note that since $(X_1, \dots, X_n)$ is an EFX allocation for an instance with identical valuations v_i, we know that $i \in N(S)$, thus $k \geq 1$. Hence, $M = \{(i_1, X_{j_1}), \dots, (i_k, X_{j_k})\}$ is a matching of size $k \geq 1$, such that $N(\{X_{j_1}, \dots, X_{j_k}\}) = \{i_1, \dots, i_k\}$. The stated claim stands proven. ☐

Theorem 3.4 ([52]). *When agents have identical monotone valuations, there always exists an* EFX *allocation.*

We are now ready to discuss our main result that constructively establishes the existence of EEFX allocation among arbitrary number of agents with monotone valuations using ALG.

Theorem 3.5. EEFX *allocations exist for any fair division instance with monotone valuations. In particular,* ALG *returns an* EEFX *allocation.*

Proof. We begin by proving that Algorithm 1 terminates. By Lemma 3.3, a matching $M = \{(i_1, X_{i_1}), \dots, (i_t, X_{i_t})\}$ of size at least 1 exists such that $N(\{X_{i_1}, \dots, X_{i_t}\}) = \{i_1, \dots, i_t\}$. Note that we can rename the bundles and the agents and without loss of generality assume that the considered matching is $M = \{(k+1, X_{k+1}), \dots, (n, X_n)\}$. Therefore, after removing $\{k+1, \dots, n\}$ from $\mathcal{N}$, the size of $\mathcal{N}$ decreases. Hence, the depth of the recursion is bounded by n (the initial number of agents).

We prove the correctness of ALG by using induction on the number of the agents. If $\mathcal{N} = \emptyset$, then $\emptyset$ is an EEFX allocation. We assume that ALG returns an EEFX allocation for any fair division instance with $n' < n$ agents with monotone valuations. Consider the matching M described in ALG. We will show the output allocation of ALG for n agents is EEFX as well. For any $i \in [n] \setminus [k]$ and any $j \in [k]$, the matching M ensures that we have $X_i \in \mathsf{EEFX}_i^n$, and $X_i \notin \mathsf{EEFX}_j^n$ (see Figure 1). By induction hypothesis $(X_1, \dots, X_k)$ is an EEFX allocation of $\mathcal{M} \setminus \bigcup_{\ell \in [k]} X_\ell$ for agents in $[k]$. Thus, by Corollary 3.2, $(X_1, X_2, \dots, X_n)$ is an EEFX allocation for agents in $[n]$. $\square$

Remark 3.6. All proofs of this section that we have for the setting when items are goods easily extend to the setting when these items are '*chores*'. Formally, when agent valuations are monotonically decreasing, then EEFX allocations are guaranteed to exist for an arbitrary number of agents.

4 Hardness Results

In this section, we complement our existential result of EEFX allocations for monotone valuations by proving computational and information-theoretic lower bounds for finding an EEFX allocation. When agents have submodular valuation functions, the way to compute the value $v(S)$ for a subset S of the items is through making value queries. Plaut and Roughgarden [52] proved that exponentially many value queries are required to compute an EFX allocation *even* for two agents with identical submodular valuations. Formally, they proved the following information-theoretic lower bounds.

Theorem 4.1 ([52])**.** *The query complexity of finding an* EFX *allocation with* $|\mathcal{M}| = 2k+1$ *many items is* $\Omega(\frac{1}{k}\binom{2k+1}{k})$*, even for two agents with identical submodular valuations.*

Moreover, Goldberg et al. [39] proved the following computational hardness for EFX allocations.

Theorem 4.2 ([39])**.** *The problem of computing an* EFX *allocation for two agents with identical submodular valuations is* PLS*-complete.*

See Section 7.2 in [11] for further discussion on the complexity class PLS. Let us now define the computational problems corresponding to finding EFX and EEFX allocations.

Definition 4.3. (ID-EFX) Given a fair division instance $\mathcal{I} = ([2], \mathcal{M}, (v, v))$ with two agents having identical submodular valuations v, find an EFX allocation.

Definition 4.4. (ID-EEFX) Given a fair division instance $\mathcal{I} = ([n], \mathcal{M}, (v, \dots, v))$ with n agents having identical submodular valuations v, find an EEFX allocation.

We reduce the problem of finding an EFX allocation for two agents with identical submodular valuations (ID-EFX) to finding an EEFX allocation for an arbitrary number of agents with identical submodular valuations (ID-EEFX), thereby establishing similar hardness results for the latter.

Our Reduction: Consider an arbitrary instance $\mathcal{I} = ([2], \mathcal{M}, (v, v))$ of ID-EFX with two agents having identical submodular valuations v. Let $\mathcal{I}' = ([n], \mathcal{M}', (v', \dots, v')\})$ be an instance of ID-EEFX with n

agents having identical valuations v' over the set of items $\mathcal{M}' = \mathcal{M} \cup \{h_1, \ldots, h_{n-2}\}$. We define the valuation v' as follows.

