Collusion in the polluted river problem

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Abstract

We study the effects of collusion in the polluted river problem. In this model, agents along a river network have to share the costs of cleaning the entire river. We define a collusion neutrality property which requires that, if all agents upstream of a given point agree to collude and act as one, then the cost to be paid by this new entity equals the sum of the costs originally allocated to the colluding agents. This property is related to the Unlimited Territorial Integrity doctrine used to settle international disputes over water streams flowing through several territories. We introduce a new cost sharing method, called the Upstream Recursive Equal Sharing (URES) method, and axiomatically characterize it using our new collusion neutrality axiom. We also provide a new characterization for the well-known Local Responsibility Sharing (LRS) method that uses the axiom and differs from the URES method only in one property that explains how costs are shared at the source of the river. Finally, we derive and characterize a family of collusion neutral methods that contains the LRS method as an extreme and the URES method as an intermediate method. All members of the family can be computed by means of a recursive algorithm.

1 Introduction

Fresh water is an increasingly important natural resource, of which rivers are the most common source. Therefore, cleaning up polluted water streams is of key importance. However, when a river passes through different agents —countries, regions, or cities— conflicts are likely to arise. As Barrett [6] mentions, 148 rivers in the world flow through two countries, 30 through three, 9 through four, and 13 through five or more.

In recent years, there has been a growing body of work on this cost sharing problem [13, 1, 17, 14]. Arguably, the two most well-studied cost sharing methods are the Local Responsibility Sharing (LRS) method and the Upstream Equal Sharing (UES) method, introduced by Ni and Wang [13] for single source rivers, and later extended to rivers with multiple sources by Dong et al. [7]. According to the LRS method, each agent is responsible for the cost of cleaning up the river segment within its own territory. On the other hand, since pollution flows along with the river, it is also reasonable to hold the agents responsible for the pollution downstream. The UES method is obtained by sharing the cost of cleaning up each segment equally among all agents upstream.

These methods are related to conflicting doctrines used in international river water disputes [11]. The doctrine of Absolute Territorial Sovereignty (ATS) states that a country can use all the water that flows through its territory at its own will. This is a very strict principle that does not favor the efficient use of natural resources. In addition, it clashes with the Unlimited Territorial Integrity (UTI) doctrine, which requires that agents should not modify the natural conditions in their territory if it harms a downstream country. Although these two doctrines seem to be in conflict with each other, several authors claim that the Upstream Equal Sharing (UES) method, which allocates the cost at some segment equally between the countries at and upstream of this segment, is aligned with the two latter doctrines [7, 17].

A weaker doctrine that is compatible with both is Territorial Integration of all Basin States (TIBS), which states that each basin state is entitled to a reasonable and equitable share of water. However, this principle does not clarify what constitutes a reasonable and equitable share of water.

In this paper we propose a refinement of the implications of the UTI doctrine. Our main contribution is the introduction of a collusion neutrality property, which requires that, if all agents upstream a given point in the river collude into a single entity, then the rule should allocate to the new entity the sum of the cost shares originally allocated to the colluded agents together. We argue that this aligns with the UTI doctrine as, under efficiency, it prevents harming downstream agents in total by these kinds of manipulations. This collusion neutrality is satisfied by the naive LRS method, but not by the UES method. We introduce a new method that shares the cost of cleaning up each segment among agents upstream and is neutral to collusion. This method, which we refer to as the Upstream Recursive Equal Sharing method (URES) method, challenges the idea that cleaning costs should be shared equally among upstream agents and proposes a recursive procedure that dilutes the responsibility of upstream agents: the further downstream pollution has been produced, the less responsibility an agent must take for it. The idea of decreasing responsibility the further away from the relevant segment is something that our approach has in common with the method introduced by Alcalde-Unzu et al. [1] and the class of methods defined by Hougaard et al. [10] but the way responsibility is shared is different in both cases.

Along with collusion neutrality, in order to characterize the URES method, we use three standard properties in the literature (efficiency, additivity and the inessential agent property) and the new Clean source property. This last property states that, if there is no pollution at one of the sources of the river, then the cost allocated to the agent at this point should be equal to that of its direct downstream neighbor. Furthermore, we propose a parallel characterization of the LRS method by replacing the clean source property by the no blind cost at sources property, which is a weakening of the no blind cost property, which Dong et al. [7] used to characterize this method. This result unveils an important difference between the URES and LRS methods. Whereas both methods have the three standard axioms and collusion neutrality in common, they differ in the way they share the cost between a source and its downstream neighbor in case the cost at the source is zero. This further invokes a family of rules that contains the LRS and URES methods and which is parametrized by a non-negative scalar which specifies the ratio between the payments assigned to a clean source and its downstream neighbor. This parameter arises naturally from the characterization of the class, which is achieved with the four common properties of the LRS and URES methods plus a generalization of both the clean source property and no blind cost at sources and by adding another weakening of the clean source property, called clean source symmetry, which requires that two zero cost sources that have the same downstream neighbor pay the same share of the cleaning costs of the river.

As mentioned before, the LRS method is one extreme (where the paramater takes the value zero) in this class while the URES method takes an intermediate position (where the paramater takes the value one). The other extreme of the class is the so-called Full Transfer Recursive Sharing method. This method allocates the cleaning costs fully to sources and is characterized using another version of the clean source property. We also show that the Full Transfer method corresponds to the limit method where the parameter goes to infinity. The LRS and Full Transfer methods also constitute the extremes of a different class of collusion neutral cost sharing methods introduced by Hougaard et al. [10].

The remainder of the paper is organized as follows. In Section 2 we formally introduce the model and the more common properties and well-known cost sharing methods. Next, in Section 3 we introduce our new URES cost sharing method, discuss its connections to the UTI doctrine, and provide a recursive algorithm to compute it, which reveals some of its desirable properties. Section 4 makes use of the characterization of the URES method to obtain a new characterization of the known LRS method. In Section 5 we derive the new class of collusion neutral cost sharing methods. Finally, Section 6 concludes and points towards directions for future research. All proofs, as well as the logical independence for the axioms in each characterization theorem, can be found in the appendix.

2 Preliminaries

A loopless directed graph is a pair (N, D), where $N \subset \mathbb{N}$ is a finite set, whose elements are called *nodes*, and D is a subset of ordered pairs of distinct nodes, that is, $D \subseteq \{(i, j) \in N \times N : i \neq j\}$. Each element $(i, j) \in D$ represents a *directed edge*, which we will shortly refer to as *edge*, from node i to node j. As long as there is no confusion about the set of nodes, we identify a graph (N, D) with its set of edges, D. We denote by $\mathcal{D}$ the set of all loopless directed graphs, shortly referred to as directed graphs.

For the remainder of this section, let $(N, D) \in \mathcal{D}$. Given a subset of nodes $N' \subset N$, we denote by $D[N'] = D \cap (N' \times N')$ the set of edges with both endpoints in N'.

Given $i \in N$, we shall denote by $N_D^+(i) = \{j \in N : (i, j) \in D\}$ the set of nodes to which there is an edge from i. We refer to the elements of $N_D^+(i)$ as the *successors* of i in D. A *bottom* node is one that has no successors; we denote the set of bottom nodes of D by $B(D) = \left\{i \in N : N_D^+(i) = \varnothing\right\}$. Similarly, we define by $N_D^-(i) = \{j \in N : (j, i) \in D\}$ the set of *predecessors* of i in D. We denote by $T(D) = \left\{i \in N : N_D^-(i) = \varnothing\right\}$ the set of *top* nodes of D, being the nodes which have no predecessors.

A *(directed) path* from $i \in N$ to $j \in N$ in D is a sequence of nodes $i_0, \ldots, i_k \in N$, where $i_0 = i$, $i_k = j$, and $(i_l, i_{l+1}) \in D$ for every $l \in \{0, \ldots, k-1\}$. We denote by $N_D^{++}(i)$ the set of nodes to which there is a path starting from i in the network. We call $N_D^{++}(i)$ the set of *subordinates* of i in D. Similarly, $N_D^{--}(i)$, the set of *superiors* of i in D, contains those nodes from which there is a directed path to i in the network.[1]

We say that $(N, D) \in \mathcal{D}$ is a *sink tree* if and only if there exists a node $b \in N$ such that (i) $B(D) = \{b\}$, (ii) $N_D^{--}(b) = N \setminus \{b\}$, and (iii) $\left|N_D^+(i)\right| = 1$ for all $i \in N \setminus \{b\}$. In words, a sink tree is a graph with a single bottom node to which there is a path from every other node, and all other nodes have exactly one successor. Then, for every $i \in N$ and $j \in N_D^{++}(i)$, there is a unique path from i to j, that we denote by $[i, j]_D$.

A *polluted river problem* is a triple (N, D, c), where $(N, D) \in \mathcal{D}$ is a sink tree and $c \in \mathbb{R}_+^{|N|}$ is an $|N|$-dimensional non-negative cost vector. In this setting, the sink tree represents a river network where water flows along the edges of D. Given $i \in N$, c_i represents the cost of cleaning the river segment between i and its sole successor.[2] In this context, we call the top nodes of D the *sources* of the river, and the sole bottom node its *mouth*. We denote by $\mathcal{R}$ the class of polluted river problems.

A *cost allocation* for $(N, D, c) \in \mathcal{R}$ is an $|N|$-dimensional non-negative vector $x \in \mathbb{R}_+^{|N|}$, whose i-th component represents the share in the total cost to be paid by node i. A *cost sharing method*, shortly referred to as a method, is a function ψ that maps every $(N, D, c) \in \mathcal{R}$ to a cost allocation for (N, D, c).

Ni and Wang [13] introduced the following two methods for single-source rivers, both extended to multiple source rivers by Dong et al. [7]. The *Local Responsibility Sharing* (LRS) method assigns to each node the cost of cleaning its own segment. The *Upstream Equal Sharing* (UES) method equally divides the cost of cleaning a segment among the nodes that are upstream from it.[3] Formally, the LRS method assigns to every $(N, D, c) \in \mathcal{R}$, the cost allocation

$$LRS_i\,(N, D, c) = c_i \qquad \text{for all } i \in N,$$

[1] Then, by construction, $N_D^{--}(i) = \left\{j \in N : i \in N_D^{++}(j)\right\}$.

[2] For the bottom node $b \in N$, which by construction does not have any successors, c_b represents the cost of cleaning the segment of the river from the last settlement it flows through and the body of water to which it leads (e.g. a sea or a lake).

