

Proportionality in Thumbs Up and Down Voting

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Abstract

In the classic committee election setting each voter approves a subset of candidates and the goal is to select k winners based on these preferences. A central focus of recent research in the area has been to achieve proportional representation, as formalized by the family of Justified Representation (JR) axioms. In this work, we explore notions of proportionality in a more expressive setting that allows voters to downvote candidates—a common feature on online polling platforms and beyond. We propose two conceptually distinct interpretations of down votes, resulting in different perspectives of proportionality. In the first, preventing the election of disapproved candidates is as important to voters as electing approved ones. In the second, approvals and disapprovals are treated separately, with each receiving its own fairness guarantees. For each approach, we introduce suitable axioms capturing proportionality and examine their satisfiability by appropriate variants of Phragmén’s rule, Proportional Approval Voting rule (PAV) and the Method of Equal Shares (MES).

1 Introduction

Consider the problem of selecting up to k winners among a set of m candidates based on voters’ preferences. Voters can *approve* (upvote) or *disapprove* (downvote) a candidate, or otherwise *abstain*. The objective is to select a committee in a *proportional* manner, which roughly means that *each group of voters with sufficiently similar preferences should have an influence on the outcome in proportion to its size*. In recent years, the proportional selection problem for approval ballots (upvotes) has become one of the most active research directions and core challenges in computational social choice [23, 15, 27]—we address the open challenge of allowing voters to also disapprove candidates.

In the classic approval voting framework, the ballot design of approval ballots lacks the nuance to capture whether not approving a candidate corresponds to indifference or disapproval towards that candidate. Incorporating down votes has been identified as an important open problem in the field [13, 23, 30]. Indeed, it introduces significant complexity, the primary conceptual challenge being to capture proportionality in the presence of down votes. To make this explicit, we consider the following simple instance with up and down votes, which will serve as a running example throughout the paper.

Example 1. Consider a *committee election in the thumbs-up/down setting* where there are 27 candidates, divided into three groups, namely C_1 , C_2 , and C_3 , and 100 voters, divided into two disjoint groups, namely V_1 and V_2 . The preferences of the voters over the candidates’ sets appear below.

	C_1 (10 candidates)	C_2 (7 candidates)	C_3 (10 candidates)
V_1 (60 voters)	neutral	approve (✓)	approve (✓)
V_2 (40 voters)	approve (✓)	disapprove (✗)	neutral

Say that the size of the committee to be elected is $k = 10$. If we ignore the negative votes and apply the naïve utilitarian rule that simply selects the k candidates with the highest approval counts, we would end up choosing candidates exclusively from $C_2 \cup C_3$. This would be unfair, as it would effectively disregard the preferences of the voters in V_2 . Moreover, not only would the voters in V_2 fail to elect any of their preferred candidates, but also (depending on the

tie-breaking¹) the selected outcome could include all the candidates they oppose. The solution that includes 6 candidates from C_2 and 4 candidates from C_1 is a more fair one since each group enjoys a satisfaction proportional to its size. However, it also disregards negative votes. Selecting candidates from C_2 causes dissatisfaction among the voters in V_2 and, hence, candidates from C_3 appear to be a better fit than those from C_2 .

How many candidates from C_3 should then be elected? This depends on how we interpret voters’ disapproval statements. One could argue that electing candidates from C_3 instead of C_2 should not affect the entitlement of group V_2 to 4 candidates from C_1 . On the other hand, one might say that V_2 gains satisfaction from blocking the election of candidates they dislike and should “pay” for this influence. Under this view, a proportional outcome might include fewer than 4 candidates from C_1 . These two intuitions reflect the two distinct models of proportionality we propose and examine. $\perp$

Motivation.

There is significant evidence that negative voting influences voting behavior and is often desired in practice. A study on the 2020 U.S. Presidential election found that approximately a third of American voters cast their ballots *more “against” a candidate than “for” one* [18]. Another example which highlights the importance of incorporating negative feedback comes from *participatory budgeting (PB)* [30], a form of citizen participation employed in cities where citizens directly decide how a portion of a public budget is spent. While some cities, like Madrid [14], allow voters to express opposition to projects, most PB elections do not. This has led to issues, as seen, for instance, when residents of a large Warsaw housing estate protested against the construction of a playground selected through the PB process [35]. Since the formal model of PB generalises committee elections, our work is a fundamental step towards incorporating negative voting in the PB model while maintaining guarantees on proportional representation. Indeed, some of our proposed rules can already be applied in PB.

Further examples where negative voting has been used in practice include *Decentralized Autonomous Organizations* (e.g., projectcatalyst.io) and *civic participation via digital platforms* (e.g., pol.is). Moreover, in constitutional AI [7], a *constitution*—a slate of ethical principles—is used to align LLMs with human values. For this purpose, Anthropic and the [Collective Intelligence Project](https://collectiveintelligenceproject.org) recently used a civic participation approach to draft a constitution based on the preferences of approximately 1000 Americans. The process involved participants voting on ethical principles by approving, disapproving or abstaining, which were later aggregated to form the final constitution [2].

Contribution.

Our main conceptual contribution consists in proposing two (incomparable) approaches to defining proportionality in the presence of down votes. The two approaches differ in how (and whether) the utilities from electing preferred candidates and from blocking the undesirable ones compare. We introduce suitable proportionality axioms for each, and we design efficient voting rules that generalize existing ones from the approval setting, specifically *Phragmén’s rule*, *Proportional Approval Voting rule (PAV)*, and the *Method of Equal Shares (MES)*. Finally, we provide positive and negative results on these rules concerning the formulated axioms. Our analysis employs several new techniques that are potentially applicable in broader contexts, e.g., in the presence of diversity constraints.

In our first interpretation of down votes, not selecting a disapproved candidate is considered just as important to a voter as selecting an approved one. We call this the *symmetric utility model*. Here, for each candidate c we introduce its virtual negative counterpart $\neg c$, and impose constraints to ensure that c and $\neg c$ are never selected together. Each voter’s utility is additive, derived either

¹A more complex version of our example which does not rely on tie-breaking is deferred to the full version of this work [22].

from including c in the winning committee if approved, or $\neg c$ if the voter disapproves of c . In this context, we first exploit the notion of *ℓ -cohesiveness*, as defined by Masařík et al. [25], which captures when a group S deserves ℓ representatives under general constraints for a committee. This is the core idea of proportionality axioms such as *Base Proportional Justified Representation* (Base PJR), *Base Extended Justified Representation* (Base EJR) and *proportionality degree* [25]. The definition of ℓ -cohesiveness for general constraints is challenging due to its abstract nature, so we first provide a characterization of it in the context of thumbs-up/down voting, almost tight for small groups of voters. Then, we introduce a variant of Phragmén’s rule, we show that it satisfies Base PJR and we provide an almost-tight guarantee for the proportionality degree of the rule. We then move to the PAV rule, and, while it follows from Masařík et al. [25] that it fails Base EJR in our setting, as our main technical contribution we show that this rule, nonetheless, gives near-optimal proportionality guarantees for ℓ -cohesive groups of voters.

Our second interpretation of down votes gives rise to a fundamentally different approach, called the *asymmetric utility model*. The distinction stems from the view that voters deserve to be represented by members of the elected committee, regardless of whether they have already influenced the outcome by blocking the selection of certain disliked candidates. Thus, we treat the right of a voter to be represented and their right to block candidates they oppose asymmetrically. Our axioms for this model are based on an adaptation of the idea of *ℓ -cohesiveness* in which a group’s voting power is determined not merely by its size but by its *effective size*—a measure that accounts for the number of opponents of the candidates the group supports. Our positive results build on priceable voting rules, where the election can be viewed as a process where voters “purchase” candidates [27]. It turns out that MES and Phragmén’s rule can be adapted to satisfy our fairness objectives, by introducing the concept of a “*virtual tax*” imposed by opponents of the candidates, which effectively increases the cost of disapproved candidates. In contrast, we show that there is no natural adaptation of PAV to this setting.

Related Work.

Proportionality in committee elections is a central topic in computational social choice [23, Chapter 4]. The election framework where voters can both up and down vote candidates or abstain (often referred to as *ternary* or *trichotomous voting*) has been already studied [36, 10, 1, 19, 37, 8, 9, 26, 24]. However, to our knowledge, the only existing work merging both topics is by Talmon and Page [33]. The authors introduce seven proportionality axioms, which differ significantly from the ones proposed in our work. They further examine variants of the Monroe, Chamberlin-Courant, STV, and PAV voting rules, all adapted to the examined setting. None of these rules satisfy any of the considered axioms; instead the authors rely on simulations to examine how often the rules produce outcomes that align with their proportionality criteria. Therefore, our work is the first to provide theoretical guarantees for the examined problem.

Our problem can be viewed as a special case of *elections with general feasibility constraints*, a setting in which the concept of proportionality has been recently explored. Masařík et al. [25] adapt PAV and Phragmén to that framework and show these rules satisfy desirable proportionality axioms provided the constraints form *matroids*. These results do not apply to our setting, as our feasibility constraints in the symmetric setting do not have a matroid structure. We defer a detailed discussion to the full version of our paper [22]. The setting we examine can also be viewed as a special case of voting under *weak preference orders* or *cardinal preferences* (cf. [3, 28]). For more details on the crucial differences among these formats and the one we examine, we also refer to the full version.

A related line of research is *public decisions*, where the voters submit ballots as in trichotomous voting and proportional considerations have been studied in this framework [17, 32, 11]. A key distinction is that in the model of public decisions there is no upper bound on the number of selected candidates; this is enough to make the two frameworks critically different, both conceptually and technically.

2 Preliminaries

An *election* E is a tuple (C, V, k, B) , where $C := \{c_1, c_2, \dots, c_m\}$ is the set of candidates, $V := \{1, 2, \dots, n\}$ is the set of voters, $k \leq m$ is an integer corresponding to an upper bound on the number of candidates to be elected, and $B := (B_i)_{i \in V}$, with B_i being the ballot submitted by the i -th voter. Since voters are allowed to express both positive and negative preferences for candidates, each ballot B_i is represented as a pair $B_i = (A_i, D_i)$, with $A_i, D_i \subseteq C$ and $A_i \cap D_i = \emptyset$. A_i denotes the set of candidates *approved by* voter i , and D_i is the set of candidates *disapproved/vetoed by* this voter. If a candidate $c \notin A_i \cup D_i$, we say that voter i is *indifferent* or *neutral* toward c .

For each candidate $c \in C$, let A_c and D_c denote the sets of voters who approve and disapprove c , respectively. Formally, $A_c := \{i \in V : c \in A_i\}$ and $D_c := \{i \in V : c \in D_i\}$. For each group of voters $S \subseteq V$, let A_S and D_S denote the sets of candidates commonly approved and disapproved by S , respectively. Formally, $A_S := \bigcap_{i \in S} A_i$ and $D_S := \bigcap_{i \in S} D_i$. An election rule is a function $\mathcal{R}$ which for each election E returns a set of at most k candidates: a *winning outcome/committee*. Clearly if $D_i = \emptyset$ for every voter $i \in V$, the considered election is a classic *approval-based committee election*.

Below we briefly describe election rules that are known to perform particularly well in terms of proportionality in the approval setting [23]. In the subsequent sections, we focus on designing generalizations of these rules that also perform well in the presence of negative ballots.

Phragmén’s rule [29, 20]. Each candidate is assumed to have a unit cost. Each voter has a virtual budget starting at 0 and increasing continuously over time. At time t , each voter has been allocated a total budget of t . When a group of voters approving a common candidate accumulates enough budget to cover the candidate’s cost, the candidate is added to the winning committee and the budgets of the voters involved in the purchase are reset to 0. This process continues until k candidates are selected.

Thiele rules [34, 20] and **Proportional Approval Voting (PAV)** [5, 6, 27]. Each rule in the class of Thiele methods is parameterized by a nondecreasing function $f : \{0, 1, \dots, k\} \rightarrow \mathbb{R}_{\geq 0}$ with $f(0) = 0$. A voter i assigns a score of $f(|W \cap A_i|)$ to a committee W . The winning outcome is the committee of size k that maximizes $\sum_{i \in V} f(|W \cap A_i|)$. PAV is the rule in the class of Thiele methods for which $f(x) = \sum_{j=1}^x 1/j$.

Method of Equal Shares (MES) [27]. The rule works in rounds. Let b_i be the virtual budget of voter i , initially set to k/n . Each candidate is assumed to have a unit cost. In each round, we consider every not yet elected candidate c whose supporters have at least a total budget that suffices to buy c . We say that such a candidate c is ρ -affordable for $\rho \in \mathbb{R}_+$ if $\sum_{i \in A_c} \min(b_i, \rho) = 1$. The candidate that is ρ -affordable for the smallest value of ρ is selected and added to the committee. The budgets of her supporters are then updated accordingly, i.e., $b_i := b_i - \min(b_i, \rho)$. The rule stops if there is no ρ -affordable candidate for any value of ρ .

3 The Symmetric Utility Model

In the *symmetric utility model*, a voter values the inclusion of supported candidates in the winning committee equally to the exclusion of disliked candidates. To formalize this notion, we introduce the concept of a virtual negative candidate. Specifically, for each candidate c , we define its negative counterpart $\neg c$. An outcome W then consists of $x \leq k$ positive candidates and at most $m - x$ negative ones, ensuring that if $c \in W$, then $\neg c \notin W$ and that if $\neg c \in W$, then $c \notin W$. These impose feasibility constraints on the election outcome.

We now introduce some additional useful notation: for each set $X \subseteq C$ we define $\neg X := \{\neg c \mid c \in X\}$. For each voter i , we define $U_i := A_i \cup \neg D_i$. Thus, U_i represents the set of both positive and negative candidates that can contribute to the voter’s satisfaction. Given a group S and a

committee W , we define the utility of a voter $i \in S$ from W as $u_i(W) := |A_i \cap W| + |\neg D_i \cap W|$, and the average satisfaction of voters from S as $\text{avgsat}_S(W) := 1/|S| \cdot \sum_{i \in S} u_i(W)$.

In this section, we demonstrate that both Phragmén's rule and PAV continue to provide strong guarantees of proportionality in the presence of negative votes, under the symmetric utility interpretation.