- For all $S \subseteq \mathcal{M}$, $v'(S) = v(S)$.
- For all $j \in [n-2]$, $v'(h_j) = 2v(\mathcal{M}) + 1$.
- For all $j \in [n-2]$ and $S \subseteq \mathcal{M}' \setminus \{h_j\}$, $v'(S \cup \{h_j\}) = v(S) + v(h_j)$.

We call items $h_1, \ldots, h_{n-2}$ heavy items. Note that we can compute $\mathcal{I}'$ from $\mathcal{I}$ in polynomial time.

Lemma 4.5. *If v is a submodular function, then v' is a submodular function as well.*

Proof. We need to prove that for all $S, T \subseteq \mathcal{M}$, $v'(S) + v'(T) \geq v'(S \cup T) + v'(S \cap T)$. Let H_S and H_T be the set of all heavy items in S and T respectively. We have

$$\begin{aligned}
&v'(S) + v'(T) \\
&= v'(S \setminus H_S) + v'(H_S) + v'(T \setminus H_T) + v'(H_T) \\
&= (v(S \setminus H_S) + v(T \setminus H_T)) + v'(H_S) + v'(H_T)) \\
&\geq v((S \setminus H_S) \cup (T \setminus H_T)) + v((S \setminus H_S) \cap (T \setminus H_T)) \\
&\quad + v'(H_S) + v'(H_T) && \text{(submodularity of } v\text{)} \\
&= v'((S \cup T) \setminus (H_S \cup H_T)) + v'((S \cap T) \setminus (H_S \cap H_T)) \\
&\quad + v'(H_S) + v'(H_T) \\
&= v'((S \cup T) \setminus (H_S \cup H_T)) + v'((S \cap T) \setminus (H_S \cap H_T)) \\
&\quad + v'(H_S \cup H_T) + v'(H_S \cap H_T) && \text{(additivity of } v' \text{ on heavy items)} \\
&= v'(S \cup T) + v'(S \cap T).
\end{aligned}$$

□

Lemma 4.6. *Given any* EEFX *allocation A in $\mathcal{I}'$, we can create an* EFX *allocation in $\mathcal{I}$ in polynomial time, where $\mathcal{I}$ and $\mathcal{I}'$ are as defined above.*

Proof. Let us assume that $A = (A_1, \ldots, A_n)$ is an EEFX allocation in instance $\mathcal{I}'$. To begin with, note that there are $n-2$ heavy items in $\mathcal{I}'$, and hence, by pigeonhole principle, there exists at least two agents, say $i, j \in \mathcal{N}'$ such that they receive no heavy item under A. Without loss of generality, let us assume that $i = 1$ and $j = 2$, and hence we have $A_1, A_2 \subseteq \mathcal{M}$. This implies that we have

$$\begin{gathered}
v'(A_1) = v(A_1),\ v'(A_2) = v(A_2), \\
\text{and, } v(A_1), v(A_2) < 2v(\mathcal{M}) + 1
\end{gathered} \tag{3}$$

Without loss of generality, let us assume $v(A_2) \geq v(A_1)$.

We will prove $(A_1, \mathcal{M} \setminus A_1)$ forms an EFX allocation in $\mathcal{I}$. Note that valuations v and v' coincide for the bundles A_1 and $\mathcal{M} \setminus A_1$. Since A is EEFX in $\mathcal{I}'$, let us denote the n-certificate for agent 1 with respect to A_1 by $C = (C_2, C_3, \ldots, C_n)$. First, we prove that no bundle C_k with a heavy item can have any other item as well. Assume otherwise. Let $\{g, h_j\} \subseteq C_k$ for some $k \in \{2, \ldots, n\}$ and some $j \in [n-2]$ and $g \neq h_j$. Then, we have

$$v'(C_k \setminus g) \geq v'(h_j) = 2v(M) + 1 > v'(A_1)$$

where, the last inequality uses equation (3). This implies that agent 1 strongly envies bundle C_k which is a contradiction to our assumption that C forms an n-certificate for bundle A_1 in instance I'. Therefore, the $n-1$ bundles in the n-certificate must look like $\{C_2, \ldots, C_n\} = \{\{h_1\}, \ldots, \{h_{n-2}\}, \mathcal{M} \setminus A_1\}$. First, note that, agent 1 with bundle A_1 must not strongly envy bundle $\mathcal{M} \setminus A_1$ since C is an n-certificate. And since, we already have $v(A_2) \geq v(A_1)$ and $A_2 \subseteq \mathcal{M} \setminus A_1$, the allocation $(A_1, \mathcal{M} \setminus A_1)$ forms an EFX allocation in $\mathcal{I}$. □

Theorem 4.7. *The query complexity of the* EEFX *allocation problem with* $|\mathcal{M}| = 2k + n - 1$ *many items is* $\Omega(\frac{1}{k}\binom{2k+1}{k})$*, for arbitrary number of agents* n *with identical submodular valuations.*

Proof. Consider any arbitrary instance $\mathcal{I} = ([2], \mathcal{M}, (v, v))$ with two agents having identical submodular valuations v and $|\mathcal{M}| = 2k + 1$ items. Create the instance $\mathcal{I}'$ as described above. Using Lemma 4.5, $\mathcal{I}'$ consists of n agents with identical submodular valuations. By Lemma 4.6, given any EEFX allocation A, we can obtain an EFX allocation for $\mathcal{I}$ in polynomial time. Finally, using Theorem 4.1, we know that the query complexity of finding an EFX allocation in $\mathcal{I}$ is $\Omega(\frac{1}{k}\binom{2k+1}{k})$. Hence, the query complexity of EEFX for n agents with identical submodular valuations admits the same lower bound. This establishes the stated claim. □

Finally, our next result follows using Lemma 4.6 and Theorem 4.2.