[3] Besides extending the LRS and UES methods to sink trees with multiple sources, Dong et al. [7] also introduced a third method, the Downstream Equal Sharing (DES) method, which equally divides the cost of cleaning a segment among the node immediately upstream and the nodes that are downstream of it.

and the UES method assigns to every $(N, D, c) \in \mathcal{R}$, the cost allocation

$$UES_i(N, D, c) = \sum_{j \in [i, b]_D} \frac{c_j}{\left|N_D^{--}(i) \cup i\right|} \qquad \text{for all } i \in N.^4$$

Later on, van den Brink et al. [17] characterized the UES method by means of five properties that a cost sharing method ψ may satisfy.

Efficiency: For every $(N, D, c) \in \mathcal{R}$, $\sum_{i \in N} \psi_i(N, D, c) = \sum_{i \in N} c_i$.

Additivity: For every $(N, D, c), (N, D, c') \in \mathcal{R}$, $\psi(N, D, c + c') = \psi(N, D, c) + \psi(N, D, c')$.

Inessential agent property: For every $(N, D, c) \in \mathcal{R}$ and $i \in N$ such that $c_j = 0$ for every $j \in [i, b]_D$, $\psi_i(N, D, c) = 0$.

Necessary agent property: For every $(N, D, c) \in \mathcal{R}$ and $i \in N$ with $c_j = 0$ for every $j \in N \setminus i$, $\psi_i(N, D, c) \geqslant \psi_j(N, D, c)$ for every $j \in N \setminus i$.

Structural monotonicity: For every $(N, D, c) \in \mathcal{R}$ and $i, j \in N$, if $i \in N_D^-(j)$, then $\psi_i(N, D, c) \geqslant \psi_j(N, D, c)$.

Theorem 2.1 (van den Brink et al., 2018)**.** The UES method is the only cost sharing method that satisfies efficiency, additivity, the inessential agent property, the necessary agent property and structural monotonicity.

3 The Upstream Recursive Equal Sharing method

In this section we introduce a new cost sharing method that, similar to the UES method, allocates the cost of a river segment among the agent on this segment and all upstream segments, but does not share the costs evenly. Instead, it reflects that responsibility decreases as the distance to the relevant segment increases. Our method has this feature in common with the methods introduced by Alcalde-Unzu et al. [1] and Hougaard et al. [10], although the former concerns a slightly different model. We discuss these connections later on.

In the URES method, which we introduce below, the share of upstream agents in the cost of downstream agents depends on the river structure in the sense that (i) the share in the cost of a downstream agent is higher the closer an agent is to this downstream agent, and (ii) in rivers with multiple springs, the costs are shared so that, at any point in the network, each tributary jointly pays the same share of the cleaning costs of the segments downstream of it. We formally define our new method below.

Definition 3.1. The *Upstream Recursive Equal Sharing* (URES) method assigns to every $(N, D, c) \in \mathcal{R}$, the cost allocation given by[5]

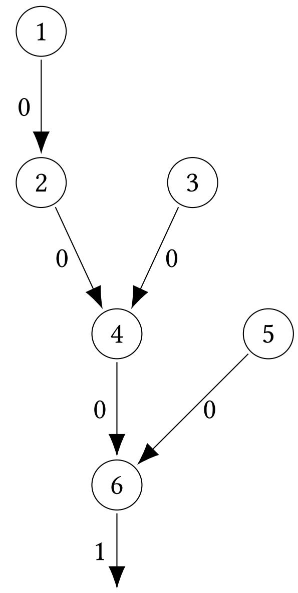
$$URES_i(N, D, c) = \sum_{j \in [i,b]_D} \frac{c_j}{\prod_{k \in [i,j]_D} \left(1 + \left|N_D^-(k)\right|\right)} \text{ for all } i \in N. \tag{1}$$

In Figure 1 we provide an example that illustrates the URES method and compares the resulting allocation with that of the LRS and UES methods. In the example, there is only one positive cleaning cost, equal to

[4]We use the short-hand notation $S \cup i$ to denote $S \cup \{i\}$, where S is a generic set. Similarly, we will use $S \setminus i$ to denote $S \setminus \{i\}$.

[5]Notice that $1 + \left|N_D^-(k)\right| = \left|N_D^-(k) \cup k\right| \ \ \forall k \in N$. Although the right-hand side expression would bring the notation closer to that of the UES method, we favor the form in (1), since it will make connections of the URES method with the class of solutions to be defined in Section 5 more clear.

one and located at the mouth of the river, node 6. The river network itself has two tributaries that flow into its mouth. Notice that, not only does the URES method allocate the same total costs to each of the tributaries, but the mouth also pays this same cost. Indeed, the mouth is held responsible for one third of the cost, while another third is collectively allocated to nodes 1 through 4, which are the nodes upstream of the "left" tributary, and the final third is assigned to node 5, the only node in the course of the "right" tributary.



i	c_i	LRS_i	UES_i	$URES_i$
1	0	0	$\frac{1}{6}$	$\frac{1}{18}$
2	0	0	$\frac{1}{6}$	$\frac{1}{18}$
3	0	0	$\frac{1}{6}$	$\frac{1}{9}$
4	0	0	$\frac{1}{6}$	$\frac{1}{9}$
5	0	0	$\frac{1}{6}$	$\frac{1}{3}$
6	1	1	$\frac{1}{6}$	$\frac{1}{3}$

Figure 1: An application of the LRS, UES and URES methods to a polluted river problem.

This pattern repeats itself across the river network: node 4 pays $\frac{1}{9}$, as does node 3, which forms the "right" tributary flowing into node 4; the payments assigned to nodes 1 and 2, the "left" tributary of node 4, also add up to $\frac{1}{9}$. In general, for every $i \in N$ and upstream neighbor $j \in N_D^-(i)$, the URES method assigns to i the same share in the cleaning costs of its downstream segments as to j and its superiors, i.e. $N_D^{--}(j) \cup j$. In particular, if $\left|N_D^-(i)\right| > 1$, then each tributary that joins the river at i pays the same share of the downstream costs. Moreover, this holds for any polluted river problem, where the allocation of the total cost is done by adding the allocations for each separate segment's cost, not only those where there is a unique non-zero cleaning cost. These observations lead to the following recursive algorithm to compute the URES method, which we will also use in the proofs of our axiomatizations below.

Lemma 3.2. Let ψ be the cost sharing method that assigns to each $(N, D, c) \in \mathcal{R}$ the output of the following algorithm:

(1) $\psi_b(N, D, c) = \frac{c_b}{1+\left|N_D^-(b)\right|}$, where $b \in N$ is the mouth of the river.

(2) For each $i \in N_D^-(b)$ (if any), let $N_D^{i-} = N_D^{--}(i) \cup i$, $D^{i-} = D\left[N_D^{i-}\right]$, and let c^{i-} be determined by

$$c_j^{i-} = \begin{cases} c_j + \frac{c_b}{1+\left|N_D^-(b)\right|} & \text{if } j = i \\ c_j & \text{if } j \in N_D^{--}(i). \end{cases}$$

For every $j \in N_D^{i-}$ with $i \in N_D^-(b)$, let $\psi_j(N, D, c) = \psi_j\left(N_D^{i-}, D^{i-}, c^{i-}\right)$.

The resulting cost sharing method is the URES method, that is, $\psi = URES$.

As we have seen, the URES method distributes the cleaning cost of each segment equally among the node immediately upstream of the segment and each of the tributaries (if any) that join the river at that point.[6] This principle is then applied recursively to share this joint cost among the agents located upstream. For this reason, we call it the Upstream Recursive Equal Sharing (URES) method.

Towards a characterization of the URES method, we introduce two new axioms for cost sharing methods. The first new axiom is the main motivation for our paper, while the second one will be relaxed in later sections. Before introducting the first new axiom, we will need some additional notation.

Given $(N, D, c) \in \mathcal{R}$ and $i \in N$, suppose i and its superiors decide to collude and act as a single entity, call it i^*. This induces a new polluted river problem $\left(N_D^{i*}, D^{i*}, c^{i*}\right) \in \mathcal{R}$, where $N_D^{i*} = N \setminus N_D^{--}(i)$, $D^{i*} = D[N_D^{i*}]$ and the cleaning cost vector c^{i*} is such that the cleaning cost of each segment in D^{i*} remains the same, except for the one that originates at i, which now also takes into account the cleaning costs of all of its superiors; that is, for $j \in N_D^{i*}$,

$$c_j^{i*} = \begin{cases} \sum\limits_{k \in N_D^{i-}} c_k & \text{if } j = i \\ c_j & \text{otherwise.} \end{cases}$$

Collusion neutrality requires that if a node reaches an agreement to represent the nodes upstream of it in the polluted river problem, then this agreement does not change the total cost allocated to these nodes.

Collusion neutrality: For every $(N, D, c) \in \mathcal{R}$ and $i \in N$, $\psi_i\left(N_D^{i*}, D^{i*}, c^{i*}\right) = \sum\limits_{j \in N_D^{i-}} \psi_j(N, D, c)$.

This axiom relates to the *Unlimited Territorial Integrity* (UTI) doctrine for settling international disputes, introduced by Ambec and Sprumont [5]. According to this doctrine, a country may not alter the natural conditions on its own territory to the disadvantage of a downstream country. Under collusion neutrality, a collection of agents cannot unilaterally override the given allocation to their benefit. Indeed, if efficiency is also satisfied, an agreement between an agent and its superiors to act as one cannot affect the *collective* cost allocated to the nodes not participating in the agreement. However, by itself, the property does not preclude the alteration of the allocation to any of these nodes individually.

We remark that Dong et al. [7] use the UTI doctrine to justify the UES method. In their argument, UTI implies Downstream Responsibility, that is, a node is responsible for a fraction of the pollution downstream from it, which for the UES method is interpreted in a specific way by attributing these responsibilities evenly. Collusion neutrality does not specify how the cost share of the set of colluded agents should be allocated among them, but only specifies that their total cost share stays the same after the collusion.

To introduce our second new axiom, consider a segment with null cleaning cost. This does not necessarily mean that the nodes upstream from it are not responsible for the pollution downstream, as it is possible that all pollution in the segment has been transferred downstream. However, this does not apply to the sources of the river since there is no water from upstream agents passing through the sources, and in such cases it seems reasonable to assume that the sources with null cleaning cost pay the same as their successors.

Clean source property: For every $(N, D, c) \in \mathcal{R}$ and $t \in T(D)$, if $c_t = 0$, then $\psi_t(N, D, c) = \psi_{t^+}(N, D, c)$, where t^+ is the sole successor of t in D.

[6] At nodes where no tributaries join the river, the URES method distributes the cleaning cost of the segment immediately downstream equally between said node and its superiors.

