Selecting Interlacing Committees

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Abstract

Polarization is a major concern for a well-functioning society. Often, mass polarization of a society is driven by polarizing political representation, even when the latter is easily preventable. The existing computational social choice methods for the task of committee selection are not designed to address this issue. We enrich the standard approach to committee selection by defining two quantitative measures that evaluate how well a given committee interconnects the voters. Maximizing these measures aims at avoiding polarizing committees. While the corresponding maximization problems are NP-complete in general, we obtain efficient algorithms for profiles in the voter-candidate interval domain. Moreover, we analyze the compatibility of our goals with other representation objectives, such as excellence, diversity, and proportionality. We identify trade-offs between approximation guarantees, and describe algorithms that achieve simultaneous constant-factor approximations.

1 Introduction

In recent years, the increasing prevalence of polarization has been a global concern, discussed not just by social scientists, but by society at large, and accompanied by extensive media coverage [22, 29]. Polarization is commonly defined as the division of a group into clusters of completely different opinions or ideologies [21]. It may result in greater ideological extremes and a reduced willingness to compromise or engage with differing views. As such, polarization is a major roadblock for the modern society, which has to work towards a consensus when resolving global challenges, such as fighting poverty, climate change, or pandemics (see [30] and the references therein).

Importantly, polarization can occur as a phenomenon concerning an entire society or only at the level of political representation, e.g., when considering the distribution of opinions among the delegates in a parliament. The former is often referred to as *mass polarization*, while the latter is known as *elite polarization* [see, e.g., 1].

Academic literature broadly agrees that the phenomenon of elite polarization is on the rise. For example, when depicting the members of the US Congress in terms of their ideology on a scale ranging from the most liberal to the most conservative, one can observe a significant shift when comparing the 87th Congress in the 1960s and the 111th Congress around 2010, see Figure 2.1 in the book by Fiorina [20]. However, whether the society as a whole is polarized as well is less clear. Fiorina et al. [22] argue that there is no conclusive evidence for mass polarization, even when considering highly sensitive topics such as abortion. For instance, they provide evidence that the elite polarization among delegates is already much higher than the polarization among party identifiers [22, Table 2.1]. They argue that the media play an important role in creating an inaccurate picture of mass polarization [22]. Indeed, the media can have a significant effect on the perception of and conclusions drawn from elite polarization [29].

This view is opposed by Abramowitz and Saunders [1], who analyze data from the American National Election Studies. They provide extensive evidence that mass polarization has increased significantly since the 1970s. Moreover, their results suggest mass polarization based on geography (i.e., different ideologies across US states) or religious beliefs.

Against this background, we aim to offer a novel perspective on the intertwined phenomena of mass polarization at the broad level of a society as a whole and elite polarization at the level of the society's political, parliamentary representation. We highlight how an election can lead to a parliament that is

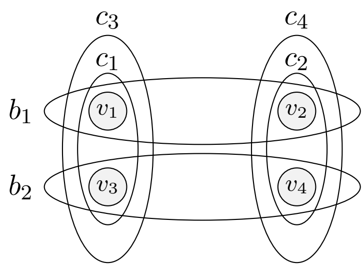

Figure 1: A preference profile with four voters $v_1, \dots, v_4$ is depicted as hypergraph, where the voters are nodes and the candidates b_i, c_j are hyperedges connecting the voters approving them. In this profile, typical multiwinner voting rules do not distinguish between selecting $\{c_1, c_2, c_3, c_4\}$ and $\{c_1, c_2, b_1, b_2\}$.

far more polarized than the society it represents, and we propose quantitative measures that evaluate a set of representatives according to how well it interlaces the electorate. We believe that our ideas can be developed to prevent societies with broadly moderate opinions from being represented by unnecessarily polarized parliaments.

We approach polarized democratic representation through the lens of social choice theory. In this line of research, parliamentary elections have been conceptualized as so-called multiwinner voting rules. Their formal study, especially in the approval-based setting, in which each voter's ballot specifies a set of approved candidates, has received extensive attention in recent years [19, 28].

Example 1.1. As a motivating example, consider the voting scenario illustrated in Figure 1. There are four voters, indicated by the gray circles, as well as six candidates. Each candidate is represented by an ellipse that covers the voters approving this candidate. For instance, candidate b_1 is approved by voters v_1 and v_2, whereas candidates c_1 and c_3 are both approved by the same set of voters, namely v_1 and v_3. In practice, this is likely to happen when c_1 and c_3 represent very similar ideologies.

Assume that we want to select a committee consisting of 4 candidates. Two reasonable choices would be to select $W = \{c_1, c_2, c_3, c_4\}$ or $W' = \{c_1, c_2, b_1, b_2\}$. Both selections lead to committees in which each voter approves exactly two selected candidates. Moreover, multiwinner voting rules typically considered in the literature, such as Thiele rules and their sequential variants [37], Phragmén's rule [35], or the more recently introduced method of equal shares [34], do not distinguish between these two choices. There is, however, a difference. While W divides the electorate into two perfectly separated subsets of voters, W' connects all voters. From the perspective of polarization, W looks polarizing while W' bridges all voters. Thus, we need novel voting rules that can tease out this distinction. In our paper, we aim to provide a principled approach that favors committees in the spirit of W'.[1] $\triangleleft$

We define two simple objectives that aim to measure how well a committee interlaces the voters. First, we consider maximizing the number of *pairs* of voters approving a common candidate (the Pairs objective). While optimizing this objective leads to the selection of W' in Example 1.1, it can still result in voters being split into large disconnected clusters (cf. Example 3.2). The reason is that Pairs only counts direct, but not indirect links. Hence, as a second objective we count the number of pairs of voters that are *connected* by a sequence of candidates (the Cons objective).

While both objectives immediately give rise to voting rules—select a committee maximizing Pairs or Cons—we primarily view them as measures of polarization. Whenever they are high, polarization in the selected committee is low. Thus, we investigate the feasibility of maximizing our objectives, both on their own and in combination with the goals of diversity and proportionality.

We first consider the computational problem of maximizing Pairs or Cons in isolation (Section 4). Unfortunately, for unrestricted preferences this problem is NP-hard. However, we obtain a polynomial-

[1] Of course, while we try to highlight the phenomenon at hand with a simple example, our construction extends to elections with many voters or candidates: e.g., each voter in the example might represent a quarter of a large electorate.

time algorithm for the structured domain of voter-candidate interval (VCI) preferences [24], where voters and candidates are represented by intervals on the real line and a voter approves a candidate if and only if their intervals intersect. Such preferences are reasonable in parliamentary elections where candidates can often be ordered on a left-right spectrum and voters approve candidates that are close to them on this spectrum.

In Section 5, we investigate whether one can select interlacing committees while achieving other desiderata. We first consider *excellence*, as measured by the *approval voting* (AV) score, i.e., the total number of approvals received by committee members. There is a straightforward way to obtain what is essentially an α-approximation of the Pairs objective together with a $(1 - \alpha)$-approximation of the AV score: one can simply use an α-fraction of the committee for the former and a $(1 - \alpha)$-fraction for the latter. Unfortunately, it turns out that this simple algorithm is essentially optimal: We prove that if a voting rule provides an α-approximation of the Pairs objective and a β-approximation of the AV score, then necessarily $\alpha + \beta \leq 1$. Next, we look at *diversity*, as captured by the *Chamberlin–Courant* (CC) score, which is the number of voters who approve at least one candidate in the committee. The CC score is closely related to the Pairs objective: the former measures the coverage of voters, while the latter measures the coverage of pairs of voters. Hence, it is quite surprising that we obtain the same trade-off as for Pairs and AV. Further, we study the compatibility with *proportionality*, as captured by the extended justified representation axiom (EJR). Again, we show the same tight trade-off: If a voting rule provides an α-approximation of the Pairs objective and a β-approximation of EJR, then $\alpha + \beta \leq 1$.

It is more challenging to combine the Cons objective with AV, CC, EJR, or even Pairs. This is due to an interesting qualitative difference between Pairs and Cons. While a constant fraction of the best candidates achieves a constant approximation of Pairs, for Cons this is not the case. Hence, we obtain worse trade-offs: If a voting rule provides an α^2-approximation of Cons and a β-approximation of AV, CC, EJR, or Pairs, then $\alpha + \beta \leq 1$. Note that since $\alpha < 1$, it holds that $\alpha^2 < \alpha$. Hence, for instance, $\alpha^2 = \frac{1}{3}$ and $\beta = \frac{1}{2}$ is already impossible. Moreover, for Cons and AV specifically, the trade-off that we obtain is even more subtle, which suggests that finding a matching lower bound might be challenging. Nevertheless, we make first steps towards this goal, by showing that under suitable domain restrictions there always exists a committee that achieves a $\frac{1}{4}$-approximation of Cons and a $\frac{1}{2}$-approximation of AV, CC, EJR, or Pairs, which matches our upper bound.

2 Related Work

In the existing literature, multiwinner voting rules usually aim to guarantee the selection of the best candidates based on their individual quality [3, 16], representation of diverse opinions [8, 14], or proportional treatment of cohesive voter groups [37, 35, 31, 34]. An overview of the most common approval-based multiwinner voting rules is given in the book by Lackner and Skowron [28]. To the best of our knowledge, no rules were proposed so far with the explicit goal of reducing polarization or connecting voters.

A line of research in multiwinner voting looks at the possibility of combining various objectives as well as their inherent trade-offs, similar to our study in Section 5. Lackner and Skowron [27] provide worst-case bounds on AV and CC scores of committees output by popular voting rules. For ordinal preferences, Kocot et al. [25] analyze the complexity of finding committees that offer an optimal combination of approximations of two objectives. Moreover, a series of works look at AV and CC scores that can be guaranteed by committees that satisfy proportionality axioms [7, 17, 18].

Several authors study the relationship between an electoral system (or, more narrowly, a voting rule) and the way the candidates choose to strategically place themselves on the political spectrum [10, 32, 5, 26]. Such an analysis can indicate whether a rule prevents, or reinforces, polarization. Our approach differs in that we analyze the direct effect of a voting rule on the polarization caused by a chosen committee, while the aforementioned works analyze how preferences evolve based on a given rule.

Delemazure et al. [11] pursue a goal that can be seen as opposite to ours: selecting a most polarizing committee of size 2; they focus on ordinal preferences. In a similar vein, Colley et al. [9] proposed measures of how *divisive*, or polarizing, a single candidate is.

3 Model

We start by introducing key notation and proposing two ways of measuring how well a committee interconnects the voters. For a positive integer $k \in \mathbb{N}$, define $[k] := \{1, \dots, k\}$.

3.1 Approval-Based Multiwinner Voting

We consider the standard setting of approval-based multiwinner voting [28]. Given a set of m candidates C, an *election instance* $\mathcal{E} = (V, A, k)$ consists of a set of n voters V, an approval profile $A = (A_v)_{v \in V}$ with $A_v \subseteq C$ for all $v \in V$, and a target committee size $k \in [m]$. For a voter $v \in V$, the set A_v captures the candidates approved by v. Throughout the paper, we view a profile A as a hypergraph with vertex set V, and, for each $c \in C$, a hyperedge $V_c = \{v \in V : c \in A_v\}$. In the remainder of this section, we consider an election instance $\mathcal{E} = (V, A, k)$ over a candidate set C.

Besides the general setting, we also consider elections with spatial one-dimensional preferences, i.e., elections where all voters and all candidates can be mapped to intervals on the real line so that a voter approves a candidate if and only if their respective intervals intersect. Formally, following Godziszewski et al. [24], we say that an election (V, A, k) belongs to the *voter-candidate interval (VCI) domain* if there exist a collection of positions $\{x_c\}_{c \in C} \cup \{x_v\}_{v \in V} \subseteq \mathbb{R}$ and a collection of nonnegative radii $\{r_c\}_{c \in C} \cup \{r_v\}_{v \in V} \subseteq \mathbb{R}^+ \cup \{0\}$ such that for all $v \in V$, $c \in C$ it holds that $c \in A_v$ if and only if $|x_c - x_v| \leq r_c + r_v$.