3.1 The Base Extended Justified Representation Axiom

We begin by recalling the axiom of Base Extended Justified Representation (Base EJR) from the work of Masařík et al. [25]. For that, we first highlight that it gives the guarantees that the prominent axiom of Extended Justified Representation (EJR) [5] would give, when we restrict attention to instances with $D_i = \emptyset$, for every voter i , i.e. in the approval setting. EJR is the concept that has received the most attention in the computational social choice literature when it comes to formalizing proportionality. Base EJR roughly says that a set of voters S deserves ℓ candidates if they can complete any reasonable selection made by the remaining voters, with ℓ commonly approved candidates. For a more detailed discussion and intuitive interpretations we refer to [25]. This axiom was introduced for the model with general feasibility constraints and we now adapt it to our setting. Let $\mathcal{F}$ be a collection of feasible outcomes. In our case, $\mathcal{F} = \{W \subseteq C \cup \neg C \text{ such that } |W \cap C| \leq k \text{ and } \neg(W \cap C) \cap (W \cap \neg C) = \emptyset\}$.

Definition 1 (Base Extended Justified Representation (Base EJR)). Consider an election E . Given a positive integer ℓ , we say that a set of voters $S \subseteq V$ is ℓ -cohesive if for each feasible solution $T \in \mathcal{F}$ either there exists a set $X \subseteq A_S \cup \neg D_S$ with $|X| \geq \ell$ such that $T \cup X$ remains feasible, or $|S|/n > \ell/(|T| + \ell)$. An outcome W is said to provide Base EJR for E if for every ℓ -cohesive set of voters S there exists a voter i with a satisfaction $u_i(W) \geq \ell$. A rule $\mathcal{R}$ satisfies Base EJR if for every election E its winning outcome provides Base EJR. $\square$

We note that this axiom is always satisfiable and generalises EJR for approval-based committee elections [25]. Observe that the definition of Base EJR requires considering all possible sets T , which often makes it difficult to interpret in terms of the actual number of representatives a group S is entitled to. Our first technical lemma addresses this limitation by providing a closed formula estimating this value.

Lemma 1. Consider an election E . A set of voters $S \subseteq V$ is ℓ -cohesive if:

$$\ell \leq \begin{cases} |D_S| - k, & \text{if } \frac{n}{n-|S|}k \leq |D_S|. \\ \frac{|S|}{2n-|S|}(|D_S| + k), & \text{if } \frac{n-|S|}{n}k \leq |D_S| \leq \frac{n}{n-|S|}k \text{ and } \frac{2n-|S|}{n}|A_S| + \frac{n-|S|}{n}|D_S| \geq k. \\ \frac{|S|}{n}k, & \text{if } |A_S| + |D_S| \geq k \text{ and } |D_S| \leq \frac{n-|S|}{n}k \text{ and } |A_S| \leq m - \frac{n-|S|}{n}k. \\ |A_S| + k - m, & \text{if } |A_S| + |D_S| \geq k \text{ and } |D_S| \leq \frac{n-|S|}{n}k \text{ and } \\ & m - \frac{n-|S|}{n}k \leq |A_S| \text{ and } |A_S| + k - m \leq \frac{|S|}{n}(|A_S| + |D_S|). \\ \frac{|S|}{n}(|D_S| + |A_S|), & \text{otherwise.} \end{cases}$$

All the bounds are tight up to the factor of $|S|/n - |S|$.

Proof Sketch. Consider an election E and a group $S \subseteq N$. Let us denote by $\text{claim}(S)$ the upper bound of ℓ for S in the statement of the lemma. To prove the statement, we will show that each group of voters $S \subseteq V$ is $\text{claim}(S)$ -cohesive and is not $(\text{claim}(S) + 1)$ -cohesive.

Consider a feasible subset of candidates $T \subseteq C$ and let T^+ and T^- denote the parts of T that consist of the positive and negative candidates, respectively. If we have that $\frac{|S|}{n} > \frac{\text{claim}(S)}{\text{claim}(S) + |T|}$,

then S is entitled to a satisfaction of $\text{claim}(S)$. From now on let us assume that the opposite inequality holds, which is equivalent to:

$$|T| \leq \frac{n - |S|}{|S|} \cdot \text{claim}(S). \quad (1)$$

Moreover, let us assume that the expression $\frac{n - |S|}{|S|} \cdot \text{claim}(S)$ is integral, that is, it is possible to find set T of exactly this size. Whenever it is true, the estimations for $\text{claim}(S)$ will be tight. Otherwise, since the space of possible sets T is smaller, the actual value of claim might be larger by less than:

$$\begin{aligned} \text{claim}(S) - \frac{|S|}{n - |S|} \cdot \left\lfloor \frac{n - |S|}{|S|} \cdot \text{claim}(S) \right\rfloor \\ = \frac{|S|}{n - |S|} \cdot \left(\frac{n - |S|}{|S|} \cdot \text{claim}(S) - \left\lfloor \frac{n - |S|}{|S|} \cdot \text{claim}(S) \right\rfloor \right) < \frac{|S|}{n - |S|}. \end{aligned}$$

So we want to show that for any such a choice of T , there is a set $X \subseteq A_S \cup \neg D_S$ that can be proposed by group S such that $T \cup X$ is feasible and $|X| = \text{claim}(S)$ and there is no such larger set X . This will give us the desired result, because such a set T certifies that group S is not $(\text{claim}(S) + 1)$ -cohesive as **Formula (1)** is also satisfied for any $\ell > \text{claim}(S)$.

Now, subject to the constraint of **Formula (1)** we will choose T adversarially, so as to minimize the size of the largest such set X . Given the set T , group S should be able to propose a set of candidates $X \subseteq A_S \cup \neg D_S$ of size $\text{claim}(S)$ (but not any larger) such that $T \cup X$ is feasible. Due to the committee size constraint, the set X contains at most $k - |T^+|$ candidates from A_S . Furthermore, $X \cup T$ must not include both a and $\neg a$ for any $a \in C$. Hence the maximum sized set X satisfies $|X \cap \neg D_S| \leq |D_S \setminus T^+|$ and $|X \cap A_S| \leq |A_S \setminus \neg T^-|$.

From these inequalities, we obtain the following formula for the maximum size of set X that S can propose so that $T \cup X$ is feasible:

$$\begin{aligned} & |D_S \setminus T^+| + \min(|A_S \setminus \neg T^-|, k - |T^+|) \\ = & |D_S| - |T^+| + |T^+ \setminus D_S| + \min(|A_S|, k - |T^+| + |A_S \cap \neg T^-|) - |A_S \cap \neg T^-|. \end{aligned}$$

Additional to the size constraint (**Formula (1)**), we now impose further restrictions on T and show that these do not decrease the size of the largest set $X \subseteq A_S \cup \neg D_S$ for which $X \cup T$ is feasible. We can assume that T has the following properties without loss of generality: $T^+ \cap A_S = \emptyset$ and $T^- \subseteq \neg A_S$. To justify this claim, observe that removing a candidate from T does not affect the satisfiability of **Formula (1)**. Removing the candidates from $T^+ \cap A_S$ does not increase the maximum possible size of X (i.e., the number of “slots” in X available for the candidates from $A_S \cup \neg D_S$). The same holds for removing the candidates from $T^- \setminus \neg A_S$. Moreover, if there is a set $X \subseteq A_S \cup D_S$ of size at least $\text{claim}(S)$, then this also holds for any subset of T . Therefore, we have that T^- contains only candidates from $\neg A_S$, and, hence, $|A_S \cap \neg T^-| = |\neg T^-| = |T^-|$. Hence, we obtain that the maximum size of X is:

$$\begin{aligned} & |D_S| - |T^+| + |T^+ \setminus D_S| + \min(|A_S|, k - |T^+| + |A_S \cap \neg T^-|) - |A_S \cap \neg T^-| \\ = & |D_S| - |T^+| + |T^+ \setminus D_S| + \min(|A_S|, k - |T^+| + |T^-|) - |T^-| \\ = & |D_S| + |T^+ \setminus D_S| + \min(|A_S|, k - |T^+| + |T^-|) - |T|. \end{aligned} \quad (2)$$

Let us now check for which sets T the **Formula (2)** is minimal. Any candidate, positive or negative, which is included in T contributes -1 to the term $-|T|$. Among those, only positive candidates included in T contribute -1 to the term $k - |T^+| + |T^-|$ and thereby potentially reduce the size of X (this is in contrast to negative candidates, which increase this term). Finally, only positive candidates in D_S additionally contribute 0 to $|T^+ \setminus D_S|$ if included in T^+ while

all other positive candidates contribute 1. Thus, to minimize the formula, we always prefer to add candidates from D_S to T^+ instead of the ones from $C \setminus D_S$ (subject to [Formula \(1\)](#) and committee constraint k). So, if possible, we choose T to be of size $\frac{n-|S|}{|S|} \cdot \text{claim}(S)$ and $T = T^+ = D_S$. If $|D_S|$ or k are too small (that is, smaller than $\frac{n-|S|}{|S|} \cdot \text{claim}(S)$, so that using up to k possible candidates from D_S is insufficient), we will sometimes fill the remaining slots in T with candidates from $C \setminus D_S \setminus A_S$ to T^+ or the ones from $\neg A_S$ to T^- .

The remainder of the proof requires an extensive case analysis. $\square$

Given an election E and a set of voters S , [Lemma 1](#) allows for a direct computation of the satisfaction that S is entitled to; this is in contrast to the definition of Base EJR.

Continuation of [Example 1](#). For $S = V_1$ we have that $|D_S| = 0$, $|A_S| + |D_S| = |A_S| = 17$, which is greater than $k = 10$ and smaller than $m - \frac{n-|S|}{n}k = 23$. Hence, V_1 is entitled at least to a satisfaction of $|S|/n \cdot k = 6$. On the other hand, for $S = V_2$ we have that $|D_S| = 7 > \frac{n-|S|}{n}k$, $|A_S| + |D_S| = 17 > k$ and $|A_S| = 10 < m - \frac{n-|S|}{n}k$. Hence, V_2 is ℓ -cohesive for $\ell \leq |S|/n \cdot (|A_S| + |D_S|) = 34/5$, which means that they are entitled at least to a satisfaction of 6. Note that this fact prevents an EJR-compliant rule from electing 6 candidates from C_2 , as then it would be impossible for V_2 to get a satisfaction of 6. $\lrcorner$

While the formula from [Lemma 1](#) appears to be quite complex—because it succeeds in precisely accounting for all edge cases, particularly those where the sizes of sets A_S and D_S are disproportionate—its core logic can be captured by a much simpler equation as our next result demonstrates. The estimation we provide is helpful for our later proofs. It is also insightful on its own as it clarifies the guarantees provided by Base EJR in our context, and affirms their alignment with the intuitive understanding of proportionality. The maximum satisfaction attainable, given the part of the preference profile on which the voters in S agree, is $(|D_S| + \min(k, |A_S|))$; thus, the group deserves a proportional share of this maximum satisfaction, subject to their agreement in preferences. Additionally, we need to include the factor of $|S|/n - |S|$ following from the potential non-tightness of the bounds in [Lemma 1](#).

Lemma 2. *Consider an election E and fix any ℓ -cohesive group of voters S . For each k -element subset $W \subseteq C$, there exists a voter $i \in S$ such that $u_i(W) \geq \ell$ or*

$$\ell < \frac{|S|}{n} \left(|D_S| + \min(k, |A_S|) + \frac{n}{n - |S|} \right).$$

3.2 Phragmén’s Rule

Recall the definition of Phragmén’s rule for approval-based committee elections from [Section 2](#). A natural first approach to extend Phragmén’s rule to the setting of negative votes could be as follows. Run two parallel elections: one (positive) election for positive candidates and one (negative) election for negative ones, with money accumulating at the same rate in both. When a candidate is elected in the positive election, their corresponding negative candidate is removed from the negative election and vice versa. This approach can fail to recognize synergies between groups that derive satisfaction differently, leading to suboptimal outcomes. Indicatively, consider an election with $n = 2\ell$ voters split into two equal groups, S_1 and S_2 . There are four disjoint candidate sets C_1, C_2, C_3, C_4 with $|C_1| = |C_3| = 2\ell$ and $|C_2| = |C_4| = \ell$. Voters in S_1 collectively approve C_1 , and each voter in S_1 down votes a unique candidate from C_4 . S_2 collectively disapproves C_3 , and each voter in S_2 approves a unique candidate from C_2 . According to the discussed variant we only select ℓ candidates from C_1 and C_2 and the negative counterparts of ℓ candidates from C_3 and all from C_4 . A solution that would better satisfy all voters is to select all candidates from C_1 along with the negative counterparts from C_3 . In response, we adapt the rule as follows:

Phragmén’s rule for thumbs-up/down voting. All voters continuously earn money at the same rate. They can spend it not only to elect candidates but also to block those they oppose. If a group of voters collectively accumulates enough funds to veto a (yet unelected) disliked candidate, they do so by purchasing the corresponding negative candidate and their budget is reset. Electing a candidate c costs one unit and simultaneously removes $\neg c$ from consideration. Similarly, purchasing $\neg c$ for one unit removes c . This process continues until no more candidates can be elected.

Phragmén’s rule does not satisfy EJR for the committee setting, hence our extension is not intended to satisfy Base EJR. However, there are two other (weaker) properties concerning ℓ -cohesive groups that are satisfied by the classic version of Phragmén, and—under the definition of ℓ -cohesiveness provided by Masařík et al. [25]—hold also for our extension. The first of them is the axiom called *Base Proportional Justified Representation* (Base PJR) [31, 25]. In contrast to Base EJR, it only ensures that the group as a whole is adequately represented, without guaranteeing that any individual benefits sufficiently. The second one is the high *proportionality degree*, guaranteeing that each ℓ -cohesive group of voters will have high representation on average [31]. The bound we offer for the latter is tight up to a constant of one [25]. Combining these results with its polynomial runtime, makes the rule particularly attractive in the presence of negative ballots as well.

Theorem 3. *Let W be an outcome returned by Phragmén’s rule. For each ℓ -cohesive set of voters S the following conditions are satisfied:*

- (1) $|(\bigcup_{i \in S} A_i \cup \bigcup_{i \in S} \neg D_i) \cap W| \geq \ell$, (Base PJR)
- (2) $\text{avgsat}_S(W) \geq \frac{\ell - 1}{2}$. (proportionality degree)

3.3 Proportional Approval Voting Rule

Another prominent rule for approval-based committee elections is Proportional Approval Voting (PAV). In contrast to Phragmén’s rule, PAV extends straightforwardly to the model with negative votes, under the symmetric interpretation of utilities.

