Method of Equal Shares with Bounded Overspending

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Abstract

Pure proportional voting rules can sometimes lead to highly suboptimal outcomes. We introduce the Method of Equal Shares with Bounded Overspending (BOS Equal Shares), a robust variant of the Method of Equal Shares that balances proportionality and efficiency. BOS Equal Shares addresses inefficiencies implied by strict proportionality, yet still provides fairness guarantees, similar to the original Equal Shares. Our extensive empirical analysis shows excellent performance of BOS Equal Shares across several metrics. In the course of the analysis, we also study a fractional variant of the Method of Equal Shares.

1 Introduction

We consider the participatory budgeting (PB) scenario, where a group of voters decides, through voting, which subset of projects to fund. The projects have varying prices, and the total cost of the selected projects cannot exceed a given budget. Voters express their preferences by casting ballots—typically either by indicating sets of approved projects (i.e., they cast approval ballots) or by assigning numerical scores to projects (so-called range voting). PB has been recently adopted by many municipalities worldwide [3] but the model applies more broadly. It extends the framework of committee elections [17, 14] and can be used even if the voters and candidates are not humans but represent abstract objects, e.g., validators in proof-of-stake blockchains [12].

Proportionality is a critical requirement in the context of PB elections. Intuitively, it says that each group of similar-minded voters should be entitled to decide about a proportional fraction of the available funds (e.g., if 30% of voters like similar projects, then roughly 30% of funds should be designated to the projects these voters support). Proportionality, among others, ensures equal treatment of minorities, geographical regions, and various project categories [15, 24]. It also ensures that groups of voters forming pluralities are not overrepresented, thus protecting elections against certain strategies employed by coordinated voters or project owners. As a result, several proportionality criteria and new voting rules have been proposed in the literature (cf. the overview of Rey and Maly [24]). One voting method, the Method of Equal Shares [22, 23], stands out by exhibiting particularly strong proportionality properties [18, 11] as well as robustness to changes in the voter participation [7]. It has also been successfully used in real-life PB [2].

Informally speaking, the Method of Equal Shares first virtually distributes the budget equally among the voters. Projects are then selected based on the vote count, but each time a project is selected, its cost is split among its supporting voters. Thus, only the projects whose supporters still have enough funds to cover their costs can be selected. Furthermore, the votes of those who have run out of money are no longer counted. This way, in subsequent rounds, the votes of minorities who have not yet influenced the decision are taken into account.

While this method offers strong proportionality guarantees, and largely exhibits desired behavior in practice, it is not without its flaws. In this work, we begin by uncovering the following drawbacks of Equal Shares (discussed in detail in Sections 3 and 6).

Underspending. As it has been already observed in the literature, the Method of Equal Shares can significantly underspend the available funds [15]. Indicatively, on real instances of PB elections it uses on average only 45% of the budget (see Section 6). Thus, in practice it heavily depends on the completion method with which it is combined (cf. Section 2).

Helenka Paradox. In Equal Shares, even a small group of voters can propose a modest project which, if unanimously supported by that group, would likely be selected. However, due to strict budget constraints, this could prevent a larger project—potentially benefiting the vast majority of voters—from being funded. We observed this issue in the Helenka district of Zabrze, Poland, where the Method of Equal Shares would have resulted in 97% of voters left with no project.

Tail Utilities. When there are large discrepancies in voters' scores assigned to projects, Equal Shares selects a project based on the score of the least satisfied voter who must cover its cost. This mechanism is egalitarian in principle, and so, the method may not properly take into account significant utility values in the decision process.

Based on these observations, we introduce and analyze a new voting rule, which we call the *Method of Equal Shares with Bounded Overspending* (in short, BOS Equal Shares, or simply BOS). The new method is a more robust variant of Equal Shares, remarkably effective in handling scenarios akin to particularly challenging instances of PB elections.

Interestingly, we argue that the first two of the problems we identify are not solely due to the Method of Equal Shares itself, but rather to the strict proportionality guarantees the method aims to provide. Specifically, it appears that the appealing axiom of Extended Justified Representation (EJR) [4, 23, 10] might enforce inefficiencies in certain scenarios. The same holds for weaker notions as well, like JR [4]. Consequently, BOS Equal Shares is not meant to generally satisfy EJR. Nevertheless, in most cases, our new method provides strong proportionality guarantees, mirroring those of the original Method of Equal Shares. To confirm this we first prove that BOS satisfies an approximate variant of EJR. This implies that even when its outcomes violate EJR—something necessary to avoid the identified issues of Equal Shares—cannot diverge significantly from it. This is an additional indicator that BOS tends to be proportional in practice. The near-tight worst-case guarantees we establish depend on the cost of the most expensive project in the instance, a dependency that is intuitive (as evident from by the Helenka case) but otherwise difficult to formalize. Consequently, our result offers a theoretical justification for placing upper bounds on project costs—a common practice in real-world PB implementations.

Our main argument for the advantages of BOS comes from our extensive empirical analysis on real-world and synthetic instances, in which it shows very good and robust performance in a number of metrics. In particular, it provides EJR+ up to one (a strong EJR-style axiom Brill and Peters [10]) in more than 95% of cases, and, in comparison to the original Method of Equal Shares, leaves less voters empty-handed. Furthermore, our rule has been recently proved to be superior in the context of selecting a representative set of influential nodes in networks [21].

Noteworthy, in the course of designing BOS, we propose and analyze a fractional variant of the Method of Equal Shares, the *Fractional Equal Shares* (FrES), which works in a model where projects are allowed to be funded partially. It extends the Generalized Method of Equal Shares [19], another rule recently proposed for the fractional model, but only for approval ballots.

2 Preliminaries

A PB election (in short, an election), is a tuple $E = (C, V, b)$, where $C = \{c_1, \dots, c_m\}$ is a set of available *candidates* (also referred to as *projects*), $V = \{v_1, \dots, v_n\}$ is a set of voters, and $b \in \mathbb{R}$ is the budget value. Each candidate $c \in C$ is associated with a cost, denoted as $\mathrm{cost}(c)$, assumed to be upper bounded by b. We extend this notation to sets of candidates, setting $\mathrm{cost}(W) = \sum_{c \in W} \mathrm{cost}(c)$ for all $W \subseteq C$. An *outcome* of an election is a subset of candidates; an outcome W is *feasible* if $\mathrm{cost}(W) \leqslant b$. An *election rule* is a function that for each election returns a nonempty collection of feasible outcomes. Typically, we are interested in a single outcome, yet we allow for ties. Each voter $v_i \in V$ has a utility function $u_i \colon C \to \mathbb{R}_{\geqslant 0}$ that assigns values to the candidates. We assume the utilities are additive, and write $u_i(W) = \sum_{c \in W} u_i(c)$ for each $W \subseteq C$. The voters' utility functions and the candidates' costs are integral parts of the election.

	cost	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
Project A	\$300k	✓	✓	✓	✓	✓	✓				
Project B	\$400k		✓	✓	✓	✓	✓				
Project C	\$300k		✓	✓	✓	✓					✓
Project D	\$240k							✓	✓	✓	✓
Project E	\$170k		✓					✓	✓	✓	
Project F	\$100k						✓			✓	✓

Table 1: An example of a PB election with 10 voters and 6 projects of varying costs. Approvals of voters towards projects are indicated by the ✓ symbol.

There are two special types of utility functions that are particularly interesting, and pivotal for certain parts of our work. In some cases it is natural to assume that the utilities directly correspond to the scores extracted from the voters' ballots; then we speak about *score utilities*. In case of approval ballots, where the voters only indicate subsets of supported candidates, we simply assume that the voter assigns scores of one to the approved candidates, and scores of zero to those she does not approve. On the other hand, the *cost utility* of a voter from a project is its score utility multiplied by the projects' cost. For example, if $W \subseteq C$ and A_i is the set of candidates that voter v_i approves, then we have $u_i(W) = |W \cap A_i|$ for score utilities (the utility is the number of selected projects the voter approves), and $u_i(W) = \mathrm{cost}(W \cap A_i)$ for cost utilities (thus, the utility is the amount of public funds allocated to projects she supports).

2.1 Method of Equal Shares

Arguably the simplest, and most commonly used voting rule in PB elections is the Utilitarian Method. This method selects the candidates by their vote count, omitting those whose selection would exceed the budget; it stops when no further candidate can be added. Since this approach is highly suboptimal from the perspective of proportionality (see for instance Example 1 that follows), we will be particularly interested in the Method of Equal Shares, a proportional election rule recently introduced in the literature [23, 22]. The rule works as follows: Let b_i be the virtual budget of voter v_i; initially $b_i := b_{\mathrm{ini}} = {}^{b}\!/_{n}$. In each round, we say that a not yet elected project c is *ρ-affordable* for $\rho \in \mathbb{R}_+$, if $\mathrm{cost}(c) = \sum_{v_i \in V} \min\left(b_i, u_i(c) \cdot \rho\right)$. In a given round the method selects the ρ-affordable candidate for the lowest possible value of ρ and updates the voters' accounts accordingly: $b_i := b_i - \min\left(b_i, u_i(c) \cdot \rho\right)$; then it moves to the next round. The rule stops if there is no ρ-affordable candidate for any value of ρ.