The VCI domain is the most general domain of one-dimensional approval preferences considered in the literature. In particular, it generalizes the voter interval (VI) and candidate interval (CI) domains, defined as follows [15]. An election belongs to the *voter interval (VI) domain* if there is an ordering of the voters $v_1, \dots, v_n$ such that each candidate is approved by some interval of this ordering, i.e., for each $c \in C$ there exist $i, j \in [n]$ such that $V_c = \{v_i, \dots, v_j\}$. Similarly, an election belongs to the *candidate interval (CI) domain* if there is an ordering of the candidates $c_1, \dots, c_m$ such that for each $v \in V$ there exist $i, j \in [m]$ such that $A_v = \{c_i, \dots, c_j\}$. It is easy to see that the VI and CI domains are contained in the VCI domain.[2]

A *committee* for an instance (V, A, k) is a subset $W \subseteq C$ with $|W| \leq k$; we say that W is *feasible* if $|W| = k$. A *(multiwinner) voting rule* f takes as input an instance (V, A, k) and outputs a feasible committee $f(V, A, k)$.

3.2 Classic Committee Selection

A popular classification of multiwinner voting rules is in terms of the main objective in electing the committee. Three most commonly studied objectives are *excellence*, *diversity*, and *proportionality* [19].

Both excellence and diversity are defined quantitatively: each of these objectives is associated with a function that assigns a numerical score to each committee, with higher scores associated with better

[2] For instance, given an election $\mathcal{E} = (V, A, k)$ in VI, as witnessed by voter ordering $v_1, \dots, v_n$, we can set $x_{v_i} = i$ and $r_{v_i} = 0$ for each $i \in [n]$. To position the candidates, for each $c \in C$ we compute $c^- = \min\{i : c \in A_{v_i}\}$ and $c^+ = \max\{i : c \in A_{v_i}\}$ and set $x_c = (c^- + c^+)/2$, $r_c = (c^+ - c^-)/2$. Clearly, these positions and radii certify that $\mathcal{E}$ belongs to the VCI domain. For CI, the construction is similar.

performance. Formally, given an instance $\mathcal{E} = (V, A, k)$ and a committee W with $|W| \le k$, we define

$$\mathrm{AV}(W, \mathcal{E}) := \textstyle\sum_{v \in V} |A_v \cap W|,$$
$$\mathrm{CC}(W, \mathcal{E}) := |\{v \in V : A_v \cap W \neq \varnothing\}|.$$

For both objectives (as well as for the two novel objectives defined in Section 3.3) we omit $\mathcal{E}$ from the notation when it is clear from the context. The quantities AV and CC are referred to as, respectively, the *approval score* and the *Chamberlin–Courant score* of committee W in election $\mathcal{E}$. Intuitively, AV counts the number of approvals received by the members of W and is viewed as a measure of excellence, while CC counts the number of voters represented by W, i.e., voters who approve at least one member of W, and is viewed as a measure of diversity. The voting rule that outputs a feasible committee maximizing AV (respectively, CC) is known as the *approval voting rule* (respectively, the *Chamberlin–Courant rule*[3]).

Consider a function S that assigns scores to committees in a given election (e.g., S = AV or S = CC). Given $\alpha \in [0, 1]$, we say that a committee W^* *satisfies* α-S for an election $\mathcal{E} = (V, A, k)$ if $S(W^*, \mathcal{E}) \ge \alpha \cdot \max_{W \subseteq C, |W| = k} S(W, \mathcal{E})$. Moreover, we say that a voting rule f *satisfies* α-S if for every election $\mathcal{E}$ it holds that $f(\mathcal{E})$ satisfies α-S for $\mathcal{E}$. For instance, the Chamberlin–Courant rule satisfies 1-CC.

A function $f : 2^X \to \mathbb{R}$ is said to be *submodular* if for every pair of sets S, T with $S \subset T \subset X$ and every $x \in X \setminus T$ it holds that $f(S \cup \{x\}) - f(S) \ge f(T \cup \{x\}) - f(T)$. It is immediate that, for a fixed election $\mathcal{E}$ over a candidate set C, the functions $\mathrm{AV}(W, \mathcal{E})$ and $\mathrm{CC}(W, \mathcal{E})$ are submodular functions from 2^C to $\mathbb{N}$. Our proofs will use the following basic fact about submodular functions (see, e.g., the seminal work of Nemhauser et al. [33]; for completeness, we provide a simple proof in the appendix).

Proposition 3.1. *Let $f : 2^X \to \mathbb{R}$ be a submodular function. For every pair of positive integers $\ell < k \le |X|$ and a set S of size k there exists a subset $S' \subseteq S$ of size ℓ with $\frac{f(S')}{\ell} \ge \frac{f(S)}{k}$.*

In contrast to excellence and diversity, proportionality is typically captured by representation axioms. A prominent axiom of this type is *extended justified representation* (EJR) [2]; intuitively, it states that sufficiently large groups of voters with similar preferences should be appropriately represented in the selected committee. We consider an approximate version in which the size of groups challenging their representation is scaled down by the approximation factor [see, e.g., 12].

Given an election (V, A, k) over C and $\alpha \in (0, 1]$, a committee $W \subseteq C$ is said to *satisfy* α*-EJR* if for every $\ell \in [k]$ and every subset $S \subseteq V$ such that $\alpha \cdot |S| \ge \frac{\ell}{k} \cdot |V|$ and $|\bigcap_{i \in S} A_i| \ge \ell$ there exists at least one voter $i \in S$ such that $|W \cap A_i| \ge \ell$. We say that a rule f *satisfies* α*-EJR* if for every election $\mathcal{E}$ it holds that $f(\mathcal{E})$ satisfies α-EJR. By setting α to 1, we obtain the standard EJR axiom.

3.3 Interlacing Committee Selection

We now define two new objectives, which assess committees based on how well they interlace voters.

Our first objective is the number of *pairs* of voters that jointly approve a selected candidate. Given an election $\mathcal{E} = (V, A, k)$, let $V^{(2)} := \{\{u, v\} \subseteq V : u \neq v\}$ be the set of all voter pairs. Then for each W with $|W| \le k$ we set

$$\textsc{Pairs}(W, \mathcal{E}) := |\{\{u, v\} \in V^{(2)} : A_u \cap A_v \cap W \neq \varnothing\}|.$$

Note that for every instance $\mathcal{E} = (V, A, k)$ one can define the *associated pair instance* $\mathcal{E}^{(2)} = (V^{(2)}, A^{(2)}, k)$, where $A^{(2)}_{\{u,v\}} = A_u \cap A_v$ for every $\{u, v\} \in V^{(2)}$. For each instance $\mathcal{E}$ and committee $W \subseteq C$ we have $\textsc{Pairs}(W, \mathcal{E}) = \mathrm{CC}(W, \mathcal{E}^{(2)})$. Moreover, $\textsc{Pairs}(W, \mathcal{E})$ is a submodular function from 2^C to $\mathbb{N}$.

[3]Originally, Chamberlin and Courant [8] proposed their rule for ordinal preferences. However, the approval variant of this rule is commonly studied in the computational social choice literature.

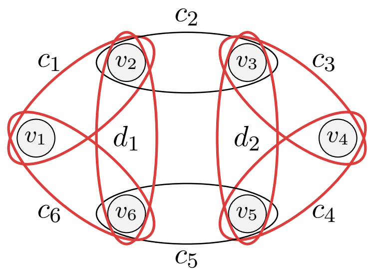

Figure 2: Illustration of Example 3.2. When the target committee size is 6, every size-6 subset of candidates maximizes the Pairs objective. However, Cons is higher for the committee $\{c_1, \dots, c_6\}$ than for the disconnected committee $\{c_1, c_3, c_4, d_1, d_2\}$ (indicated by thick red lines).

While the Pairs objective only considers direct links between voters, our second objective takes into account indirect connections as well. Given an instance $\mathcal{E} = (A, V, k)$ and a subset of candidates $W \subseteq C$, we say that two voters $u, v \in V$ are *connected by* W (and write $u \sim_W v$) if there is a sequence of voters $u = v_0, v_1, \dots, v_s = v$ with $A_{v_{i-1}} \cap A_{v_i} \cap W \neq \varnothing$ for every $i \in [s]$. To evaluate a committee W, we count pairs of voters connected by W. Formally,

$$\textsc{Cons}(W, \mathcal{E}) := \left| \{\{u, v\} \in V^{(2)} : u \sim_W v\} \right|.$$

Since both Pairs and Cons assign scores to committees, we also consider their approximate versions, i.e., α-Pairs and α-Cons.

Our interest in Cons is motivated by the following example.

Example 3.2. Consider a profile with six voters $v_1, \dots, v_6$, six *cycle* candidates $c_1, \dots, c_6$, and two *diagonal* candidates d_1 and d_2, whose hypergraph is depicted in Figure 2. Each cycle candidate is approved by two consecutive voters: for $i = 1, \dots, 5$ candidate c_i is approved by v_i and v_{i+1}, while c_6 is approved by v_1 and v_6. Also, d_1 is approved by v_2 and v_6 and d_2 by v_3 and v_5. Let $k = 6$.

Consider the following two committees: $W = \{c_1, c_2, c_3, c_4, c_5, c_6\}$ contains all cycle candidates, whereas in $W' = \{c_1, c_3, c_4, c_6, d_1, d_6\}$ two cycle candidates are exchanged for the diagonal candidates (W' is marked by thick red hyperedges in Figure 2). Common voting rules, including the approval voting rule and the Chamberlin–Courant rule, do not distinguish between W and W', as each voter approves exactly two candidates in either committee. Moreover, the rule that maximizes Pairs is also unable to distinguish them, as both W and W' cover exactly 6 pairs of voters. However, intuitively, W' seems more polarizing: under W', there are two disconnected groups of voters, each supporting (though not fully) their own set of candidates.

In contrast, a rule that maximizes Cons is sensitive to the differences between the two committees. Under W, all 15 pairs of voters are connected, while W' only achieves 6 connections. $\triangleleft$

Note that, in contrast to AV, CC and Pairs, the Cons objective is not submodular: e.g., in Example 3.2 adding c_2 to $\{c_1\}$ creates two additional connected pairs, while adding c_2 to $\{c_1, c_3\}$ creates four additional connected pairs.

4 Computation of The New Objectives

In this section, we show that maximizing Pairs and Cons is NP-hard in general, but tractable on well-structured domains. Missing proofs from this section can be found in Appendix A.

4.1 General Preferences

Our hardness proofs are based on the NP-complete problem Exact Cover by 3-Sets (X3C) [23]. An instance of X3C is a pair $(R, \mathcal{S})$, where R is a ground set of size 3ρ and $\mathcal{S}$ is a collection of 3-element subsets of R; it is a Yes-instance if and only if there exists a subset $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| = \rho$ that covers R.

The proof idea for the Pairs objective is to represent every element in the ground set of an X3C instance by a pair of voters.

Theorem 4.1. *It is* NP*-complete to decide, given an election $\mathcal{E} = (V, A, k)$ and a threshold $q \in \mathbb{N}$, whether there exists a committee W of size at most k such that* Pairs$(W, \mathcal{E}) \geq q$.

A similar hardness result holds for Cons. The proof idea is to introduce an auxiliary voter that is the focal point in connecting all voters.

Theorem 4.2. *It is* NP*-complete to decide, given an election $\mathcal{E} = (V, A, k)$ and a threshold $q \in \mathbb{N}$, whether there exists a committee W of size k such that* Cons$(W, \mathcal{E}) \geq q$. *The hardness result holds even if $q = \binom{n}{2}$, i.e., if the goal is to connect all n voters.*

4.2 One-Dimensional Preferences

We will now complement our hardness results by arguing that, on elections in the VCI domain, we can maximize Pairs and Cons in polynomial time.

We start by observing that, for the objectives we consider, a VCI instance can be transformed into a CI instance without changing the value of these objectives. To this end, we define a notion of dominance among candidates and prove that, in the absence of dominated candidates, every VCI instance is a CI instance.

4.2.1 From VCI to CI

Given an election $\mathcal{E} = (V, A, k)$ over a candidate set C, we say that a candidate $c' \in C$ is *dominated* by a candidate $c \in C$ if every voter approving c' also approves c, and some voter approves c but not c', i.e., $V_{c'}$ is a proper subset of V_c.

It turns out that if an election in the VCI domain contains no dominated candidates, it belongs to the much simpler to analyze CI domain; this observation, which is implicit in the work of Elkind et al. [17, Lemma 4.7], may be of independent interest. Indeed, removal of dominated candidates from a winning committee does not affect the Pairs and Cons objectives, so we can simply remove all dominated candidates from the input instance.

Proposition 4.3. *Let $\mathcal{E}$ be an instance in the VCI domain. If $\mathcal{E}$ contains no dominated candidates, then it belongs to the CI domain.*

Proof. Consider an election $\mathcal{E} = (V, A, k)$ over the candidate set C that belongs to the VCI domain, as witnessed by positions $\{x_c\}_{c \in C} \cup \{x_v\}_{v \in V}$ and radii $\{r_c\}_{c \in C} \cup \{r_v\}_{v \in V}$. Renumber the candidates so that $x_{c_1} \leq x_{c_2} \leq \cdots \leq x_{c_m}$.