PAV rule for thumbs-up/down voting. It selects a feasible outcome W that maximizes:

$$\sum_{i \in V} f(|W \cap (A_i \cup \neg D_i)|), \quad \text{where} \quad f(x) = \sum_{j=1}^x 1/j.$$

It follows from the work of Masařík et al. [25] (see the construction proving the negative statement of Theorem 7 there) that PAV does not satisfy Base EJR in our setting. However, we prove that it provides a guarantee closely approximating ℓ on average for voters of every ℓ -cohesive set S (the smaller the size of S is, the closer is the guarantee). This means that, even in the case where there is no single voter from S having a utility of at least ℓ (as required by EJR), on average, the voters from S have a utility close to ℓ . Hence we once again establish a strong proportionality guarantee. This is the main result of this section, and the main technical contribution of our paper.

Theorem 4. *Consider an election E and let W be an outcome of PAV. For each ℓ -cohesive $S \subseteq V$:*

$$\text{avgsat}_S(W) \geq (1 - \varepsilon) \left(\ell - \frac{|S|}{n - |S|} \right) - 3/2, \quad \text{where} \quad \varepsilon := \frac{2}{k + 4}.$$

Proof Sketch. We assume, for the sake of contradiction, that there exists an outcome W selected by PAV where a group of voters S achieves a lower satisfaction than stated in the theorem. Typically in the literature, analogous results (c.f., [5, 25]) are proved by considering “one-to-one” swaps between carefully chosen candidates from W and $C \setminus W$. It is then shown that there exists such a swap which increases the PAV score, leading to a contradiction. Instead, in our setting, adding a candidate c to W may require not just removing an arbitrary candidate from W , but also addressing the presence of $\neg c$. As a result, we consider a much broader space of (non-disjoint) exchanges that involve “one-to-many” and “many-to-many” swaps. This consideration of swaps between groups of candidates makes the proof significantly more complex and technically demanding compared to results on proportionality for PAV in the absence of down votes. $\square$

On the negative side, PAV is NP-hard to compute [4]. Nevertheless, for approval-based committee elections, its local-search variant runs in polynomial time while also achieving strong proportionality guarantees [6, 21]. The proof of [Theorem 4](#) also manages to reveal that a local-search approach based on the swaps we employed in the course of the proof works for our setting as well. In general, Base EJR is satisfiable, but by a rather technical algorithm running in exponential time in the number of voters and candidates [25]. In contrast, the guarantee in [Theorem 4](#) also applies to a polynomial-time algorithm. To conclude, it remains a major open problem to develop a suitable generalization of the Method of Equal Shares for the symmetric setting—a question already posed by Masařík et al. [25].

We conclude by noting that Base EJR, Base PJR, and proportionality degree are closely related. In all these definitions, the key element is how cohesiveness is defined, and this element is common for the three definitions. Thus, our results stated for proportionality degree imply analogous results for Base EJR and Base PJR. Therefore, all the results in this section revolve around the notion analyzed in [Section 3.1](#).

4 The Asymmetric Utility Model

The utility model examined in [Section 3](#) enforces symmetry, treating voters as equally concerned with electing approved candidates and preventing the selection of disliked ones. However, this symmetry may not always reflect voters’ true feelings. In some scenarios, voters might rightfully feel entitled to a certain number of representatives in the elected body, regardless of whether their negative votes contributed to blocking certain candidates. To account for this, we introduce the *asymmetric utility model*, where voters’ satisfaction depends only on the election of approved candidates, while disapprovals serve to hinder a candidate’s selection. This imbalance requires introducing separate guarantees with respect to (i) approved, and (ii) disapproved candidates.

4.1 Axioms for Positive Representation and the Group Veto Axiom

In this section, we propose two axioms which formalize voters’ entitlement to (i) representation in the committee and (ii) the ability to veto candidates. Our first axiom ensures guarantees for voters with shared *positive preferences*. It is inspired by the classic Extended Justified Representation (EJR) for approval-based committee elections and reduces to it when $D_i = \emptyset$ for all $i \in V$. We propose a definition of cohesiveness in the examined setting which defines how large a group S must be, in the presence of voters with opposing preferences, to justify electing ℓ representatives of their liking. Specifically, we say that S must have enough voters to outweigh the effect of vetoing voters—meaning that for each down vote, there must be a voter in S to cancel it out—and still remain large enough to claim a fair share.

Definition 2 (ℓ -positive-cohesiveness). Consider an election E . Given a positive integer $\ell \leq k$, we say that a set of voters $S \subseteq V$ is ℓ -positively-cohesive if there exists a set of at least ℓ candidates T such that $T \subseteq A_S$ and $|S| - |D_c| \geq \ell \cdot n/k$ for every candidate $c \in T$. $\dashv$

After defining ℓ -positive-cohesiveness, we can provide the analogs of EJR and PJR in this setting. We say that an outcome W for an election E provides **Extended Justified Positive Representation (EJPR)** if for each ℓ -positively-cohesive group of voters S there exists a voter $i \in S$ such that $|A_i \cap W| \geq \ell$, and **Proportional Justified Positive Representation (PJPR)** if for each ℓ -positively-cohesive group of voters S , it holds that $|\cup_{i \in S} A_i \cap W| \geq \ell$. A rule $\mathcal{R}$ satisfies EJPR (resp. PJPR) if for every election E its winning outcome provides EJPR (resp. PJPR).

Having defined a guarantee for commonly approved candidates, we now aim to define a *negative guarantee* as well. This will ensure that sufficiently large groups of voters also have a right to block candidates they dislike from being selected. The necessity of such an axiom is evident as EJPR can be easily satisfied by disregarding negative ballots—hence dissolving any hopes for negative guarantees—and then applying any rule that satisfies EJR in the approval-based committee elections setting. On the flip side, the naïve rule that always returns the empty committee fully respects down votes, but offers no positive guarantees. Thus, from a rule-design standpoint, the goal is to develop a rule that offers both positive and negative guarantees simultaneously. The axiom we propose, called *Group Veto*, aims to ensure the exclusion from the committee of candidates based on the vetoes they receive from sufficiently large groups of voters. In other words, a strong opposition limits the number of candidates that can be elected from the opposed set.

Definition 3 (Group Veto). Consider an election E . For a set of candidates T , say that $ap(T)$ is the set of voters that approve at least one candidate from T . Given a non-negative integer $\ell \leq k$ and a set of at least ℓ candidates T , we say that a set of voters $S \subseteq V$ is (ℓ, T) -negatively-cohesive if $T \subseteq D_S$ and $|S| \geq |ap(T)| - \ell \cdot n/k$. An outcome W is said to provide Group Veto for E if for each (ℓ, T) -negatively-cohesive group it holds $|W \cap T| \leq \ell$. A rule $\mathcal{R}$ satisfies the Group Veto axiom if for every election E its winning outcome provides Group Veto. $\lrcorner$

We return to our running example to illustrate the axioms we introduced.

Continuation of Example 1. Since for both groups of voters V_1, V_2 , we can find approved sets (respectively) C_3, C_1 that are not opposed by the other group, the guarantees given by both EJPR and PJPR are 6 for V_1 and 4 for V_2 . Note that by themselves they do not say whether the guarantee for V_1 should be satisfied by candidates from C_2 or from C_3 .

Setting $S = V_2$ and $T = C_2$ in the Group Veto definition, we find that $\ell = (|ap(T)| - |S|) \cdot k/n = (|V_1| - |V_2|) \cdot k/n = 2$. As a result, any rule satisfying the Group Veto axiom should elect at most 2 candidates from C_2 . $\lrcorner$

In the remainder of our work we propose rules that fit to the interpretation of utilities in the asymmetric model and we analyze them in terms of whether they satisfy the proposed axioms simultaneously.

4.2 Method of Equal Shares and Phragmén’s Rule with Opposition Tax

According to the original definitions of the Method of Equal Shares and Phragmén’s rule (see [Section 2](#)) each candidate is assumed to have a unit cost. In our generalizations, we increase this cost by introducing an *opposition tax*. In turn, we call the rules *Method of Equal Shares* and *Phragmén with Opposition Tax* (*Tax-MES* and *Tax-Phragmén* for short). The tax captures the idea that the more voters veto a candidate c , the higher the price her supporters need to pay to elect c . As a result, candidates’ prices become unequal, setting our methods apart from their standard analogs. We select the tax appropriately to ensure that (i) among candidates with equal number of supporters, the one with higher net approval is prioritized, and (ii) among those with equal net approval, the one with more supporters is prioritized. Moreover, we focus on electing only candidates with more supporters than opponents; note that this would not necessarily hold for rules designed to fit the symmetric model.

Tax-MES/Tax-Phragmén. Given an election E , we define $C' = \{c \in C : |A_c| > |D_c|\}$. For each $c \in C'$ we set its price to $p(c) := \frac{|A_c|}{|A_c| - |D_c|}$. We create a ballot profile B' such that for each voter $i \in V$, we set $B'_i := (A_i, \emptyset)$, $\forall i \in V$. Then, in (C', V, k, B') we execute MES (respectively, Phragmén) in their standard form. In the case of Phragmén, we stop increasing a voter's budget once she has received k/n in total.

It is straightforward to check that in the instance of [Example 1](#) both tax-MES and tax-Phragmén elect outcomes which provide the earlier defined proportionality axioms. Specifically, since the price is 1 for candidates in $C_1 \cup C_3$ and 3 for C_2 , both rules elect 6 candidates from C_3 and 4 from C_1 . This observation can be generalized as our next result shows. Especially for Tax-Phragmén, we note that it must fail EJPR since for $D_i = \emptyset$ for all $i \in V$ it reduces to classic Phragmén, which fails EJR [12].

Theorem 5. *Tax-MES satisfies EJPR and Group Veto. Tax-Phragmén satisfies PJPR and Group Veto.*

Proof Sketch. The proofs regarding positive representation for tax-MES and tax-Phragmén are both by contradiction: after assuming that the required axiom is violated for an ℓ -positively-cohesive group S , we obtain that at the end of the procedure the group has too much money left in total for that to be possible.

The fact that both rules satisfy Group Veto, follows by lower-bounding the price of each $c \in T$ by a function of $|ap(T)|$ and $|S|$ and by observing that the total budget that the voters in $ap(T)$ may spend is limited. Hence, there is an upper bound in the number of candidates from T which can be selected. We note that the part of the proof regarding Group Veto actually applies to a broad family of (priceable) rules [27] which includes Tax-MES and Tax-Phragmén. $\square$

One could wonder why the definition of tax-Phragmén we propose halts the procedure at time k/n —a condition non-existent in the standard definition of Phragmén's rule. Without it, the rule would always elect k candidates (provided there are at least k candidates with more in-favor than against ballots). This is not always the case under tax-Phragmén, and this is intentional. An adaptation that wouldn't stop the procedure would not satisfy Group Veto and its negative guarantees would be much weaker (see [Appendix B](#)). On the other hand, tax-MES might select fewer than k candidates even in the absence of negative ballots, i.e., for classic MES [16]. Therefore, classic MES is typically paired with a completion strategy [16] to increase the number of selected candidates. However, in the model with negative votes (where adding a candidate can worsen the committee for some voters), the seek for exhaustiveness has a much weaker justification than in the standard setting.

4.3 Generalized Thiele Rules

In [Section 3.3](#) we proved that PAV exhibits strong proportionality guarantees in the symmetric utility model. Interestingly, the rule is not suitable if we take the asymmetric interpretation. In particular, below, we show that PAV cannot be adapted to satisfy EJPR while accounting for negative ballots. This limitation extends to the entire family of Thiele rules when adapted for this setting: the class of *generalized Thiele rules*. Each rule is defined via a mapping f , where $f(z, s)$ specifies the score that a committee obtains from a voter that approves z candidates in it while disapproving s .

Generalized Thiele Rules. A generalized Thiele rule induced by the scoring function $f: \mathbb{N}^2 \rightarrow \mathbb{R}$ selects for each election E a feasible outcome W that maximizes:

$$\sum_{i \in V} f(|W \cap A_i|, |W \cap D_i|).$$

Theorem 6. No generalized Thiele rule induced by a scoring function f such that $f(z, s) < f(z, 0)$ for some $(z, s) \in \mathbb{N}^2$, satisfies EJPR.

Proof Sketch. Consider a generalized Thiele rule induced by a function f for which the desired pair of values $(z, s) \in \mathbb{N}^2$ exists. Assume for the sake of contradiction that the rule satisfies EPJR. Since in the standard setting EJPR is equivalent to standard EJR, and PAV is the only Thiele rule satisfying EJR [5], it holds that $f(x, 0) = \sum_{1 \leq i \leq x} 1/i$ for every $x \in \mathbb{N}$. We construct a family of instances parameterized by z and s . Specifically, for any value of $t > s$, we define an election $E_t = (C, V, k, B)$ as follows: The set of candidates C consists of $3(t + z)$ candidates in total, $2(t + z)$ of which are dummy candidates who are neither approved nor disapproved by any voter. Apart from the dummy candidates, we have 3 disjoint sets of candidates: T_1 of $t - s$ candidates, T_2 of s candidates and T_3 of z candidates. There are two disjoint groups of voters: V_1 approves $T_1 \cup T_2 \cup T_3$, and V_2 approves T_3 and disapproves T_2 . We set $k = 3(t + z)$. EJPR requires that voters in V_1 deserve at least $t + z$ approved candidates in the committee, hence the considered rule should elect all candidates from $T_1 \cup T_2 \cup T_3$. Nevertheless, using the obtained relation for $f(x, 0)$, we show that there exists a value of t such that, in the election E_t , the committee $T_1 \cup T_3$ is better according to f . Such an election E_t serves as the counterexample establishing the theorem. $\square$

The assumption appearing in Theorem 6 simply says that no voter should prefer a committee containing candidates they disapprove over one without (as long as both include the same number of candidates they approve)—a natural restriction. A direct consequence of Theorem 6 is that no generalized Thiele rule satisfies any form of a negative guarantee, such as Group Veto, while also satisfying EJPR. In particular, there does not exist a reasonable (i.e., not neglecting negative ballots) extension of PAV to this setting. Importantly, Theorem 6 also applies to PJPR without any changes to the construction used in its proof. The markedly different behavior of the two PAV variants we study in Section 3 and Section 4 reinforces the intuition that the two interpretations of down votes we employ necessitate for different approaches—both conceptually and technically—when designing proportional voting rules for each utility model.