The concept of ρ-affordability is crucial to our work. Intuitively, voters supporting a ρ-affordable candidate c can cover its cost in such a way that each of them pays ρ per unit of utility or all of their remaining funds. In simpler terms, ρ represents the rate (price per unit of utility) at which the least advantaged supporter of the project would "purchase" their satisfaction, if the project is selected. Also, note that a situation in which a candidate is not ρ-affordable for all $\rho \in \mathbb{R}_+$ happens only if its supporters do not have enough money to cover its cost.

Example 1. Consider the PB election depicted in Table 1, and assume cost-utilities. The Utilitarian Method would select projects solely based on their vote count, thus choosing Projects A, B and C. This seems unfair since a large fraction of the voters (namely voters v_7 to v_9 making up 30% of the electorate) would not approve any of the selected projects.

The Method of Equal Shares, on the other hand, would first assign \$100k to each voter. Initially, the method would select Project A, splitting its cost equally among voters v_1 to v_6 ($\rho = {}^{1}\!/_{6}$). As a result, these voters would spend half the money they are entitled to towards purchasing Project A, so in the subsequent rounds, they would not be able to afford Project B. Furthermore, funding Project C would only be affordable if voter v_{10} spent all their money on it, thus paying

$\rho = 1/3$ of its price. The Method of Equal Shares would not select it, preferring Projects D and E for which the whole cost can be equally spread among four supporters ($\rho = 1/4$). Regardless of the choice, both Projects D and E would be bought in the last two rounds and the method would stop. See Appendix B.1 for a step-by-step illustration of its execution. Clearly, the selection made by the Method of Equal Shares is less discriminatory than the one by Utilitarian, as each voter approves at least one of the selected projects. Note that the bundle purchased by the Method of Equal Shares comes at a total cost of \$710k, leaving as much as \$250k in virtual accounts of voters v_1–v_6. While it is possible to additionally fund Project F, its supporters do not have enough money to fund it, so the project is not selected. ⌟

Example 1 shows that Equal Shares is non-exhaustive: an outcome $W \subseteq C$ is *exhaustive* if it utilizes the available funds in a way that no further project can be funded, in other words if for each unelected candidate $c \notin W$ it holds that $\text{cost}(W \cup \{c\}) > b$. While non-exhaustiveness itself may not be a critical flaw, a more concerning issue is that the method tends to significantly underspend the available funds. In real instances of participatory budgeting elections, Equal Shares allocates, on average, only 45% of the available budget (see Section 6).

To deal with this issue, Equal Shares is typically used together with a completion strategy. An example of a well-performing strategy suggested in the literature is Add1U. According to it, we gradually increment the initial endowment b_{ini} by one unit and rerun the Method of Equal Shares from scratch, until it produces an outcome that exceeds the budget. Then we return the outcome computed for the previous value of b_{ini}, hence the feasible outcome produced for the highest tested value of b_{ini}. Since the result may still be non-exhaustive (though typically at this point most of the funds are already spent), as the final step, we select affordable projects with the highest vote count until no further candidate can be added [15].

3 Limitations of the Method of Equal Shares

In this section, we present concrete case studies which indicate that using the Method of Equal Shares in its basic form may lead to intuitively suboptimal solutions.

3.1 Helenka Paradox

The instance that follows comes from the PB elections held in 2020 in the Polish city of Zabrze, in district Helenka [1]. Two projects were proposed in this district, namely an expansion and modernization of sports facilities (to be called project A), and a plant sculpture (project B). Their costs and number of supporters are as follows.

	cost	403 voters	11 voters
Project A	\$310k	✓	
Project B	\$6k		✓

The budget is $b = \$310,000$ and assume cost utilities. The second group of 11 voters should intuitively be entitled to $\$(11/414 \cdot 310,000) \approx \$8,000$. In fact, any rule that satisfies EJR must select project B and so does Equal Shares. However, selecting project B precludes the inclusion of project A within the budget constraint. This is highly counterintuitive since it leaves a great majority of voters (over 97% of the electorate) empty handed, despite the fact that they commonly approve an affordable project. Thus, the Helenka Paradox serves not only as a critique of the Method of Equal Shares, but also of the prominent axiom of EJR itself. The same holds for the issue of undespending which is evident in the presented example as well.

In order to solve the indicated problem, additional strategies could be employed. Cities may compare the outcomes returned by the Method of Equal Shares and by the standard Utilitarian Method. If ballots indicate that more voters prefer the outcome of the standard Utilitarian Method, then it could be selected. An alternative solution is to put an upper bound on the initial cost of the projects. Interestingly, while working on the experimental part of our work, we observed no evident paradoxes like the discussed one in data from elections where the cost of the projects did not exceed 30% of the budget. While these two solutions can work well in practice, neither is perfect. The runoff approach might potentially result in a utilitarian solution where some groups of voters are underrepresented. Moreover, imposing an upper limit on project costs might exclude some worthwhile and highly popular ideas, especially in small-scale elections.

3.2 Tail Utilities

Consider the following election with $m = 2$ candidates, and $n = 100$ voters casting ballots via range voting, as follows (the values in the table indicate the assigned scores).

	cost	99 voters	1 voter
Project A	\$1	100	1
Project B	\$1	2	2

Assume that the budget is $b = \$1$. Under the Method of Equal Shares, all voters have to pay all their virtual money to cover the cost of the one project that will be selected. As a result, if project A is selected, 99 voters will pay \$0.0001 per unit of utility and 1 voter will pay \$0.01 per unit of utility. In turn, for project B, all voters will pay \$0.005 per unit of utility. Thus, project A is $1/100$-affordable while project B is $1/200$-affordable, and the rule selects project B, even though 99% of voters consider project A as a much better option. This is because the Method of Equal Shares is in some sense egalitarian: when assessing the quality of a candidate, it essentially considers the utility assigned to the candidate by the least satisfied voter among those covering its cost. Note that the presented problem does not appear in approval elections. We observed this issue when applying Equal Shares to certain range voting committee elections.

Unlike the other issues presented, this one isn't rooted in proportionality but rather in the rule's internal mechanics. Still, all three limitations ultimately arise from the indivisibility of projects in PB. BOS addresses them simultaneously by relying on fractional allocations.

4 Method of Fractional Equal Shares

To develop intuitions required for the introduction of BOS Equal Shares, we first present Fractional Equal Shares (FrES)—an adaptation of Equal Shares to fractional PB, where projects can be partially funded. The idea is simple: a fraction α of a candidate c can be bought for the corresponding fraction of its cost: $\alpha \cdot \mathrm{cost}(c)$. If such a purchase is made, each voter $v_i \in V$ receives the utility of $\alpha \cdot u_i(c)$. Let us start by extending the notion of ρ-affordability to project fractions. For $\alpha \in (0, 1]$, we say that a candidate c is *(α, ρ)-affordable* if its α fraction can be bought with ratio ρ such that

$$\alpha \cdot \mathrm{cost}(c) = \textstyle\sum_{v_i \in V} \min(b_i, \alpha \cdot u_i(c) \cdot \rho). \tag{1}$$

The rule works as follows: Let b_i be the virtual budget of voter v_i, initially set to $b_i := b/n$. In each round, the method selects the candidate c which is (α, ρ)-affordable for the lowest possible value of ρ and buys the largest possible fraction α for which the candidate remains (α, ρ)-affordable. Then, the accounts of the supporters of c are updated accordingly: $b_i := b_i - \min(b_i, \alpha \cdot u_i(c) \cdot \rho)$. The method stops when no further fraction of a project can be selected within the budget.

Let us explain this method in more detail. Consider a partially funded candidate c and let S be the set of its supporters who still have money. Note that as α increases, the ratio ρ cannot decrease, meaning the minimum value of ρ occurs at small α. If α is small enough, the candidate can be funded with all voters contributing proportionally to their utilities, since no one exhausts their funds, i.e., in $\min(b_i, \alpha \cdot u_i(c) \cdot \rho)$, it is always $\alpha \cdot u_i(c) \cdot \rho$ that is smaller (or equal). Then, from Eq. (1) we obtain that $\rho = \mathrm{cost}(c)/\sum_{v_i \in S} u_i(c)$. Hence, Fractional Equal Shares selects the project with the lowest such ρ value and covers the fraction of the candidate's cost with payments proportional to the voters' utilities. This fraction is determined by the first moment a supporter exhausts their funds or the selected project is fully funded, whichever comes first. Therefore it runs in polynomial time. Moreover, $\alpha = \min\left(1 - W_c, \min_{v_i \in S} b_i/\rho \cdot u_i(c)\right)$, where W_c is the fraction of the project c bought already. Since FrES may also not spend the whole budget we can complete its outcomes in the utilitarian fashion, i.e., buying projects that maximize total utility per cost until the budget is exhausted. A pseudo-code of the method is provided in Appendix A.

In the discrete model, it is established that, unless P = NP, no election rule computable in (strongly) polynomial time can satisfy EJR. This hardness result stems from a reduction from the KNAPSACK problem [23], and holds even for instances with a single voter. However, this does not extend to the fractional setting as a simple greedy algorithm can solve fractional KNAPSACK optimally—in fact, FrES applied to an instance with a single voter is equivalent to such an algorithm and returns the optimal solution. This opens the possibility that a polynomial-time rule, like FrES, could indeed satisfy (the fractional analog of) EJR. This is indeed the case.