Suppose for the sake of contradiction that this ordering of the candidates does not witness that $\mathcal{E}$ belongs to CI. Then, there exists a voter $v \in V$ and $h < i < j$ such that v approves c_h and c_j, but not c_i. For readability, we will refer to the positions and radii of c_h, c_i and c_j as x_h, x_i, x_j and r_h, r_i, r_j, respectively. Since v does not approve c_i, we have $x_v \neq x_i$; we can then assume without loss of generality that $x_v < x_i \leq x_j$. To obtain a contradiction, we will show that c_i is dominated by c_j.

To this end, we will argue that $[x_i - r_i, x_i + r_i] \subseteq [x_j - r_j, x_j + r_j]$. Indeed, $c_i \notin A_v$ implies $x_i - r_i > x_v + r_v$, whereas $c_j \in A_v$ implies $x_j - r_j \leq x_v + r_v$. Combining these inequalities, we obtain $x_i - r_i > x_j - r_j$. It then follows that $x_i + r_i < x_i + (x_i - x_j + r_j) \leq x_j + r_j$, where the last inequality follows from $x_i \leq x_j$. Thus, the interval of c_i is subsumed by that of c_j, and hence every voter who approves c_i also approves c_j. Moreover, v approves c_j, but not c_i. We have shown that c_i is dominated, concluding the proof. □

In what follows, we state our results for the VCI domain, but assume that the input election belongs to the CI domain, and we are explicitly given the respective candidate order. It will also be convenient to assume that this order is $c_1, \dots, c_m$. This requires two preprocessing steps: first, we eliminate all dominated candidates (which, by Proposition 4.3, results in a CI election), and second, we compute an ordering of the candidates witnessing that our instance belongs to the CI domain. Both steps can be implemented in polynomial time [for the second step, see, e.g., 15].

4.2.2 Efficient Algorithms

We are ready to present polynomial-time algorithms for Pairs and Cons on the VCI domain. Since Pairs is identical to CC on the associated pair instance, we can compute Pairs by leveraging an existing algorithm for CC in the CI domain [4, 15].

Proposition 4.4. *In the VCI domain, a committee that maximizes Pairs can be found in polynomial time.*

Proof. Fix an election instance $\mathcal{E}$. As argued earlier, we can assume that $\mathcal{E}$ is in the CI domain with respect to candidate ordering $c_1, \dots, c_m$. Recall that $\text{Pairs}(W, \mathcal{E}) = \text{CC}(W, \mathcal{E}^{(2)})$. Now, note that for all $\{u, v\} \in V^{(2)}$, it holds by definition that $A_{\{u,v\}} = A_u \cap A_v$ is the intersection of two intervals of that ordering and hence itself an interval. Thus, $\mathcal{E}^{(2)}$ is in the CI domain with respect to the same candidate ordering. For instances in the CI domain, CC can be maximized in polynomial time [4, 15]. □

In the VCI domain, we can also compute a committee that maximizes Cons in polynomial time; however, the argument is significantly more complicated. Again, we assume that the input profile belongs to the CI domain, as witnessed by the candidate ordering $c_1, \dots, c_m$. A natural idea, then, is to use dynamic programming to compute, for each $b \in [k]$ and $i \in [m]$, an optimal subcommittee of size b with rightmost candidate c_i. For $b = 1$, the computation is straightforward, and for $b = k$, one of the resulting m committees globally maximizes Cons. However, computing the value of adding c_i to a committee of size $b - 1$ that has c_j, $j < i$, as its rightmost candidate is a challenging task: this is because the number of connections that c_i adds depends on the size of the connected component associated with c_j. To handle this, we add a third dimension to the dynamic program: the number of voters $x \in [n]$ in the connected component of the last selected candidate. The resulting dynamic program has $\mathcal{O}(mnk)$ cells, and each cell can be filled in polynomial time given the values of the already-filled-in cells. We present the proof in Appendix A.2.

Theorem 4.5. *In the VCI domain, a committee that maximizes Cons can be computed in polynomial time.*

5 Combining Objectives

While interlacing objectives can be viewed in isolation, in many cases, standard objectives of excellence, diversity, or proportionality continue to be important for the selection of a committee. In this section, we investigate to what extent we can select committees that simultaneously perform well with respect to both interlacing and standard objectives. Many of our results will show that there are inherent trade-offs between the objectives. For this purpose, given two objectives, we construct instances whose hypergraph representation can be partitioned into two components, each corresponding to one objective.

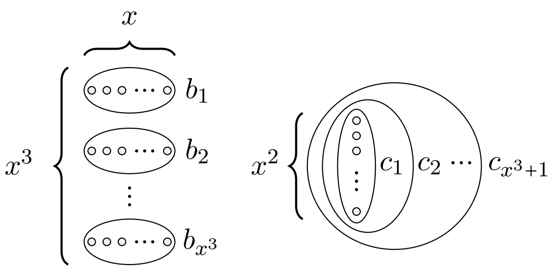

Figure 3: Illustration of the profile constructed in the proof of Proposition 5.2. The block voters are on the left and the central voters are on the right. Each block candidate is approved by x block voters, whereas each central candidate is approved by all central voters.

By design, candidates selected in one component mainly contribute to this component's objective while having a negligible effect on the other objective. Missing proofs can be found in Appendix B.

5.1 Pairs Objective

First, we consider combining the Pairs objective with individual excellence of the committee members, as measured by AV. For every $\alpha \in [0,1]$ and every election $\mathcal{E} = (V, A, k)$, there is a simple way to obtain a simultaneous $\lceil \alpha k \rceil / k$-approximation of Pairs and $\lfloor (1-\alpha)k \rfloor / k$-approximation of AV. Indeed, we can split the k positions on the committee into two parts of size $k_1 = \lceil \alpha k \rceil$ and $k_2 = \lfloor (1-\alpha)k \rfloor$, respectively, and then select k_1 candidates so as to maximize Pairs and k_2 candidates so as to maximize AV (if some candidate is selected both times, we replace their second copy by an arbitrary unselected candidate). Since Pairs and AV are submodular functions, Proposition 3.1 implies that this procedure provides the desired guarantees. We can use the same technique to combine Pairs with the goal of diverse representation, as measured by CC (recall that CC is submodular, too). Note that Lackner and Skowron [27] propose a similar method for combining AV and CC.

Proposition 5.1. *For every $\alpha \in [0,1]$ and election $\mathcal{E}$, there exist committees that satisfy $\lceil \alpha k \rceil / k$-Pairs and one of $\lfloor (1-\alpha)k \rfloor / k$-AV or $\lfloor (1-\alpha)k \rfloor / k$-CC.*

Perhaps surprisingly, it turns out that, for both combinations, this is the best we can hope for.

Proposition 5.2. *For every $\alpha, \beta \in [0,1]$, if a voting rule satisfies α-Pairs and β-AV, then $\alpha + \beta \leq 1$.*

Proof. We will construct a family of instances that allows us to bound the sum of approximation ratios. For a given constant $x \in \mathbb{N}$, consider the election $\mathcal{E} = (V, A, k)$ defined as follows (see Figure 3 for an illustration). The set C consists of x^3 *block candidates* $b_1, \ldots, b_{x^3}$, and $x^3 + 1$ *central candidates* $c_1, \ldots, c_{x^3+1}$, so that $|C| = 2x^3 + 1$. The set V consist of x^3 groups of *block voters* $(B_i)_{i \in [x^3]}$ of size x each, and a single group of x^2 *central voters*. For $i \in [x^3]$, each voter in block B_i approves candidate b_i only, whereas each central voter approves all central candidates $c_1, \ldots, c_{x^3+1}$. The target committee size is set to $k = x^3 + 1$.

Since there are fewer than k block candidates, every committee W contains at least one central candidate, who is approved by all x^2 central voters, and covers all $(x^2-1)x^2/2$ pairs of central voters. Then, every additional central candidate contributes x^2 to the AV objective and 0 to the Pairs objective, whereas every additional block candidate contributes x to the AV objective and $x(x-1)/2$ to the Pairs objective.

Assume that for some $\gamma \in \{0, 1/x^3, 2/x^3, \ldots, 1\}$ our rule selects a committee $W_\gamma \subseteq C$ with $\gamma x^3 + 1$

central candidates and $(1-\gamma)x^3$ block candidates. Then, we obtain the following AV and Pairs scores:

$$\mathrm{AV}(W_\gamma,\mathcal{E}) = (\gamma x^3+1)x^2 + (1-\gamma)x^4 = \gamma x^5 + \mathcal{O}(x^4), \text{ and}$$
$$\textsc{Pairs}(W_\gamma,\mathcal{E}) = \frac{x^4-x^2}{2} + (1-\gamma)\frac{x^5-x^4}{2} = \frac{1-\gamma}{2}x^5 + \mathcal{O}(x^4),$$

where the $\mathcal{O}(\cdot)$ upper bound holds for all values of γ. Observe that the maximum AV score is obtained when we take $\gamma = 1$, and the maximum Pairs score is obtained when $\gamma = 0$. Also,

$$\frac{\mathrm{AV}(W_\gamma,\mathcal{E})}{\mathrm{AV}(W_1,\mathcal{E})} + \frac{\textsc{Pairs}(W_\gamma,\mathcal{E})}{\textsc{Pairs}(W_0,\mathcal{E})} \leq 1 + \mathcal{O}(1/x).$$

Hence, when x tends to infinity, the sum of the approximation ratios gets arbitrarily close to 1. □

One may expect the Pairs and CC objectives to be more aligned than Pairs and AV. Indeed, recall that Pairs is the same as computing CC on the associated pair instance. However, surprisingly, the worst-case trade-off for these objectives is the same as for Pairs and AV.

Proposition 5.3. *For every $\alpha,\beta \in [0,1]$, if a voting rule satisfies α-Pairs and β-CC, then $\alpha+\beta \leq 1$.*

Finally, we investigate how to combine the Pairs objective with proportional representation, as captured by the EJR axiom. Again, we can use the committee-splitting technique to show that for every election $\mathcal{E} = (V,A,k)$ there is a committee that satisfies $\lceil \alpha k \rceil/k$-Pairs and $(1-\alpha)$-EJR. For this, we first need to show that we can guarantee $(1-\alpha)$-EJR with a $(1-\alpha)$-fraction of the committee seats. To this end, we employ a variant of the *method of equal shares* (MES) by Peters and Skowron [34]. Briefly, this rule gives each voter k/n units of money; it then sequentially selects candidates that are best for voters that still have money, and subtracts money from the supporters of the selected candidates. By adapting the proof that MES satisfies EJR [34], we show that, by executing MES while scaling the voters' budgets by α, we obtain α-EJR for the original instance; we believe that this result is of independent interest.[4] We defer a formal definition of MES and the proof of Lemma 5.4 to Appendix B.2.

Lemma 5.4. *Let $\alpha \leq 1$ be given. For every election $\mathcal{E} = (V,A,k)$, executing MES on $(V,A,\alpha k)$ returns a committee of size $\lfloor \alpha k \rfloor$ satisfying α-EJR in polynomial time.*

We remark that the committee obtained as described in Lemma 5.4 satisfies an even stronger notion of proportionality, namely α-EJR+ [6]. Using Lemma 5.4, we now easily obtain the desired guarantees.

Proposition 5.5. *For every $\alpha \in [0,1]$ and election $\mathcal{E}$, there exists a committee that satisfies α-Pairs and $(1-\alpha)$-EJR.*

Proof. Consider an election $\mathcal{E}$. By Lemma 5.4, we can satisfy $(1-\alpha)$-EJR using $\lfloor (1-\alpha)k \rfloor$ candidates. With the remaining $k - \lfloor (1-\alpha)k \rfloor = \lceil \alpha k \rceil$ candidates, we can guarantee α-CC on the associated pair instance $\mathcal{E}^{(2)}$. This is equivalent to satisfying α-Pairs on $\mathcal{E}$, concluding the proof. □

As before, we provide a matching upper bound. We note that our proof works even if, instead of EJR, we consider the much weaker axiom of *justified representation* (JR) [2].