5 Concluding Discussion

We have presented two formal approaches capturing the idea of proportional representation in the presence of down votes. These approaches are fundamentally different, and the choice between them may depend on the context. Specifically, the symmetric approach is more appropriate in contexts such as selecting a set of blog posts or comments, where the primary goal is to fairly aggregate voters' *opinions* rather than to provide direct representation within the elected committee. In contrast, the asymmetric approach appears better suited for participatory budgeting elections or elections to representative bodies, where ensuring proportional *representation* of voters is a key objective. Moreover, the symmetric setting is appropriate when vote budgets are limited or candidates cannot be duplicated easily. In contrast, the asymmetric setting gives voters incentives to cast negative votes against neutral candidates, making it necessary to constrain the allowed number of down votes. A major conceptual open problem for future work is designing voting rules and axiomatic properties that lie between the two settings. Our adaptations of Phragmén's rule and MES can be directly applied to PB; establishing proportionality guarantees for this setting, in the presence of both up and down votes, is a clear direction for future work.

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Technical Appendix

A Proofs Omitted from the Main Text

Lemma 1. Consider an election E . A set of voters $S \subseteq V$ is ℓ -cohesive if:

$$\ell \leq \begin{cases} |D_S| - k, & \text{if } \frac{n}{n-|S|}k \leq |D_S|. \\ \frac{|S|}{2n-|S|}(|D_S| + k), & \text{if } \frac{n-|S|}{n}k \leq |D_S| \leq \frac{n}{n-|S|}k \text{ and } \frac{2n-|S|}{n}|A_S| + \frac{n-|S|}{n}|D_S| \geq k. \\ \frac{|S|}{n}k, & \text{if } |A_S| + |D_S| \geq k \text{ and } |D_S| \leq \frac{n-|S|}{n}k \text{ and } |A_S| \leq m - \frac{n-|S|}{n}k. \\ |A_S| + k - m, & \text{if } |A_S| + |D_S| \geq k \text{ and } |D_S| \leq \frac{n-|S|}{n}k \text{ and } \\ & m - \frac{n-|S|}{n}k \leq |A_S| \text{ and } |A_S| + k - m \leq \frac{|S|}{n}(|A_S| + |D_S|). \\ \frac{|S|}{n}(|D_S| + |A_S|), & \text{otherwise.} \end{cases}$$

All the bounds are tight up to the factor of $|S|/n-|S|$.

Proof. Consider an election E and a group $S \subseteq N$. Let us denote by $\text{claim}(S)$ the upper bound of ℓ for S in the statement of the lemma. To prove the statement, we will show that each group of voters $S \subseteq V$ is $\text{claim}(S)$ -cohesive and is not $(\text{claim}(S) + 1)$ -cohesive.

Consider a feasible subset of candidates $T \subseteq C$ and let T^+ and T^- denote the parts of T that consist of the positive and negative candidates, respectively. If we have that $\frac{|S|}{n} > \frac{\text{claim}(S)}{\text{claim}(S) + |T|}$, then S is entitled to satisfaction $\text{claim}(S)$. From now on let us assume that the opposite inequality holds, which is equivalent to:

$$|T| \leq \frac{n - |S|}{|S|} \cdot \text{claim}(S). \quad (3)$$

Moreover, let us assume that the expression $\frac{n-|S|}{|S|} \cdot \text{claim}(S)$ is integral, that is, it is possible to find set T of exactly this size. Whenever it is true, the estimations for $\text{claim}(S)$ will be tight. Otherwise, since the space of possible sets T is smaller, the actual value of claim might be larger by less than:

$$\begin{aligned} \text{claim}(S) - \frac{|S|}{n - |S|} \cdot \left\lfloor \frac{n - |S|}{|S|} \cdot \text{claim}(S) \right\rfloor \\ = \frac{|S|}{n - |S|} \cdot \left(\frac{n - |S|}{|S|} \cdot \text{claim}(S) - \left\lfloor \frac{n - |S|}{|S|} \cdot \text{claim}(S) \right\rfloor \right) < \frac{|S|}{n - |S|}. \end{aligned} \quad (4)$$

So we want to show that for any choice of such a set T , there is a set $X \subseteq A_S \cup \neg D_S$ that can be proposed by group S such that $T \cup X$ is feasible and $|X| = \text{claim}(S)$ and there is no such larger set X . This will give us the desired result, because such a set T certifies that group S is not $(\text{claim}(S) + 1)$ -cohesive as [Formula \(3\)](#) is also satisfied for any $\ell > \text{claim}(S)$.

Now, subject to the constraint of [Formula \(3\)](#) we will choose T adversarially, so as to minimize the size of the largest such set X .

Given a set T , group S should be able to propose a set of candidates $X \subseteq A_S \cup \neg D_S$ of size $\text{claim}(S)$ (but not any larger) such that $T \cup X$ is feasible. Due to the committee size constraint, the set X contains at most $k - |T^+|$ candidates from A_S . Furthermore, $X \cup T$ must not include both a and $\neg a$ for any $a \in C$. Hence the maximum sized set X satisfies $|X \cap \neg D_S| \leq |D_S \setminus T^+|$ and $|X \cap A_S| \leq |A_S \setminus \neg T^-|$. From these inequalities, we obtain the following formula for the maximum size of set X that S can propose so that $T \cup X$ is feasible:

$$|D_S \setminus T^+| + \min(|A_S \setminus \neg T^-|, k - |T^+|)$$

$$= |D_S| - |T^+| + |T^+ \setminus D_S| + \min(|A_S|, k - |T^+| + |A_S \cap \neg T^-|) - |A_S \cap \neg T^-|.$$

In addition to the size constraint (Formula (3)), we now impose further restrictions on T and show that these do not decrease the size of the largest set $X \subseteq A_S \cup \neg D_S$ for which $X \cup T$ is feasible. We can assume that T has the following properties without loss of generality: $T^+ \cap A_S = \emptyset$ and $T^- \subseteq \neg A_S$. To justify this claim, observe that removing a candidate from T does not affect the satisfiability of Formula (3). Removing the candidates from $T^+ \cap A_S$ does not increase the maximum possible size of X (i.e., the number of “slots” in X available for the candidates from $A_S \cup \neg D_S$). The same holds for removing the candidates from $T^- \setminus \neg A_S$. Moreover, if for T there is a set $X \subseteq A_S \cup D_S$ of size at least $\text{claim}(S)$, then this also holds for any subset of T .

Therefore, we have that T^- contains only candidates from $\neg A_S$, and, $|A_S \cap \neg T^-| = |\neg T^-| = |T^-|$. Hence, the maximum size of X is:

$$\begin{aligned} & |D_S| - |T^+| + |T^+ \setminus D_S| + \min(|A_S|, k - |T^+| + |A_S \cap \neg T^-|) - |A_S \cap \neg T^-| \\ &= |D_S| - |T^+| + |T^+ \setminus D_S| + \min(|A_S|, k - |T^+| + |T^-|) - |T^-| \\ &= |D_S| + |T^+ \setminus D_S| + \min(|A_S|, k - |T^+| + |T^-|) - |T|. \end{aligned} \quad (5)$$

Let us now check for which sets T the Formula (5) is minimal. Any candidate, positive or negative, included in T , contributes -1 to the last term $-|T|$. Among those, only positive candidates included in T contribute -1 to the term $k - |T^+| + |T^-|$ and thereby potentially reduce the size of X (this is in contrast to negative candidates, which increase this term). Finally, only positive candidates in D_S additionally contribute 0 to $|T^+ \setminus D_S|$ if included in T^+ while all other positive candidates contribute 1. Thus, to minimize the formula, we always prefer to add candidates from D_S to T^+ instead of the ones from $C \setminus D_S$ (subject to Formula (3) and committee constraint k). So, if possible, we choose T to be of size $\frac{n-|S|}{|S|} \cdot \text{claim}(S)$ and $T = T^+ = D_S$. If $|D_S|$ or k are too small (that is, smaller than $\frac{n-|S|}{|S|} \cdot \text{claim}(S)$, so that using up to the k maximum possible candidates from D_S is insufficient), we will sometimes fill the remaining slots in T with candidates from $C \setminus D_S \setminus A_S$ to T^+ or the ones from $\neg A_S$ to T^- . The remainder of the proof requires an extensive case analysis.

Case 1: Assume that $|D_S| \geq \frac{n}{n-|S|} \cdot k$.

In this case we have that $\text{claim}(S) = |D_S| - k$. The condition in the assumption can be rewritten as:

$$|D_S| \geq \frac{n}{n-|S|} \cdot k \iff |D_S| - k \geq \frac{|S|}{n-|S|} \cdot k \iff \frac{n-|S|}{|S|} \cdot \text{claim}(S) \geq k.$$

Then it is possible to take $T = T^+ \subseteq D_S$ such that T^+ has the maximal possible size of k . As argued in the main text, this minimizes Formula (5). Then we can transform Formula (5) as:

$$|D_S| + \min(|A_S|, 0) - k = |D_S| - k = \text{claim}(S),$$

which shows that indeed Base EJR gives precisely a guarantee of $\text{claim}(S)$ in this case.

Case 2: Assume that $\frac{n-|S|}{n} \cdot k \leq |D_S| \leq \frac{n}{n-|S|} \cdot k$ and $\frac{2n-|S|}{n} |A_S| + \frac{n-|S|}{n} |D_S| \geq k$.

In this case we have that $\text{claim}(S) = \frac{|S|}{2n-|S|} \cdot (|D_S| + k)$. The assumption $\frac{n-|S|}{n} \cdot k \leq |D_S| \leq \frac{n}{n-|S|} \cdot k$ implies both $\frac{n-|S|}{|S|} \cdot \text{claim}(S) \leq \frac{n-|S|}{2n-|S|} \left(\frac{n}{n-|S|} k + \frac{n-|S|}{n-|S|} k \right)$ and $\frac{n-|S|}{|S|} \cdot \text{claim}(S) \leq \frac{n-|S|}{2n-|S|} \left(\frac{n-|S|}{n-|S|} D_S + \frac{n}{n-|S|} D_S \right)$. Therefore,

$$\frac{n-|S|}{|S|} \cdot \text{claim}(S) \leq |D_S|, \quad \text{and} \quad \frac{n-|S|}{|S|} \cdot \text{claim}(S) \leq k.$$

Hence, it is possible to take $T = T^+ \subseteq D_S$ such that T has the maximal possible size of $\frac{n-|S|}{|S|} \cdot \text{claim}(S)$. Now we can rewrite **Formula (5)** as:

$$|D_S| + \min(|A_S|, k - \frac{n-|S|}{|S|} \cdot \text{claim}(S)) - \frac{n-|S|}{|S|} \cdot \text{claim}(S). \quad (6)$$

Then, from the assumptions of the considered case we obtain

$$\frac{2n-|S|}{n}|A_S| + \frac{n-|S|}{n}|D_S| \geq k,$$

which can be equivalently written as:

$$\begin{aligned} |A_S| &\geq \frac{k - \frac{n-|S|}{n}|D_S|}{\frac{2n-|S|}{n}} \\ \iff |A_S| &\geq k - \frac{n-|S|}{2n-|S|}(|D_S| + k) \\ \iff |A_S| &\geq k - \frac{n-|S|}{|S|} \cdot \frac{|S|}{2n-|S|}(|D_S| + k) \\ \iff |A_S| &\geq k - \frac{n-|S|}{|S|} \text{claim}(S). \end{aligned}$$

So, $\min(|A_S|, k - \frac{n-|S|}{|S|} \text{claim}(S)) = k - \frac{n-|S|}{|S|} \text{claim}(S)$ in **Formula (6)**, which can be further simplified as follows

$$\begin{aligned} &|D_S| + k - \frac{n-|S|}{|S|} \cdot \text{claim}(S) - \frac{n-|S|}{|S|} \cdot \text{claim}(S) \\ &= |D_S| + k - \frac{2(n-|S|)}{|S|} \cdot \frac{|S|}{2n-|S|}(|D_S| + k) = \frac{|S|}{2n-|S|} \cdot (|D_S| + k) = \text{claim}(S), \end{aligned}$$

which shows that indeed Base EJR gives a guarantee of $\text{claim}(S)$ in this case.

Case 3: Assume that $|A_S| + |D_S| \geq k$ and $|D_S| \leq \frac{n-|S|}{n} \cdot k$ and $|A_S| \leq m - \frac{n-|S|}{n} \cdot k$.

In this case, we have that $\text{claim}(S) = \frac{|S|}{n}k$. Hence, $|D_S| \leq \frac{n-|S|}{n} \cdot k = \frac{n-|S|}{|S|} \cdot \frac{|S|}{n} \cdot k \leq \frac{n-|S|}{|S|} \cdot \text{claim}(S)$. For the purpose of analysis consider $T = T^+ = D_S$, we will enlarge T thereafter. Under this assumption the term $\min(|A_S|, k - |T^+| + |T^-|)$ in **Formula (5)** equals $k - |D_S|$ using that $|A_S| + |D_S| \geq k$. We now add further $x = \frac{n-|S|}{n} \cdot k - |D_S|$ to T so as to minimize **Formula (5)**. Since for our temporary choice of T it holds that $k - |T^+| + |T^-| \leq |A_S|$, it is true that the *min* term can be further reduced by adding x positive candidates from $C \setminus D_S \setminus A_S$ to T^+ , upon which term $\min(|A_S|, k - |T^+| + |T^-|)$ decreases by x while $|T^+ \setminus D_S|$ increases by x and the part $-|T|$ decreases by x . Overall, **Formula (5)** decreases by x . Note that if we increased the size of T in any other way (i.e., by adding candidates to T^-) the decrease would be smaller, so that our choice of T minimizes **Formula (5)**. It remains to justify that there are at least $x = \frac{n-|S|}{n} \cdot k - |D_S|$ candidates in $C \setminus D_S \setminus A_S$ or equivalently, whether $m - |D_S| - |A_S| \geq \frac{n-|S|}{n} \cdot k - |D_S|$. The latter condition is equivalent to $|A_S| \leq m - \frac{n-|S|}{n} \cdot k$, which holds by assumption. Finally, we simplify **Formula (5)** as:

$$\begin{aligned} &|D_S| + |T^+ \setminus D_S| + \min(|A_S|, k - |T^+| + |T^-|) - |T| = \\ &|D_S| + \frac{n-|S|}{|S|} \text{claim}(S) - |D_S| + k - |D_S| - \left(\frac{n-|S|}{|S|} \text{claim}(S) - |D_S| \right) - \frac{n-|S|}{|S|} \text{claim}(S) = \\ &\frac{n-|S|}{|S|} \text{claim}(S) + k - \frac{n-|S|}{|S|} \text{claim}(S) - \frac{n-|S|}{|S|} \text{claim}(S) = k - \frac{n-|S|}{|S|} \text{claim}(S) = \text{claim}(S), \end{aligned}$$

which shows that indeed Base EJR gives a guarantee of $\text{claim}(S)$ in this case.