This clean source property follows the Downstream Responsibility principle implied by the UTI doctrine, saying that agents should bear some responsibility for cleaning the downstream segments. Specifically, it holds the source equally responsible for the pollution at its downstream neighbor even if its own cost is zero. We will weaken this assumption in Section 5.

Together with the known axioms of efficiency, additivity and the inessential agent property, the two new axioms characterize the URES method.

Theorem 3.3. The URES method is the unique cost sharing method satisfying efficiency, additivity, the inessential agent property, collusion neutrality, and the clean source property.

Remark 3.4. Earlier, we remarked that collusion neutrality reflects the UTI doctrine since, under efficiency, it implies that collusion of an agent with all its superiors does not change the total contribution of all other agents together. A stronger interpretation of UTI could be to require that such a collusion does not have an impact on the contribution of any of the other agents individually. On its turn, under efficiency, this *Collusion invariance*[7] axiom implies collusion neutrality. Since the URES method also satisfies collusion invariance, as seen in the proof of Theorem 3.3, we can replace collusion neutrality by collusion invariance in that characterization of the URES method.

It is easy to check that, while the UES method satisfies the clean source property, the LRS method does not. On the contrary, collusion neutrality holds for the LRS method, while it fails for the UES method. In fact, as we discuss later, the latter property sets the UES and URES methods apart.

In contrast with the UES methods, the newly defined URES method dilutes the responsibility of a node to clean up a downstream segment as its distance from the segment increases. In a related model, instead of the river network, Alcalde-Unzu et al. [1] use the cost vector to determine the responsibility of each node. In their model, the *transfer rate* of pollution, that is, the share of pollution transferred from one segment of the river to the next, plays an important role. In fact, the specification of a problem in said model requires an interval that reflects a social planner's initial knowledge of the transfer rate. The solution proposed by Alcalde-Unzu et al. [1], named the *Upstream Responsibility* method, estimates the transfer rate in a given polluted network by means of the cost vector[8] and computes the responsibility of each node using this information.

On the other hand, in a slightly more general model than polluted river problems, Hougaard et al. [10] introduce the class of *geometric* cost sharing methods. When applied to polluted river problems, the elements in this class also dilute the responsibility of a node for downstream costs as the distance increases, similarly to the URES method. The difference lies in how rivers with multiple springs are treated, hence, in how downstream costs are shared among tributaries that join the river at the same node. We defer this discussion to the end of Section 5, where we compare the geometric methods to a newly defined class of methods.

4 A new axiomatization of the LRS method

We devote the remainder of this paper to defining a class of collusion neutral cost sharing methods that includes the URES method. As a stepping stone, we will first provide a new characterization of the LRS method using collusion neutrality. In doing so, we will also showcase the technique we use to generalize the URES method.

[7] Formally, a cost sharing method ψ satisfies *collusion invariance* if, for every $(N, D, c) \in \mathcal{R}$ and $i \in N$, $\psi_j\left(N_D^{i*}, D^{i*}, c^{i*}\right) = \psi_j(N, D, c)$ for every $j \in N \setminus N_D^{i-}$.

[8] The transfer rate is estimated as the average between the lower bound of the given interval and the minimum ratio of the cleaning costs of a node and one of its predecessors. If this minimum is greater than the upper bound of the interval, the midpoint of the initial interval is taken instead.

As seen in the appendix, specifically in the examples showing logical independence of axioms, out of the properties used to characterize the URES method in Theorem 3.3, the LRS method satisfies all of them except for the clean source property. However, this property is just a specific way to interpret the UTI doctrine. To put it in words, by itself it only implies that if a segment at the source of the river is clean, then the source node is to be held equally responsible for the pollution downstream as its immediate successor. The UES method extends this interpretation to the whole network so that the responsibility for cleaning each segment is evenly distributed among the agents upstream of it.

On the other hand, Ni and Wang [13] argue that the LRS method is an implementation of the Absolute Territorial Sovereignty (ATS) principle. Indeed, Ni and Wang [13] regard ATS as "a statement that people living in a segment of the river have an absolute sovereignty to ask any polluter located within said segment to pay the costs of cleaning pollutants [in the segment]". Hence, irrespective of the network, the responsibility for the costs of cleaning pollutants in a segment should be assigned to the node where this segment originates. Ni and Wang [13] formalize these ideas into the no blind cost axiom.[9]

No blind cost: For every $(N, D, c) \in \mathcal{R}$ and $i \in N$, if $c_i = 0$, then $\psi_i(N, D, c) = 0$.

We propose a significant weakening of the No blind cost property by applying it at the sources of the river only.

No blind cost at sources: For every $(N, D, c) \in \mathcal{R}$ and $t \in T(D)$, if $c_t = 0$, then $\psi_t(N, D, c) = 0$.

As such, no blind cost at sources serves as a variant of the clean source property: it relaxes the no blind cost property and the ATS doctrine in the same way that the clean source property relaxes some of the principles of the UES method and the UTI doctrine, and applies them only at the top of the river network. The following result shows that, by replacing the clean source property by no blind cost at sources we obtain a new characterization for the LRS method.

Theorem 4.1. The LRS method is the unique cost sharing method that satisfies efficiency, additivity, the inessential agent property, collusion neutrality, and no blind cost at sources.

5 A class of collusion neutral cost sharing methods

As mentioned earlier, the new characterization of the LRS method we present in the previous section provides some insight on how to derive a class of cost sharing methods that satisfy collusion neutrality. The key lies in the differences between the URES and LRS methods.

In summary, from a normative point of view, we have seen that the two methods only differ in how they allocate costs at the top of the river network. The URES method satisfies the clean source property, meaning any source is as responsible as its successor for downstream pollution, while the LRS method satisfies no blind cost at sources, which implies that sources bear no responsibility for downstream pollution. Thus, each method applies a particular doctrine: to reiterate, the LRS method follows ATS, while we justify through the UTI doctrine the equal sharing of costs that the URES method implements at the top of the river network.

On the other hand, we already established in the previous section that the manner in which the URES method applies the UTI doctrine is a specific interpretation of it. The new characterization of the LRS method reveals a range of ways to divide the cleaning responsibilities among the nodes at the top of the river network. On one extreme, the LRS method assigns all downstream responsibilities to downstream nodes. The URES method assigns the same responsibility to a source and its successor, but one could argue that sources should hold more responsibility than their successors, or less, while still being responsible for a fraction of the pollution downstream. The following axiom, defined in terms of a parameter, covers the different possibilities.

[9] Ni and Wang [13] characterize the LRS method using efficiency, additivity and no blind cost.

α-clean source property: Given $\alpha \geqslant 0$, for every $(N, D, c) \in \mathcal{R}$ and $t \in T(D)$, if $c_t = 0$, then $\psi_t(N, D, c) = \alpha\psi_{t^+}(N, D, c)$, where t^+ is the sole successor of t in D.

It is clear that for $\alpha = 0$ the axiom becomes the no blind cost at sources property, and for $\alpha = 1$ we retrieve the clean source property introduced in Section 3.

We have also observed that in both the URES and LRS methods, collusion neutrality extends the principles used to divide costs at the top of the river network to any set of tributaries of the river. The URES method makes the *collective* responsibility of a tributary for downstream pollution equal to that of the node where it joins the main river; the LRS method assigns zero responsibility for downstream pollution to any tributary. It is natural to conclude that a method that satisfies collusion neutrality and the α-clean source property yields a similar pattern.

In what follows, we will see that each of the α-Upstream Recursive Sharing methods defined below are characterized by replacing the clean source property by its parametric counterpart.

Definition 5.1. Let $\alpha \geqslant 0$. The *α-Upstream Recursive Sharing (α-URS)* assigns to each $(N, D, c) \in \mathcal{R}$ the cost allocation given by

$$URS_i^{\alpha}(N, D, c) = \sum_{j \in [i, b]_D} \alpha^{|[i, j]_D|-1} \frac{c_j}{\prod_{k \in [i, j]_D} \left(1 + \alpha \left|N_D^-(k)\right|\right)} \qquad \text{for all } i \in N. \tag{2}$$

for every $(N, D, c) \in \mathcal{R}$ and $i \in N$.

It is straightforward to see that the α-URS methods for $\alpha = 0$ and $\alpha = 1$ are the LRS method and the URES method, respectively. Figure 2 shows the allocations given by these and other members of the class on the same example as in Figure 1.

i	$URS_i^0 = LRS_i$	$URS_i^{\frac{1}{2}}$	$URS_i^1 = URES_i$	URS_i^2
1	0	$\frac{1}{48}$	$\frac{1}{18}$	$\frac{8}{75}$
2	0	$\frac{1}{24}$	$\frac{1}{18}$	$\frac{4}{75}$
3	0	$\frac{1}{16}$	$\frac{1}{9}$	$\frac{4}{25}$
4	0	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{2}{25}$
5	0	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{2}{5}$
6	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$

Figure 2: The allocations given by some members of the class URS^{α} applied to the example in Figure 1.

Indeed, we see that the $\frac{1}{2}$-URS method assigns to node 5 half the payment as to node 6; the tributary formed by nodes 1 through 4 also collectively pays half as much as node 6. Similarly, under the 2-URS method, the payment of node 5 and the sum of payments of nodes 1 through 4 are double the payment of node 6. Furthermore, $URS_1^{\frac{1}{2}} + URS_2^{\frac{1}{2}} = URS_3^{\frac{1}{2}} = \frac{1}{16} = \frac{1}{2} \cdot URS_4^{\frac{1}{2}}$ and $URS_1^2 + URS_2^2 = URS_3^2 = \frac{4}{25} = 2 \cdot URS_4^2$. This is in line with the idea that, for every node i, the α-URS method assigns α times as much responsibility for downstream pollution to each tributary leading to i than to i itself. Then, we can generalize the recursive algorithm presented in Lemma 3.2 to compute any α-URS method. As was the case with the URES method, not only is this useful for calculations, but also to show some of the properties satisfied by the α-URS methods.

Lemma 5.2. Given $\alpha \geqslant 0$, let ψ^{α} be the cost sharing method that assigns to each $(N, D, c) \in \mathcal{R}$ the output of the following algorithm:

(1) $\psi_b^\alpha(N, D, c) = \frac{c_b}{1+\alpha\left|N_D^-(b)\right|}$, where $b \in N$ is the mouth of the river.