Definition 1 (**Fractional EJR**). A group of voters $S \subseteq V$ is (T, β, γ)-cohesive for $T \subseteq C$, $\beta : C \to [0, 1]$, and $\gamma : C \to \mathbb{R}_{\geqslant 0}$ if

1. $\sum_{c \in T} \mathrm{cost}(c) \cdot \beta(c) \leqslant b \cdot (|S|/n)$, and
2. $u_i(c) \cdot \beta(c) \geqslant \gamma(c)$ for all $c \in T$ and $v_i \in S$.

A fractional outcome W satisfies Fractional EJR if for every (T, β, γ)-cohesive group of voters S there is a voter $v_i \in S$ for which $\sum_{c \in C} u_i(c) \cdot W_c \geqslant \sum_{c \in T} \gamma(c)$. ⌟

By fixing $\beta(c) = 1$ for all $c \in C$, and by admitting only the integral solutions (that is, assuming $W_c \in \{0, 1\}$), we get the standard definition of EJR for the integral PB model and general utilities [23]. If we also fix $\gamma(c) = 1$ and $u_i(c) \in \{0, 1\}$ for each voter v_i and candidate c, we get the classic definition of EJR for approval-based committee voting [4].

Additionally, if applied to approval ballots with cost utilities, Definition 1 is equivalent to Cake EJR proposed by Bei et al. [6]. Fractional Equal Shares under approval ballots with cost utilities is equivalent to *Generalized Method of Equal Shares* introduced by Lu et al. [19], who showed that their rule satisfies Cake EJR. In the following theorem, we generalize this result and show that FrES satisfies Fractional EJR under arbitrary additive utilities.[1]

Theorem 1. *FrES satisfies Fractional EJR.*

Aspects of fairness and proportionality in models related to our fractional setting have also been examined by Fain et al. [13], Kroer and Peters [16], Aziz et al. [5], Bogomolnaia et al. [8], Brandl et al. [9], Munagala et al. [20], Suzuki and Vollen [25].

5 Method of Equal Shares with Bounded Overspending

In this section we build upon the idea behind Fractional Equal Shares, and design a new method for the standard (integral) model of participatory budgeting. The Method of Equal Shares with Bounded Overspending, in short, BOS Equal Shares, or just BOS, can be viewed as a rounding procedure for FrES, but also as a variant of Equal Shares, where voters may occasionally spend more than their initial entitlement. Under the fractional rule, an α-fraction of a candidate can be purchased for a corresponding fraction of its cost; then each voter receives a proportional fraction of the utility. However, in the integral model, fractional purchases are not possible. In BOS, we simulate buying an α-fraction of a candidate, still assuming the voters gain fractions of utilities. However, the voters are now required to cover the full cost. The distribution of the payments is proportional to the one computed for the α-fraction. Specifically, for ρ satisfying

[1] Proofs of our results are deferred to Appendix C.

$$\alpha \cdot \mathrm{cost}(c) = \sum_{v_i \in V} \min(b_i, \alpha \cdot u_i(c) \cdot \rho),$$

a voter v_i supporting c has to pay $p_i(c) = \min(b_i, \alpha \cdot u_i(c) \cdot \rho)/\alpha$. Notice that we divide by α as we now need to cover the entire cost of the project, not only its α fraction. Assuming $\alpha \cdot u_i(c)$ is the utility from candidate c, the highest payment per unit of utility is then ρ/α. Observe that $p_i(c)$ can be actually greater than b_i, i.e., the remaining budget of voter v_i. In such a case we say that voter v_i is *overspending* and we set its account to zero.

The rule works as follows: Let b_i be the virtual budget of voter v_i, initially set to $b_i := b/n$. Each round, among all candidates that fit within the remaining budget, BOS buys the (α, ρ)-affordable candidate c with the lowest possible value of ρ/α. The accounts of the supporters of c are updated accordingly: $b_i := \max(0, b_i - u_i(c) \cdot \rho)$. The method stops if no remaining candidate fitting within the budget is (α, ρ)-affordable for some $\alpha \in (0, 1]$ and $\rho \in \mathbb{R}_+$. Voters without money and those that overspent before, do not have an impact on the further decisions. We provide a pseudo-code of the Method of Equal Shares with Bounded Overspending in Appendix A.

Continuation of Example 1. Each voter has initially the same amount of money (\$100k) and the cost of every project can be covered by its supporters. For every project, $\alpha = 1$ is now optimal, which corresponds to buying the project in full. Hence, BOS, as the Method of Equal Shares, selects Project A, being the one with the highest number of votes and equally distributes its cost among the supporters. As a result, voters v_1 to v_6 are left with \$50k, while voters v_7 to v_{10} still have \$100k. The values of α and ρ for the second round follow:

	A	B	C	D	E	F
$\alpha = 1, \rho$	–	$+\infty$	$1/3$	$1/4$	$1/4$	$1/3$
$\alpha = 5/6, \rho$	–	$+\infty$	$1/5$	$1/4$	$1/4$	$1/3$
$\alpha = 5/8, \rho$	–	$1/5$	$1/5$	$1/4$	$1/4$	$1/3$

For instance, consider Project B. Since voters v_1 to v_6 have only \$50k each, Project B cannot be purchased in full. However, its supporters can cover $\alpha = 250/400 = 5/8$ of its cost. This cost would be equally spread among five voters, so $\rho = 1/5$ and the ratio ρ/α equals $8/25$. Now, consider Project C. This project can be bought in full, but only if voter v_{10} pays $\rho = 1/3$ of the cost. The Method of Equal Shares rejects this option as imbalanced and selects a project with a smaller ρ parameter. In turn, BOS considers also paying for a fraction of the project with balanced payments. Specifically, to maintain equal payments, we can simulate buying a fraction $\alpha = 5/6$ of the project. This would result in $\rho = 1/5$ and ratio $\rho/\alpha = 6/25$. Since this ratio is smaller than the ratio of Project B as well as the ratios of Projects D, E and F which remained unchanged, BOS selects Project C with $\alpha = 5/6$. To cover $5/6$ of its cost, its supporters would have to pay \$50k each. Hence, in BOS, each supporter should pay $6/5 \cdot \$50\text{k} = \$60\text{k}$. The supporters who do not have enough funds simply pay all their remaining money.

In the third round, BOS would buy Project D with $\alpha = 1$ and $\rho = 5/18$. Finally, in the last round, Project F would be bought for $\alpha = 5/6$ and $\rho = 3/5$. We refer the reader to Appendix B.3 for the detailed explanation of the example. ⌟

As we have noted in the above example, we do not have to consider all values of α. In fact, it can be shown that it is optimal to either consider $\alpha = 1$ or a fraction α that, for some voter, is the smallest amount for which she runs out of money, assuming proportional payments, i.e., in which $b_i = u_i(c) \cdot \rho$ for some voter $v_i \in V$. In such a case, it holds that $\rho = b_i/u_i(c)$. Since there are at most n such values, the outcome of BOS can be computed in polynomial time.

Theorem 2. *BOS runs in polynomial time.*

5.1 Addressing the Limitations of the Method of Equal Shares

We now discuss how BOS addresses each of the limitations of Equal Shares that we have identified (Section 3). For underspending, observe that BOS Equal Shares leads to a non-exhaustive outcome only if all voters that still have remaining funds have all of their supported projects selected (otherwise there is an (α, ρ)-affordable project for some, possibly very small, α). Based on our empirical analysis (see Section 6) such a situation is extremely rare in practice. As a result, BOS spends on average a similar fraction of a budget as the Utilitarian Method and does not need to be combined with a completion mechanism, in contrast to the Method of Equal Shares. However, if an exhaustive rule is required, a simple modification can be made to BOS Equal Shares: whenever all projects supported by a voter are already selected, we remove this voter from the election and redistribute her remaining funds equally among the remaining voters. We note that similar modification for the original Method of Equal Shares would still lead to a non-exhaustive rule.

Example 2: BOS on Helenka Paradox. Consider the Helenka Paradox election instance presented in Section 3. The initial endowment is $\$(310{,}000/414) \approx \$750$. Only the $\alpha = 403/414$ fraction of the project A can be paid for by its supporters. This gives $\rho = 1/403$, which results in $\rho/\alpha = 414/403^2$. In turn, project B can be fully bought, thus we have $\alpha = 1$ and $\rho = 1/11 = \rho/\alpha$. Hence, BOS, in contrast to the Method of Equal Shares, would select project A.