Case 4: Assume that $|A_S| + |D_S| \geq k$ and $|D_S| \leq \frac{n-|S|}{n} \cdot k$ and $m - \frac{n-|S|}{n} \cdot k \leq |A_S|$.

In this case we can have that either $\text{claim}(S) = |A_S| + k - m$ or $\text{claim}(S) = \frac{|S|}{n}(|A_S| + |D_S|)$, depending on which value is smaller. Alternatively, we can say that $\text{claim}(S) = \min(|A_S| + k - m, \frac{|S|}{n}(|A_S| + |D_S|))$. We have that:

$$\frac{n-|S|}{|S|}(|A_S| + k - m) \geq \frac{n-|S|}{|S|}(m - \frac{n-|S|}{n} \cdot k + k - m) = \frac{n-|S|}{n} \cdot k \geq |D_S|,$$

$$\frac{n-|S|}{n}(|A_S| + |D_S|) \geq \frac{n-|S|}{n}k \geq |D_S|,$$

which means that $|D_S| \leq \frac{n-|S|}{|S|}\text{claim}(S)$ and, as in Case 3, we first include all candidates from D_S in T . Now, the question is whether set T should be completed by the candidates from $C \setminus A_S \setminus D_S$ or by the candidates from $\neg A_S$. Note that from our assumptions, we have that:

$$m - |A_S| \leq \frac{n-|S|}{n}k \leq k,$$

$$m - |A_S| \leq \frac{n-|S|}{n}k \leq \frac{n-|S|}{n}(|A_S| + |D_S|),$$

$$m - |A_S| \leq \frac{n-|S|}{n}k \iff \frac{n}{|S|}m - \frac{n}{|S|}|A_S| \leq \frac{n-|S|}{|S|}k \iff m - |A_S| \leq \frac{n-|S|}{|S|}(|A_S| + k - m),$$

which means that $m - |A_S| \leq \min(k, \frac{n-|S|}{|S|\text{claim}(S)})$ and it is possible to include all the candidates from $C \setminus A_S \setminus D_S$ in T . On the other hand, we have that $|A_S| + |D_S| \geq \frac{|S|}{n}(|A_S| + |D_S|)$ and $|A_S| + |D_S| \geq k \geq |A_S| + k - m$, which means that it is not possible to include all the candidates from $\neg A_S$ in T .

Looking at the [Formula \(5\)](#), we can prove the following claim:

Claim 7. *An optimal set T minimizing [Formula \(5\)](#), either contains all the candidates from $C \setminus A_S \setminus D_S$, or none of them.*

Proof of Claim 7. Indeed, consider a set T in which there is some x ($0 < x < m - |A_S| - |D_S|$) candidates from $C \setminus A_S \setminus D_S$ and some y candidates from $\neg A_S$. We will show that T does not minimize [Formula \(5\)](#).

We know that $y < |A_S|$ (as we noted above, it is not possible to include all the candidates from $\neg A_S$ in T). First of all, consider the case when $y = 0$. Then, as noted above, $|T| < m - |A_S| \leq \frac{n-|S|}{|S|}\text{claim}(S)$ and it is possible to add a candidate from $\neg A_S$ to T . From the perspective of minimizing [Formula \(5\)](#), such an operation clearly is (weakly) profitable—it always decreases the value of the part $-|T|$ by one, and it can increase the value of part $\min(|A_S|, k - |T^+| + |T^-|)$ only by at most one. Further, let us assume that $y > 0$.

Observe now that the part $\min(|A_S|, k - |T^+| + |T^-|)$ is either equal to $|A_S|$ or to $k - |T^+| + |T^-|$. In the first case, we consider T' obtained from T by making a swap between $c \in T \cap (C \setminus A_S \setminus D_S)$ and a candidate $c' \in A_S \setminus T$. In the second case, we consider T' obtained from T by making a swap between a candidate $c \in (C \setminus A_S \setminus D_S) \setminus T$ and a candidate $c' \in T \cap A_S$. It is straightforward to check that in both cases the value of [Formula \(5\)](#) under T' decreases, which proves that T does not minimize [Formula \(5\)](#). $\square$

Hence, the optimal T in this case either contains all the candidates from D_S and $\frac{n-|S|}{|S|}\text{claim}(S) - |D_S|$ candidates from $\neg A_S$ (let us denote such T by T_1) or all the candidates from $C \setminus A_S$ and $\frac{n-|S|}{|S|}\text{claim}(S) - (m - |A_S|)$ candidates from $\neg A_S$ (let us denote such T by T_2). In the first case, [Formula \(5\)](#) can be transformed as follows:

$$|D_S| + \min(|A_S|, k - |D_S| + \frac{n-|S|}{|S|}\text{claim}(S) - |D_S|) - \frac{n-|S|}{|S|}\text{claim}(S). \quad (7)$$

In the second case, **Formula (5)** can be transformed as follows:

$$\begin{aligned} m - |A_S| + \min(|A_S|, k - (m - |A_S|)) + \frac{n - |S|}{|S|} \text{claim}(S) - (m - |A_S|) - \frac{n - |S|}{|S|} \text{claim}(S) \\ = m - |A_S| + \min(|A_S|, k - 2m + 2|A_S| + \frac{n - |S|}{|S|} \text{claim}(S)) - \frac{n - |S|}{|S|} \text{claim}(S). \end{aligned} \quad (8)$$

Let us now consider the following subcases:

Subcase 4.1: Assume that $|A_S| + k - m \geq \frac{|S|}{n}(|A_S| + |D_S|)$.

In this subcase, we have that $\text{claim}(S) = \frac{|S|}{n}(|A_S| + |D_S|)$. From our assumption we have that:

$$\begin{aligned} \frac{|S|}{n}(|A_S| + |D_S|) &\leq |A_S| + k - m \\ \iff \frac{|S|}{n}(|A_S| + |D_S|) + |D_S| &\leq |A_S| + |D_S| + k - m \\ \implies \frac{|S|}{n}|A_S| + \frac{n + |S|}{n}|D_S| &\leq k \\ \iff |A_S| + 2|D_S| &\leq k + \frac{n - |S|}{n}(|A_S| + |D_S|) \\ \iff |A_S| &\leq k - |D_S| + \frac{n - |S|}{|S|} \text{claim}(S) - |D_S|. \end{aligned}$$

Hence, taking $T = T_1$, we can transform **Formula (7)** as:

$$|D_S| + |A_S| - \frac{n - |S|}{n}(|A_S| + |D_S|) = \frac{|S|}{n}(|A_S| + |D_S|) = \text{claim}(S),$$

and when we take $T = T_2$, then, depending on the value of the min part, we can transform **Formula (8)** either as:

$$m - |A_S| + |A_S| - \frac{n - |S|}{n}(|A_S| + |D_S|) \geq \frac{|S|}{n}m \geq \text{claim}(S)$$

or as:

$$\begin{aligned} m - |A_S| + k - 2m + 2|A_S| + \frac{n - |S|}{|S|} \text{claim}(S) - \frac{n - |S|}{|S|} \text{claim}(S) \\ = |A_S| + k - m \geq \frac{|S|}{n}(|A_S| + |D_S|) = \text{claim}(S). \end{aligned}$$

In all cases group S has a guarantee of at least $\text{claim}(S)$ and for $T = T_1$, the inequality is tight, which shows that indeed Base EJR gives a guarantee of $\text{claim}(S)$ in this subcase.

Subcase 4.2: Assume that $|A_S| + k - m \leq \frac{|S|}{n}(|A_S| + |D_S|)$.

In this subcase we have that $\text{claim}(S) = |A_S| + k - m$. Note that our assumption, together with the fact that $|A_S| + |D_S| \leq m$, implies that $|A_S| + k - m \leq \frac{|S|}{n}m$. From that we further obtain:

$$\begin{aligned} \frac{|S|}{n}m &\geq |A_S| + k - m \\ \iff \frac{n + |S|}{n}m - k &\geq |A_S| \\ \iff |A_S| &\geq k + 2|A_S| - \frac{n + |S|}{n}m \\ \iff |A_S| &\geq k - 2m + 2|A_S| + \frac{n - |S|}{n}m \end{aligned}$$

$$\begin{aligned} &\implies |A_S| \geq k - 2m + 2|A_S| + \frac{n-|S|}{|S|}(|A_S| + k - m) \\ &\iff |A_S| \geq k - 2m + 2|A_S| + \frac{n-|S|}{|S|}\text{claim}(S). \end{aligned}$$

Therefore, taking $T = T_2$, we can transform [Formula \(8\)](#) as follows:

$$m - |A_S| + k - 2m + 2|A_S| + \frac{n-|S|}{|S|}\text{claim}(S) - \frac{n-|S|}{|S|}\text{claim}(S) = |A_S| + k - m = \text{claim}(S).$$

If we take $T = T_1$, then, depending on the value of the min part, we can transform [Formula \(7\)](#) either as:

$$\begin{aligned} |D_S| + |A_S| - \frac{n-|S|}{|S|}(|A_S| + k - m) &\geq |D_S| + |A_S| - \frac{n-|S|}{n}(|A_S| + |D_S|) \\ &= \frac{|S|}{n}(|A_S| + |D_S|) \geq |A_S| + k - m = \text{claim}(S), \end{aligned}$$

or as:

$$|D_S| + k - 2|D_S| + \frac{n-|S|}{|S|}\text{claim}(S) - \frac{n-|S|}{|S|}\text{claim}(S) = k - |D_S| \geq |A_S| + k - m = \text{claim}(S).$$

In all cases group S has a guarantee of at least $\text{claim}(S)$ and for $T = T_2$, the inequality is tight, which shows that indeed Base EJR gives a guarantee of $\text{claim}(S)$ in this subcase.

Case 5: Assume that none of the conditions examined in the previous cases hold.

In this case we have that $\text{claim}(S) = \frac{|S|}{n} \cdot (|A_S| + |D_S|)$. We start by proving that now that at least one of the following is true:

$$\begin{aligned} &\left(\frac{2n-|S|}{n}|A_S| + \frac{n-|S|}{n}|D_S| \leq k \text{ and } |A_S| \leq \frac{|S|}{n-|S|}|D_S| \right) \text{ or} \\ &\left(|A_S| + |D_S| \leq k \text{ and } |A_S| \geq \frac{|S|}{n-|S|}|D_S| \right). \end{aligned} \tag{9}$$

Suppose first that $\frac{2n-|S|}{n}|A_S| + \frac{n-|S|}{n}|D_S| > k$. Then, since the conditions for Cases 1 and 2 are not satisfied, it has to hold that $|D_S| < \frac{n-|S|}{n}k$. Since the conditions for Cases 3 and 4 are not satisfied, it has to hold that $|A_S| + |D_S| < k$. But then $|A_S| + |D_S| < \frac{2n-|S|}{n}|A_S| + \frac{n-|S|}{n}|D_S|$, from which we obtain $|A_S| > \frac{|S|}{n-|S|}|D_S|$. Eventually, the second part of [Formula \(9\)](#) is satisfied.

Suppose now that $\frac{2n-|S|}{n}|A_S| + \frac{n-|S|}{n}|D_S| \leq k$ and $|A_S| > \frac{|S|}{n-|S|}|D_S|$. Then we have that $k \geq \frac{2n-|S|}{n}|A_S| + \frac{n-|S|}{n}|D_S| = |A_S| + \frac{n-|S|}{n}|A_S| + \frac{n-|S|}{n}|D_S| > |A_S| + \frac{|S|}{n}|D_S| + \frac{n-|S|}{n}|D_S| = |A_S| + |D_S|$. Therefore, once again, we have that the second part of [Formula \(9\)](#) is satisfied.

We will now split the further analysis into two subcases, based on [Formula \(9\)](#).

Subcase 5.1: Assume that $\frac{2n-|S|}{n}|A_S| + \frac{n-|S|}{n}|D_S| \leq k$ and $|A_S| \leq \frac{|S|}{n-|S|}|D_S|$.

In this subcase, the following two relations hold:

$$\begin{aligned} |A_S| + |D_S| \leq k &\implies \frac{n-|S|}{n}(|A_S| + |D_S|) \leq k \iff \frac{n-|S|}{|S|}\text{claim}(S) \leq k \\ |A_S| \leq \frac{|S|}{n-|S|}|D_S| &\iff \frac{n-|S|}{n}(|D_S| + |A_S|) \leq |D_S| \iff \frac{n-|S|}{|S|}\text{claim}(S) \leq |D_S| \end{aligned}$$

Hence, in this case it is possible to take $T = T^+ \subseteq D_S$ such that $|T| = \frac{n-|S|}{|S|}\text{claim}(S)$. The [Formula \(5\)](#) then can be written as:

$$|D_S| + \min(|A_S|, k - \frac{n-|S|}{|S|}\text{claim}(S)) - \frac{n-|S|}{|S|}\text{claim}(S).$$

Since $\frac{2n-|S|}{n}|A_S| + \frac{n-|S|}{n}|D_S| \leq k \iff |A_S| \leq k - \frac{n-|S|}{n}(|A_S| + |D_S|) \iff |A_S| \leq k - \frac{n-|S|}{|S|}\text{claim}(S)$, we may further transform [Formula \(5\)](#) as:

$$|D_S| + |A_S| - \frac{n-|S|}{n}(|A_S| + |D_S|) = \frac{|S|}{n}(|A_S| + |D_S|) = \text{claim}(S),$$

which shows that indeed Base EJR gives a guarantee of $\text{claim}(S)$ in this case.

Subcase 5.2: Assume that $|A_S| + |D_S| \leq k$ and $|A_S| \geq \frac{|S|}{n-|S|}|D_S|$.

In this subcase the following equivalence holds:

$$|A_S| \geq \frac{|S|}{n-|S|}|D_S| \iff |D_S| \leq \frac{n-|S|}{n}(|D_S| + |A_S|) \iff |D_S| \leq \frac{n-|S|}{|S|}\text{claim}(S).$$

Hence, we can add all the candidates from D_S to T . Then, [Formula \(5\)](#) becomes:

$$|D_S| + \min(|A_S|, k - |D_S|) - |D_S|$$

However, we can still add $\frac{n-|S|}{|S|}\text{claim}(S) - |D_S|$ candidates more to T . Since we assumed that $|A_S| + |D_S| \leq k$, it also holds that $\min(|A_S|, k - |D_S|) = |A_S|$ and we can add more candidates to T^- without increasing this part of the expression. Note that $|A_S| + |D_S| = \frac{n}{|S|}\text{claim}(S) \geq \frac{n-|S|}{|S|}\text{claim}(S)$, hence, we can add $\frac{n-|S|}{|S|}\text{claim}(S) - |D_S|$ candidates from $\neg A_S$ to T . After that, we may further transform [Formula \(5\)](#) as:

$$u_i(W) = |D_S| + |A_S| - \frac{n-|S|}{|S|}\text{claim}(S) = \text{claim}(S),$$

which shows that indeed Base EJR gives a guarantee of $\text{claim}(S)$ in this case.