Proposition 5.6. *For every $\alpha,\beta \in [0,1]$, if a voting rule satisfies α-Pairs and β-EJR, then $\alpha+\beta \leq 1$.*

To conclude this section, we note that the guarantees offered by Propositions 5.1 and 5.5 are established by combining separate algorithms for the two objectives in question. Some of these objectives (in particular, CC and Pairs) do not admit polynomial-time algorithms unless P=NP. To achieve a polynomial runtime, we can instead use well-known approximation algorithms for CC and Pairs (for Pairs, we use the sequential Chamberlin–Courant rule on the associated pair instance). This comes, however, at the expense of a factor of up to $1-\frac{1}{e}$ in the approximation ratio [27].

[4] A similar observation was made by Dong and Peters [13], but they require $\lceil \alpha k \rceil$ candidates, which in our case would allow only for a rounded-down Pairs guarantee.

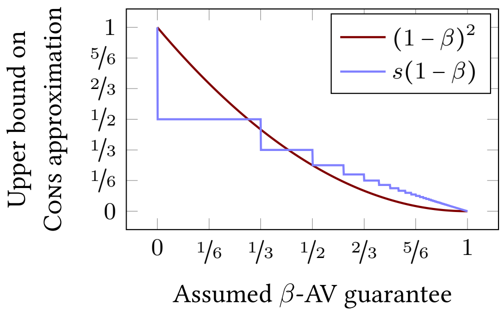

Figure 4: Two different upper bounds on the possible α-approximation of Cons for rules that satisfy β-AV. The $(1-\beta)^2$ upper bound is the result of Proposition 5.7 and $s(1-\beta)$ is implied by Proposition 5.9.

5.2 Cons Objective

An important reason why we obtained good approximations of Pairs, AV, and CC was that these objectives are submodular. In contrast, the Cons objective is not submodular (or even subadditive), so we cannot use Proposition 3.1. In fact, the following result shows that the trade-off between Cons and any of AV, CC, or Pairs is strictly worse (on the side of the Cons) than the trade-offs we have established in Section 5.1. Notably, our bound applies even to instances that belong to the VI domain.

Proposition 5.7. *For every $\alpha, \beta \in [0,1]$, if a voting rule satisfies α^2-Cons and β-AV, β-CC, or β-Pairs, then $\alpha + \beta \le 1$. This already holds in the VI domain.*

Proof sketch. For the proof of all three statements, consider an instance with x^3 blocks, with each block consisting of x voters approving respective block candidate. Further, we have $x^3 + 1$ central voters ordered on a line; each pair of adjacent central voters approves a designated central candidate. This instance is in the VI domain: we can place the voters within each block on the line, followed by the central voters. To satisfy β-Pairs, β-CC, or β-AV, we require at least $\beta x^3 - \mathcal{O}(x^2)$ block candidates; with the remaining candidates, we can obtain at most a $(1-\beta)^2$-approximation of Cons. $\square$

We derive a similar result for EJR by reducing the number of blocks from x^3 to slightly fewer than βx^3.

Proposition 5.8. *For every $\alpha, \beta \in [0,1]$, if a voting rule satisfies α^2-Cons and β-EJR, then $\alpha + \beta \le 1$. This already holds in the VI domain.*

For the Pairs objective, our trade-offs were complemented with straightforward algorithms matching the limitations of the trade-offs (up to rounding). However, in case of Cons, there is no straightforward method to use an α fraction of candidates to achieve α^2-Cons. In fact, we will now present a result showing that trade-offs can be even worse than the ones in Propositions 5.7 and 5.8.

Consider a stepwise function $s\colon (0,1] \to [0,1]$ given by $s(\alpha) = 1/(\lceil 2/\alpha \rceil - 1)$, see Figure 4 for an illustration. Intuitively, it finds the smallest $p \in \mathbb{N}$ such that $\alpha \ge 2/p$ and returns $1/(p-1)$. We then have the following trade-off between Cons and AV.

Proposition 5.9. *For every $\beta \in [0,1]$, if a voting rule satisfies β-AV, then it satisfies at most $s(1-\beta)$-Cons.*

Proposition 5.9 indicates that the trade-offs involving Cons may be rather complex: for some parameters β, Proposition 5.9 leads to steeper trade-offs than Proposition 5.7 (see Figure 4). This is in contrast to the bound obtained in Proposition 5.2, where the trade-off between Pairs and AV is tight up to rounding (see Proposition 5.1).

In fact, our two upper bounds for achieving an α-approximation of Cons given that a voting rule satisfies β-AV are based on functions intersecting several times (cf. Figure 4) and they are of different nature (stepwise versus smooth). Hence, none of them can yield a tight trade-off; indeed obtaining tight trade-offs appears to be a challenging problem. In particular, because Cons is not submodular, finding a general lower bound for these trade-offs seems non-trivial. Nevertheless, we conclude this section with a positive result on guarantees that we can obtain for a combination of the Cons objective with our other objectives in the VI domain. It matches a particular intersection point of both of our upper bounds.

Proposition 5.10. *For every instance (V, A, k) in the VI domain with even k, there exists a committee that satisfies $\frac{1}{4}$-Cons and any one of the criteria $\frac{1}{2}$-AV, $\frac{1}{2}$-CC, $\frac{1}{2}$-EJR, or $\frac{1}{2}$-Pairs.*

For $\frac{1}{4}$-Cons and $\frac{1}{2}$-EJR, the result of Proposition 5.10 extends to odd k. Indeed, it suffices to use $\lfloor \frac{k}{2} \rfloor$ candidates to achieve $\frac{1}{2}$-EJR (Lemma 5.4). Moreover, for odd k, the proof of Proposition 5.10 implies that choosing $\lceil \frac{k}{2} \rceil$ candidates yields $\frac{1}{4}$-Cons in the modified election where we are allowed to select $k + 1$ instead of k candidates, i.e., we achieve an approximation of a value that can only be larger than the maximum Cons score in the original election.

6 Conclusion

Our paper sheds new light on the interdependency of mass and elite polarization. We observe that the selection of a representative committee can significantly influence elite polarization independently of mass polarization. With the aim of avoiding polarization at the level of the representation, we have introduced Pairs and Cons, two numerical objectives that measure how well a committee interlaces the electorate.

We show that, while maximizing both objectives is NP-hard, a committee maximizing either of them can be computed in polynomial time on the voter-candidate interval domain. Also, we study the compatibility of our objectives with measures of excellence, diversity, and proportionality. We identify approximation trade-offs suggesting that, in the worst case, one cannot improve over the simple strategy of dividing the committee seats among different objectives and maximizing each objective with its designated share of the seats: there are instances on which the synergies are negligible. For almost all objectives we study, a subcommittee yields a fraction of the optimal value that is proportional to its size. Only for Cons, the dependency is quadratic (or even worse), leading to inferior guarantees.

For future work, an immediate open question is to determine the exact trade-off between Cons and other objectives. While we have a bound for α^2-Cons and $(1 - \alpha)$-approximations of other objectives, Proposition 5.9 shows that the picture is more nuanced.

Another direction is to consider our objectives in the broader context of participatory budgeting (PB), where each candidate has a cost, and the committee needs to stay within a given budget [36]. In this setting, candidates are usually projects, such as a playground, a community garden, or a cycling path. Interlacing voters by projects in PB has an additional interpretation: the funded projects may lead to interaction among the agents who use them (e.g., working together in a community garden). This seems quite desirable in the context of PB, where one of the goals is community building.

Moreover, it would be interesting to explore the compatibility of our objectives and the canonical desiderata in the context of real-life instances: It is plausible that on realistic data one can achieve much better trade-offs than in the worst case.

Finally, while Pairs and Cons offer some insight into the polarization induced by a committee, there are settings where they fail to provide useful information: For example, if some candidate is approved by all voters, any committee containing this candidate maximizes both objectives. Therefore, further insights could be gained by studying refined versions of our objectives; e.g., one can consider the strength of the connections or, in case of the Cons objective, the length of the (shortest) path between a pair of voters.

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Appendix

In the appendix, we present proofs missing from the main part of the paper.

A Missing Proofs from Section 4

In this appendix, we provide details about the computability of our objectives.

A.1 Hardness of Computation

Theorem 4.1. *It is* NP*-complete to decide, given an election $\mathcal{E} = (V, A, k)$ and a threshold $q \in \mathbb{N}$, whether there exists a committee W of size at most k such that* Pairs$(W, \mathcal{E}) \geq q$.

Proof. Membership in NP is immediate: for a given committee, its size and the number of pairs of voters approving a common candidate can be checked in polynomial time.

To show NP-hardness, we present a reduction from X3C. Given an instance $(R, \mathcal{S})$ of X3C with $|R| = 3\rho$, we construct an instance of our problem as follows. We create one candidate for each set in $\mathcal{S}$ and two voters for each element of the ground set, i.e., we set $C = \{c_S : S \in \mathcal{S}\}$ and $V = \{v_r, v'_r : r \in R\}$. For each $S \in \mathcal{S}$, candidate c_S is approved by voters $\{v_r, v'_r : r \in S\}$. We set the target committee size k to ρ and the threshold q to 15ρ. We will show that we can cover q pairs of voters if and only if the source instance is a Yes-instance of X3C.

Suppose first there exists a feasible committee W that covers q pairs of voters. Each $c \in W$ is approved by exactly 6 voters, so it can cover at most $\binom{6}{2} = 15$ pairs of voters. Moreover, the candidates' support sets are either disjoint or overlap in at least two voters. As $q = 15k$, this means that candidates in W have pairwise disjoint support sets. Since $|W| = k$, it follows that $\{S \in \mathcal{S} : c_S \in W\}$ forms a cover of R, i.e., our instance of X3C is a Yes-instance.

Conversely, assume that there exists a subset $\mathcal{S}' \subseteq \mathcal{S}$ of size k that covers R. Consider the committee $W = \{c_S : S \in \mathcal{S}'\}$. Then, $|W| = |\mathcal{S}'| = \rho = k$. Moreover, since, all of the sets in $\mathcal{S}'$ are pairwise disjoint, the support sets of the candidates in W are pairwise disjoint and contain exactly 6 voters each. Hence, there are $k \cdot \binom{6}{2} = 15\rho = q$ pairs of voters who approve a common candidate. □

Theorem 4.2. *It is* NP*-complete to decide, given an election $\mathcal{E} = (V, A, k)$ and a threshold $q \in \mathbb{N}$, whether there exists a committee W of size k such that* Cons$(W, \mathcal{E}) \geq q$. *The hardness result holds even if $q = \binom{n}{2}$, i.e., if the goal is to connect all n voters.*

Proof. Membership in NP is immediate: given an election instance (V, A, k), a committee W and a target q, we (1) construct a graph G with vertex set V where there is an edge between v and v' if and only if there is a candidate $c \in W$ that is approved by both v and v'; (2) identify the connected components of G, which we denote by $G_1, \ldots, G_t$; (3) return "yes" if and only if $\sum_{s \in [t]} \frac{|G_s|(|G_s|-1)}{2} \geq q$, where $|G_s|$ denotes the number of vertices of G_s.

To show NP-hardness, we present a reduction from X3C. Given an instance $(R, \mathcal{S})$ of X3C with $|R| = 3\rho$, we construct an instance of our problem as follows. We create a candidate for each set in $\mathcal{S}$ and a voter for each element of the ground set R, as well as one additional voter, i.e., we set $C = \{c_S : S \in \mathcal{S}\}$, $V = \{v\} \cup \{v_r : r \in R\}$. For each $S \in \mathcal{S}$, c_S is approved by $\{v\} \cup \{v_r : r \in S\}$. We want to select a committee $W \subseteq C$ of size $k = \rho$ and set the threshold q to $\binom{n}{2}$, where $n = |V|$.

Consider a collection $\mathcal{S}' \subseteq \mathcal{S}$ of size k and the respective committee $W = \{c_S : S \in \mathcal{S}'\}$. If $\mathcal{S}$ covers R, each voter in V approves a candidate in W, and v approves all candidates, so all voters are connected via v. Conversely, if all pairs of voters are connected, then each voter must approve some candidate in W and hence $\mathcal{S}'$ covers R. This completes the proof. $\square$

A.2 Efficient Maximization of the Cons Objective in the VCI Domain

In this section, we provide a full specification of the dynamic program for maximizing Cons in the VCI domain. We then prove Theorem 4.5, restated as follows.