(2) For each $i \in N_D^-(b)$ (if any), and $j \in N_D^{i-}$, let $\psi_j^\alpha(N, D, c) = \psi_j^\alpha\left(N_D^{i-}, D^{i-}, c^{i-}(\alpha)\right)$, where $c^{i-}(\alpha)$ is determined by

$$c_j^{i-}(\alpha) = \begin{cases} c_j + \frac{1}{\left|N_D^-(b)\right|}\left(c_b - \frac{c_b}{1+\alpha\left|N_D^-(b)\right|}\right) & \text{if } j = i \\ c_j & \text{if } j \in N_D^{--}(i). \end{cases}$$

The resulting cost sharing method is the α-URS method, that is, $\psi^\alpha = URS^\alpha$.

Theorem 5.3. For each $\alpha \geqslant 0$, the α-URS method is the only cost sharing method that satisfies efficiency, additivity, the inessential agent property, collusion neutrality and the α-clean source property.

From the fact that each α-URS method satisfies the α-clean source property, on top of the intuition gained from Figure 2, it follows that, as α increases, more responsibility is attributed to upstream nodes. Then, it is natural to define a cost sharing method that divides cleaning costs exclusively between the sources of the river, following the same principles as the α-URS methods.

Definition 5.4. The *Full Transfer Recursive Sharing (∞-URS)* method assigns to each $(N, D, c) \in \mathcal{R}$ the cost allocation given by

$$URS_i^\infty(N, D, c) = \begin{cases} \sum\limits_{j \in [i, b]_D} \frac{c_j}{\prod\limits_{k \in [i^+, j]_D} \left|N_D^-(k)\right|} & \text{if } i \in T(D) \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

This method fits naturally within the class of α-URS methods, as it coincides with the limit of these methods as α grows to infinity.

Lemma 5.5. The Full Transfer Recursive Sharing method satisfies $URS^\infty = \lim\limits_{\alpha \to \infty} URS^\alpha$.

Out of the axioms used in the characterization for the α-URS methods, it is clear that the Full Transfer method does not satisfy the α-Clean source property for any $\alpha \geqslant 0$. Instead, it satisfies the following axiom.

Clean source full-transfer: For every $(N, D, c) \in \mathcal{R}$ and $t \in T(D)$, if $c_t = 0$, then $\psi_{t^+}(N, D, c) = 0$.

This property formalizes the fact that all responsibility lies on the sources of the river. Thus, clean source full-transfer takes the opposite view from no blind cost at sources regarding how costs are divided at the top of the river network. Namely, no blind cost at sources implies that clean sources need not pay any cleaning costs, while clean source full-transfer states that their successors pay nothing. The reasoning is that all pollution upstream has been *transferred.* Therefore, pollution downstream is considered to have originated in its entirety at the sources of the sources of the river, hence the resulting allocation.

Then, it is natural to take the Full Transfer method as the polar opposite of the LRS method, since each satisfies one of the two aforementioned properties. In the case of the LRS method, each node takes responsibility for the pollution immediately downstream. Thus, as seen in Theorem 4.1, no blind cost at sources, together with collusion neutrality and the three basic properties, is enough to achieve a characterization. For the Full Transfer method, since responsibilities are transferred upstream, and there may be multiple source nodes, we will need to establish how the costs are shared among them. We will see that the following weak symmetry axiom suffices.

Clean source symmetry: For every $(N, D, c) \in \mathcal{R}$, if $s, t \in T(D)$ have the same successor, that is, $s^+ = t^+$, and $c_s = c_t = 0$, then $\psi_t(N, D, c) = \psi_s(N, D, c)$.

As is the case with several axioms we have introduced, clean source symmetry only applies to the top of the river network. It is also worth noting that all of the cost sharing methods discussed throughout the paper satisfy this property.

Theorem 5.6. The ∞-URS method is the only cost sharing method that satisfies efficiency, additivity, the inessential agent property, collusion neutrality, clean source full-transfer, and clean source symmetry.

As a limiting case of the α-URS methods, the Full Transfer method shares some properties with those methods. In particular, although it does not satisfy the α-clean source property for any $\alpha \geqslant 0$, it still allocates downstream costs between to a source and its successor in the same ratio across all networks, a key feature we observed in the class of α-URS methods. We formalize this in the following property.

Clean source proportionality: For every $(N, D, c), (N', D', c') \in \mathcal{R}$ and $t \in T(D)$, $s \in T(D')$ such that $c_t = c'_s = 0$, $\psi_t(N, D, c) \cdot \psi_{s^+}(N', D', c') = \psi_{t^+}(N, D, c) \cdot \psi_s(N', D', c')$.

This property settles a particular way to allocate downstream costs at the top of each network; collusion neutrality extends it so that it applies at any point of a network, in particular when distributing responsibilities between a tributary and the node where it joins the main river. As for the responsibilities of different tributaries, clean source symmetry guarantees they are the same for tributaries of length one that join the river at the same point. Analogously to proportionality, collusion neutrality extends this symmetry so that any two tributaries that join the river at the same point are responsible for the same share of the downstream cleaning costs.

These observations lead to the following characterization of the class of α-URS methods with the Full Transfer method.

Theorem 5.7. A cost sharing method satisfies efficiency, additivity, the inessential agent property, collusion neutrality, clean source symmetry, and clean source proportionality if and only if it is an α-URS method with $\alpha \geq 0$ or $\alpha = \infty$.

To conclude, we discuss some similarities between the newly defined class and the so-called geometric methods introduced by Hougaard et al. [10]. In a setting where a directed graph is used to convey that some agents are responsible for others (e.g. in the polluted river problem, upstream agents are held responsible for downstream pollution), Hougaard et al. [10] define a class of sharing methods in which part of the revenue generated by an agent is transferred to the agent or agents responsible for them.

Definition 5.8 (Hougaard et al., 2017)**.** Given $\lambda \in [0, 1]$, the geometric sharing method γ^λ is defined by

$$\gamma_i^\lambda(N, D, c) = \begin{cases} \sum\limits_{j \in [i, b]_D} c_j \prod\limits_{k \in [i^+, j]_D} \frac{1-\lambda}{|N_D^-(k)|} & \text{if } i \in T(D) \\ \lambda \sum\limits_{j \in [i, b]_D} c_j \prod\limits_{k \in [i^+, j]_D} \frac{1-\lambda}{|N_D^-(k)|} & \text{otherwise.} \end{cases}$$

Note that the geometric methods γ^0 and γ^1 are the Full-Transfer method and the Local Responsibility Sharing method, respectively. In general, the γ^λ geometric method, each node is held responsible for a fraction λ of the cleaning cost for its segment. The rest is shared among the nodes upstream in the same way as in our Upstream Recursive Sharing methods.

It can also be shown that, on line networks, the α-URS method coincides with the $\gamma^{\frac{1}{1+\alpha}}$ geometric method. Then, the two families of methods coincide on this particular class of river networks. In particular, the class of geometric sharing methods also satisfies collusion neutrality; in fact, this is true for all networks.

Thus, the main difference between the two families is the share of the local cost each node is allocated. In the geometric methods, $\lambda \in [0, 1]$ represents the share of the cost immediately downstream each node is assigned to pay. In the α-URS methods, this fraction depends on both the parameter and the number of predecessors of the node at issue. Arguably, this implies that the α-URS methods are more sensitive to the structure of the network. The parameter $\alpha \geqslant 0$ instead conveys the ratio between the responsibility of a node and its predecessors and the node immediately below it for the pollution downstream. Hence, as λ grows, more responsibility is assumed locally, while the opposite occurs as α increases.

6 Final remarks

In this paper, we study the effects of collusion in the distribution of costs to clean a polluted river network. Our first main contribution in this regard is the Upstream Recursive Equal Sharing (URES) method, which is collusion neutral in the sense that the colluding nodes cannot collectively gain from acting as one. This property is implied by the Unlimited Territorial Integrity (UTI) doctrine. Since the UES method introduced by Dong et al. [7] is also derived from the UTI principle, although it does not satisfy collusion neutrality, we argue that the URES method refines the UES method.

Dong et al. [7] also introduced the LRS method, which we show to be collusion neutral as well. In Theorem 4.1 we provide a new characterization for this method using collusion neutrality. Furthermore, exploiting the differences between the URES and LRS methods, we characterize a class of cost sharing methods that satisfy collusion neutrality. For every solution in the class, the ratio between the payment of a source with null cleaning cost and that of its successor is the same in every polluted river problem. This generalizes the clean source full transfer property and the α-clean source property for every $\alpha \geqslant 0$ (of which the clean source property and the no blind cost at sources property are the particular cases $\alpha = 1$ and $\alpha = 0$, respectively), which pin down this ratio to achieve a characterization of an element of the class.

Our approach is similar to that of Hougaard et al. [10], whose geometric methods are also collusion neutral. The novelty lies in the treatment of tributaries: at any node i where multiple tributaries join the river, the λ-geometric method allocates to i a λ share of the downstream costs, and divides the rest equally among the tributaries, while the α-URS method divides the cost so that each tributary contributes α times as i in the payment of downstream costs. On the other hand, the two families of methods are defined for slightly different models. The geometric methods are defined on general hierarchies, which include the polluted river problems we analyze in this paper. Following van den Brink et al. [17], our results can be extended to the more general setting.

It is worth noting that the most well-known cost sharing methods in the literature, the UES and LRS methods, can be obtained as the Shapley value of a game associated to a polluted river problem [7, 17]. For future work, we aim to relate the URES method and the family of URS methods as a whole to solutions of cooperative games. This has also not been done for the class of geometric methods; moreover, it would be interesting to explore the connections between the two classes further.

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A Proofs for Section 3

Lemma 3.2. Let ψ be the cost sharing method that assigns to each $(N, D, c) \in \mathcal{R}$ the output of the following algorithm:

(1) $\psi_b(N, D, c) = \frac{c_b}{1+|N_D^-(b)|}$, where $b \in N$ is the mouth of the river.

(2) For each $i \in N_D^-(b)$ (if any), let $N_D^{i-} = N_D^{--}(i) \cup i$, $D^{i-} = D\left[N_D^{i-}\right]$, and let c^{i-} be determined by

$$c_j^{i-} = \begin{cases} c_j + \frac{c_b}{1+|N_D^-(b)|} & \text{if } j = i \\ c_j & \text{if } j \in N_D^{--}(i). \end{cases}$$

For every $j \in N_D^{i-}$ with $i \in N_D^-(b)$, let $\psi_j(N, D, c) = \psi_j\left(N_D^{i-}, D^{i-}, c^{i-}\right)$.

The resulting cost sharing method is the URES method, that is, $\psi = URES$.