To conclude the case analysis, we can see than in each case Base EJR provides the guarantee of exactly $\text{claim}(S)$ for the worst-case adversarial set, which completes the proof. $\square$

Lemma 2. Consider an election E and fix any ℓ -cohesive group of voters S . For each k -element subset $W \subseteq C$, there exists a voter $i \in S$ such that $u_i(W) \geq \ell$ or

$$\ell < \frac{|S|}{n} \left(|D_S| + \min(k, |A_S|) + \frac{n}{n-|S|} \right).$$

Proof. The proof strategy is as follows: we consider the cases in the expression of the statement of [Lemma 1](#) to show that all of them are less than or equal $\frac{|S|}{n}(|D_S| + \min(k, |A_S|))$. After that, it is sufficient to use [Relation \(4\)](#) to prove the statement of the lemma.

First, observe that the voters from S will always have the satisfaction of at least $|D_S| - k$ no matter what is the result of the election. Similarly, if we select k candidates then, we will not select $(m-k)$ of them. Thus, the voters from S will have at least the satisfaction of $|A_S| + k - m$. Hence, we can focus on the remaining three cases only, namely the second, third, and the last one.

Consider the second case. Clearly, it always holds that:

$$\frac{2n-|S|}{n}k + \frac{n-|S|}{n}|D_S| = k + \frac{n-|S|}{n}(k + |D_S|) \geq k.$$

Consequently, since, by assumption, it also holds that $\frac{2n-|S|}{n}|A_S| + \frac{n-|S|}{n}|D_S| \geq k$, we get that:

$$\frac{2n-|S|}{n} \min(k, |A_S|) + \frac{n-|S|}{n}|D_S| \geq k.$$

After reformulation, we get that:

$$\frac{2n - |S|}{n} \min(k, |A_S|) + \frac{2n - |S|}{n} |D_S| \geq |D_S| + k,$$

from which our statement follows immediately.

Let us now move to the third case. Here we have that:

$$|A_S| + |D_S| \geq k \implies \min(|A_S|, k) + |D_S| \geq k,$$

from which our statement follows immediately.

Regarding the last case, say, first, that $|A_S| + k - m > \frac{|S|}{n}(|A_S| + |D_S|)$ (that is, the negated last condition in the fourth case). Using the previous observation that every k -element set W provides to every voter $i \in S$ at least the satisfaction $|A_S| + k - m$, we immediately obtain that $u_i(W) \geq \frac{|S|}{n}(|A_S| + |D_S|)$. Finally, we simply need to show that in the remaining part of the last case we have that $|A_S| \leq k$, or equivalently, that $|A_S| > k$ is covered by the previous cases. Indeed, if $|A_S| \geq k$ then all previous conditions that involve A_S are satisfied. At the same time the conditions involving only D_S cover the whole space of possibilities. $\square$

Theorem 3. *Let W be an outcome returned by Phragmén's rule. For each ℓ -cohesive set of voters S the following conditions are satisfied:*

- (1) $|(\bigcup_{i \in S} A_i \cup \bigcup_{i \in S \setminus D_i} \neg D_i) \cap W| \geq \ell$, (Base PJR)
- (2) $\text{avgsat}_S(W) \geq \frac{\ell - 1}{2}$. (proportionality degree)

Proof. We begin with Statement (1) on Base PJR. Consider a group S that deserves ℓ candidates and suppose $|\bigcup_{i \in S} (A_i \cup \neg D_i) \cap W| < \ell$ where W is the outcome returned by the proposed rule. Consider the first moment, t , when all the candidates from $A_S \cup \neg D_S$ are either elected or removed. Note that $t \leq \frac{\ell}{|S|}$. Indeed, if $t > \frac{\ell}{|S|}$, then at time $\frac{\ell}{|S|}$ the group S would collect in total ℓ dollars, and would buy at least ℓ candidates from $\bigcup_{i \in S} A_i \cup \neg D_i$ (the possibility of buying such candidates comes from the fact that there would always be a candidate from $A_S \cup \neg D_S$ available for purchase). As in the proof of Theorem 8 in the paper by Masařík et al. [25], the outcome W selected by time t , can be decomposed as $W = T \cup A$ where $A \subseteq \bigcup_{i \in S} (A_i \cup \neg D_i)$ and $T = W \setminus \bigcup_{i \in S} (A_i \cup \neg D_i)$, and there exists $X \subseteq A_S \cup D_S$ of size ℓ such that $T \cup X$ is feasible.

Note that for every positive candidate $a \in A$ there is $i \in S$ that approves a and so $\neg a$ is not in X . Similarly, for every negative candidate $\neg a \in A$, there is $i \in S$ that vetoes a and so $a \notin X$. From this and the fact that $|X| > |A|$ it follows that there exists $c \in X \setminus A$ such that $W \cup \{c\}$ is feasible. But then Phragmén should have selected c instead of terminating; a contradiction.

We now move to Statement (2) on proportionality degree. Consider a group $S \subset N$ deserving $\ell = \text{claim}(S)$ candidates. Toward a contradiction assume that the average satisfaction that group S has for W , the outcome returned by Phragmén, is less than $\frac{\ell-1}{2}$. As in the proof of Theorem 9 in the paper of Masařík et al. [25] we define

$$t = \frac{\ell}{|S|} + \frac{\Delta - 1}{n},$$

where Δ is the smallest non-negative value such that at t the voters from S have at most Δ unspent dollars (if such Δ does not exist, then we simply set $\Delta = 0$). Observe that $t \geq \frac{\ell}{|S|} - \frac{1}{n} \geq \frac{\ell-1}{|S|}$. Consider the set W of candidates elected by Phragmén up to time t and observe that it follows from Statement (1) on Base PJR that W contains at least $\ell - 1$ candidates from $\bigcup_{i \in S} A_i$. We can therefore take a set $A \subseteq W$ of exactly $\ell - 1$ candidates from the union of approval sets of voters

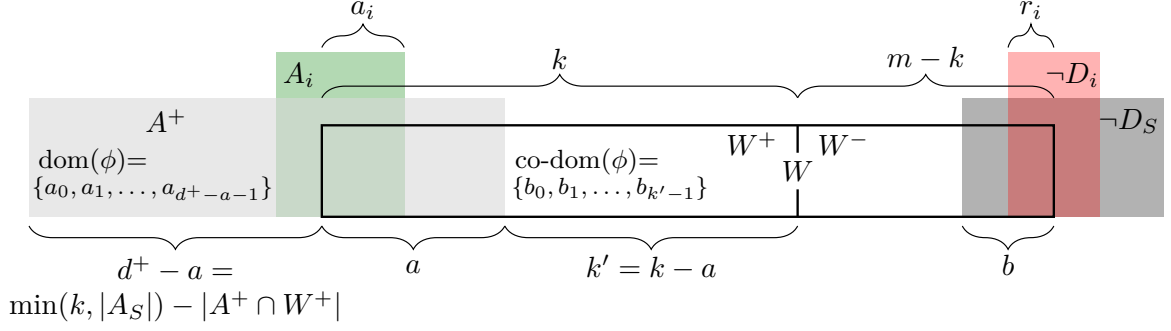


Figure 1: Illustration of sets and parameters involved in the proof of [Theorem 4](#). We denote by $\text{dom}(\phi)$ the domain of an injection ϕ and by $\text{co-dom}(\phi)$ its co-domain.

S , selecting maximally many candidates from their intersection, A_S , and define $T = W \setminus A$. Note that $T \cap A_S = \emptyset$. Since S deserves ℓ candidates, there exists a set X of size ℓ such that $T \cup X$ is feasible (if X does not exist, then the same argument as in the proof of Theorem 9 in [25] holds since A has size exactly $\ell - 1$). Now comparing $W = T \cup A$ and $T \cup X$, we argue similarly as in the poof of Statement (1). For every $a \in A$ there is some voter v in S that approves a since A is a subset of the union of approval sets of voters in S . So the negation of a is not in X , as voter v does not vote both for and against the candidate a . Since $|A| < |X|$ and $|T \cap X| = \emptyset$, we have that $|W| = |T \cup A| < |T \cup X|$, and as we argued the restricted exchange property applies, implying the existence of $x \in X \setminus A$ such that $W \cup \{x\}$ is feasible. The remaining proof continues as in [25]. $\square$

Theorem 4. Consider an election E and let W be an outcome of PAV. For each ℓ -cohesive $S \subseteq V$:

$$\text{avgsat}_S(W) \geq (1 - \varepsilon) \left(\ell - \frac{|S|}{n - |S|} \right) - 3/2, \quad \text{where } \varepsilon := \frac{2}{k + 4}.$$

Proof. We begin by outlining the strategy of the proof. Assume, for the sake of contradiction, that there exists an outcome W selected by PAV where the group of voters S achieves a lower satisfaction than stated in the theorem. To address this, we consider a series of possible swaps that replace certain candidates in the committee W with candidates outside W . Instead of analyzing each individual swap, we aggregate them to compute the overall change in the PAV-score of W resulting from these swaps. It is crucial that these swaps are applied independently to the original committee W , rather than sequentially. It suffices to demonstrate that the aggregated change in the PAV-score is positive. Then, by the pigeonhole principle, this implies that there is at least one swap that can improve the PAV-score of the outcome W , leading to a contradiction. This line of reasoning was employed in the original proof that PAV satisfies EJR without negative votes. However, in the original setting, swaps simply involve replacing candidates one for one; an approach that fails in our setting. This is because adding a candidate c to the committee may require not just removing an arbitrary candidate from W , but also addressing the presence of a “virtual candidate” $\neg c$, which represents the exclusion of c from W . Hence, our analysis has to be much more nuanced, involving the consideration of swaps between groups of candidates rather than individual candidates, which makes it more technically demanding. [Figure 1](#) illustrates several sets and parameters that will be defined and utilized subsequently.

Let us now start by introducing some additional notation. Let A^+ be an arbitrary subset of A_S of size $d^+ = \min(k, |A_S|)$. Let $A = \neg D_S \cup A^+$. These are the candidates we will be trying to add to the committee W , while removing others to maintain feasibility. We denote the size of A as d .

$$d = |D_S| + \min(|A_S|, k).$$

By [Lemma 2](#) we know that if $|W| = k$, then we can assume:

$$\text{claim}(S) < \frac{|S|}{n} \cdot \left(d + \frac{n}{n - |S|} \right) \implies \text{claim}(S) - \frac{|S|}{n - |S|} < \frac{|S|}{n} \cdot d.$$

If $W < k$ then in order to add any candidate from $U_i \setminus W$ to W we need to remove at most one candidate from W . Then the proof simplifies, and the reasoning from the proof of Theorem 6 by Masařík et al. [25] can be directly applied.

We split the committee W into positive candidates W^+ and negative candidates W^- . We assume in the following that $|W^+| = k$; if not, the outcome can simply be extended by adding dummy candidates that are neither approved nor disapproved by any voter. Let $a = |A^+ \cap W^+|$. We assume that $a < k$ as otherwise $W^+ \subseteq A_S$ and the committee $W \cup \{\neg c \mid c \in D_S\}$ is feasible. This committee would already imply the desired result. Further, we assume without loss of generality that for every $c \in C$, either $c \in W^+$ or $\neg c \in W^-$. Indeed, if this is not the case, then $\neg c$ can be always added to the outcome, and the PAV score will not decrease after such a change. Finally,

$$k' := k - a = |W^+ \setminus A^+|.$$

We will now define k' injections from $A^+ \setminus W^+$ to $W^+ \setminus A^+$. Note that

$$|A^+ \setminus W^+| = d^+ - a \leq k - a = |W^+ \setminus A^+|.$$

For convenience we label the candidates in $A^+ \setminus W^+$ as $a_0, \dots, a_{d^+ - a - 1}$, and the candidates in $W^+ \setminus A^+$ as $b_0, \dots, b_{k' - 1}$. We define ϕ_i for $i = 0, \dots, k' - 1$ as follows

$$\phi_i(a_j) = b_{j+i \bmod k'}, \quad \text{for } j = 0, \dots, d^+ - a - 1.$$

We begin by considering a concrete example of how to construct the exchanges from a single fixed injection ϕ before exhibiting our general construction. Let $k = 3$ be the committee size, and let $W = \{c_1, c_2, c_3, \neg c_4, \neg c_5, \neg c_6\}$, i.e. $W^+ = \{c_1, c_2, c_3\}$ and $W^- = \{\neg c_4, \neg c_5, \neg c_6\}$. Suppose that group S jointly approves $A_S = \{c_4, c_5, c_6\}$ and $D_S = \{\neg c_1, \neg c_2\}$. Then, we want to add elements from $A_S \cup D_S$ and remove elements from W . For feasibility, adding a negative candidate (i.e., one from D_S) only requires removing the corresponding positive. However, adding a positive candidate (i.e., one from A_S) requires removing both the corresponding negative and an additional positive from W , as determined by the injection. Note that $W^+ \cap A^+ = \emptyset$ and consider a sample injection ϕ from A^+ to W^+ : $\phi(c_4) = c_1$, $\phi(c_5) = c_2$ and $\phi(c_6) = c_3$. The primary swaps associated with ϕ are depicted in [Figure 2](#) (Left); to add x we remove $\neg x$ and $\phi(x)$. The set of swaps associated with the injection has the following property: if we restrict our attention to rows where positive candidates are being added, any positive candidate is removed in at most one such swap. Consequently, each positive candidate is removed in at most two swaps across the entire table. However, for our proof we will need that each candidate (positive or negative) is either removed or added in at most one swap; and, clearly any candidate might be considered for addition or removal but not both. To achieve this, we iteratively identify pairs of swaps that involve the removal of the same positive candidate x . These pairs have the following structure: in one swap, x is removed to make room for $\neg x$; in the other, x and some $\neg y$ are removed to accommodate y . We merge these into a single exchange where $\neg x$ and y are added, and x and $\neg y$ are removed. The merged exchanges are illustrated in [Figure 2](#) (Right). It is evident that in the merged table every candidate is removed or added at most once, our desired property. In the general case, let ϕ be an arbitrary injection from $A^+ \setminus W^+$ to $W^+ \setminus A^+$. Let $T \subseteq D_S$ be exactly the subset of negative candidates $\neg c$ from $\neg D_S$ for which there exists some $a \in A^+$ such that $\phi(a) = c$ (in our example these are candidates $\neg c_1$ and $\neg c_2$). The collection of swaps $\mathcal{E} = \{(X_1, Y_1), \dots, (X_r, Y_r)\}$, $r \leq d$ associated with ϕ (the first element of a swap corresponds to candidates to be added and the second to those to be removed) is defined as follows:

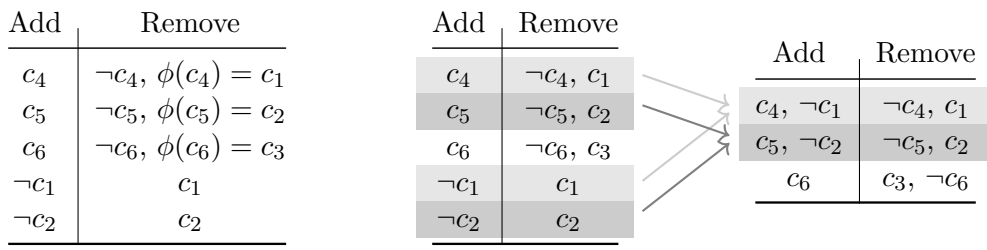


Figure 2: Illustration of the example presented in the proof of [Theorem 4](#). The table on the left depicts the primary swaps associated with ϕ . The right part of the figure illustrates the operation of merging two swaps.