The reader may wonder what would happen if the support for project B was higher. Clearly, as the support for project B that is affordable at full increases and that of project A taking the whole budget decreases, at some point we should switch from selecting A to selecting B. At what point would BOS start selecting project B? Assume that x voters support A and $n - x$ voters support B. Then, for A we have $\rho = 1/x$ and $\alpha = x/n$, thus $\rho/\alpha = n/x^2$. In turn, for B we have $\rho = 1/(n - x)$ and $\alpha = 1$, thus $\rho/\alpha = 1/(n - x)$. Hence, we can select project B if $1/(n - x) \leqslant n/x^2$ or equivalently $n^2 - xn \geqslant x^2$. Solving the quadratic equation, we get that if $x > 0$, then this is equivalent to $x/n \leqslant (\sqrt{5} - 1)/2 = 1/\varphi$. It turns out that the switching point is when the supports of the projects are in the golden ratio to each other, i.e., the project using the entire budget is supported by roughly 61.8% of the electorate. This is close to the value of the thresholds required for a supermajority in many democratic systems (e.g., 3/5 or 2/3). ⌟

Example 3: BOS on Tail Utilities. Consider the Tail Utilities instance from Section 3. All voters have the same utility from project B, hence, every fraction α of the project can be bought with equal payments and $\rho = \mathrm{cost}(c)/\sum_{v_i \in V} u_i(c) = 1/200$. Therefore, we obtain the minimal ratio $\rho/\alpha = 1/200$, for $\alpha = 1$. Let us focus now on project A. If the project is bought in full, the last voter would have to pay \$0.01 per unit of utility, giving $\rho = 1/100$. However, it is also possible to simulate the purchase of the fraction of the project that ensures the payments are proportional to utilities. This happens if 99 voters pay \$0.01 and 1 voter pays \$0.0001. The voters pay for a fraction of $\alpha = 9901/10000$ of the project, achieving a much better ratio: $\rho = \mathrm{cost}(c)/\sum_{v_i \in V} u_i(c) = 1/9901$. Thus, $\rho/\alpha = 10000/9901^2$, and so BOS would select project A. ⌟

5.2 Proportionality Guarantees of BOS Equal Shares

As previously noted, Extended Justified Representation (EJR) is a well-established axiom of proportionality for participatory budgeting. In Example 2, we illustrate that BOS fails to satisfy EJR under cost utilities in the PB setting. Importantly, this failure is not a deficiency but an intended behavior. Nevertheless, BOS still provides strong proportionality guarantees. We primarily demonstrate this through extensive experiments on real data (see Section 6), and we additionally prove theoretically that the extent of violation of EJR is limited.

In line with the definition of EJR under cost utilities, given a subset of projects $T \subseteq C$, we say that a group of voters S *deserves* the satisfaction of $\mathrm{cost}(T)$ if $|S| \geqslant \mathrm{cost}(T)\frac{n}{b}$ and all voters in S approve all projects in T, that is $T \subseteq \bigcap_{i \in S} A_i$.

Definition 2. Assuming cost utilities and fixing a function from (E, S) to $\mathcal{R}_{\geqslant 0}$, where E is an election (C, V, b) and $S \subseteq V$, we say that a rule satisfies EJR up to t, if for every E such that W is the outcome of the rule on E and every $(S, T) \subseteq V \times C$ such that S deserves the satisfaction of $\mathrm{cost}(T)$, there is a voter v_i such that $u_i(W) \geqslant \mathrm{cost}(T) - t - \mathrm{cost}(c)$, for all $c \in T \setminus W$. ⌟

The main theoretical contribution of our work follows. It reassures that BOS does not excessively deviate from EJR, even when it violates it. Our guarantees are asymptotically tight.

Theorem 3. *For cost utilities, BOS satisfies EJR up to* $\frac{n-|S|}{2|S|} \cdot c^*$, *where* $c^* := \max_{c \in C} \mathrm{cost}(c)$.

Consequently, for approval-based committee elections, BOS satisfies EJR up to $\left\lceil \frac{k-\ell}{2\ell} \right\rceil$ candidates, where k is the size of the committee, and $\ell = |S| \cdot k/n$ is the number of representatives that voters in S deserve. To illustrate this, consider a group entitled to $\sqrt{k}$ representatives. BOS ensures that at least one voter in the group receives no fewer than $\sqrt{k}/2$ candidates she likes. As the group size increases, this guarantee becomes stronger.

Proposition 4. *For each* ℓ *there exists an approval-based committee election where a group of voters deserves* $\left\lceil \frac{k-\ell}{4\ell} \right\rceil$ *candidates, but they all get no representatives under BOS.*

Cities employing PB often find the paradoxes associated with classic Equal Shares unacceptable. Although it is rarely mentioned in the literature, all cities currently employing Equal Shares include in their regulations a final step that compares its outcome with the utilitarian one. If a majority of voters derive higher utility from the outcome of Equal Shares, it is accepted; otherwise, the utilitarian prevails. This comparison renders the entire procedure essentially majoritarian in the worst-case. Therefore, the solution used in practice has worse worst-case proportionality guarantees than BOS. The comparison with the utilitarian outcome is only an imperfect workaround and, in contrast, BOS provides a principled alternative.

5.3 Method of Equal Shares with Bounded Overspending Plus

While BOS solves multiple problems of Equal Shares, one can still argue that it is not always evident that it produces the most desirable outcome.

Example 4: Selection of Unpopular Projects. Consider the following election with $m = 310$ unit-cost candidates, and $n = 1000$ voters. The budget is $b = \$10$. Voters $v_1, \ldots, v_{700}$ approve ten projects A_1 to A_{10}, and each voter from v_{701} to v_{1000} approves a single project from B_1 to B_{300}; each such project is approved by only one voter.

	cost	v_1	$\ldots$	v_{700}	v_{701}	$\ldots$	v_{1000}
Project A_1	\$1	✓	✓	✓			
$\ldots$	\$1	✓	✓	✓			
Project A_{10}	\$1	✓	✓	✓			
Project B_1	\$1				✓		
$\ldots$	\$1					✓	
Project B_{300}	\$1						✓

Consider how BOS operates on this instance. First, it would buy seven A-projects. After that, voters v_1 to v_{700} would not have any money left, so BOS would additionally buy three B-projects, which is arguably not the most effective allocation of the funds. On the other hand, the Method of Equal Shares would buy only seven A-projects, but its Add1U variant would select all ten A-projects, i.e., it would return the outcome for the initial endowment equal to $b_{\mathrm{ini}} = \$(1 - \varepsilon)$. ⌟

Motivated by this observation we introduce a rule combining the key ideas of BOS and Add1U completion for Equal Shares, which works as follows: In each round we first find the (α, ρ)-affordable candidate c that minimizes ρ/α, as in standard BOS. If buying candidate c requires overspending, we do not buy it, but look for a better candidate that would not overspend more. Specifically, we first compute the maximal overspending assuming all voters overspend equally:

$$\Delta b = \frac{\mathrm{cost}(c) - \alpha \mathrm{cost}(c)}{|\{v_i \in V : \rho u_i(c) \geqslant b_i > 0\}|},$$

we temporarily set $b_i := b_i + \Delta b$ and pick the $(1, \rho)$-affordable project that fits within the budget and minimizes ρ. We charge the voters', revoke the increase of their entitlements, and proceed. The pseudo-code of BOS+ is given in Appendix A.

Observe that BOS+ would select ten A-projects in the instance from Example 4: every attempt to buy a B-project would increase initial endowments of voters v_1–v_{700} and allow them to buy yet another A-project. Note also that this example highlights the fact that the optimal outcome may depend on the specific situation it is used in. Indicatively, if the election method is used for selecting validators in a blockchain protocol, there is particular concern about not over-representing groups of voters or giving them too much voting power [12]. BOS seems better suited for such applications. If the instance comes from PB elections conducted by municipalities, the outcome produced by BOS+ appears to be much more aligned with expectations.

For BOS+ we obtain the same proportionality guarantee as for BOS. Moreover, the hard instance from Proposition 4 also applies to BOS+.

Theorem 5. *For cost utilities, BOS+ satisfies EJR up to* $\frac{n-|S|}{2|S|} \cdot c^*$, *where* $c^* := \max_{c \in C} \mathrm{cost}(c)$.

6 Empirical Analysis

We now evaluate our rules on real-world PB data and on synthetic Euclidean elections.

6.1 Pabulib Instances

We computed the outcomes of the Utilitarian Method, Equal Shares (with Add1U completion), FrES (with utilitarian completion), BOS, and BOS+ across all 1274 participatory budgeting instances in Pabulib [15]. To compare performance, we analyzed six statistics: *score satisfaction*, *cost satisfaction*, *exclusion ratio*, *running time*, *EJR+ violations*, and *exhaustiveness*. For the first four, we observed that the behavior largely depends on the instance size, defined as the number of projects within it. Thus, we partitioned the instances into four size ranges, aiming for ranges with an almost equal number of instances. Figure 1 shows the average of each metric by rule and size range; detailed values and statistical significance appear in Appendix D.

Score and Cost Satisfaction. These metrics assess the total utility from the selected projects based on score and cost utility measures. In Figure 1, the values are normalized against Utilitarian. For score satisfaction, in medium and large instances, Equal Shares, BOS, and BOS+ yield similar and significantly higher results than FrES and Utilitarian; for small instances, the performance of all is comparable. Regarding cost satisfaction, Utilitarian outperforms all others across all instances. It can be considered a greedy method maximizing cost satisfaction objective. Yet, BOS and FrES outperform Equal Shares in small and medium instances. Notably, BOS+ performs as good as BOS for the smaller instances and slightly surpasses it for the larger.

Exclusion Ratio. This metric represents the fraction of voters who do not support any of the selected projects. For medium and large instances Utilitarian excludes significantly more voters on average than the other rules, which perform similarly. However, for the smallest instances, Equal Shares performs considerably worse than the others. This is due to small instances often resembling the Helenka Paradox. For FrES, the exclusion ratio is always zero.