Proof. Let $(N, D, c) \in \mathcal{R}$. First of all, note that ψ is well-defined, since the algorithm is guaranteed to stop. indeed, if $N_D^-(b) \neq \varnothing$, for every $i \in N_D^-(b)$ we have $\left|N_D^{i-}\right| < |N|$, that is, each of the problems to be considered in the following step of the algorithm is "smaller" than the original problem. If $N_D^-(b) = \varnothing$, then $N = \{b\}$, and ψ fully assigns the only cost, c_b, to the only agent, b.

It follows that $\psi = URES$ if $|N| = 1$. Inductively, suppose the result is true for networks with up to $n \geqslant 1$ nodes. For an arbitrary $(N, D, c) \in \mathcal{R}$ such that $|N| = n + 1$, we will show that for every $j \in N$, $\psi_j(N, D, c) = URES_j(N, D, c)$.

This is immediate if $j = b$. Otherwise, let i be the unique predecessor of b contained in $[j, b]_D$. By definition, $\psi_j(N, D, c) = \psi_j\left(N_D^{i-}, D^{i-}, c^{i-}\right)$. Since $\left|N_D^{i-}\right| \leqslant n$, we can apply the induction hypothesis,

$$\begin{aligned}
\psi_j(N, D, c) &= URES_j\left(N_D^{i-}, D^{i-}, c^{i-}\right) \\
&= \sum_{l \in [j, i]_{D^{i-}}} \frac{c_l^{i-}}{\prod_{k \in [j, l]_{D^{i-}}} \left(1 + \left|N_{D^{i-}}^-(k)\right|\right)} \\
&= \sum_{\substack{l \in [j, i]_{D^{i-}} \\ l \neq i}} \frac{c_l}{\prod_{k \in [j, l]_{D^{i-}}} \left(1 + \left|N_{D^{i-}}^-(k)\right|\right)} + \frac{c_i + \frac{c_b}{1+|N_D^-(b)|}}{\prod_{k \in [j, i]_{D^{i-}}} \left(1 + \left|N_{D^{i-}}^-(k)\right|\right)},
\end{aligned}$$

where the third equality follows from the definition of c^{i-}, since i is the bottom node of D^{i-}. On the other hand, D^{i-} is, by construction, the restriction of network D to i and its superiors. Then, in particular, every such node has the same predecessors in D^{i-} as in D, and so for every $k \in N_D^{--}(i)$, the path from k to i in D contains the same elements. That is, for every $k \in N_D^{i-}$, $N_{D^{i-}}^-(k) = N_D^-(k)$, and $[k, i]_{D^{i-}} = [k, i]_D$. Therefore,

$$\psi_j(N,D,c) = \sum_{\substack{l\in[j,i]_D \\ l\neq i}} \frac{c_l}{\prod\limits_{k\in[j,l]_D}\left(1+\left|N_D^-(k)\right|\right)} + \frac{c_i + \frac{c_b}{1+\left|N_D^-(b)\right|}}{\prod\limits_{k\in[j,i]_D}\left(1+\left|N_D^-(k)\right|\right)}$$
$$= \sum_{l\in[j,i]_D} \frac{c_l}{\prod\limits_{k\in[j,l]_D}\left(1+\left|N_D^-(k)\right|\right)} + \frac{c_b}{\prod\limits_{k\in[j,i]_D}\left(1+\left|N_D^-(k)\right|\right)}.$$

Finally, sicne we have chosen i so that $i\in N_D^-(b)$, we obtain

$$\psi_j(N,D,c) = \sum_{l\in[j,b]_D} \frac{c_l}{\prod\limits_{k\in[j,l]_D}\left(1+\left|N_D^-(k)\right|\right)} = URES_j(N,D,c).$$

□

Theorem 3.3. The URES method is the unique cost sharing method satisfying efficiency, additivity, the inessential agent property, collusion neutrality, and the clean source property.

Proof. It is straightforward from Definition 3.1 that the URES method satisfies additivity. The inessential agent property is also trivially satisfied, as agents only pay shares of their own cost and costs downstream. To check that the URES method satisfies the clean source property, consider $(N,D,c)\in\mathcal{R}$ and $t\in T(D)$ such that $c_t=0$. By Definition 3.1 and the null cost at node t, it follows that $\frac{URES_t(N,D,c)}{URES_{t^+}(N,D,c)} = \frac{1}{1+\left|N_D^-(t)\right|}$, which equals one because, by definition, $t\in T(D)$ if and only if $N_D^-(t)=\varnothing$.

To show efficiency, we proceed by induction on $n=|N|$. The result is trivial when $n=1$. Suppose it holds for networks with up to $n\geqslant 1$ nodes, and let $(N,D,c)\in\mathcal{R}$ be such that $|N|=n+1$. Using Lemma 3.2, we have

$$\sum_{i\in N} URES_i(N,D,c) = \frac{c_b}{1+\left|N_D^-(b)\right|} + \sum_{i\in N_D^-(b)}\sum_{j\in N_D^{i-}} URES_j\left(N_D^{i-},D^{i-},c^{i-}\right),$$

where $b\in N$ is the mouth of the river and, for every $i\in N_D^-(b)$, the polluted river problem $\left(N_D^{i-},D^{i-},c^{i-}\right)$ is defined as in Lemma 3.2. Since $\left|N_D^{i-}\right|\leqslant n$ for every $i\in N_D^-(b)$, the induction hypothesis applies to yield

$$\sum_{i\in N} URES_i(N,D,c) = \frac{c_b}{1+\left|N_D^-(b)\right|} + \sum_{i\in N_D^-(b)}\sum_{j\in N_D^{i-}} c_j^{i-}$$
$$= \frac{c_b}{1+\left|N_D^-(b)\right|} + \sum_{i\in N_D^-(b)}\left(\frac{c_b}{1+\left|N_D^-(b)\right|} + \sum_{j\in N_D^{i-}} c_j\right),$$

by definition of c^{i-}. Then,

$$\sum_{i\in N} URES_i(N,D,c) = c_b + \sum_{i\in N_D^-(b)}\sum_{j\in N_D^{i-}} c_j = \sum_{i\in N} c_i,$$

so efficiency is shown.

For collusion neutrality, let $(N,\, D,\, c) \in \mathcal{R}$ and $i \in N$. Note that if $i \in B(D)$, that is, i is the mouth of the river, collusion neutrality follows directly from efficiency. If $i \in T(D)$, collusion neutrality is trivial.

Suppose then $i \in N \setminus (T(D) \cup B(D))$ is neither a source nor the mouth of the river, and consider the problem $\left(N^{i*},\, D^{i*},\, c^{i*}\right)$, where i and its superiors collude. The key observation is that, for every $j \in N^{i^*}$, the path from j to the mouth of the river contains the same nodes in network D^{i*} as it does in D, that is, $[j,\, b]_{D^{i*}} = [j,\, b]_D$. Furthermore, for each $k \in [j,\, b]_D$, $N_D^-(k) = N_{D^{i*}}^-(k)$. Finally, since for every $j \in N^{i*} \setminus i$, $c_j^{i*} = c_j$, the definition of the URES method in (1) implies that

$$URES_j\,(N,\, D,\, c) = URES_j\left(N^{i*},\, D^{i*},\, c^{i*}\right) \qquad \text{for every } j \in N^{i*} \setminus i.$$

Collusion neutrality follows from this observation and the fact that the URES method is efficient, as previously shown.

To show uniqueness, let ψ be a cost sharing method satisfying the properties in the statement, and let $(N,\, D,\, c) \in \mathcal{R}$. If $|N| = 1$, efficiency alone implies that ψ is uniquely defined. Let $n \geqslant 1$ be such that ψ is uniquely defined on problems with up to n nodes, and suppose $|N| = n + 1$.

Due to additivity, it suffices to show uniqueness for problems where only one segment has positive cleaning cost. Suppose $i \in N$ is such that $c_i > 0$ and for every $j \in N \setminus i$, $c_j = 0$. The inessential agent property forces

$$\psi_j\,(N,\, D,\, c) = 0 \qquad \text{for every } j \in N \setminus N_D^{i-}. \tag{4}$$

Then, by efficiency,

$$\sum_{j \in N_D^{i-}} \psi_j\,(N,\, D,\, c) = c_i. \tag{5}$$

If $N_D^{--}(i) = \varnothing$, i.e. $i \in T(D)$, ψ is uniquely determined by (4) and (5).

Therefore, suppose $i \in N \setminus T(D)$. The clean source property implies that

$$\psi_j\,(N,\, D,\, c) = \psi_{j^+}\,(N,\, D,\, c) \qquad \text{for every } j \in N_D^{--}(i) \cap T(D), \tag{6}$$

where j^+ is the sole successor of j in D. By collusion neutrality, we obtain

$$\sum_{k \in N_D^{j-}} \psi_k\,(N,\, D,\, c) = \psi_j\left(N_D^{j*},\, D^{j*},\, c^{j*}\right) \qquad \text{for every } j \in N_D^{--}(i) \setminus T(D). \tag{7}$$

Since, by construction, $j \in N_D^{--}(i)$ cannot be the mouth of the river, we have $\left|N_D^{j*}\right| \leqslant n$ for every $j \in N_D^{--}(j) \setminus T(D)$ and thus the induction hypothesis implies that, for each such j, $\psi\left(N_D^{j*},\, D^{j*},\, c^{j*}\right)$ is uniquely defined. Hence, the right-hand side of each equation (7) is a constant.

In summary, (5), (6) and (7) together give $1 + \left|N_D^{--}(i) \cap T(D)\right| + \left|N_D^{--}(i) \setminus T(D)\right| = 1 + \left|N_D^{--}(i)\right| = \left|N_D^{i-}\right|$ equations in the same number of unknowns $x_j = \psi_j\,(N,\, D,\, c)\,,\ j \in N_D^{i-}$. Therefore, to end the proof it is necessary and sufficient to show that the equations are linearly independent.

First, observe that the set of equations of the form (6) is linearly independent, since each of them contains a variable associated to a top node, which does not appear in any other equation of this form.

Now, let $d(j) = \max_{t \in T(D)} |[t, j]_D|$ be the maximum length of a path from a top node of D to $j \in N$. We will show that for every $j \in N_D^{i-}$, the system of equations that only contain the variables $\{x_k\}_{k \in N_D^{j-}}$ are linearly independent. First, let $j \in N_D^{i-}$ be such that $d(j) = 1$, that is, all superiors of j are top nodes. For such j, there is only equation of the form (7). Moreover, it is the only equation in the set that contains variable x_j; hence, it is linearly independent to the other equations, all of the form (6).