1. Trivial swaps: If $x \in A \cap W$, $(\{x\}, \{x\}) \in \mathcal{E}$.
2. If $\neg x \in (D_S \setminus W) \setminus T$, then $(\{\neg x\}, \{x\}) \in \mathcal{E}$.
3. If for $x \in A^+ \setminus W^+$, $\neg \phi(x) \in A$, $(\{x, \neg \phi(x)\}, \{\neg x, \phi(x)\}) \in \mathcal{E}$.
4. If for $x \in A^+ \setminus W^+$, $\neg \phi(x) \notin A$, $(\{x\}, \{\neg x, \phi(x)\}) \in \mathcal{E}$.

Let $w_i = |U_i \cap W|$. Further, let p be the number of swaps $(X, Y) \in \mathcal{E}$ such that $|X| = 2$ (denote the set of these swaps as $\mathcal{E}_2$) and let x be the number of such swaps where $|X| = 1$ (with the set of these swaps denoted as $\mathcal{E}_1$). Then, clearly, the following holds:

$$2p + x = |A| = d \quad (10)$$

Note that for swaps with $|X| = 2$, it holds that $Y \cap A = \emptyset$, i.e., no voter approves the removed candidates. Let $\Delta_i(X, Y)$ denote the change of the PAV score that voter i assigns to committee W due to performing the swap (X, Y) . We have that:

$$\begin{aligned}
\sum_{(X,Y) \in \mathcal{E}} \sum_{i \in S} \Delta_i(X, Y) &= \sum_{(X,Y) \in \mathcal{E}_2} \sum_{i \in S} \left(\frac{1}{w_i + 1} + \frac{1}{w_i + 2} \right) + \sum_{(X,Y) \in \mathcal{E}_1} \left(\sum_{i \in S} \frac{1}{w_i + 1} - \sum_{\substack{i \in S, \\ |U_i \cap Y| = 1}} \frac{1}{w_i + 1} \right) \\
&= \sum_{i \in S} \left(\frac{p}{w_i + 1} + \frac{p}{w_i + 2} \right) + \sum_{i \in S} \frac{x}{w_i + 1} - \sum_{i \in S} \sum_{\substack{(X,Y) \in \mathcal{E}_1, \\ |U_i \cap Y| = 1}} \frac{1}{w_i + 1} \\
&\geq \sum_{i \in S} \left(\frac{p+x}{w_i + 1} + \frac{p}{w_i + 2} \right) - \sum_{i \in S} \frac{w_i}{w_i + 1} \\
&= \sum_{i \in S} \left(\frac{p+x}{w_i + 1} + \frac{p}{w_i + 2} \right) + \sum_{i \in S} \frac{1}{w_i + 1} - |S| \\
&\geq \frac{(2p+x+1)}{2} \cdot \sum_{i \in S} \left(\frac{1}{w_i + 1} + \frac{1}{w_i + 2} \right) - |S|
\end{aligned}$$

Now consider the swaps $\mathcal{E}(\phi_j)$ we obtain from each ϕ_j by the above-mentioned construction. By summing up over all swaps that arise by all possible defined injections, we obtain:

$$\sum_{j \in [k']} \sum_{(X,Y) \in \mathcal{E}(\phi_j)} \sum_{i \in S} \Delta_i(X, Y) \geq k' \cdot \left(\frac{(2p+x+1)}{2} \cdot \sum_{i \in S} \left(\frac{1}{w_i + 1} + \frac{1}{w_i + 2} \right) - |S| \right). \quad (11)$$

Let us introduce some further notation that will prove highly useful in the careful analysis that follows. For each voter $i \in N$, we define:

$$b = |\neg D_S \cap W^-| \quad \quad b_i = |\neg D_i \cap (\neg D_S \cap W^-)|$$

$$\begin{aligned}
a &= |A^+ \cap W^+| & a_i &= |A_i \cap (A^+ \cap W^+)| \\
r_i &= |\neg D_i \cap W^-| & w_i &= |U_i \cap W| \\
p_i &= |\{c \in D_S \mid c \in A_i \cap W^+\}|
\end{aligned}$$

Note that $p_i \leq w_i - r_i - a_i$ and that $b_i = b, p_i = 0$ for $i \in S$. Our goal is to lower bound the following term, being the lower bound on the change in the PAV score of the voters from $N \setminus S$ due to performing the swaps:

$$\Delta_{N \setminus S} = - \sum_{\phi_j} \sum_{(X,Y) \in \mathcal{E}(\phi_j)} \left(\sum_{\substack{i \in N \setminus S, \\ |U_i \cap Y|=2}} \left(\frac{1}{w_i} + \frac{1}{w_i - 1} \right) + \sum_{\substack{i \in N \setminus S, \\ |U_i \cap Y|=1}} \frac{1}{w_i} \right).$$

We also note the following observation, which will become useful later on.

Claim 8. *We may assume wlog that if some $i \in N \setminus S$ approves $\neg c \in W^-$ then $c \in A^+$ or $\neg c \in D_S$.*

Proof of Claim 8. If some $i \in N \setminus S$ approves $\neg c \in W^-$ for which neither $c \in A^+$ nor $\neg c \in D_S$ holds, we may simply delete such an approval. This only results in decreasing the above term and so lower bounds the one for the original instance. $\square$

Let $\mathcal{E} = \cup_{j=0}^{k'-1} \mathcal{E}(\phi_j)$. For each agent i the trivial swaps contribute $-\frac{k'(a_i+b_i)}{w_i}$ to $\Delta_{N \setminus S}$. Then, the counting of the number of swaps of the form $(\neg c, c)$ with $\neg c \in D_S$ and $c \in W^+ \setminus A^+$ can be done as follows. Among the k' injections $\phi_1, \dots, \phi_{k'}$, whenever c is in the image of a ϕ_i , the initial exchange $(\neg c, c)$ will be merged into an exchange of the form $(\{\neg c, \phi_i^{-1}(c)\}, \{\phi_i(c), c\})$. So we need to count the number of these injections for which the exchange $(\neg c, c)$ does not get merged this way. Since each of the k' injections has an image of size $d^+ - a$ and each of the k' candidates $c \in W^+ \setminus A^+$ is in the image of the same number of such injections, each of them is in the image $\frac{k'(d^+ - a)}{k'} = d^+ - a$ times. Hence, the candidate c is not in the image of $k' - (d^+ - a) = k - d^+$ injections. Thus, the swaps of the form $(\neg c, c)$ for the voter $i \in N \setminus S$ contribute $-(k - d^+) \cdot \frac{p_i}{w_i}$ to the above negative term $\Delta_{N \setminus S}$. We have now covered all the swaps $(X, Y) \in \mathcal{E}$ for which $|Y| = 1$.

We next count the swaps $(X, Y) \in \mathcal{E}$ where $|Y| = 2$. These are swaps that involve removing a negation $\neg c$ and another positive candidate g in $W^+ \setminus A^+$ as well as adding candidate $c \in A^+$ (and potentially adding $\neg g$). So the set Y has the form $\{\neg c, g\}$ where $g \in W^+ \setminus A^+$ and $c \in A^+ \setminus W^+$.

We will split these swaps further into these for which agent i approves (1) only $\neg c$, (2) only g or (3) both of them. (It is also possible that i approves neither of them; however, then the removal of those candidates does not decrease the score the voter assigns to W .)

The number of candidates $g \in W^+ \setminus A^+$ that i approves is $w_i - r_i - a_i$. The number of candidates $g \in W^+ \setminus A^+$ that i does not approve is the rest of them, i.e., $(k' - (w_i - r_i - a_i))$. The number of candidates $c \in A^+ \setminus W^+$ such that $\neg c \in W^-$ and i disapproves c is $(r_i - b_i)$ (where we use Claim 8 that if $i \in N \setminus S$ disapproves c then either $c \in A^+$ or $c \in D_S$). The number of candidates $c \in A^+ \setminus W^+$ such that i does not disapprove c is the rest of them i.e. $|A^+ \setminus W^+| - (r_i - b_i) = d^+ - a - (r_i - b_i)$. Consequently, the number of pairs in which i both disapproves c and approves g is $(r_i - b_i)(w_i - r_i - a_i)$, the number of pairs in which i disapproves c but does not approve g is $(r_i - b_i)(k - a - (w_i - r_i - a_i))$ and finally the number of pairs in which i approves g but does not disapprove c is $(d^+ - a - (r_i - b_i))(w_i - r_i - a_i)$.

Now, observe that the reason why $|Y| = 2$ is that for a positive candidate c from $A^+ \setminus W$ we are adding we also need to remove $\neg c$ which is in the committee. In each of the k' injections c is

matched with a different candidate from $W^+ \setminus A^+$ to form such a swap (X, Y) . Thus, each set Y appears at most once in all the injections.

Collecting all the terms we get the following equality:

$$\begin{aligned} \Delta_{N \setminus S} &= - \sum_{\phi_j} \sum_{(X, Y) \in \mathcal{E}(\phi_j)} \sum_{\substack{i \in N \setminus S, \\ |U_i \cap Y| = 2}} \left(\frac{1}{w_i} + \frac{1}{w_i - 1} \right) + \sum_{\substack{i \in N \setminus S, \\ |U_i \cap Y| = 1}} \frac{1}{w_i} \\ &\geq - \sum_{i \in N \setminus S} (r_i - b_i)(w_i - r_i - a_i) \left(\frac{1}{w_i} + \frac{1}{w_i - 1} \right) - \sum_{i \in N \setminus S} ((d^+ - a) - (r_i - b_i))(w_i - r_i - a_i) \frac{1}{w_i} \\ &\quad - \sum_{i \in N \setminus S} (r_i - b_i)(k' - (w_i - r_i - a_i)) \frac{1}{w_i} - \sum_{i \in N \setminus S} k'(a_i + b_i) \frac{1}{w_i} - \sum_{i \in N \setminus S} (k - d^+) p_i \frac{1}{w_i}. \end{aligned}$$

We now proceed to simplify the summations. The proof of the corresponding lemma is quite technical and so we defer it to [Appendix A](#). To show how close our analysis is to being tight, we note that the proof relies on inequalities only twice: first we use the approximation that $-p_i \geq -(w_i - r_i)$ and later we also use that $\frac{1}{w_i - 1} > \frac{1}{w_i}$ for $w_i > 1$.

Lemma 9. *It holds that $\Delta_{N \setminus S} \geq -(k' + 1)(n - |S|)$.*

Proof of Lemma 9. In the main proof of [Theorem 4](#) we derived the following estimation:

$$\begin{aligned} \Delta_{N \setminus S} &= - \sum_{\phi_j} \sum_{(X, Y) \in \mathcal{E}_j} \sum_{\substack{i \in N \setminus S, \\ |U_i \cap Y| = 2}} \left(\frac{1}{w_i} + \frac{1}{w_i - 1} \right) + \sum_{\substack{i \in N \setminus S, \\ |U_i \cap Y| = 1}} \frac{1}{w_i} \\ &\geq - \sum_{i \in N \setminus S} (r_i - b_i)(w_i - r_i - a_i) \left(\frac{1}{w_i} + \frac{1}{w_i - 1} \right) \\ &\quad - \sum_{i \in N \setminus S} ((d^+ - a) - (r_i - b_i))(w_i - r_i - a_i) \frac{1}{w_i} \\ &\quad - \sum_{i \in N \setminus S} (r_i - b_i)(k' - (w_i - r_i - a_i)) \frac{1}{w_i} \\ &\quad - \sum_{i \in N \setminus S} k'(a_i + b_i) \frac{1}{w_i} - \sum_{i \in N \setminus S} (k' - (d^+ - a)) p_i \frac{1}{w_i} \end{aligned}$$

Adding up the highlighted terms and using the fact that $p_i \leq w_i - r_i - a_i$ we get that:

$$\begin{aligned} \Delta_{N \setminus S} &\geq - \sum_{i \in N \setminus S} (r_i - b_i)(w_i - r_i - a_i) \left(\frac{1}{w_i} + \frac{1}{w_i - 1} \right) \\ &\quad - \sum_{i \in N \setminus S} (-(r_i - b_i))(w_i - r_i - a_i) \frac{1}{w_i} \\ &\quad - \sum_{i \in N \setminus S} (r_i - b_i)(k' - (w_i - r_i - a_i)) \frac{1}{w_i} \\ &\quad - \sum_{i \in N \setminus S} k'(a_i + b_i) \frac{1}{w_i} - \sum_{i \in N \setminus S} k'(w_i - r_i - a_i) \frac{1}{w_i} \\ &= - \sum_{i \in N \setminus S} (r_i - b_i)(w_i - r_i - a_i) \frac{1}{w_i} \\ &\quad - \sum_{i \in N \setminus S} (r_i - b_i)(w_i - r_i - a_i) \frac{1}{w_i - 1} \end{aligned}$$

$$\begin{aligned}
& - \sum_{i \in N \setminus S} (-(r_i - b_i))(w_i - r_i - a_i) \frac{1}{w_i} \\
& - \sum_{i \in N \setminus S} (r_i - b_i)(k' - (w_i - r_i - a_i)) \frac{1}{w_i} \\
& - \sum_{i \in N \setminus S} k'(a_i + b_i) \frac{1}{w_i} - \sum_{i \in N \setminus S} k'(w_i - r_i - a_i) \frac{1}{w_i}
\end{aligned}$$