Running Time. We observe that BOS and BOS+ are an order of magnitude faster than Equal Shares. This is because Equal Shares with Add1U completion, requires computing the outcome multiple times until the final value of voter endowments is established. Computations has been conducted in Python on a MacBook Air laptop with an Apple M3 processor and 16 GB of RAM.

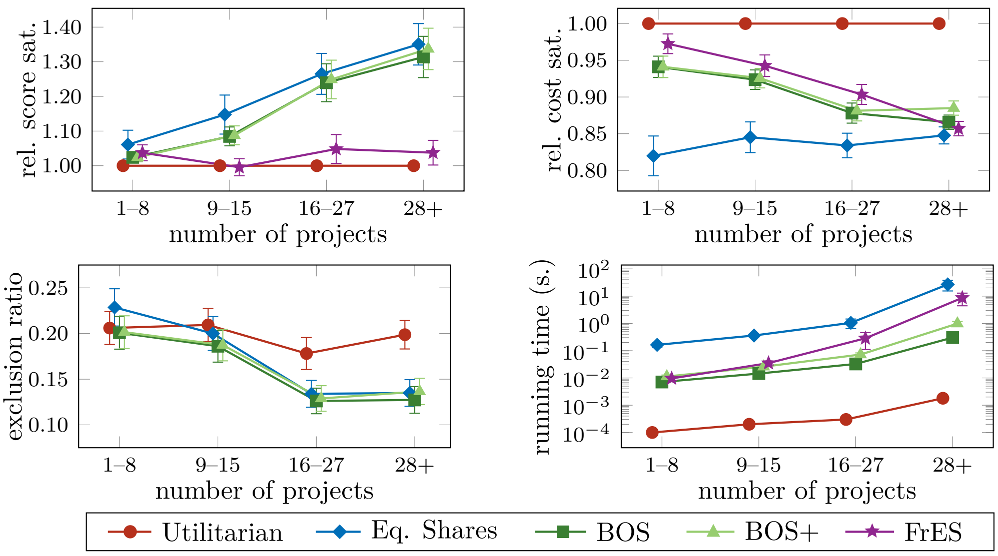

Figure 1: Average score satisfaction (top-left), cost satisfaction (top-right), exclusion ratio (bottom-right), and running time in seconds (bottom-left) of the outputs of the considered rules based on Pabulib data by the number projects in an instance. Cost and score satisfactions are reported as fractions of the respective satisfaction under utilitarian rule. Thin vertical lines mark 90% confidence intervals. Note that the y-axis in the running time plot is in a logarithmic scale.

EJR+ Violations. EJR+ is one of the strongest satisfiable proportionality axioms, strictly stronger than EJR [10]. Its satisfaction can be efficiently verified, making it a suitable metric for empirical comparison of the proportionality of different rules. Unfortunately, to date, it is defined only for approval ballots, thus we limit our analysis to such instances. On a high level, for each unselected candidate in an election, we say that it violates EJR+ up to one (we actually count violations of this established relaxation [10]), if it is approved by a sufficiently large group of voters each of whom is inadequately satisfied by the outcome of a rule. For Equal Shares, it is known that its outcome cannot have any violations. In contrast, Utilitarian averages 0.933 violations per instance, setting it apart from BOS and BOS+, averaging only 0.061 and 0.06, respectively. This indicates that although our rules, in theory, do not guarantee EJR+ up to one, they almost always yield proportional solutions according to this strong axiom.

Exhaustiveness Recall that the outcome of a rule is exhaustive, if there is no unselected project with a cost not exceeding the remaining budget (for FrES, we require that it either spends the whole budget or selects all projects fully). By definition, the Utilitarian Method is always exhaustive. In contrast, if we run Equal Shares without any completion, then it is exhaustive in only 7% of the instances and on average spends less than 45% of the available budget. On the other hand, BOS is non-exhaustive in only 2 instances out of 1274 in the dataset. Moreover, in both of these instances, it spends more than 99% of the budget (only very cheap additional projects can fit the budget in both cases). BOS+ is non-exhaustive in a few more instances, but spends similar fraction of the budget on average. Therefore, while Equal Shares should not be used without a completion mechanism, BOS and BOS+ can safely be deployed without one.

6.2 Euclidean Instances

We further illustrate the differences between the examined rules using synthetic Euclidean elections. Our results are presented in Figure 2. This analysis again highlights that BOS offers more desirable results than Equal Shares.

In the examined setting, each voter and candidate is represented as a point in a 2D space. The

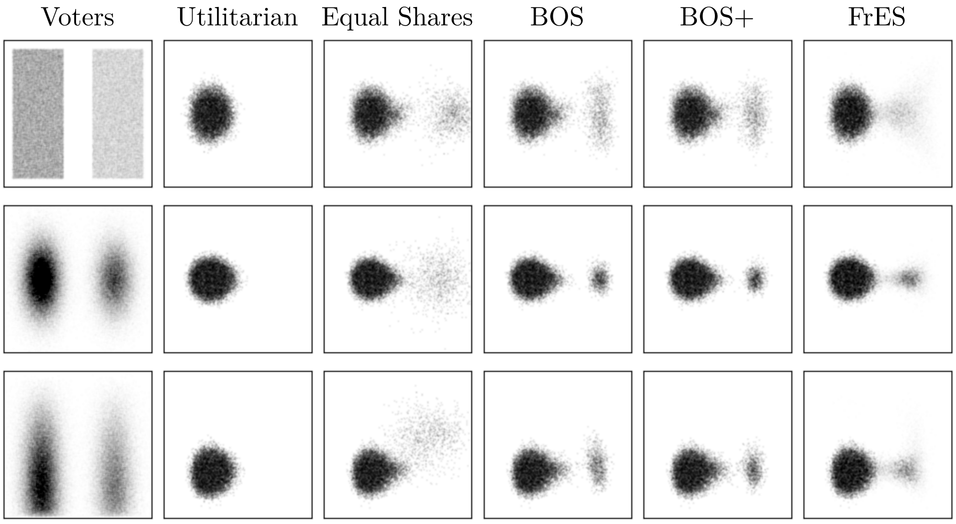

Figure 2: Results on Euclidean elections. The first column presents the superimposed positions of voters in all generated elections and the following columns show positions of candidates selected by respective rules (for FrES, the opacity reflects the selected fraction of candidates).

utility of the voter v_i for candidate c is defined as $u_i(c) = (\mathrm{dist}(v_i, c) + \lambda)^{-1}$, where $\mathrm{dist}(v_i, c)$ is the Euclidean distance between them, and the denominator is shifted by the constant of λ to bound the maximal utility. Here, we present the results for $\lambda = 1$; the results for $\lambda = 1/2$ and $\lambda = 2$ (in Appendix D) lead to similar conclusion. For each election, we placed 150 candidates uniformly at random in a unit square and considered three different voter distributions. In the first, 100 voters were drawn uniformly from the rectangle $(0.05, 0.05) - (0.4, 0.95)$ and 50 from $(0.6, 0.05) - (0.95, 0.95)$. The second and third used Gaussian distributions for x-coordinates: 100 voters centered at 0.25 and 50 centered at 0.75. In the second, y-coordinates were drawn from a Gaussian centered at 0.5, while in the third, they were drawn from a beta distribution with $a = 1.5$ and $b = 3$, pushing more voters to the bottom of the square. The candidates had unit costs, and the budget was set to 10. We sampled 1000 elections per distribution.

The Utilitarian Method selects only candidates from the left-hand side of the square. Equal Shares, BOS, BOS+, and FrES also select candidates from the right-hand side but less than 1/3 of the total, which is the proportion of the voters there. This is because the right-hand side voters gain some utility from left-hand side candidates, thus spending part of their budget on them. In the third distribution, Equal Shares selects candidates from the top-right part of the square, which has fewer voters, while BOS and BOS+ choose candidates from the bottom-right, where there is greater overall support. BOS+ and BOS output very similar sets of candidates, with BOS+ giving a bit more concentrated outputs. FrES' selection resembles that of BOS, particularly in the last two distributions, reinforcing the view that BOS is a rounding of FrES.

7 Conclusion

We have introduced the Method of Equal Shares with Bounded Overspending (BOS), a robust variant of the Method of Equal Shares, along with its refined version, BOS+. We have identified inefficiencies in the original method, illustrated by simple well-structured examples and supported by empirical analysis. Our new rule maintains strong fairness properties, which we have confirmed both theoretically and in experiments. In the process of developing BOS, we have also introduced and analyzed FrES, a fractional variant of Equal Shares for the PB model with additive utilities.

The practical relevance of our work is exemplified by the fact that the council of a medium-sized city has already chosen BOS for the upcoming participatory budgeting vote. This will be the largest MES-like allocated budget to date, amounting to approximately half a million euros.

Acknowledgements

T. Wąs was partially supported by UK Engineering and Physical Sciences Research Council (EPSRC) under grant EP/X038548/1. The rest of the authors were partially supported by the European Union (ERC, PRO-DEMOCRATIC, 101076570). Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them.

References

[1] Election votes in Zabrze (Helenka), Poland. https://pabulib.org/?search=Poland%20Zabrze%202021%20Helenka, 2021. Accessed: 2025-05-04.

[2] Election results. https://equalshares.net/elections, 2024. Accessed: 2025-05-04.