Proceeding by induction, suppose then that the result holds for every $j \in N_D^{i-}$ with $d(j) \leqslant d$, for some $d \geqslant 1$. Let $k \in N_D^{i-}$ be such that $d(k) = d + 1$. By the induction hypothesis, for every $l \in N_D^{-}(k)$, the equations that only contain the variables $\{x_r\}_{r \in N_D^{l-}}$ are linearly independent, since for every $l \in N_D^{-}(k)$, $d(l) \leqslant d$. Furthermore, these sets are pairwise linearly independent, since they involve disjoint sets of variables. Note that the union of these sets contains all equations of interest except one of the form (7), the only one that contains variable x_k. By the same reasoning as before, this equation is linearly independent from the rest.

Summarizing, it is shown then that, for every $j \in N_D^{i-}$, the set of equations with variables $\{x_k\}_{k \in N_D^{j-}}$ is linearly independent. In particular, this is true for $j = i$, which is what we needed to prove. □

B Proofs for Section 4

Theorem 4.1. The LRS method is the unique cost sharing method that satisfies efficiency, additivity, the inessential agent property, collusion neutrality, and no blind cost at sources.

Proof. It is straightforward from the definitions that the LRS method satisfies the five properties. To show uniqueness, it suffices to follow the same process described in the proof of Theorem 3.3, just changing the part that uses the clean source property. To be specific, at the induction step of the proof, consider a polluted river problem $(N, D, c) \in \mathcal{R}$ where $c_j = 0 \ \forall j \in N \setminus i$ and $N_D^{--}(i) \neq \varnothing$. In this case, no blind cost at sources forces

$$\psi_j(N, D, c) = 0 \qquad \text{for every } j \in N_D^{--}(i) \cap T(D), \tag{8}$$

instead of (6).

Together with (5) and (7), which follow from efficiency and collusion neutrality, respectively, we once again obtain a set of $\left|N_D^{i-}\right|$ equations with as many variables. The same arguments as in Theorem 3.3 can be used to show that the equations are linearly independent, which ends the proof. □

C Proofs for Section 5

Lemma 5.2. Given $\alpha \geqslant 0$, let ψ^α be the cost sharing method that assigns to each $(N, D, c) \in \mathcal{R}$ the output of the following algorithm:

(1) $\psi_b^\alpha(N, D, c) = \frac{c_b}{1+\alpha\left|N_D^{-}(b)\right|}$, where $b \in N$ is the mouth of the river.

(2) For each $i \in N_D^{-}(b)$ (if any), and $j \in N_D^{i-}$, let $\psi_j^\alpha(N, D, c) = \psi_j^\alpha\left(N_D^{i-}, D^{i-}, c^{i-}(\alpha)\right)$, where $c^{i-}(\alpha)$ is determined by

$$c_j^{i-}(\alpha) = \begin{cases} c_j + \frac{1}{\left|N_D^{-}(b)\right|}\left(c_b - \frac{c_b}{1+\alpha\left|N_D^{-}(b)\right|}\right) & \text{if } j = i \\ c_j & \text{if } j \in N_D^{--}(i). \end{cases}$$

The resulting cost sharing method is the α-URS method, that is, $\psi^{\alpha} = URS^{\alpha}$.

Proof. First of all, note that, by the same reasoning used in Lemma 3.2, the algorithm is guaranteed to stop, and so ψ^{α} is well-defined. Moreover, if $|N| = 1$, the result holds trivially. As an induction hypothesis, suppose $\psi^{\alpha} = URS^{\alpha}$ on problems with up to $n \geqslant 1$ nodes.

Let $(N, D, c) \in \mathcal{R}$ be such that $|N| = n+1$. It will suffice to show that $\psi_j^{\alpha}(N, D, c) = URS_j^{\alpha}(N, D, c)$ for every $j \in N$. If $j = b$, using (2), it is straightforward to see that $\psi_j^{\alpha}(N, D, c) = \frac{c_j}{1+\alpha\left|N_D^-(j)\right|} = URS_j^{\alpha}(N, D, c)$. If $j \in N \setminus b$, let i be the unique predecessor of b in D that belongs to the path $[j, b]_D$. In this case, the induction hypothesis and the definition of ψ^{α} yield

$$\begin{aligned}
\psi_j^{\alpha}(N, D, c) &= \psi_j^{\alpha}\left(N_D^{i-}, D^{i-}, c^{i-}(\alpha)\right) \\
&= \sum_{l\in[j,\, i]_{D^{i-}}} \alpha^{\left|[j,\, l]_{D^{i-}}\right|-1} \frac{c_l^{i-}(\alpha)}{\prod\limits_{k\in[j,\, l]_{D^{i-}}} \left(1+\alpha\left|N_{D^{i-}}^-(k)\right|\right)} \\
&= \sum_{\substack{l\in[j,\, i]_{D^{i-}} \\ l\neq i}} \alpha^{\left|[j,\, l]_{D^{i-}}\right|-1} \frac{c_l}{\prod\limits_{k\in[j,\, l]_{D^{i-}}} \left(1+\alpha\left|N_{D^{i-}}^-(k)\right|\right)} \\
&\quad + \alpha^{\left|[j,\, i]_{D^{i-}}\right|-1} \frac{c_i + \frac{1}{\left|N_D^-(b)\right|}\left(c_b - \frac{c_b}{1+\alpha\left|N_D^-(b)\right|}\right)}{\prod\limits_{k\in[j,\, i]_{D^{i-}}} \left(1+\alpha\left|N_{D^{i-}}^-(k)\right|\right)} \\
&= \sum_{\substack{l\in[j,\, i]_{D^{i-}} \\ l\neq i}} \alpha^{\left|[j,\, l]_{D^{i-}}\right|-1} \frac{c_l}{\prod\limits_{k\in[j,\, l]_{D^{i-}}} \left(1+\alpha\left|N_{D^{i-}}^-(k)\right|\right)} \\
&\quad + \alpha^{\left|[j,\, i]_{D^{i-}}\right|-1} \frac{c_i + \frac{\alpha c_b}{1+\alpha\left|N_D^-(b)\right|}}{\prod\limits_{k\in[j,\, i]_{D^{i-}}} \left(1+\alpha\left|N_{D^{i-}}^-(k)\right|\right)}.
\end{aligned}$$

Finally, using as in Lemma 3.2 that, for every $k \in N_D^{i-}$ we have $N_{D^{i-}}^-(k) = N_D^-(k)$ and $[k, i]_{D^{i-}} = [k, i]_D$, and also taking into account that $i \in N_D^-(b)$, we obtain

$$\begin{aligned}
\psi_j^{\alpha}(N, D, c) &= \sum_{\substack{l\in[j,\, i]_D \\ l\neq i}} \alpha^{\left|[j,\, l]_D\right|-1} \frac{c_l}{\prod\limits_{k\in[j,\, l]_D} \left(1+\alpha\left|N_D^-(k)\right|\right)} \\
&\quad + \alpha^{\left|[j,\, i]_D\right|-1} \frac{c_i + \frac{\alpha c_b}{1+\alpha\left|N_D^-(b)\right|}}{\prod\limits_{k\in[j,\, i]_D} \left(1+\alpha\left|N_D^-(k)\right|\right)} \\
&= \sum_{l\in[j,\, i]_D} \alpha^{\left|[j,\, l]_D\right|-1} \frac{c_l}{\prod\limits_{k\in[j,\, l]_D} \left(1+\alpha\left|N_D^-(k)\right|\right)} \\
&\quad + \alpha^{\left|[j,\, b]_D\right|-1} \frac{c_b}{\prod\limits_{k\in[j,\, b]_D} \left(1+\alpha\left|N_D^-(k)\right|\right)} \\
&= \sum_{l\in[j,\, b]_D} \alpha^{\left|[j,\, l]_D\right|-1} \frac{c_l}{\prod\limits_{k\in[j,\, l]_D} \left(1+\alpha\left|N_D^-(k)\right|\right)} = URS_i^{\alpha}(N, D, c).
\end{aligned}$$

□

Theorem 5.3. For each $\alpha \geqslant 0$, the α-URS method is the only cost sharing method that satisfies efficiency, additivity, the inessential agent property, collusion neutrality and the α-clean source property.

Proof. Let $\alpha \geqslant 0$. The proof follows closely that of Theorem 3.3. It is clear from Definition 5.1 that URS^{α} is additive and satisfies the inessential agent property. For the α-clean source property, let $(N, D, c) \in \mathcal{R}$ and $s \in T(D)$ with $c_s = 0$. Since by definition $N_D^-(s) = \varnothing$,

$$
\begin{aligned}
URS_s^{\alpha}(N, D, c) &= \sum_{j \in [s, b]_D} \alpha^{\left|[s, j]_D\right|-1} \frac{c_j}{\prod_{k \in [s, j]_D} \left(1 + \alpha \left|N_D^-(k)\right|\right)} \\
&= \sum_{j \in [s^+, b]_D} \alpha^{\left|[s, j]_D\right|-1} \frac{c_j}{\prod_{k \in [s^+, j]_D} \left(1 + \alpha \left|N_D^-(k)\right|\right)} \\
&= \alpha \sum_{j \in [s^+, b]_D} \alpha^{\left|[s^+, j]_D\right|-1} \frac{c_j}{\prod_{k \in [s^+, j]_D} \left(1 + \alpha \left|N_D^-(k)\right|\right)} \\
&= \alpha\, URS_{s^+}^{\alpha}(N, D, c)\,.
\end{aligned}
$$

It is also trivial that the method is efficient for networks with only one node. Suppose URS^{α} satisfies efficiency on problems with up to $n \geqslant 1$ nodes, and let $(N, D, c) \in \mathcal{R}$ be such that $|N| = n + 1$. Then, by Lemma 5.2,

$$
\sum_{i \in N} URS_i^{\alpha}(N, D, c) = \frac{c_b}{1 + \alpha \left|N_D^-(b)\right|} + \sum_{i \in N_D^-(b)} \sum_{j \in N_D^{i-}} URS_j^{\alpha}\left(N_D^{i-}, D^{i-}, c^{i-}(\alpha)\right),
$$

where $b \in N$ is the mouth of the river and, for every $i \in N_D^-(b)$, the polluted river problem $\left(N_D^{i-}, D^{i-}, c^{i-}(\alpha)\right)$ is defined as in Lemma 5.2. By construction, $\left|N_D^{i-}\right| < n + 1$ for every $i \in N_D^-(b)$. Then, by the induction hypothesis and the definition of $c^{i-}(\alpha)$,

$$
\begin{aligned}
\sum_{i \in N} URS_i^{\alpha}(N, D, c) &= \frac{c_b}{1 + \alpha \left|N_D^-(b)\right|} + \sum_{i \in N_D^-(b)} \sum_{j \in N_D^{i-}} c_j^{i-}(\alpha) \\
&= \frac{c_b}{1 + \alpha \left|N_D^-(b)\right|} + \sum_{i \in N_D^-(b)} \left(\frac{\alpha c_b}{1 + \alpha \left|N_D^-(b)\right|} + \sum_{j \in N_D^{i-}} c_j \right),
\end{aligned}
$$

which in turn yields efficiency, since it follows that

$$
\sum_{i \in N} URS_i^{\alpha}(N, D, c) = c_b + \sum_{i \in N_D^-(b)} \sum_{j \in N_D^{i-}} c_j = \sum_{i \in N}.
$$

Again, the same logic we applied to the URES method in Theorem 3.3 shows that for every $(N, D, c) \in \mathcal{R}$ and $i \in N$ we have $URS_j^{\alpha}(N, D, c) = URS_j^{\alpha}\left(N^{i*}, D^{i*}, c^{i*}\right)$ for every $j \in N^{i*} \setminus i$. Since URS^{α} is also efficient, it follows that it satisfies collusion neutrality.