In the last step, we split up the first term and obtained further cancellations. Next we simplify the $\frac{1}{w_i - 1}$ term and also obtain further cancellations. Hence, we have that:

$$\begin{aligned}
\Delta_{N \setminus S} & \geq - \sum_{i \in N \setminus S} (r_i - b_i)(w_i - r_i - a_i) \left(\frac{1}{w_i - 1} \right) \\
& - \sum_{i \in N \setminus S} (r_i - b_i)(k' - (w_i - r_i - a_i)) \frac{1}{w_i} \\
& - \sum_{i \in N \setminus S} k'(a_i + b_i) \frac{1}{w_i} - \sum_{i \in N \setminus S} k'(w_i - r_i - a_i) \frac{1}{w_i} \\
& = - \sum_{i \in N \setminus S} (r_i - b_i) - \sum_{i \in N \setminus S} - \left(\frac{(r_i - b_i)(r_i + a_i - 1)}{w_i - 1} \right) \\
& - \sum_{i \in N \setminus S} (r_i - b_i)(k' - (w_i - r_i - a_i)) \frac{1}{w_i} \\
& - \sum_{i \in N \setminus S} k' b_i \frac{1}{w_i} - \sum_{i \in N \setminus S} k'(w_i - r_i) \frac{1}{w_i}
\end{aligned}$$

Finally, it remains to iteratively collect terms that cancel.

$$\begin{aligned}
& > - \sum_{i \in N \setminus S} (r_i - b_i) - \sum_{i \in N \setminus S} - \left(\frac{(r_i - b_i)(r_i + a_i - 1)}{w_i} \right) \\
& - \sum_{i \in N \setminus S} \frac{(r_i - b_i)k'}{w_i} - \sum_{i \in N \setminus S} - \frac{(r_i - b_i)(w_i - r_i)}{w_i} - \sum_{i \in N \setminus S} - \frac{(r_i - b_i)(-a_i)}{w_i} \\
& - \sum_{i \in N \setminus S} k' b_i \frac{1}{w_i} - \sum_{i \in N \setminus S} k'(w_i - r_i) \frac{1}{w_i} \\
& = - \sum_{i \in N \setminus S} (r_i - b_i) - \sum_{i \in N \setminus S} - \left(\frac{(r_i - b_i)(r_i + a_i - 1)}{w_i} \right) \\
& - \sum_{i \in N \setminus S} - \frac{(r_i - b_i)(w_i - r_i)}{w_i} - \sum_{i \in N \setminus S} - \frac{(r_i - b_i)(-a_i)}{w_i} \\
& - \sum_{i \in N \setminus S} k w_i \frac{1}{w_i} \\
& = - \sum_{i \in N \setminus S} - \left(\frac{(r_i - b_i)(r_i + a_i - 1)}{w_i} \right) \\
& - \sum_{i \in N \setminus S} - \frac{(r_i - b_i)(-r_i)}{w_i} - \sum_{i \in N \setminus S} - \frac{(r_i - b_i)(-a_i)}{w_i} \\
& - \sum_{i \in N \setminus S} k' w_i \frac{1}{w_i}
\end{aligned}$$

$$\begin{aligned}
&= - \sum_{i \in N \setminus S} - \left(\frac{(r_i - b_i)(a_i - 1)}{w_i} \right) - \sum_{i \in N \setminus S} - \frac{(r_i - b_i)(-a_i)}{w_i} - \sum_{i \in N \setminus S} k' w_i \frac{1}{w_i} \\
&= - \sum_{i \in N \setminus S} \left(k' + \frac{r_i - b_i}{w_i} \right) \geq - \sum_{i \in N \setminus S} \left(k' + \frac{r_i}{w_i} \right) \\
&\geq - \sum_{i \in N \setminus S} (k' + 1) = -(n - |S|)(k' + 1),
\end{aligned}$$

which concludes the proof of [Lemma 9](#). $\square$

We now combine [Relation \(11\)](#) with the result from [Lemma 9](#), taking into account that W is the outcome selected by PAV, and we get the following:

$$\begin{aligned}
&k' \frac{(2p + x + 1)}{2} \cdot \sum_{i \in S} \left(\frac{1}{w_i + 1} + \frac{1}{w_i + 2} \right) - k'|S| - (k' + 1)(n - |S|) \leq 0 \stackrel{(10)}{\iff} \\
&k' \cdot \frac{(d + 1)}{2} \cdot \sum_{i \in S} \left(\frac{1}{w_i + 1} + \frac{1}{w_i + 2} \right) - k'n - n + |S| \leq 0 \iff \\
&0 \geq \frac{(d + 1)}{2} \cdot \sum_{i \in S} \left(\frac{1}{w_i + 1} + \frac{1}{w_i + 2} \right) - n - \frac{n - |S|}{k'} \iff \\
&0 \geq (d + 1) \cdot \sum_{i \in S} \frac{1}{w_i + 3/2} - n - \frac{n - |S|}{k'}. \tag{12}
\end{aligned}$$

Now observe that from A^+ we did not select at most k' candidates. Also, D_S contains at most k' negative candidates that have a positive counterpart selected. All other candidates from D_S were selected. Thus, we know that the average satisfaction of the voters from S equals at least $d - 2k'$. If $d - 2k' \geq d \cdot \frac{s}{n}$, then the proof follows immediately. Thus, from now on we can assume that $d - 2k' < d \cdot \frac{s}{n}$, hence:

$$k' > \frac{1}{2}d \cdot \frac{n - |S|}{n} \stackrel{(12)}{\iff} 0 \geq (d + 1) \sum_{i \in S} \frac{1}{w_i + 3/2} - n - \frac{2n}{d}.$$

From the inequality between harmonic and arithmetic mean we get that:

$$\sum_{i \in S} \frac{1}{w_i + 3/2} \geq \frac{|S|}{\text{avgsat}_S(W) + 3/2},$$

and combining the last two relations we get the following:

$$\frac{|S|}{\text{avgsat}_S(W) + 3/2} < \frac{2n + nd}{d(d + 1)} \iff \text{avgsat}_S(W) > \frac{d + 1}{d + 2} \cdot d \frac{|S|}{n} - \frac{3}{2}.$$

If $d \geq k/2$, then the proof is complete. Now, we consider the case when $d < k/2$. In this case however, we know that $k' = k - a \geq k - d > k/2$. From [Relation \(12\)](#) we get that:

$$0 \geq (d + 1) \cdot \sum_{i \in S} \frac{1}{w_i + 3/2} - n \cdot \frac{k' + 1}{k'} \iff \sum_{i \in S} \frac{1}{w_i + 3/2} \leq n \cdot \frac{k' + 1}{k'(d + 1)}.$$

By using again the inequality between harmonic and arithmetic mean we get:

$$\begin{aligned}
&\frac{|S|}{\text{avgsat}_S(W) + 3/2} \leq n \cdot \frac{k' + 1}{k'(d + 1)} \iff \\
&\text{avgsat}_S(W) \geq \frac{|S|}{n} \cdot (d + 1) \cdot \frac{k'}{k' + 1} - \frac{3}{2} \geq \frac{|S|}{n} \cdot d \cdot \frac{k' + 1}{k' + 2} - \frac{3}{2},
\end{aligned}$$

where the last inequality follows from $d < k/2$. In all cases we got:

$$\text{avgsat}_S(W) \geq \frac{|S|}{n} \cdot d \cdot \frac{k/2 + 1}{k/2 + 2} - \frac{3}{2} = \frac{|S|}{n} \cdot d \cdot \left(1 - \frac{2}{k+4}\right) - \frac{3}{2}.$$

This completes the proof. $\square$

Theorem 5. *Tax-MES satisfies EJPR and Group Veto. Tax-Phragmén satisfies PJPR and Group Veto.*

Proof. For proving that Tax-Phragmén satisfies PJPR consider an ℓ -positively-cohesive group S , as witnessed by $T \subseteq A_S$, $|T| \geq \ell$. Fix a candidate $c \in T$. Any candidate $c \in T$ costs $\frac{|A_c|}{|A_c| - |D_c|}$. By the fact that $A_c \supseteq S$ we have the following:

$$|A_c||D_c| \geq |S||D_c| \iff |A_c|(|S| - |D_c|) \leq |S|(|A_c| - |D_c|).$$

Therefore, the cost of c is upper bounded by $\frac{|S|}{|S| - |D_c|} \leq |S| \frac{k}{\ell n}$. Hence, ℓ candidates from T cost in total at most $|S| \frac{k}{n}$. Thus, voters in S have participated in buying at least ℓ candidates in total, i.e. $|W \cap \cup_{i \in S} A_i| \geq \ell$.

Regarding the Group Veto axiom, we will show the following: Let $\mathcal{R}$ be a priceable rule such that $p(c) \geq \frac{|A_c|}{|A_c| - |D_c|}$ whenever $|A_c| > |D_c|$, and $p(c) = \infty$, otherwise. Additionally, assume that the voters in $\mathcal{R}$ are initially endowed with the budget of at most k/n . Then, $\mathcal{R}$ satisfies the Group Veto. Notice that both Tax-MES and Tax-Phragmén belong to the aforementioned family of rules.

Consider a candidate $c \in T$. Since $A_c \subseteq ap(T)$, it holds that $|A_c| \leq |ap(T)|$ and since $S \subseteq D_c$ it holds that $|D_c| \geq |S|$, and so $|ap(T)| - |S| \geq |A_c| - |D_c|$. We get:

$$\begin{aligned} \frac{1}{|S|}(|ap(T)| - |S|) &\geq \frac{1}{|D_c|}(|A_c| - |D_c|) \iff \frac{|ap(T)|}{|S|} - 1 \geq \frac{|A_c|}{|D_c|} - 1 \iff \\ \frac{|D_c|}{|A_c|} &\geq \frac{|S|}{|ap(T)|} \iff 1 - \frac{|S|}{|ap(T)|} \geq 1 - \frac{|D_c|}{|A_c|} \iff \frac{|A_c|}{|A_c| - |D_c|} \geq \frac{|ap(T)|}{|ap(T)| - |S|}. \end{aligned}$$

We note that all fractions that appear above are well defined. This is simply because, by the definition of $\mathcal{R}$, it holds that $|D_c| < |A_c|$, and similarly $\frac{|S|}{|ap(T)|} \leq \frac{|D_c|}{|A_c|} < 1$.

Since $p(c) \geq \frac{|A_c|}{|A_c| - |D_c|}$, we have that $p(c) \geq \frac{|ap(T)|}{|ap(T)| - |S|}$. It holds that only voters from $ap(T)$ can pay for candidates from T and jointly they have at most $|ap(T)| \frac{k}{n}$ money. Since the purchase of a candidate from T costs at least $\frac{|ap(T)|}{|ap(T)| - |S|}$, they can afford a number of candidates that is at most

$$\frac{|ap(T)| \frac{k}{n}}{\frac{|ap(T)|}{|ap(T)| - |S|}} = (|ap(T)| - |S|) \frac{k}{n} \leq \ell \cdot \frac{n}{k} \cdot \frac{k}{n} = \ell.$$

This implies in particular that at most ℓ candidates from T are selected. $\square$

B Negative Guarantee for Exhaustive Priceable Rules

Recall that the proof of [Theorem 5](#) relies on voters having a limited budget. This, combined with the high cost of electing strongly opposed candidates, ensures that such candidates need significant support to be selected. When voter funds are unlimited, the proposed rules could continue adding candidates as long as they have more supporters than opposers (respecting the committee size constraint). However, even in this case, vetoing preferences are still being respected. Candidates with less opposition are prioritized, and others are included only if no better options exist. Below we formulate an axiom that sets the requirement on the level of support vetoed candidates must have in order to be elected.

Definition 4. Consider an election $E = (C, V, k, B)$ in which there is a set of voters S' such that $A_{S'} = \{c^*\}$ and $D_{c^*} = \emptyset$. In such an election, given a positive integer $\ell \leq k$ and a set of at least ℓ candidates T , we say that a set of voters $S \subseteq V$ is weakly (ℓ, T) -negatively-cohesive if $T \subseteq D_S$ and $|S| \geq |ap(T)| - \ell \cdot |S'|$. An outcome W is said to provide weak Group Veto for E if when $c^* \notin W$ then every weakly (ℓ, T) -negatively-cohesive group satisfies $|W \cap T| < \ell$. A rule $\mathcal{R}$ satisfies the weak Group Veto axiom if for every election E it outputs an outcome that provides weak Group Veto.

Theorem 10. Let $\mathcal{R}$ be a priceable rule such that $p(c) \geq \frac{|A_c|}{|A_c| - |D_c|}$ whenever $|A_c| > |D_c|$, and $p(c) = \infty$, otherwise. Then, $\mathcal{R}$ satisfies weak Group Veto.

Proof. Let S be a group of voters satisfying the conditions of weak Group Veto, specifically, there exists a set of candidates $T \subseteq \cap_{i \in S} D_i$ of cardinality ℓ and $|S| \geq |ap(T)| - \ell |S'|$. Following the arguments from the proof of [Theorem 5](#) we get that $p(c) \geq \frac{|ap(T)|}{|ap(T)| - |S|}$.

We now turn on computing $|W \cap T|$. Note that only voters from $ap(T)$ paid for including a candidate from T into W . Voters from $ap(T)$ have spent at most $|ap(T)|^{1/|S'|}$ until the end of the execution of $\mathcal{R}$ as we will now justify. Say that until the end of the execution of the rule, each voter has been allocated and has been allowed to spend at most q . Voters from S' didn't buy c^* by assumption. Therefore, $|S'| \cdot q < 1 \Rightarrow q < \frac{1}{|S'|}$. Hence, voters from $ap(T)$ had at most $|ap(T)|^{1/|S'|}$ to spend. By the fact that each purchase cost at least $\frac{|ap(T)|}{|ap(T)| - |S|}$, voters in $ap(T)$ can afford a number of candidates that is strictly less than

$$\frac{|ap(T)|^{1/|S'|}}{\frac{|ap(T)|}{|ap(T)| - |S|}} = (|ap(T)| - |S|) \frac{1}{|S'|} \leq \ell \cdot |S'| \frac{1}{|S'|} = \ell.$$

This completes the proof. □

We note that the term “exhaustive” has been used in the committee elections literature for rules that always elect k candidates. In our setting, fewer than k candidates may be elected even with unlimited voter budgets. This occurs when fewer than k candidates have more supporters than opposers.