Theorem 4.5. *In the VCI domain, a committee that maximizes Cons can be computed in polynomial time.*

Proof. Consider an election $\mathcal{E} = (V, A, k)$. By Proposition 4.3, we may assume without loss of generality that $\mathcal{E}$ belongs to the CI domain. Moreover, by polynomial-time preprocessing, we obtain a candidate order $c_1, \dots, c_m$ witnessing membership in the CI domain [15]. For each voter $v \in V$, let $\ell(v) := \min\{i : c_i \in A_v\}$ and $r(v) := \max\{i : c_i \in A_v\}$ be the leftmost and the rightmost approved candidates of voter v, respectively. Given candidate indices $1 \leq j < i \leq m$, let $V(\neg j, i)$ be the set of voters that approve c_i, but not c_j, and let $n(\neg j, i)$ be the size of this set. Formally, we define

$$V(\neg j, i) := \{v \in V : j < \ell(v) \leq i \leq r(v)\} \quad \text{and} \quad n(\neg j, i) := |V(\neg j, i)|.$$

Further, we introduce an indicator variable $\mathbb{1}(j \wedge i)$ defined as

$$\mathbb{1}(j \wedge i) := \begin{cases} \text{true} & \text{if } V_{c_j} \cap V_{c_i} \neq \varnothing \\ \text{false} & \text{otherwise.} \end{cases}$$

That is, $\mathbb{1}(j \wedge i)$ is true if and only if there is a voter that approves both c_j and c_i.

Calculating the number of connected pairs after adding a candidate. Consider adding c_i to a committee $W \subseteq \{c_1, \dots, c_{i-1}\}$, i.e., c_i is to the right of all candidates in W with respect to the candidate order. Let $j^* \in [i-1]$ be the index of the rightmost candidate in W. We will now discuss how to update $\textsc{Cons}(W \cup \{c_i\})$. The update procedure depends on the value of $\mathbb{1}(j^* \wedge i)$, i.e, on whether there is a voter that approves both c_{j^*} and c_i.

First, suppose that $W = \varnothing$ or $\mathbb{1}(j^* \wedge i)$ is false. Then we claim that no voter in V_{c_i} approves any candidate in W. This is obvious if $W = \varnothing$. On the other hand, if $\mathbb{1}(j^* \wedge i)$ is false, suppose for contradiction that some voter $v \in V_{c_i}$ approves some candidate $c_j \in W$. Then by the choice of j^* we have $j < j^*$, and in the CI domain $c_j, c_i \in A_v$ implies $c_{j^*} \in A_v$, a contradiction.

Thus, adding c_i interconnects the voters approving c_i, but does not connect any of them to any other voters. Hence, in both cases, the number of connected pairs after adding c_i to W is

$$\textsc{Cons}(W \cup \{c_i\}) = \textsc{Cons}(W) + \binom{|V_{c_i}|}{2}.$$

Now, suppose that $\mathbb{1}(j^* \wedge i)$ is true. By CI, if a voter approves both c_i and some $c_j \in W$, then they also approve c_{j^*}. The update now depends on the connected component containing c_{j^*} in the hypergraph induced by $W \cup \{c_i\}$. Given a set of candidates W and a candidate $c \in W$, we define $K_W(c) := \{u \in V : u \sim_W v \text{ for some } v \in V_c\}$. Note that $K_W(c)$ is well-defined, because $\sim_W$ is an equivalence relation and $v \sim_W v'$ for all $v, v' \in V_c$. In fact, $K_W(c)$ is exactly the set of voters in the connected component containing the hyperedge c.

Now, adding c_i creates two types of connections: those among the newly connected voters in $V(\neg j^*, i)$ and those between $V(\neg j^*, i)$ and the voters in the connected component of c_{j^*}, i.e., $K_W(c_{j^*})$. Therefore, we have

$$K_{W \cup \{c_i\}}(c_i) = V(\neg j^*, i) \uplus K_W(c_{j^*}),$$

where $\uplus$ denotes the disjoint union of two sets. Consequently,

$$\textsc{Cons}(W \cup \{c_i\}) = \textsc{Cons}(W) + \binom{n(\neg j^*, i)}{2} + n(\neg j^*, i) \cdot |K_W(c_{j^*})|.$$

Defining the dynamic program. For each $i \in [m]$, $b \in [k]$, $x \in \{0\} \cup [n]$, let $\mathbf{opt}[i, x, b]$ denote the maximum number of voter pairs that can be connected by a committee of size at most b that has c_i as its rightmost candidate, with c_i being in a connected component that contains x voters. We use the convention that $\mathbf{opt}[i, x, b] = -1$ if there is no such committee. We will define functions $\mathbf{dp}[i, x, b]$ and $W[i, x, b]$ and argue that for all i, x, b it holds that $\mathbf{dp}[i, x, b] = \mathbf{opt}[i, x, b]$ and, moreover, if this value is nonnegative, $W[i, x, b]$ is a committee of size at most b with rightmost candidate c_i being in a connected component that contains x voters, which satisfies $\textsc{Cons}(W[i, x, b]) = \mathbf{opt}[i, x, b]$.

We initialize the dynamic program by handling the case $b = 1$. For convenience, we also deal with the case $i = 1$ at this point.

- For each $i \in [m]$, the number of pairs connected by $\{c_i\}$ is $\binom{|V_{c_i}|}{2}$. Hence, we set $\mathbf{dp}[i, |V_{c_i}|, 1] = \binom{|V_{c_i}|}{2}$ and $W[i, |V_{c_i}|, 1] = \{c_i\}$. For $x \neq |V_{c_i}|$, we set $\mathbf{dp}[i, x, 1] = -1$.
- For each $b \in [k]$, we set $\mathbf{dp}[1, |V_{c_1}|, b] = \binom{|V_{c_1}|}{2}$: this is the number of pairs connected by $\{c_1\}$. Also, set $W[1, |V_{c_1}|, b] = \{c_1\}$, and $\mathbf{dp}[1, x, b] = -1$ for $x \neq |V_{c_1}|$.

Clearly, it holds that $\mathbf{dp}[i, x, b] = \mathbf{opt}[i, x, b]$ for $b = 1$ and all $i \in [m]$, $x \in \{0\} \cup [n]$ as well as for $i = 1$ and all $x \in \{0\} \cup [n]$, $b \in [k]$. Moreover, for triples (i, x, b) that satisfy $b = 1$ or $i = 1$ it holds that if $\mathbf{dp}[i, x, b] \geq 0$, then $W[i, x, b]$ is a committee that provides $\mathbf{opt}[i, x, b]$ connections.

Having computed $\mathbf{dp}[i, x, 1]$ and $W[i, x, 1]$ for all $i \in [m]$, $x \in \{0\} \cup [n]$, we proceed in $k - 1$ stages: in stage $t \in [k-1]$, we compute $\mathbf{dp}[i, x, t+1]$ and $W[i, x, t+1]$ for all $i \in [m]$, $x \in \{0\} \cup [n]$. Within each stage, we proceed in increasing order of i (note that the case $i = 1$ was handled during the initialization stage). Thus, it remains to explain how to fill out the cells $\mathbf{dp}[i, x, b]$ and $W[i, x, b]$ with $i, b \geq 2$, given that we have already filled out $\mathbf{dp}[j, y, b-1]$ and $W[j, y, b-1]$ for all $j < i$ and all $y \in \{0\} \cup [n]$.

For a given triple (i, x, b) and each $j \in [i-1]$, $y \in \{0\} \cup [n]$, we define $\mathbf{score}(i, x, b, j, y)$ as follows:

- If $\mathbb{1}(j \wedge i)$ is true, $y + n(\neg j, i) = x$, and $\mathbf{dp}[j, y, b-1] \geq 0$, set $\mathbf{score}(i, x, b, j, y) = \mathbf{dp}[j, y, b-1] + \binom{x-y}{2} + y(x-y)$.
- If $\mathbb{1}(j \wedge i)$ is false, $|V_{c_i}| = x$, and $\mathbf{dp}[j, y, b-1] \geq 0$, set $\mathbf{score}(i, x, b, j, y) = \mathbf{dp}[j, y, b-1] + \binom{x}{2}$.
- In all other cases, set $\mathbf{score}(i, x, b, j, y) = -1$.

Note that $\mathbf{score}(i, x, b, j, y)$ calculates the number of pairs that can be obtained by starting with the committee $W[j, y, b-1]$ and then adding c_i so that the connected component of c_i has x voters. We then define $\mathbf{dp}[i, x, b]$ and $W[i, x, b]$ so as to maximize this quantity.

- Pick (j^*, y^*) from $\arg\max_{(j,y): j \in [i-1], y \in \{0\} \cup [n]} \mathbf{score}(i, x, b, j, y)$.
- Set $\mathbf{dp}[i, x, b] = \mathbf{score}(i, x, b, j^*, y^*)$.
- Set $W[i, x, b] = W[j^*, y^*, b-1] \cup \{c_i\}$ if $\mathbf{dp}[i, x, b] \geq 0$.

Correctness of the dynamic program. By construction, it holds that if $\mathbf{dp}[i,x,b]$ is nonnegative, then we have $\textsc{Cons}(W[i,x,b]) = \mathbf{dp}[i,x,b]$. It remains to prove correctness of the update formulas for the dynamic program, i.e., to show that $\mathbf{dp}[i,x,b] = \mathbf{opt}[i,x,b]$ for all $i \in [m]$, all $x \in \{0\} \cup [n]$ and all $b \in [k]$. To this end, we proceed by induction on b, and for each fixed value of b, by induction on i (note that we have already argued that our dynamic program is correct for $b = 1$ and for $i = 1$). Thus, we fix a triple (i,x,b) and assume that $\mathbf{dp}[j,y,b-1] = \mathbf{opt}[j,y,b-1]$ for all $j < i$ and all $y \in [0] \cup \{n\}$; our goal is to show that $\mathbf{dp}[i,x,b] = \mathbf{opt}[i,x,b]$. We split the proof into two parts.

First, we will argue that $\mathbf{dp}[i,x,b] \le \mathbf{opt}[i,x,b]$. Clearly, this is true if $\mathbf{dp}[i,x,b] = -1$. Otherwise, $W[i,x,b] = W[j,y,b-1] \cup \{c_i\}$ for some $j < i$, $y \in \{0\} \cup [n]$ that maximize $\mathbf{score}(i,x,b,j,y)$. Since $\mathbf{dp}[i,x,b]$ is positive, the quantity $\mathbf{score}(i,x,b,j,y)$ is positive as well. Consequently, if $\mathbb{1}(j \wedge i)$ is false, then necessarily $x = |V_{c_i}|$, while if $\mathbb{1}(j \wedge i)$ is true, then necessarily $x = n(\neg j, i) + y$.

By our previous observations, if $\mathbb{1}(j \wedge i)$ is true, the number of pairs interconnected by $W[i,x,b]$ is equal to

$$\begin{aligned}\textsc{Cons}(W[i,x,b]) &= \textsc{Cons}(W[j,y,b-1]) + \binom{x-y}{2} + (x-y)y \\ &= \mathbf{dp}[j,y,b-1] + \binom{x-y}{2} + y(x-y) \\ &= \mathbf{score}(i,x,b,j,y) = \mathbf{dp}[i,x,b],\end{aligned}$$

and if $\mathbb{1}(j \wedge i)$ is false, the number of pairs interconnected by $W[i,x,b]$ is equal to

$$\begin{aligned}\textsc{Cons}(W[i,x,b]) &= \textsc{Cons}(W[j,y,b-1]) + \binom{x}{2} = \mathbf{dp}[j,y,b-1] + \binom{x}{2} \\ &= \mathbf{score}(i,x,b,j,y) = \mathbf{dp}[i,x,b];\end{aligned}$$

in both cases, the second transition uses the inductive hypothesis. As $W[i,x,b]$ is a committee of size at most b that has c_i as its rightmost candidate, with c_i being in a connected component that contains x voters, we conclude that $\mathbf{dp}[i,x,b] \le \mathbf{opt}[i,x,b]$.

Next, we will show that $\mathbf{dp}[i,x,b] \ge \mathbf{opt}[i,x,b]$. First, if there is no committee of size at most b that has c_i as its rightmost candidate, with c_i being in a connected component containing x voters, then $\mathbf{opt}[i,x,b] = -1$, and the inequality is true. Otherwise, consider some such committee W^* with $\textsc{Cons}(W^*) = \mathbf{opt}[i,x,b]$. Since $b \ge 2$ and $i \ge 2$, we may assume without loss of generality that $|W| \ge 2$. We set $W' = W^* \setminus \{c_i\}$, let j be the rightmost candidate in W', and let y be the size of j's connected component with respect to W'. If $\mathbb{1}(j \wedge i)$ is true, we have

$$\mathbf{dp}[j,y,b-1] = \mathbf{opt}[j,y,b-1] \ge \textsc{Cons}(W') = \textsc{Cons}(W^*) - \binom{x-y}{2} - (x-y)y = \mathbf{opt}[i,x,b] - \binom{x-y}{2} - (x-y)y,$$

and if $\mathbb{1}(j^* \wedge i)$ is false, we have

$$\mathbf{dp}[j,y,b-1] = \mathbf{opt}[j,y,b-1] \ge \textsc{Cons}(W') = \textsc{Cons}(W^*) - \binom{x}{2} = \mathbf{opt}[i,x,b] - \binom{x}{2}.$$

In the first case, we have $\mathbf{dp}[i,x,b] \ge \mathbf{score}(i,x,b,j,y) = \mathbf{dp}[j,y,b-1] + \binom{x-y}{2} + (x-y)y \ge \mathbf{opt}[i,x,b]$. Similarly, in the second case, we have $\mathbf{dp}[i,x,b] \ge \mathbf{score}(i,x,b,j,y) = \mathbf{dp}[j,y,b-1] + \binom{x}{2} \ge \mathbf{opt}[i,x,b]$.