For uniqueness, we modify the process described in the proof of Theorem 3.3 as follows. In this case, at the induction step, given a polluted river problem $(N, D, c) \in \mathcal{R}$ where $c_j = 0$ for every $j \in N \setminus i$ and $N_D^-(i) \neq \varnothing$, the α-clean source property forces

$$\psi_j\left(N,\,D,\,c\right)=\alpha\psi_{j^+}\left(N,\,D,\,c\right)\qquad\text{for every } j\in N_D^{--}(i)\cap T(D).\tag{9}$$

These equations, along with (5) and (7), which follow from efficiency and collusion neutrality, respectively, yield a system of $\left|N_D^{i-}\right|$ equations with that same number of variables. The same arguments as in Theorem 3.3 can be used to show that the equations are linearly independent, which ends the proof. □

Lemma 5.5. The Full Transfer Recursive Sharing method satisfies $URS^{\infty}=\lim\limits_{\alpha\to\infty}URS^{\alpha}$.

Proof. First of all we show that for every $(N,\,D,\,c)\in\mathcal{R}$ and $i\in N$, the limit $\lim\limits_{\alpha\to\infty}URS_i^{\alpha}\left(N,\,D,\,c\right)$ exists. Let $(N,\,D,\,c)\in\mathcal{R}$ and $i\in N$. For every $k\in[i^+,\,b]_D$ we have $\lim\limits_{\alpha\to\infty}\frac{\alpha}{1+\alpha\left|N_D^-(k)\right|}=\frac{1}{\left|N_D^-(k)\right|}$. If $i\in T(D)$, that is, $N_D^-(i)=\varnothing$, then $\lim\limits_{\alpha\to\infty}\frac{1}{1+\alpha\left|N_D^-(i)\right|}=1$. Otherwise, $i\in N\setminus T(D)$ and $\lim\limits_{\alpha\to\infty}\frac{1}{1+\alpha\left|N_D^-(i)\right|}=0$. By the product law of limits, for every $j\in[i,\,b]_D$,

$$\begin{aligned}\lim_{\alpha\to\infty}\alpha^{\left|[i,\,j]_D\right|-1}\prod_{k\in[i,\,j]_D}\frac{1}{1+\alpha\left|N_D^-(k)\right|}&=\left(\lim_{\alpha\to\infty}\frac{1}{1+\alpha\left|N_D^-(i)\right|}\right)\cdot\prod_{k\in[i^+,\,j]_D}\lim_{\alpha\to\infty}\frac{\alpha}{1+\alpha\left|N_D^-(k)\right|}\\&=\begin{cases}\prod\limits_{k\in[i^+,\,j]_D}\frac{1}{\left|N_D^-(k)\right|}&\text{if } i\in T(D)\\0&\text{otherwise.}\end{cases}\end{aligned}$$

By the sum law of limits, it follows that the limit $\lim\limits_{\alpha\to\infty}URS_i^{\alpha}\left(N,\,D,\,c\right)$ exists for every $(N,\,D,\,c)\in\mathcal{R}$ and $i\in N$ and it satisfies

$$\begin{aligned}\lim_{\alpha\to\infty}URS_i^{\alpha}\left(N,\,D,\,c\right)&=\lim_{\alpha\to\infty}\sum_{j\in[i,\,b]_D}c_j\cdot\alpha^{\left|[i,\,j]_D\right|-1}\prod_{k\in[i,\,j]_D}\frac{1}{1+\alpha\left|N_D^-(k)\right|}\\&=\sum_{j\in[i,\,b]_D}c_j\cdot\lim_{\alpha\to\infty}\left(\alpha^{\left|[i,\,j]_D\right|-1}\prod_{k\in[i,\,j]_D}\frac{1}{1+\alpha\left|N_D^-(k)\right|}\right).\end{aligned}$$

Moreover, if $i\in T(D)$, $\lim\limits_{\alpha\to\infty}URS_i^{\alpha}\left(N,\,D,\,c\right)=\sum\limits_{j\in[i,\,b]_D}\frac{c_j}{\prod\limits_{k\in[i^+,\,j]_D}\left|N_D^-(k)\right|}=URS_i^{\infty}\left(N,\,D,\,c\right)$. If $i\in N\setminus T(D)$, then $\lim\limits_{\alpha\to\infty}URS_i^{\alpha}\left(N,\,D,\,c\right)=\sum\limits_{i\in N}c_j\cdot 0=0=URS_i^{\infty}\left(N,\,D,\,c\right)$.

□

Theorem 5.6. The ∞-URS method is the only cost sharing method that satisfies efficiency, additivity, the inessential agent property, collusion neutrality, clean source full-transfer, and clean source symmetry.

Proof. It is immediate from Definition 5.4 that the ∞-URS method satisfies additivity, the inessential agent property, clean source symmetry and the clean source full-transfer property.

Furthermore, since every α-URS method is efficient, by the sum law of limits, for every $(N,\,D,\,c)\in\mathcal{R}$,

$$\begin{aligned}\sum_{i\in N}URS_i^{\infty}\left(N,\,D,\,c\right)&=\sum_{i\in N}\lim_{\alpha\to\infty}URS_i^{\alpha}\left(N,\,D,\,c\right)\\&=\lim_{\alpha\to\infty}\sum_{i\in N}URS_i^{\alpha}\left(N,\,D,\,c\right)=\lim_{\alpha\to\infty}\sum_{i\in N}c_i=\sum_{i\in N}c_i.\end{aligned}$$

Similarly, since the α-URS methods are collusion neutral, for any $(N,\, D,\, c) \in \mathcal{R}$ and $i \in N$, it follows that

$$\begin{aligned}\sum_{j \in N_D^{i-}} URS_j^{\infty}\left(N,\, D,\, c\right) &= \sum_{j \in N_D^{i-}} \lim_{\alpha \to \infty} URS_j^{\alpha}\left(N,\, D,\, c\right) = \lim_{\alpha \to \infty} \sum_{j \in N_D^{i-}} URS_j^{\alpha}\left(N,\, D,\, c\right)\\ &= \lim_{\alpha \to \infty} URS_i^{\alpha}\left(N^{i*},\, D^{i*},\, c^{i*}\right) = URS_i^{\infty}\left(N^{i*},\, D^{i*},\, c^{i*}\right).\end{aligned}$$

To show uniqueness, we proceed as in Theorem 3.3 up to the induction step. Suppose ψ is a cost sharing method that satisfies efficiency, additivity, the inessential agent property, collusion neutrality, clean source full-transfer and clean source symmetry, and let $(N,\, D,\, c) \in \mathcal{R}$ be such that there is some $i \in N$ such that $c_i > 0$, $N_D^{-}(i) \neq \varnothing$ and $c_j = 0$ for every $j \in N \setminus i$. Given $j \in N_D^{--}(i) \cap T(D)$, consider its set of "siblings", that is, the nodes with which it shares its successor, $T_j(D) = \{t \in T(D) :\, t^+ = j^+\}$. Notice that this defines a partition of the set of top nodes. By clean source symmetry, for every $j \in T(D)$ and $t \in T_j(D)$, we have $\psi_j\left(N,\, D,\, c\right) = \psi_t\left(N,\, D,\, c\right)$. Furthermore, by construction, $c_j = 0\ \forall j \in T(D)$, hence the clean source full-transfer property implies $\psi_{j^+}\left(N,\, D,\, c\right) = 0$.

Thus, for each $j \in N_D^{--} \cap T(D)$ we obtain $|T_j(D)|$ linearly independent equations. If $j,\, k \in N_D^{--} \cap T(D)$ and $k \in T_j(D)$, then the equations derived from j and k are the same. On the contrary, if $k \notin T_j(D)$, then the sets of equations derived from j and k are linearly independent from each other, since they have no variable in common. Hence, we have $\left|N_D^{--} \cap T(D)\right|$ linearly independent equations, as well as $\left|N_D^{--} \setminus T(D)\right|$ of the form (7), and (5). By the same arguments as in Theorem 3.3, the set of all $\left|N_D^{i-}\right|$ equations is linearly independent. Since the system has that same number of unknowns, one for the cost allocated to each $j \in \left|N_D^{i-}\right|$, uniqueness follows. □

Theorem 5.7. A cost sharing method satisfies efficiency, additivity, the inessential agent property, collusion neutrality, clean source symmetry, and clean source proportionality if and only if it is an α-URS method with $\alpha \geq 0$ or $\alpha = \infty$.

Proof. By Theorems 5.3 and 5.6 we already know that URS^{∞} and every URS^{α} with $\alpha \geqslant 0$ satisfy efficiency, additivity, the inessential agent property and collusion neutrality; we also know that the ∞-URS method satisfies clean source symmetry. Now, let $\alpha \geqslant 0$. Since URS^{α} satisfies the α-clean source property, it follows that it also verifies clean source symmetry. It also follows that $\forall\, (N,\, D,\, c)\,,\, (N',\, D',\, c') \in \mathcal{R}$ and $t \in T(D),\ s \in T(D')$ such that $c_t = c'_s = 0$,

$$\begin{aligned}URS_t^{\alpha}\left(N,\, D,\, c\right) \cdot URS_{s^+}^{\alpha}\left(N',\, D',\, c'\right) &= \alpha URS_{t^+}^{\alpha}\left(N,\, D,\, c\right) \cdot URS_{s^+}^{\alpha}\left(N,\, D,\, c\right)\\ &= URS_s^{\alpha}\left(N,\, D,\, c\right) \cdot URS_{t^+}^{\alpha}\left(N',\, D',\, c'\right).\end{aligned}$$

Similarly, for URS^{∞}, due to the clean source full-transfer property,

$$URS_t^{\infty}\left(N,\, D,\, c\right) \cdot URS_{s^+}^{\infty}\left(N',\, D',\, c'\right) = 0 = URS_s^{\infty}\left(N,\, D,\, c\right) \cdot URS_{t^+}^{\infty}\left(N',\, D',\, c'\right).$$

In words, clean source proportionality is also satisfied in all cases.