[3] List of participatory budgeting votes. https://en.wikipedia.org/wiki/List_of_participatory_budgeting_votes, 2024. Accessed: 2025-05-04.

[4] H. Aziz, M. Brill, V. Conitzer, E. Elkind, R. Freeman, and T. Walsh. Justified representation in approval-based committee voting. *Social Choice and Welfare*, 48(2):461–485, 2017.

[5] H. Aziz, A. Bogomolnaia, and H. Moulin. Fair mixing: the case of dichotomous preferences. In *Proceedings of the 20th ACM Conference on Economics and Computation (EC'19)*, pages 753–781, 2019.

[6] X. Bei, X. Lu, and W. Suksompong. Truthful cake sharing. *Social Choice and Welfare*, 64: 309–343, 2024.

[7] G. Benade, Kobi K. Gal, and R. Fairstein. Participatory budgeting design for the real world. In *Proceedings of the 37th AAAI Conference on Artificial Intelligence (AAAI'23)*, pages 5633–5640., 2023.

[8] A. Bogomolnaia, H. Moulin, and R. Stong. Collective choice under dichotomous preferences. *Journal of Economic Theory*, 122(2):165–184, 2005.

[9] F. Brandl, F. Brandt, D. Peters, and C. Stricker. Distribution rules under dichotomous preferences: two out of three ain't bad. In *Proceedings of the 22nd ACM Conference on Economics and Computation (EC'21)*, pages 158–179, 2021.

[10] M. Brill and J. Peters. Robust and verifiable proportionality axioms for multiwinner voting. In *Proceedings of the 24th ACM Conference on Economics and Computation (EC'23)*, page 301, 2023.

[11] M. Brill, S. Forster, M. Lackner, J. Maly, and J. Peters. Proportionality in approval-based participatory budgeting. In *Proceedings of the 37th AAAI Conference on Artificial Intelligence (AAAI'23)*, pages 5524–5531, 2023.

[12] A. Cevallos and A. Stewart. A verifiably secure and proportional committee election rule. In *Proceedings of the 3rd ACM Conference on Advances in Financial Technologies (AFT'21)*, pages 29–42, 2021.

[13] B. Fain, A. Goel, and K. Munagala. The core of the participatory budgeting problem. In *Proceedings of the 12th International Conference on Web and Internet Economics (WINE'16)*, pages 384–399, 2016.

[14] P. Faliszewski, P. Skowron, A. Slinko, and N. Talmon. Multiwinner voting: A new challenge for social choice theory. In U. Endriss, editor, *Trends in Computational Social Choice*. AI Access, 2017.

[15] P. Faliszewski, J. Flis, D. Peters, G. Pierczynski, P. Skowron, D. Stolicki, S. Szufa, and N. Talmon. Participatory budgeting: Data, tools and analysis. In *Proceedings of the 32nd International Joint Conference on Artificial Intelligence (IJCAI'23)*, pages 2667–2674, 2023.

[16] Christian Kroer and Dominik Peters. Computing lindahl equilibrium for public goods with and without funding caps. *Proceedings of the 26th ACM Conference on Economics and Computation (EC'25)*, 2025.

[17] M. Lackner and P. Skowron. *Multi-winner voting with approval preferences.* Springer Briefs in Intelligent Systems, Springer. Springer, 2023.

[18] M. Los, Z. Christoff, and D. Grossi. Proportional budget allocations: Towards a systematization. In *Proceedings of the 31st International Joint Conference on Artificial Intelligence (IJCAI'22)*, pages 398–404, 2022.

[19] X. Lu, J. Peters, H. Aziz, X. Bei, and W. Suksompong. Approval-based voting with mixed goods. *Social Choice and Welfare*, 62:643–677, 2024.

[20] K. Munagala, Y. Shen, K. Wang, and Z. Wang. Approximate core for committee selection via multilinear extension and market clearing. In *Proceedings of the Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'22)*, pages 2229–2252, 2022.

[21] G. Papasotiropoulos, O. Skibski, P. Skowron, and T. Wąs. Proportional selection in networks. *arXiv preprint arXiv:2502.03545*, 2025.

[22] D. Peters and P. Skowron. Proportionality and the limits of welfarism. In *Proceedings of the 21st ACM Conference on Economics and Computation (EC'20)*, pages 793–794, 2020.

[23] D. Peters, G. Pierczyński, and P. Skowron. Proportional participatory budgeting with additive utilities. In *Proceedings of the 35th Conference on Neural Information Processing Systems (NeurIPS'21)*, pages 12726–12737, 2021.

[24] S. Rey and J. Maly. The (computational) social choice take on indivisible participatory budgeting. *arXiv preprint arXiv:2303.00621*, 2023.

[25] M. Suzuki and J. Vollen. Maximum flow is fair: A network flow approach to committee voting. In *Proceedings of the 25th ACM Conference on Economics and Computation (EC'24)*, pages 964–983, 2024.

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Appendix A. Pseudo-Codes of the Proposed Rules

ALGORITHM 1: Pseudo-code of Fractional Equal Shares.

```
Input: A PB election (C, V, b)
W_c ← 0 for each c ∈ C
b_i ← b/n for each v_i ∈ V
S ← V
while exists c such that W_c ≠ 1 and Σ_{v_i∈S} u_i(c) > 0 do
    c ← argmin_{c∈C:W_c≠1} (cost(c)/ Σ_{v_i∈S} u_i(c))
    ρ ← cost(c)/ Σ_{v_i∈S} u_i(c)
    α ← min(1 − W_c, min_{v_i∈S} b_i/(ρ · u_i(c)))
    W_c ← W_c + α
    for v_i ∈ S do
        b_i ← b_i − α · ρ · u_i(c)
        if b_i = 0 then
            S ← S \ {v_i}
return W
```

ALGORITHM 2: Pseudo-code of the Method of Equal Shares with Bounded Overspending.

```
Input: A PB election (C, V, b)
W ← ∅
b_i ← b/n for each v_i ∈ V
while C' = {c ∈ C \ W : cost(c) ⩽ b − cost(W) and Σ_{v_i∈V:u_i(c)>0} b_i > 0} is nonempty do
    (α*, ρ*, c*) ← (1, +∞, c)
    for c ∈ C' do
        λ' ← λ satisfying cost(c) = Σ_{i=1}^{n} min(b_i, u_i(c) · λ) or +∞ if there is no such λ, i.e.,
          cost(c) > Σ_{v_i∈V:u_i(c)>0} b_i
        for λ ∈ {b_i/u_i(c) : v_i ∈ V, b_i > 0, u_i(c) > 0} ∪ {λ'} do
            α ← min((Σ_{i=1}^{n} min(b_i, u_i(c) · λ)) /cost(c), 1)
            ρ ← λ/α
            if ρ/α < ρ*/α* then
                (α*, ρ*, c*) ← (α, ρ, c)
    W ← W ∪ {c*}
    for v_i ∈ V such that b_i > 0 and u_i(c*) > 0 do
        b_i ← max(0, b_i − u_i(c*) · ρ*)
return W
```

ALGORITHM 3: Pseudo-code of BOS Equal Shares Plus.

Input: *A PB election* (C, V, b)

$W \leftarrow \emptyset$, $\quad b_i \leftarrow b/n$ for each $v_i \in V$

$\text{over}_i \leftarrow 0$ for each $v_i \in V$ `/* Overspending for each voter */`

while exists $c \in C \setminus W$ s.t. $\text{cost}(c) \leqslant b - \text{cost}(W)$ **do**

  `/* The part of computing the values` α^*, ρ^*, c^* `is the same as in pure BOS. */`

  $(\alpha^*, \rho^*, c^*) \leftarrow (1, +\infty, c)$

  for $c \in C \setminus W$ s.t. $\text{cost}(c) \leqslant b - \text{cost}(W)$ **do**

    $\lambda' \leftarrow \lambda$ satisfying $\text{cost}(c) = \sum_{i=1}^{n} \min(b_i, u_i(c) \cdot \lambda)$ or $+\infty$ if there is no such λ, i.e., $\text{cost}(c) > \sum_{v_i \in V : u_i(c) > 0} b_i$

    for $\lambda \in \{b_i/u_i(c) : v_i \in V, b_i > 0, u_i(c) > 0\} \cup \{\lambda'\}$ **do**

      $\alpha \leftarrow \min\left(\left(\sum_{i=1}^{n} \min(b_i, u_i(c) \cdot \lambda)\right) / \text{cost}(c), 1\right)$

      $\rho \leftarrow \lambda/\alpha$

      if $\rho/\alpha < \rho^*/\alpha^*$ **then**

        $(\alpha^*, \rho^*, c^*) \leftarrow (\alpha, \rho, c)$

  `/* Temporary increase of the budgets. */`

  $\Delta b = \frac{\text{cost}(c^*) - \alpha^* \text{cost}(c^*)}{|\{v_i \in V :\, \rho^* u_i(c^*) \geqslant b_i > 0\}|}$