Together, we obtain $\mathbf{dp}[i,x,b] = \mathbf{opt}[i,x,b]$. Finally, to compute a feasible committee that maximizes $\textsc{Cons}$, we output an arbitrary committee $W \in \arg\max_{i \in [m], x \in \{0\} \cup [n]} \textsc{Cons}(W[i,x,k])$.

Note that our dynamic program has $\mathcal{O}(mnk)$ cells, as $i \in [m]$, $x \in \{0\} \cup [n]$, and $b \in [k]$. Moreover, every cell can be filled in polynomial time given the values of previously computed cells. Hence, we have obtained a polynomial-time algorithm to compute a committee that maximizes $\textsc{Cons}$. $\square$

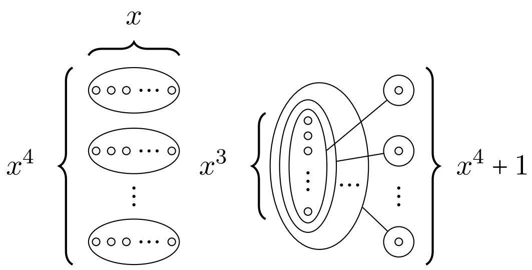

Figure 5: An illustration of the profile constructed in the proof of Proposition 5.3. Block voters are on the left, central voters in the middle, and arm voters on the right. Each block candidate is approved by x block voters, whereas each central candidate is approved by all central voters and one arm voter.

B Missing Proofs from Section 5

In this section, we provide the missing proofs for all results on tradeoffs between classic and interlacing committee selection objectives.

B.1 Basic Tradeoff for the Pairs Objective

Proposition 5.3. *For every $\alpha, \beta \in [0,1]$, if a voting rule satisfies α-Pairs and β-CC, then $\alpha + \beta \leq 1$.*

Proof. For a given constant $x \in \mathbb{N}$, consider an election instance $\mathcal{E} = (V, A, k)$ defined as follows (see Figure 5 for an illustration).

The candidate set consists of x^4 *block candidates* $b_1, \dots, b_{x^4}$ and $x^4 + 1$ *arm candidates* $a_1, \dots, a_{x^4+1}$, i.e., $2x^4 + 1$ candidates in total. The set V consists of x^5 *block voters*, split into x^4 blocks $B_1, \dots, B_{x^4}$ of size x each, $x^4 + 1$ *arm voters*, and x^3 *central voters*, i.e., $x^5 + x^4 + x^3 + 1$ voters in total.

For each $i \in [x^4]$ the voters in block B_i approve the block candidate b_i. For each $i \in [x^4 + 1]$, the ith arm voter approves the ith arm candidate a_i, and all central voters approve all arm candidates.

The target committee size is set to $k = x^4 + 1$. As there are fewer than k block candidates, every committee of size k contains at least one arm candidate. Thus, by symmetry, without loss of generality, we can assume that for some $\gamma \in \{0, 1/x^4, 2/x^4, \dots, 1\}$ we select a committee $W_\gamma \subseteq C$ that consists of arm candidates $a_1, a_2, \dots, a_{\gamma x^4+1}$ and block candidates $b_1, b_2, \dots, b_{(1-\gamma)x^4}$. Every selected block candidate covers x voters and $x(x-1)/2$ pairs of voters. Moreover, the first selected arm candidate covers $x^3 + 1$ voters and $x^3(x^3+1)/2$ pairs of voters, whereas every subsequent arm candidate covers 1 voter and x^3 pairs of voters. Thus, when we select $\gamma x^4 + 1$ arm candidates and $(1-\gamma)x^4$ block candidates, we obtain the following CC and Pairs scores:

$$\mathrm{CC}(W_\gamma, \mathcal{E}) = (1-\gamma)x^5 + \gamma x^4 + x^3 + 1, \text{ and}$$
$$\textsc{Pairs}(W_\gamma, \mathcal{E}) = \gamma x^7 + \frac{2-\gamma}{2}x^6 - \frac{1-\gamma}{2}x^5 + \frac{1}{2}x^3.$$

Observe that the maximum number of covered voters is obtained when we take $\gamma = 0$, whereas the maximum number of covered pairs of voters is obtained when $\gamma = 1$. Also, we have

$$\frac{\mathrm{CC}(W_\gamma, \mathcal{E})}{\mathrm{CC}(W_0, \mathcal{E})} + \frac{\textsc{Pairs}(W_\gamma, \mathcal{E})}{\textsc{Pairs}(W_1, \mathcal{E})} = 1 + \mathcal{O}(1/x).$$

Hence, when x tends to infinity, the sum of approximation ratios for CC and Pairs gets arbitrarily close to 1. $\square$

B.2 Approximation of EJR with the Method of Equal Shares

We now define the method of equal shares for a reduced budget governed by a parameter $\alpha \leq 1$. Recall that $n = |V|$. At the start, every voter $v \in V$ is assigned a *budget* $\text{bud}(v) = \alpha \cdot \frac{k}{n}$. The committee W is initialized as the empty set, and every candidate is assumed to have a cost of 1. Informally, at each step, we consider the candidates in $C \setminus W$ that can be afforded by the voters approving them while sharing the costs equally (with the caveat that a voter who cannot afford to pay an equal share can contribute her entire remaining budget instead). We define a candidate's *price* ρ as the maximum amount that a supporter of this candidate contributes to its cost. We then add a candidate with a minimum price to W, and proceed to the next step.

More formally, at the start of each step we let

$$C^* = \{c \in C \setminus W : \sum_{v : c \in A_v} \text{bud}(v) \geq 1\}.$$

If $C^* = \varnothing$, the algorithm terminates and returns W. Otherwise, for each $c \in C^*$ we set

$$\rho(c) = \min\{\rho \geq 0 : \sum_{v : c \in A_v} \min(\text{bud}(v), \rho) \geq 1\}.$$

We then pick c^* in $\arg\min_{c \in C^*} \rho(c)$ and add it to W. In addition, we update the budget of each voter v with $c^* \in A_v$ as $\text{bud}(v) - \min\{\text{bud}(v), \rho(c^*)\}$.

We are ready to prove the EJR guarantee achieved by this method.

Lemma 5.4. *Let $\alpha \leq 1$ be given. For every election $\mathcal{E} = (V, A, k)$, executing MES on $(V, A, \alpha k)$ returns a committee of size $\lfloor \alpha k \rfloor$ satisfying α-EJR in polynomial time.*

Proof. At the start of the procedure, the sum of voters' budgets is αk, and in each round the total budget is reduced by 1. Hence, MES terminates after at most $\lfloor \alpha k \rfloor$ rounds, returning a committee of size at most $\lfloor \alpha k \rfloor$. Assume without loss of generality that $W = \{c_1, \dots, c_r\}$, where $r \leq \lfloor \alpha k \rfloor$ and for each $t \in [r]$ it holds that candidate c_t is added to W at the tth round.

Assume for contradiction that W violates α-EJR. Hence, there is a subset of voters $S \subseteq V$ and a positive integer $\ell \leq k$ with $|S| \geq \frac{\ell n}{\alpha k}$ and $|\bigcap_{v \in S} A_v| \geq \ell$ such that $|A_v \cap W| < \ell$ for all $v \in S$. Observe that the set $\bigcap_{v \in S} A_v \setminus W$ is non-empty, and let c^* be some candidate in this set.

We claim that whenever a voter $v \in S$ makes a positive contribution towards the cost of some candidate during the execution of the algorithm, she pays at most $\frac{\alpha k}{\ell n}$. Indeed, suppose this is not the case, and consider the first round t in which a voter $v \in S$ spends strictly more than $\frac{\alpha k}{\ell n}$. By definition of MES, this means that in round t we have $\rho(c_t) > \frac{\alpha k}{\ell n}$. On the other hand, in each round $i < t$, each voter in S spends at most $\frac{\alpha k}{\ell n}$. Moreover, by our assumption, each voter in S approves at most $\ell - 1$ candidates from W. Thus, before round t, the remaining budget of each voter in S is at least $\frac{\alpha k}{n} - (\ell - 1) \cdot \frac{\alpha k}{\ell n} = \frac{\alpha k}{\ell n}$. Thus, together they have a budget of at least $|S| \cdot \frac{\alpha k}{n \ell} \geq \frac{\ell n}{\alpha k} \cdot \frac{\alpha k}{\ell n} = 1$. This means that the candidate $c^* \in \bigcap_{v \in S} A_v \setminus W$ belongs to the set C^* at the start of round t and, moreover, $\rho(c^*) \leq \frac{\alpha k}{\ell n} < \rho(c_t)$, a contradiction with the choice of c_t.

Now, since each voter in S contributes at most $\frac{\alpha k}{\ell n}$ towards the cost of each candidate in W, and she pays for at most $\ell - 1$ candidates in W, at the end of the rth round, the remaining budget of each voter in S is at least $\frac{\alpha k}{\ell n}$, so voters in S can still afford candidate c^*, a contradiction with the assumption that MES terminates after r rounds. □

Proposition 5.6. *For every $\alpha, \beta \in [0, 1]$, if a voting rule satisfies α-PAIRS and β-EJR, then $\alpha + \beta \leq 1$.*

Proof. Clearly, the statement is true for $\beta = 0$. Let f be a voting rule that satisfies β-EJR for some $\beta \in (0, 1]$. Fix an $\varepsilon > 0$ so that $\varepsilon < \beta$ and $\beta - \varepsilon \in \mathbb{Q}$. We modify the election $\mathcal{E}$ from the proof of

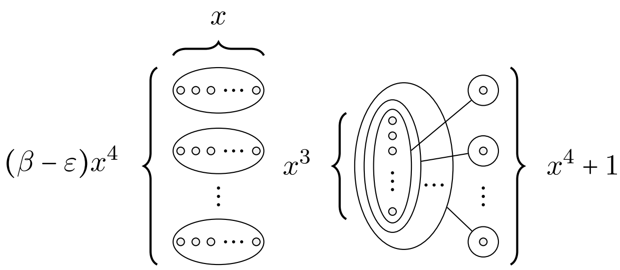

Figure 6: Illustration of the profile constructed in the proofs of Proposition 5.6. Block voters are on the left, central voters in the middle, and arm voters on the right. Block candidates are approved by x block voters each, whereas arm candidates are approved by all central voters and one arm voter each.

Proposition 5.3 by reducing the number of block candidates (resp., block voters) from x^4 to $(\beta - \varepsilon)x^4$ (resp., $(\beta - \varepsilon)x^5$) and only considering values of x for which $(\beta - \varepsilon)x$ is integer (as $\beta - \varepsilon$ is rational, there are infinitely many such values). As in the proof of Proposition 5.3, we want to select $k = x^4 + 1$ candidates. Denote the resulting election by $\mathcal{E}'$.

Fix a committee W that satisfies β-EJR. We claim that for large enough x, committee W contains all block candidates. To see this, note that there are $(\beta - \varepsilon)x^5$ block voters, x^3 central voters, and $x^4 + 1$ arm voters, so $n = (\beta - \varepsilon)x^5 + \mathcal{O}(x^4)$. Thus,

$$\frac{n}{\beta k} = \frac{(\beta - \varepsilon)x^5 + \mathcal{O}(x^4)}{\beta(x^4 + 1)} = \frac{\beta - \varepsilon}{\beta} \cdot x + \mathcal{O}(1).$$

Hence, we have $\frac{n}{\beta k} < x$ for large enough x. Consequently, β-EJR (with $\ell = 1$) demands that a group of x voters who all approve the same candidate has to be represented in W, i.e., W must contain a candidate approved by some of these voters. In particular, this applies to each block of block voters. Since the only way to represent these voters is to include their respective block candidate, W has to contain all block candidates.