Conversely, let ψ be a cost sharing method that satisfies the six properties and consider a polluted river problem with just two nodes and a clean source, i.e. $(N,\, D,\, c) \in \mathcal{R}$, with $N = \{t, b\}$, $D = \{(t, b)\}$, $c_t = 0$, and $c_b > 0$. If $\psi_b\left(N,\, D,\, c\right) > 0$, then there is some $\alpha \geq 0$ such that $\psi_t\left(N,\, D,\, c\right) = \alpha\psi_b\left(N,\, D,\, c\right)$. By clean source proportionality, this implies that ψ satisfies the α-clean source property; in turn, by Theorem 5.3, $\psi = URS^{\alpha}$.

If $\psi_b(N, D, c) = 0$, then by efficiency $\psi_t(N, D, c) = c_b > 0$. By clean source proportionality, for every $(N, D, c) \in \mathcal{R}$ and $t \in T(D)$, $\psi_{t^+}(N, D, c) = 0$. That is, ψ satisfies the clean source full-transfer property and, by Theorem 5.6, $\psi = URS^{\infty}$. □

D Logical independence of axioms

The following cost sharing methods show the properties in Theorem 3.3 are logically independent.

(1) The cost sharing method that assigns zero to every node, for every polluted river problem, satisfies all properties but efficiency.

(2) Given $(N, D, c) \in \mathcal{R}$, let $N^{0-}(D)$ be the set of nodes with null cleaning cost such that all of their superiors also have null cleaning cost, that is,

$$N_D^{0-} = \left\{ i \in N : \, c_i = 0 \; \forall j \in N_D^{--}(i) \right\}.$$

Furthermore, for $i \in N_D^{0-}$, let $i^{+0} \in [i, b]_D$ be the node where the first segment with non-zero cleaning cost in the path $[i, b]_D$ starts. If no such segment exists, let $\bar{c}_i = 0$ and $i^{+0} = \varnothing$.[10]

Define the cost sharing method $\overline{LRS}$ for every $(N, D, c) \in \mathcal{R}$ and $i \in N$ by

$$\overline{LRS}_i(N, D, c) = \begin{cases} \frac{\bar{c}_i}{1 + \left| N_D^-(i^{+0}) \cap N^{0-}(D) \right|} \prod\limits_{\substack{j \in \left[i, i^{+0}\right]_D \\ j \neq i^{+0}}} \frac{1}{1 + \left| N_D^-(j) \right|} & \text{if } i \in N^{0-}(D) \\ \frac{c_i}{1 + \left| N_D^-(i) \cap N^{0-}(D) \right|} & \text{otherwise.} \end{cases}$$

The $\overline{LRS}$ method satisfies all the properties in Theorem 3.3 but additivity.[11]

(3) Consider the following modification of the URES method. Given $(N, D, c) \in \mathcal{R}$, let c' be the cost vector where $c'_b = \sum\limits_{i \in N} c_i$, and $c'_i = 0$ otherwise. Let $\overline{URES}(N, D, c) = URES(N, D, c')$ for every $(N, D, c) \in \mathcal{R}$. The $\overline{URES}$ method satsifies all the properties in the theorem except the inessential agent property.

(4) The LRS method satisfies all properties but the clean source property.

(5) The UES method satisfies all properties but collusion neutrality.

The following cost sharing methods show the properties in Theorem 4.1 are logically independent.

(1) The cost sharing method that assigns zero to every node, for every polluted river problem, satisfies all properties but efficiency.

(2) Consider the following modification of the LRS method, call it $\widetilde{LRS}$:

$$\widetilde{LRS}_i(N, D, c) = \begin{cases} 0 & \text{if } i \in T^0(D) \\ \frac{c_i}{1 + \left| N_D^-(i) \cap (T(D) \setminus T^0(D)) \right|} & \text{otherwise.} \end{cases}$$

The $\widetilde{LRS}$ method satisfies all properties in the Theorem except for additivity.

[10] In such case, we will abuse notation and set $N_D^-(\varnothing) = \varnothing$ and $[i, \varnothing]_D = \varnothing$. This will ensure that the inessential agent property is not violated.

[11] Note that on every $(N, D, c) \in \mathcal{R}$, the $\overline{LRS}$ method coincides with the LRS method on nodes that are not successors of a node in $N^{0-}(D)$, and applies the URES method with only the closest downstream cost on the rest of nodes.

(3) The method that for every polluted river problem assigns all cleaning costs to the bottom node satisfies all properties but the inessential agent property.

(4) Consider the following modification of the UES method, call it $\widetilde{UES}$:

$$\widetilde{UES}_i(N,\,D,\,c)=\begin{cases} c_i & \text{if } i\in T(D)\\ \sum\limits_{j\in[i,\,b]_D}\frac{c_j}{\left|N_D^{j-}\backslash T(D)\right|} & \text{otherwise.}\end{cases}$$

The $\widetilde{UES}$ method satisfies all properties but collusion neutrality.

(5) The URES method satisfies all properties but no blind cost at sources.

For a given $\alpha \geqslant 0$, the following cost sharing methods show the properties in Theorem 5.3 are logically independent.

(1) The cost sharing method that for every $(N,\,D,\,c)\in\mathcal{R}$ assigns $\frac{1}{2}URS_i^{\alpha}(N,\,D,\,c)$ to each $i\in N$ satisfies all properties but efficiency.

(2) Consider the following modification of the $\overline{LRS}$ method introduced earlier in this appendix, call it $\overline{LRS}^{\alpha}$. Following the same notation used for the $\overline{LRS}$, for every $(N,\,D,\,c)\in\mathcal{R}$ and $j\in N$, let

$$\overline{LRS}_j^{\alpha}(N,\,D,\,c)=\begin{cases} \frac{\overline{c_i}}{1+\alpha\left|N_D^-(i^{+0})\cap N^{0-}(D)\right|}\prod\limits_{\substack{j\in\left[i,\,i^{+0}\right]_D\\ j\neq i^{+0}}}\frac{\alpha}{1+\alpha\left|N_D^-(j)\right|} & \text{if } i\in N^{0-}(D)\\ \frac{c_i}{1+\alpha\left|N_D^-(i)\cap N^{0-}(D)\right|} & \text{otherwise.}\end{cases}$$

The $\overline{LRS}^{\alpha}$ method satisfies all the properties in Theorem 5.3 but additivity.

(3) Consider the following modification of the α-URS method. Given $(N,\,D,\,c)\in\mathcal{R}$, let c' be the cost vector where $c'_b=\sum\limits_{i\in N}c_i$ and $c'_i=0$ otherwise. Let $\overline{URS}^{\alpha}(N,\,D,\,c)=URS^{\alpha}(N,\,D,\,c')$ for every $(N,\,D,\,c)\in\mathcal{R}$. The $\overline{URS}^{\alpha}$ method satisfies all the properties except the inessential agent property.

(4) Consider the following modification of the UES method, call it $\widetilde{UES}^{\alpha}$:

$$\widetilde{UES}_i^{\alpha}(N,\,D,\,c)=\begin{cases} c_i+\frac{\alpha}{1+\alpha\left|N_D^-(i^+)\right|}\sum\limits_{j\in[i^+,\,b]_D}\frac{c_j}{\left|N_D^{j-}\backslash T(D)\right|} & \text{if } i\in T(D)\\ \frac{1}{1+\alpha\left|N_D^-(i)\right|}\sum\limits_{j\in[i,\,b]_D}\frac{c_j}{\left|N_D^{j-}\backslash T(D)\right|} & \text{if } N_D^-(i)\cap T(D)\neq\varnothing\\ \sum\limits_{j\in[i,\,b]_D}\frac{c_j}{\left|N_D^{j-}\backslash T(D)\right|} & \text{otherwise.}\end{cases}$$

The $\widetilde{UES}^{\alpha}$ method satisfies all properties but collusion neutrality.

(5) For any $\beta\geqslant 0$, $\beta\neq\alpha$, the β-URS method satisfies all properties but the α-clean source property.

The following cost sharing methods show the properties in Theorem 5.6 are logically independent.

(1) The cost sharing method that assigns zero to every node, for every polluted river problem, satisfies all properties but efficiency.

(2) Let $\overline{LRS}^{\infty} = \lim_{\alpha \to \infty} \overline{LRS}^{\alpha}$. It can be shown that this limit exists and for every $(N, D, c) \in \mathcal{R}$ and $i \in N$ satisfies

$$\overline{LRS}_i^{\infty}(N, D, c) = \begin{cases} \frac{\overline{c_i}}{\left|N_D^-(i^{+0}) \cap N^{0-}(D)\right|} \prod\limits_{\substack{j \in \left[i,\, i^{+0}\right]_D \\ j \neq i^{+0}}} \frac{1}{\left|N_D^-(j)\right|} & \text{if } i \in N^{0-}(D) \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, the $\overline{LRS}^{\infty}$ method satisfies all the properties in Theorem 5.6 but additivity.

(3) The method that for every polluted river problem assigns all cleaning costs to the bottom node satisfies all properties but the inessential agent property.

(4) The method that equally divides all costs among the sources of the river satisfies all properties but collusion neutrality.

(5) For any $\alpha \geqslant 0$, the α-URS method satisfies all properties except clean source full-transfer.

(6) Let l be a function that maps each network $(N, D) \in \mathcal{D}$ to one of its top nodes so that, for every $(N, D) \in \mathcal{D}$, $i \in N$,

$$l\left(N_D^{i-}, D\left[N_D^{i-}\right]\right) = \begin{cases} i \text{ if } i \in [l(N, D), b]_D \\ l(N, D) \text{ otherwise.} \end{cases}$$

The method that assigns all costs to $l(N, D)$ satisfies all properties but clean source symmetry.

For efficiency, additivity, the inessential agent property, collusion neutrality and clean source symmetry, the counterexamples for Theorem 5.6 above show each of these axioms is logically independent from the rest. Finally, any geometric sharing method γ^{λ} with $0 < \lambda < 1$ satisfies all properties except for clean source proportionality.