Theorem 4.8. *The problem of computing an* EEFX *allocation for for any number* $n \geq 2$ *of agents with identical submodular valuations is* PLS*-hard.*

Since our reduction works even for three agents, Theorems 4.7 and 4.8 hold true for the problem of computing EEFX allocations even for three agents with identical submodular valuations. Note that the set of EFX and EEFX allocations coincide for the case of two agents and hence it inherits the same computational hardness guarantees as that of EFX here.

4.1 Description of Polynomial Local Search (PLS)

The following description of the complexity class PLS is taken from Section 7.2 in [11].

The class PLS (Polynomial Local Search) was defined by Johnson et al. [44] to capture the complexity of finding local optima of optimization problems. Here, a generic instance $\mathcal{I}$ of an optimization problem has a corresponding finite set of solutions $S(\mathcal{I})$ and a potential $c(s)$ associated with each solution $s \in S(\mathcal{I})$. The objective is to find a solution that maximizes (or minimizes) this potential. In the local search version of the problem, each solution $s \in S(\mathcal{I})$ additionally has a well-defined neighborhood $N(s) \in 2^{S(\mathcal{I})}$ and the objective is to find a local optimum, i.e., a solution $s \in S(\mathcal{I})$ such that no solution in its neighborhood $N(s)$ has a higher potential.

Definition 4.9 (PLS)**.** Consider an optimization problem $\mathcal{X}$, and for all input instances $\mathcal{I}$ of $\mathcal{X}$ let $S(\mathcal{I})$ denote the finite set of feasible solutions for this instance, $N(s)$ be the neighborhood of a solution $s \in S(\mathcal{I})$, and $c(s)$ be the potential of solution s. The desired output is a local optimum with respect to the potential function.

Specifically, $\mathcal{X}$ is a polynomial local search problem (i.e., $\mathcal{X} \in$ PLS) if all solutions are bounded in the size of the input $\mathcal{I}$ and there exists polynomial-time algorithms $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ such that:

1. $\mathcal{A}_1$ tests whether the input $\mathcal{I}$ is a legitimate instance of $\mathcal{X}$ and if yes, outputs a solution $s_{\text{initial}} \in S(\mathcal{I})$.

2. $\mathcal{A}_2$ takes as input instance $\mathcal{I}$ and candidate solution s, tests if $s \in S(\mathcal{I})$ and if yes, computes $c(s)$.
3. $\mathcal{A}_3$ takes as input instance $\mathcal{I}$ and candidate solution s, tests if s is a local optimum and if not, outputs $s' \in N(s)$ such that $c(s') > c(s)$ (the inequality is reversed for the minimization version).

Each PLS problem comes with an associated local search algorithm that is implicitly described by the three algorithms mentioned above. The first algorithm is used to find an initial solution to the problem and the third algorithm is iteratively used to find a potential-improving neighbor until a local optimum is reached.

5 Conclusion and Open Problems

In this work, we establish the existence of EEFX allocations for an arbitrary number of agents with general monotone valuations. Furthermore, we also prove that the problem of computing an EEFX allocation for instances with an arbitrary number of agents with submodular valuations is PLS-hard and requires an exponential number of valuations queries as well. Our existential result of EEFX allocations for monotone valuations has opened a variety of major problems in discrete fair division. We list three of them here, that we believe should be explored first.

The first interesting question is, for submodular or monotone valuations, explore the possibility of a PTAS for computing an EEFX allocation, or otherwise prove its APX-hardness. An equally exciting problem would be to explore the compatibility of EEFX and EF1 allocations. Even for instances with additive valuations, does there always exist an allocation that is simultaneously both EEFX and EF1? If yes, can we compute it? What about similar compatibility question of EEFX with *Nash social welfare*? We know that a *maximum Nash welfare* (MNW) allocation is both EF1 and *Pareto-optimal* [24]. What kind of a relation[7] exist between EEFX and MNW allocations?

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[7] We know that MNW allocations may not be EEFX. Consider the following example with two agents and three items $\{a, b, c\}$. Agent 1 values item a at 11, item b at 1, and item c at $\varepsilon > 0$. Agent 2 values item a at 10, item b at ε, and item c at 1. Here, the unique MNW allocation (that assigns items a and b to agent 1, and item c to agent 2) is not EEFX.

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