It then follows that W contains $1 + (1 - \beta + \varepsilon)x^4$ arm candidates. Consequently, we have

$$\textsc{Pairs}(W, \mathcal{E}') = (\beta - \varepsilon)x^4\binom{x}{2} + \binom{x^3}{2} + x^3\big(1 + (1 - \beta + \varepsilon)x^4\big) = (1 - \beta + \varepsilon)x^7 + \mathcal{O}(x^6).$$

By contrast, when selecting all arm candidates, we obtain a Pairs score for $\mathcal{E}'$ of $\binom{x^3}{2} + x^3(1 + x^4) = x^7 + O(x^6)$. Considering the ratio of these two values and having x tend to infinity shows that f is at most a $(1 - \beta + \varepsilon)$-approximation of Pairs. Since $\varepsilon > 0$ with $\varepsilon < \beta$ was chosen arbitrarily, the assertion follows. □

B.3 Trade-offs for the Cons Objective

Proposition 5.7. *For every $\alpha, \beta \in [0, 1]$, if a voting rule satisfies α^2-Cons and β-AV, β-CC, or β-Pairs, then $\alpha + \beta \leq 1$. This already holds in the VI domain.*

Proof. For the proof of all three statements, we consider the identical instance (V, A, x^3) depicted in Figure 7. The candidate set consists of x^3 *block candidates* $b_1, \dots, b_{x^3}$ and x^3 *central candidates* $c_1, \dots, c_{x^3}$, i.e., $2x^3$ candidates in total. The voter set V consists of x^4 *block voters*, split into x^3 blocks $B_1, \dots, B_{x^3}$ of size x each, and $x^3 + 1$ *central voters*, i.e., $x^5 + x^3 + 1$ voters in total. For $i \in [x^3]$, each voter in block B_i approves candidate b_i only. Moreover, the first central voter approves candidate c_1, the last central voter approves candidate c_{x^3}, and for $i = 2, \dots, x^3$ the ith central voter approves candidates c_i and c_{i-1}. We set $k = x^3$.

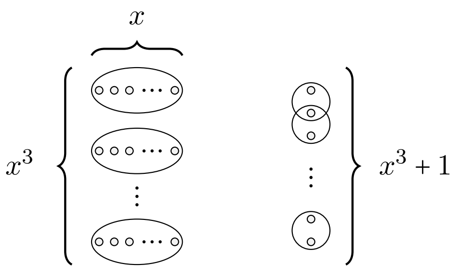

Figure 7: An illustration of the profile constructed in the proof of Proposition 5.7. Block voters are on the left, central voters on the right. Each central candidate covers one pair of voters, but choosing all central candidates yields a large connected component.

To show that this instance is in VI, we first enumerate the voters in each block, and then the central voters from first to last. By construction, each candidate is approved by an interval of voters.

The remainder of the proof consists of two parts. The first part shows that, to satisfy β-AV, β-CC, or β-Pairs, we require at least $\beta x^3 - \mathcal{O}(x^2)$ block candidates. The second part shows that, with the remaining candidates, we can obtain at most a $(1-\beta)^2$ approximation of Cons.

AV: Each block candidate contributes x to the AV-score, whereas each central candidate only contributes 2. Thus, the maximum AV-score is x^4, achieved by choosing all block candidates. If we choose $y \le x^3$ block candidates, then the AV-score is $yx + (x^3 - y)2$. Hence, for a committee of this form to guarantee β-AV, it has to be the case that $yx + (x^3 - y)2 \ge \beta x^4$, or, equivalently,

$$y \ge \frac{\beta x^4 - 2x^3}{x-2} \ge \frac{\beta x^4 - 2x^3}{x} = \beta x^3 - \mathcal{O}(x^2).$$

CC: Each block candidate contributes x to the CC-score, whereas each central candidate contributes at most 2. Thus, the maximum CC-score is x^4, achieved by choosing all block candidates. If we choose $y \le x^3$ block candidates, then the CC-score is at most $yx + (x^3 - y)2$. Hence, for a committee of this form to guarantee β-CC, it has to be the case that $yx + (x^3 - y)2 \ge \beta x^4$. As argued in our analysis for AV, this implies $y \ge \beta x^3 - \mathcal{O}(x^2)$.

Pairs: Each block candidate contributes $\frac{x(x-1)}{2}$ to the Pairs-score, while each central candidate contributes 1. Thus, the maximum Pairs-score is $x^3(\frac{x^2}{2} - \frac{x}{2})$, achieved by choosing all block candidates. If we choose $y \le x^3$ block candidates, the Pairs score is at most $x^3 + y(\frac{x^2}{2} - \frac{x}{2})$. Hence, for a committee of this form to guarantee β-Pairs, it has to be the case that $x^3 + y(\frac{x^2}{2} - \frac{x}{2}) \ge \beta x^3(\frac{x^2}{2} - \frac{x}{2})$ and thus

$$y \ge \beta x^3 - \frac{x^3}{\frac{x^2}{2} - \frac{x}{2}} = \beta x^3 - \mathcal{O}(x).$$

Cons: We now show that with the remaining $(1-\beta)x^3 + \mathcal{O}(x^2)$ candidates, we obtain at best a $(1-\beta)^2$-approximation of Cons. Note that each block candidate contributes $\binom{x}{2}$ to Cons, hence there are a total of at most $x^3\binom{x}{2}$ connections through block candidates. Moreover, y central candidates can connect at most $y+1$ voters. Hence, a set of $(1-\beta)x^3 + \mathcal{O}(x^2)$ central candidates can cause at most $\binom{(1-\beta)x^3 + \mathcal{O}(x^2)}{2}$ connections. Thus, a committee containing $\beta x^3 - \mathcal{O}(x^2)$ block candidates can achieve a Cons score of at most

$$x^3\binom{x}{2} + \binom{(1-\beta)x^3 + \mathcal{O}(x^2)}{2} = \frac{(1-\beta)^2}{2} \cdot x^6 + \mathcal{O}(x^5).$$

By contrast, consider the committee selecting all x^3 central candidates. This connects all pairs of central voters, and, therefore, achieves a Cons score of $\binom{x^3+1}{2} = \frac{x^6}{2} + +\mathcal{O}(x^5)$. Hence, for x tending to infinity, a committee containing $\beta x^3 - \mathcal{O}(x^2)$ block candidates achieves at most $(1-\beta)^2$-Cons. □

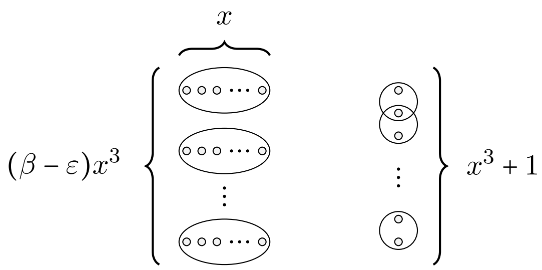

Figure 8: An illustration of the profile constructed in the proof of Proposition 5.8. By reducing the number of blocks in comparison to Figure 7, each block enforces via β-EJR that their block candidate is elected.

Proposition 5.8. *For every $\alpha, \beta \in [0,1]$, if a voting rule satisfies α^2-*Cons *and β-EJR, then $\alpha + \beta \leq 1$. This already holds in the VI domain.*

Proof. Clearly, the statement is true for $\beta = 0$. Let f be a voting rule that satisfies β-EJR for some $\beta \in (0,1]$. Fix an $\varepsilon > 0$ so that $\varepsilon < \beta$ and $\beta - \varepsilon \in \mathbb{Q}$. We modify the election $\mathcal{E}$ from the proof of Proposition 5.7 by reducing the number of block candidates (resp., block voters) from x^3 to $(\beta-\varepsilon)x^3$ (resp., $(\beta-\varepsilon)x^4$) and only considering values of x for which $(\beta-\varepsilon)x$ is integer (as $\beta-\varepsilon$ is rational, there are infinitely many such values). The instance is illustrated in Figure 8. As in the proof of Proposition 5.3, we want to select $k = x^3$ candidates. Denote the resulting election by $\mathcal{E}'$. Clearly, $\mathcal{E}'$ remains in VI.

Fix a committee W that satisfies β-EJR. We claim that for large enough x, committee W contains all block candidates. Each block candidate has a support of x voters and the total number of voters is $n = (\beta-\varepsilon)x^4 + \mathcal{O}(x^3)$. Hence, for large enough x, we have

$$\frac{n}{\beta k} = \frac{(\beta-\varepsilon)x^4 + \mathcal{O}(x^3)}{\beta x^3} = \frac{\beta-\varepsilon}{\beta}\cdot x + \mathcal{O}(1) < x.$$

Consequentially, β-EJR demands that in each block at least one voter approves a candidate of the winning committee. Since the voters in each block exclusively approve of the corresponding block candidate, all $(\beta-\varepsilon)x^3$ block candidates have to be contained in W.

It follows that W contains $(1-\beta+\varepsilon)x^3$ central candidates. The highest number of connections caused by central candidates is achieved when they yield one large connected component in the hypergraph induced by the central candidates, i.e., connecting $(1-\beta+\varepsilon)x^3+1$ voters. Adding this to the connections by block candidates results in

$$\textsc{Cons}(W, \mathcal{E}') = (\beta-\varepsilon)x^3\binom{x}{2} + \binom{(1-\beta+\varepsilon)x^3+1}{2} = \frac{(1-\beta+\varepsilon)^2}{2}\cdot x^6 + \mathcal{O}(x^5).$$

By contrast, when selecting all central candidates, we obtain a Cons score of $\frac{x^6}{2} + \mathcal{O}(x^5)$. Considering the ratio of these two values and having x tend to infinity shows that f is at most a $(1-\beta+\varepsilon)^2$-approximation Cons. Since $\varepsilon > 0$ with $\varepsilon < \beta$ was chosen arbitrarily, the assertion follows. ☐

Proposition 5.9. *For every $\beta \in [0,1]$, if a voting rule satisfies β-AV, then it satisfies at most $s(1-\beta)$-*Cons*.*

Proof. Let $y = \frac{1}{s(1-\beta)} = \lceil \frac{2}{1-\beta} \rceil - 1$. Note that y is an integer. For an arbitrary constant $x \in \mathbb{N}$, consider the election $\mathcal{E} = (V, A, k)$ defined as follows (see Figure 9 for an illustration). The candidate set contains yx^2+1 *block* candidates $b_1, \dots, b_{yx^2+1}$, y *arm* candidates $a_1, \dots, a_y$, and yx^2 *chain* candidates $(c_{i,j})_{i\in[x^2], j\in[y]}$. The voter set V consists of x^2 *block* voters, yx^3 *arm* voters split into y arms $A_1, \dots, A_y$ of size x^3 each, $y(x^2-1)$ *chain* voters $(h_{i,j})_{i\in[x^2-1], j\in[y]}$ split into y arms $H_1, \dots, H_y$ of size x^2-1 each, and one *central* voter v.

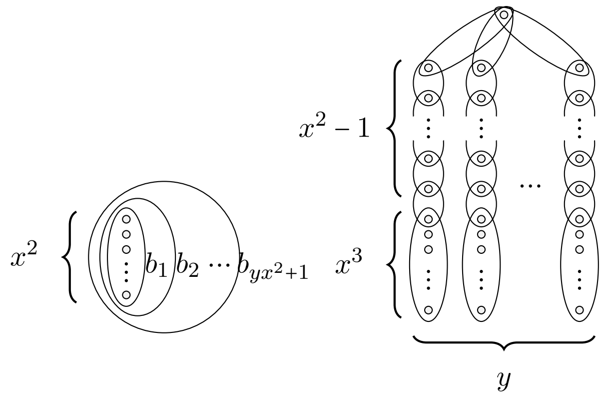

Figure 9: An illustration of the profile constructed in the proof of Proposition 5.9.

The voters have the following preferences. All block voters approve all block candidates. For each $j \in [y]$, each voter in arm A_j approves the arm candidate a_j, and additionally, exactly one voter in A_j approves the chain candidate $c_{x^2,j}$. For each $i \in [x^2 - 1]$ and $j \in [y]$, the chain voter $h_{i,j}$ approves the chain candidates $c_{i,j}$ and $c_{i+1,j}$. Finally, the central voter approves the chain candidates $c_{1,j}$ for each $j \in [y]$.

We set the target committee size to $k = y(x^2 + 1) + 1$. The high-level idea of the proof is that for Cons it is important to connect the arm voters through the selection of chain candidates. However, if we select a β-fraction of block candidates in order to guarantee β-AV, then we cannot connect arm voters from different arms.