  $b_i^* = b_i + \max(0, \Delta b - \text{over}_i)$ for each $v_i \in V$

  `/* Finding the optimal candidate is the same as in the Method of Equal Shares. */`

  for $c \in C \setminus W$ **do**

    if $\sum_{v_i \in V : u_i(c) > 0} b_i^* < \text{cost}(c)$ **then**

      $\rho_{\text{MES}}(c) \leftarrow \infty$ `/* Project not affordable */`

    else

      Let $v_{i_1}, \ldots, v_{i_t}$ be a list of all voters $v_{i_j} \in V$ with $u_{i_j}(c) > 0$, ordered so that $b_{i_1}^*/u_{i_1}(c) \leqslant \cdots \leqslant b_{i_t}^*/u_{i_t}(c)$.

      for $s = 1, \ldots, t$ **do**

        $\rho_{\text{MES}} \leftarrow (\text{cost}(c) - (b_{i_1}^* + \cdots + b_{i_{s-1}}^*))/(u_{i_s}(c) + \cdots + u_{i_t}(c))$

        if $\rho_{\text{MES}}(c) \cdot u_{i_s} \leqslant b_{i_s}^*$ **then**

          break `/* we have found the optimal` ρ_{MES}`-value */`

  $c^* \leftarrow \text{argmin}_{c \in C \setminus W} \rho_{\text{MES}}(c)$

  $W \leftarrow W \cup \{c^*\}$

  `/* Updating budgets. */`

  for $v_i \in V$ **do**

    $\text{pay}_i \leftarrow \min(b_i^*, \rho_{\text{MES}}(c^*) \cdot u_i(c^*))$

    if $b_i \geqslant \text{pay}_i$ **then**

      $b_i \leftarrow b_i - \text{pay}_i$

    else

      $\text{over}_i \leftarrow \text{over}_i + (\text{pay}_i - b_i)$

      $b_i \leftarrow 0$

return W

Appendix B. Complete Examples Illustrating the Rules' Execution

In this section, we provide detailed description of all steps in the runs of Equal Shares, FrES, and BOS on the instance from Example 1. Let us repeat the table with the projects and voters for convenience.

	cost	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
Project A	\$300k	✓	✓	✓	✓	✓	✓				
Project B	\$400k		✓	✓	✓	✓	✓				
Project C	\$300k		✓	✓	✓	✓					✓
Project D	\$240k							✓	✓	✓	✓
Project E	\$170k		✓					✓	✓	✓	
Project F	\$100k						✓			✓	✓

The Utilitarian Method would select projects solely based on their vote count, thus choosing Projects A, B and C. This seems unfair since a large fraction of the voters (namely voters v_7 to v_9 making up 30% of the electorate) would not approve any of the selected projects.

B.1 Method of Equal Shares

At the beginning, the Method of Equal Shares assigns \$100k to each voter. The table that appears below indicates the available (virtual) budget of each voter (in thousands of dollars).

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
b_i	100	100	100	100	100	100	100	100	100	100

We first need to determine how affordable each project is. Project A is $1/6$-affordable, as it can be funded if each of its supporters pays \$50k, which is $1/6$ of its cost (recall that we assumed cost-utilities for this example). Analogously, each other project that received x votes is $1/x$-affordable.

	A	B	C	D	E	F
ρ	$1/6$	$1/5$	$1/5$	$1/4$	$1/4$	$1/3$

Thus, in the first round the rule simply selects the project with the highest vote count, namely project A. After paying its cost, voters v_1 to v_6 are left with \$(100k − 300k/6) = \$50k. Voters' remaining budget follows.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
b_i	50	50	50	50	50	50	100	100	100	100

In the second round, project B is no longer affordable, as its supporters do not have a total of at least \$400k to fund it. Project C is $1/3$-affordable. Indeed, to fund it, voters v_2, v_3, v_4, v_5 and v_{10} would have to use all their money. This means, in particular, that voter v_{10} would pay \$100k out of \$300k which is $1/3$ of the cost of the project. The rest of the projects remain ρ-affordable for the same values of ρ, as their cost can still be spread equally among their supporters.

	A	B	C	D	E	F
ρ	–	–	$1/3$	$1/4$	$1/4$	$1/3$

Hence, projects D and E are both ρ-affordable for the smallest value of $\rho = 1/4$. Let us assume the former project is selected, by breaking ties lexicographically. After paying its cost, voters v_7 to v_{10} are left with \$(100k − 240k/4) = \$40k.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
b_i	50	50	50	50	50	50	40	40	40	40

In the third round, project C is no longer affordable, as its supporters have together only \$240k. Project E is $5/17$-affordable: voters v_7, v_8 and v_9 have only \$120k, which means that voter v_2 needs to pay \$50k to cover the cost of \$170k. Project F remains $1/3$-affordable.

	A	B	C	D	E	F
ρ	–	–	–	–	$5/17$	$1/3$

As a result, project E is selected and voters v_2, v_7, v_8, and v_9 run out of money.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
b_i	50	0	50	50	50	50	0	0	0	40

At this point, no project is affordable since the supporters of projects B, C and F have in total \$200k, \$150k and \$90k, respectively, therefore, the procedure stops having selected the outcome $\{A, D, E\}$. Note that the purchased bundle comes at a total cost of \$710k, which is \$290k less than the initially available budget. Thus, in principle, we could afford to additionally fund project F. However, the supporters of this project do not have enough (virtual) money to fund it, and so the project is not selected by the Method of Equal Shares.

Clearly, the selection made by the Method of Equal Shares is less discriminatory than the one by Utilitarian, as each voter approves at least one of the selected projects.

B.2 Fractional Equal Shares

Note that for cost utilities, in each round Fractional Equal Shares selects the project with the most supporters who still have money left. The method purchases the largest portion of the project that can be covered with equal payments of the supporters.

In the first round, as in the Method of Equal Shares, project A is chosen. It is bought in full, as its whole price can be split equally between supporters, each paying \$50k. Voters' remaining funds appear below.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
b_i	50	50	50	50	50	50	100	100	100	100

In the second round, projects B or C can be selected. Let us assume project C is selected. Since voters v_2 to v_5 have only \$50k left, only $5/6$ of it is bought and each voter pays \$50k as the following table indicates.

	A	B	C	D	E	F
ρ	–	$1/5$	$1/5$	$1/4$	$1/4$	$1/3$
α	1		$5/6$			

The following table depicts the remaining amount of money of each voter.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
b_i	50	0	0	0	0	50	100	100	100	50

Since voters v_2 to v_5 run out of money, in the third round project B has only one vote and project E has 3 votes instead of 4. Hence, project D with 4 votes is selected.

	A	B	C	D	E	F
ρ	–	$1/1$	$1/1$	$1/4$	$1/3$	$1/2$
α	1		$5/6$	$5/6$		

For buying project D all voters pay the maximal equal price: \$50k and their remaining budget appears below:

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
b_i	50	0	0	0	0	50	50	50	50	0

In the fourth round, project D lost one vote as v_{10} run out of money, but still has the highest number of votes (ex aequo with project E). Hence, Fractional Equal Shares buys the remaining $1/6$ of project D asking voters v_7 to v_9 to pay $\$13.\bar{3}$ each.

	A	B	C	D	E	F
ρ	–	$1/1$	–	$1/3$	$1/3$	$1/2$
α	1		$5/6$	1		

After the purchase, the remaining funds are as follows.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
b_i	50	0	0	0	0	50	$36.\bar{6}$	$36.\bar{6}$	$36.\bar{6}$	0

Since the fourth round did not change the numbers of supporters as no new voters ran out of budget, a fraction of $11/17$ of project E is bought next.

	A	B	C	D	E	F
ρ	–	$1/1$	–	–	$1/3$	$1/2$
α	1		$5/6$	1	$11/17$	

Regarding voters' currently available budget, only v_1 and v_6 have a positive amount, each still having \$50k.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
b_i	50	0	0	0	0	50	0	0	0	0

As a result, projects B and F have one supporter each whose money can be used to buy a portion of one of these projects. Assume $1/2$ of project F is bought and v_6 pays for it.

	A	B	C	D	E	F
ρ	–	$1/1$	–	–	–	$1/1$
α	1		$5/6$	1	$11/17$	$1/2$

Finally, the only voter with positive amount of money is v_1 who is left with \$50k. However, the only project that v_1 supports has already been bought. Thus, Fraction Equal Shares concludes.

As a result, in our example, FrES allocated \$550k to projects A–C and \$400k to projects D–F, while the Method of Equal Shares allocated \$300k to the former and \$410k to the latter. In what follows, we will propose a modification of Equal Shares that is more aligned with the outcomes of FrES and in this example spends more funds on projects A–C than on D–F.

B.3 BOS Equal Shares

Consider the first round. At the beginning, each voter has the same amount of money:

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
b_i	100	100	100	100	100	100	100	100	100	100

Note that the cost of every project can be covered by its supporters. As observed in Example 2, for every project, $\alpha = 1$ is optimal, which corresponds to buying the project in full.

	A	B	C	D	E	F
$\alpha = 1, \rho$	$1/6$	$1/5$	$1/5$	$1/4$	$1/4$	$1/3$

Hence, BOS, as the Method of Equal Shares, selects project A, being the one with the highest number of votes and equally distributes its cost among the supporters. After the purchase, voters have the following amount of money:

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
b_i	50	50	50	50	50	50	100	100	100	100

In the second round the remaining budget is \$700k. Reasonable values of α and the corresponding ρ are as follows:

	A	B	C	D	E	F
$\alpha = 1, \rho$	–	–	$1/3$	$1/4$	$1/4$	$1/3$
$\alpha = 5/6, \rho$	–	–	$1/5$	$1/4$	$1/4$	$1/3$
$\alpha = 5/8, \rho$	–	$1/5$	$1/5$	$1/4$	$1/4$	$1/3$

Since voters v_1 to v_6 have only \$50k each, project B cannot be purchased in full. However, its supporters can cover $\alpha = 250/400 = 5/8$ of its cost. This cost would be equally spread among five voters, so $\rho = 1/5$ and the ratio (ρ scaled by α) equals $\rho/\alpha = 8/25$.