Consider a committee W of size k. Note that W contains at least one block candidate, as there are only $y + yx^2 = k - 1$ arm and chain candidates. Moreover, suppose there is an arm candidate a not included in W. Then removing a block candidate from W and adding a instead increases both AV and Cons. Thus, we can assume that W contains all y arm candidates. For each $z \in \{0, 1, \ldots, yx^2\}$, let W_z be a committee that selects y arm candidates, $z + 1$ block candidates, and $yx^2 - z$ chain candidates.

Each arm candidate in W_z contributes x^3 to the AV score, each block candidate contributes x^2, and each chain candidate contributes 2. Thus, we obtain

$$\text{AV}(W_z, \mathcal{E}) = yx^3 + zx^2 + x^2 + 2(yx^2 - z).$$

Observe that AV is maximized when $z = yx^2$; thus, for W_z to provide β-AV, it has to be the case that $z \geq \beta yx^2 - \mathcal{O}(x)$.

Now, for Cons, we claim that for large enough x, with the remaining $yx^2 - z$ chain candidates, we cannot connect arm voters from two different arms. Indeed, recall that $y = \frac{1}{s(1-\beta)} = \lceil \frac{2}{1-\beta} \rceil - 1 < \frac{2}{1-\beta}$. Hence, $(1-\beta)y < 2$. Consequently, there is an $\varepsilon > 0$ such that $(1-\beta)y = 2 - \varepsilon$. Since $z \geq \beta yx^2 - \mathcal{O}(x)$, we choose at most $yx^2 - z \leq (1-\beta)yx^2 + \mathcal{O}(x) < 2x^2 - \varepsilon x^2 + \mathcal{O}(x)$ chain candidates. Thus, for large enough x we have strictly fewer than $2x^2 - 2$ chain candidates. This proves the claim, as it takes $2(x^2 - 1)$ chain candidates to connect two arm voters from different arms.

Let us now calculate the value of the Cons objective. The connections among block voters contribute at most $\binom{x^2}{2}$, and the connections among chain voters contribute at most $\binom{yx^2}{2}$. Connections between chain voters and arm voters contribute at most yx^5, as each chain voter can only be connected to arm voters in a single arm. We connect all arm voters within the same arm, but, as argued above, we do not connect arm voters from different arms. Hence, connections among arm voters contribute $y\binom{x^3}{2}$. The central voter can contribute $\mathcal{O}(yx^3)$ connections. In total,

$$\text{Cons}(W_z, \mathcal{E}) = \frac{yx^6}{2} + \mathcal{O}(x^5).$$

On the other hand, the maximum value of Cons is obtained when we select all $yx^2 - y$ chain voters: in this case, all yx^3 arm voters are connected, and Cons is at least

$$\frac{y^2x^6}{2} + \mathcal{O}(x^5).$$

Thus, the fraction of Cons we can obtain while satisfying β-AV converges to $\frac{1}{y} = s(1-\beta)$ for $x \to \infty$. □

B.4 Combination of Cons Objective with other Objectives

Proposition 5.10. *For every instance (V, A, k) in the VI domain with even k, there exists a committee that satisfies $\frac{1}{4}$-Cons and any one of the criteria $\frac{1}{2}$-AV, $\frac{1}{2}$-CC, $\frac{1}{2}$-EJR, or $\frac{1}{2}$-Pairs.*

Proof. Let (V, A, k) be an election instance in the VI domain as certified by the voter ordering $v_1, \dots, v_n$ and let k be even. It suffices to show that with $\frac{1}{2}k$ candidates, we can guarantee $\frac{1}{4}$-Cons. With the other $\frac{1}{2}k$ candidates, we can use the methods in Proposition 5.1 and Lemma 5.4 to obtain $\frac{1}{2}$-AV, $\frac{1}{2}$-CC, $\frac{1}{2}$-EJR, or $\frac{1}{2}$-Pairs.

The proof idea is to partition the candidates of the optimal solution for Cons into subsets that form intervals. Each interval formed by an even number of candidates is split into two halves and we choose the larger one. If some intervals are formed using an odd number of candidates, there must be an even number of such intervals. We show that for each pair of such intervals, we can always assign one candidate more than half to one interval and one candidate less than half to the other in a way that the $\frac{1}{4}$-approximation remains intact.

First, let $d_1, \dots, d_t \in C$ be candidates such that the voters approving at least one of them form an interval. Since for Cons it is never desirable to select candidates approved by the identical set of voters or dominated candidates, we can assume that in W there are no such candidates. Let $\ell(i)$ and $r(i)$ denote the index of the leftmost and rightmost voter approving d_i, respectively. By renaming candidates, we may assume without loss of generality that $\ell(1) \le \ell(2) \le \cdots \le \ell(t)$. By the choice of our candidates, it follows that in fact $\ell(1) < \ell(2) < \cdots < \ell(t)$, $r(1) < r(2) < \dots r(t)$ (no candidates approved by identical set of voters, no dominated candidates), and $r(i) \ge \ell(i+1)$ (the candidates form a connected subinterval). Without loss of generality, we assume that $\ell(1) = 1$ and $r(t) = x$ for some $x \in \mathbb{N}$. Hence, the connected subinterval connects x voters.

There are two cases. First, if $t = 2s$ is even, then we simply consider $r(s)$. If $r(s) > \frac{x}{2}$, then clearly we can choose $\{d_1, \dots, d_s\}$ which create an interval containing more than $\frac{x}{2}$ voters. Else, $\ell(s+1) \le r(s) \le \frac{x}{2}$. Hence, the candidates $\{d_{s+1}, \dots, d_t\}$ create an interval containing more than $\frac{x}{2}$ voters. Note that $\binom{y}{2} \ge \frac{1}{4}\binom{x}{2}$ for all $y \ge \frac{x}{2} + 1$, concluding the first part of the proof.

The remaining case is that $t = 2s + 1$ is odd, where we say that candidate d_{s+1} is the *central candidate*. Since k is even, there must be a second interval consisting of y voters $\{w_1, \dots, w_y\}$, also formed by an odd number of candidates $e_1, \dots, e_{2s'+1}$. In both intervals we can only choose strictly less or strictly more than half of the candidates. E.g., in the first interval, we can choose s or $s + 1$ candidates. Without loss of generality, we may assume that the left half of the interval including the central candidate contains strictly more than half of the voters. Hence, we can consider $h_x \in 0.5\mathbb{N}$ such that $\frac{x}{2} + h_x = r(s+1)$. Then, choosing $\{d_1, \dots, d_{s+1}\}$ yields an interval containing $\frac{x}{2} + h_x$ voters and consequently choosing $d_{s+2}, \dots, d_t$ yields an interval containing at least $x - \frac{x}{2} - h_x + 1 = \frac{x}{2} - h_x + 1$ voters, where the $+1$ comes from the fact that $\ell(s+2) \le r(s+1)$. Analogously, we define h_y such that half of the candidates e_i rounded up we connect a subinterval containing $\frac{y}{2} + h_y$ voters, and with half of the candidates rounded down we can still include $\frac{y}{2} - h_y + 1$ voters in our subinterval.

We now claim that if choosing $d_1, \dots d_s, d_{s+1}$ and $e_1, \dots e_{s'}$ does not yield a $\frac{1}{4}$-approximation of Cons, then we can choose the other half of candidates to achieve this. Clearly, the number of pairs that needs to be connected is

$$\frac{x^2}{8} - \frac{x}{8} + \frac{y^2}{8} - \frac{y}{8}.$$

First, the number of pairs induced by $d_1, \dots d_s, d_{s+1}$ is $\binom{\frac{x}{2}+h_x}{2}$, and the number of pairs induced by $e_1, \dots e_{s'}$, is $\binom{\frac{y}{2}-h_y+1}{2}$. Note that

$$\binom{\frac{x}{2}+h_x}{2} = \frac{(\frac{x}{2}+h_x)^2 - (\frac{x}{2}+h_x)}{2} = \frac{\frac{x^2}{4} + xh_x + h_x^2 - \frac{x}{2} - h_x}{2} = \frac{x^2}{8} + \frac{(x+h_x-1)h_x}{2} - \frac{x}{4}.$$

We want the sum of these to be at least as large as the number of pairs that needs to be connected, so we subtract the former from the latter obtaining

$$\binom{\frac{x}{2}+h_x}{2} - \left(\frac{x^2}{8} - \frac{x}{8}\right) = \frac{(x+h_x-1)h_x}{2} - \frac{x}{8}.$$

Doing the same for terms involving y, we obtain

$$\begin{aligned}\binom{\frac{y}{2}-h_y+1}{2} &= \frac{(\frac{y}{2}-h_y+1)^2 - (\frac{y}{2}-h_y+1)}{2} = \frac{\frac{y^2}{4} + h_y^2 + 1 - yh_y - 2h_y + y - \frac{y}{2} + h_y - 1}{2} \\ &= \frac{y^2}{8} - \frac{yh_y}{2} + \frac{y}{4} + \frac{h_y^2}{2} - \frac{h_y}{2}\end{aligned}$$

and hence

$$\binom{\frac{y}{2}-h_y+1}{2} - \left(\frac{y^2}{8} - \frac{y}{8}\right) = -\frac{yh_y}{2} + \frac{3}{8}y + \frac{h_y^2}{2} - \frac{h_y}{2}.$$

In total, by summing the two differences together we obtain

$$\binom{\frac{x}{2}+h_x}{2} + \binom{\frac{y}{2}-h_y+1}{2} - \left(\frac{x^2}{8} - \frac{x}{8} + \frac{y^2}{8} - \frac{y}{8}\right) = \frac{(x+h_x-1)h_x}{2} - \frac{x}{8} - \frac{yh_y}{2} + \frac{3}{8}y + \frac{h_y^2}{2} - \frac{h_y}{2}.$$

If this term is at least zero, then we can guarantee $\frac{1}{4}$ of the pairs that the two intervals connect by electing the subcommittee $\{d_1, \dots, d_s, d_{s+1}, e_1, \dots e_{s'}\}$, where $s+1+s'$ is precisely half of the $2s+2s'+2$ candidates required to connect the two intervals.

Else, the difference is strictly less than zero. By isolating yh_y in the inequality, we obtain

$$yh_y > (x+h_x-1)h_x - \frac{x}{4} + \frac{3}{4}y + h_y^2 - h_y. \tag{$*$}$$

We use this to claim that with the other half of the candidates, we can obtain the desired guarantee. Formally, consider the subcommittee $\{d_{s+2}, \dots, d_{2s+1}, e_{s'+1}, \dots, e_{2s'+1}\}$. Now, we only obtain $\frac{x}{2} - h_x + 1$ voters from the chosen candidates d_i, but in return obtain $\frac{y}{2} + h_y$ voters from the chosen e_i. The calculations are precisely symmetric to the ones we did before, just with x and y replaced by each other.

Thus, if we denote the difference between actually connected pairs and the desired approximation by D, we obtain that

$$\begin{aligned} D &= \binom{\frac{x}{2} - h_x + 1}{2} + \binom{\frac{y}{2} + h_y}{2} - \left(\frac{x^2}{8} - \frac{x}{8} + \frac{y^2}{8} - \frac{y}{8}\right) \\ &= \frac{(y + h_y - 1)h_y}{2} - \frac{y}{8} - \frac{xh_x}{2} + \frac{3}{8}x + \frac{h_x^2}{2} - \frac{h_x}{2}. \end{aligned}$$

Then, we obtain

$$\begin{aligned} 2D &= (y + h_y - 1)h_y - \frac{y}{4} - xh_x + \frac{3}{4}x + h_x^2 - h_x = yh_y + h_y^2 - h_y - \frac{y}{4} - xh_x + \frac{3}{4}x + h_x^2 - h_x \\ &> (x + h_x - 1)h_x - \frac{x}{4} + \frac{3}{4}y + h_y^2 - h_y + h_y^2 - h_y - \frac{y}{4} - xh_x + \frac{3}{4}x + h_x^2 - h_x && \text{(by } (*)\text{)} \\ &= 2h_x^2 - 2h_x + \frac{1}{2}x + \frac{1}{2}y + 2h_y^2 - 2h_y \\ &\geq 1 + 2h_x^2 - 2h_x + 2h_y^2 - 2h_y && \text{(since } x, y \geq 1\text{)} \\ &\geq 0. && \text{(as } z^2 - z \geq -\tfrac{1}{4} \text{ for all } z \in \mathbb{R}\text{)} \end{aligned}$$

Thus, we get $D > 0$, which concludes the proof. □

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