Now, let us consider project C. This project can be bought in full, but only if voter v_{10} pays $\rho = 1/3$ of the cost. The Method of Equal Shares rejects this option as imbalanced and selects a project with a smaller ρ parameter. In turn, Bounded Overspending considers also buying a fraction of the project with balanced payments. Specifically, to maintain equal payments, $\alpha = 5/6$ of the project can be bought. This would result in $\rho = 1/5$ and ratio $\rho/\alpha = 6/25$. Since this ratio is smaller than the ratio of project B and the ratio of projects D, E and F which remain unchanged, Bounded Overspending selects project C with $\alpha = 5/6$. To cover $5/6$ of its cost, its supporters would have to pay \$50k each. Hence, in Bounded Overspending, each voter pays $6/5 \cdot \$50\text{k} = \$60\text{k}$ (or their whole budget).

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
b_i	50	0 (-10)	0 (-10)	0 (-10)	0 (-10)	50	100	100	100	40

In the third round, the remaining budget is \$400k. Consider the following α and ρ values:

	A	B	C	D	E	F
$\alpha = 1, \rho$	–	–	–	$5/18$	$1/3$	$1/3$
$\alpha = 2/3, \rho$	–	–	–	$1/4$	$1/3$	$1/3$
$\alpha = 1/8, \rho$	–	$1/1$	–	$1/4$	$1/3$	$1/3$

Note that only $1/8$ of the project B can be bought (with $\rho = 1$). Project E and F can still be bough in full, but since voter v_2 run out of money, the value of ρ for project E dropped to $1/3$. Now, project D can either be bought in full (with unequal payments $(66.\bar{6}, 66.\bar{6}, 66.\bar{6}, 40)$ and $\rho = 66.\bar{6}/240 = \frac{5}{18}$) or in part: for $\alpha = 2/3$ with equal payments $(40, 40, 40, 40)$ and $\rho = \frac{1}{4}$. Since the ratio ρ/α for the former option is lower and it is also lower than the ratio for other projects, Bounded Overspending selects project D with unequal payments.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
b_i	50	0 (-10)	0 (-10)	0 (-10)	0 (-10)	50	$33.\bar{3}$	$33.\bar{3}$	$33.\bar{3}$	0

In the fifth round, the remaining budget is \$160k. Consider the following values of α and ρ:

	A	B	C	D	E	F
$\alpha = 1, \rho$	–	–	–	–	–	–
$\alpha = 5/6, \rho$	–	–	–	–	–	$3/5$
$\alpha = 2/3, \rho$	–	–	–	–	–	$1/2$

Hence, only project F can be bought which is done for $\alpha = {}^{5}\!/\!{}_{6}$, $\rho = {}^{3}\!/\!{}_{5}$ (voter v_6 pays 50 out of $83.\bar{3}$ that supporters can cover) and ratio $\rho/\alpha = \frac{18}{25}$. As a result, the remaining budget is \$60k, no project can be added and BOS terminates.

Appendix C. Proofs Omitted from the Main Text

Theorem 1. *FrES satisfies Fractional EJR.*

Proof. Consider an $(T, \beta, \gamma)-$cohesive set of voters S. We will prove that there exists a voter in S for which the outcome of FrES results in the satisfaction of at least $\tilde{\gamma} = \sum_{c \in T} \min_{v_i \in S} u_i(c)\beta(c) \geqslant \sum_{c \in T} \gamma(c)$. Without loss of generality, let us assume that $\gamma(c) > 0$ for every $c \in T$. Our proof follows a similar strategy to the proof that MES satisfies EJR up-to-one [23]. We examine runs of the following variants of FrES:

(A) FrES in the original instance and formulated as in Algorithm 1.

(B) FrES where voters from S can go with their money b_i below zero when paying for projects in T (so they do not have a budget constraint when they pay for these projects, but they do have when paying for the rest), and each project $c \in T$ can be bought up to an amount of $\beta(c)$.

(C) FrES in the instance truncated to the voters from S and projects from T. The utility of a voter v_i from a project c is set to $u_i(c) = \min_{j \in S} u_j(c)$. Additionally, each voter from S can go with their money b_i below zero and each project $c \in T$ can be bought up to an amount of $\beta(c)$.

For each variant of the rule $\mathcal{R}$ and each iteration t, we denote the value of the variables from Algorithm 1 after t by putting $\mathcal{R}$ and t as superscripts; for example, $b_i^{A,t}$ or $W_c^{B,t}$. The initial values are indicated with $t = 0$, while the final values are written without an iteration number.

Observe that in variants (B) and (C) the utility of all voters from S towards projects in T is strictly positive and no voters from S have budget constraint for these projects. Thus, at the end of those procedures each project c from T will be bought up to $\beta(c)$. If at the end of (B) no voter v_i from S has negative b_i, then the solution of (B) coincides with the solution of (A). Thus, in such a case, for every project $c \in T$, at least $\beta(c)$ of this project will be bought in W^A. Therefore, every voter v_i in S would have a total satisfaction of at least

$$\sum_{c \in T} u_i(c)\beta(c) \geqslant \sum_{c \in T} \min_{i \in S} u_i(c)\beta(c) = \tilde{\gamma}.$$

Hence, in the remainder of the proof, we will consider the case in which a voter from S overspent in (B) and denote the first such a voter by v_i.

Let $f_A(x)$ be the total amount of money spent by v_i during the execution of variant (A) at the point when v_i has exactly the utility of x. Formally, for every $x \in (0, u_i(W^A)]$,

$$f_A(x) = (b/n - b_i^{A,t-1}) + \rho^{A,t}(x - u_i(W^{A,t-1})), \quad \text{where } t \text{ is such that } x \in (u_i(W^{A,t-1}), u_i(W^{A,t})].$$

Here, $(b/n - b_i^{A,t-1})$ is the money spent in the first $t-1$ rounds and $\rho^{A,t}(x - u_i(W^{A,t-1}))$ is the money spent in the t-th round just to the point of obtaining utility x. Let us analogously define $f_B(x)$ and $f_C(x)$. In what follows we will prove that (1) $f_C(\tilde{\gamma}) \leqslant b/n$, (2) $f_B(x) \leqslant f_C(x)$ for every $x \leqslant \tilde{\gamma}$, and (3) $f_B(x) = f_A(x)$ for x such that $f_B(x) = b/n$. Together, these three statements will imply the thesis.

(1) Since in (C) the voters do not have budget constraints we buy all candidates from T, up to the level of β in increasing order of ρ. In every iteration $t \in [|T|]$, we buy candidate $c^{C,t}$ with $\rho^{C,t}$ equal to :

$$\rho^{C,t} = \frac{\text{cost}(c^{C,t})}{\sum_{v_j \in S} \min_{v_j \in S} u_j(c)} = \frac{\text{cost}(c^{C,t})}{|S| \min_{v_j \in S} u_j(c)}.$$

As we buy $\beta(c^{C,t})$ fraction of this candidate, we have

$$b_i^{C,t} - b_i^{C,t-1} = \beta(c^{C,t}) \cdot \min_{v_j \in S} u_j(c) \cdot \rho = \beta(c^{C,t}) \cdot \frac{\text{cost}(c^{C,t})}{|S|}.$$

Summing this up for all $t \in [|T|]$, we have

$$b_i^{C,|T|} - b_i^{C,0} = \frac{1}{|S|} \sum_{c \in T} \beta(c) \cdot \text{cost}(c),$$

which by the first assumption of Definition 1, is not greater than b/n. Since at the end of iteration $|T|$ in (C), the utility of voter v_i is exactly $\tilde{\gamma}$, we have that indeed, $f_C(\tilde{\gamma}) \leqslant b/n$.

(2) To show that $f_B(x) \leqslant f_C(x)$ for every $x \leqslant \tilde{\gamma}$, we will show that the left derivative of $f_B(x)$ is not greater than that of $f_C(x)$, for every such x. Intuitively, this means that under (B) the money of voter v_i is spent at least as effectively as under (C). Observe that the left derivative of $f_B(x)$ is just $\rho^{B,t}$ for t such that $x \in (u_i(W^{B,t-1}), u_i(W^{B,t})]$ and analogously for $f_C(x)$. Thus, for a contradiction, assume that there is $x \leqslant \tilde{\gamma}, t$, and t' such that $x \in (u_i(W^{B,t-1}), u_i(W^{B,t})]$, $x \in (u_i(W^{C,t'-1}), u_i(W^{C,t'})]$, and $\rho^{B,t} > \rho^{C,t'}$. Observe that for each candidate $c \in T$, its ρ in variant (B) is not greater than in variant (C) as utilities for c of voters in S can only be larger than in (C) and additionally, other voters can also pay for it. Thus, since $\rho^{B,t} > \rho^{C,t'}$, this means that at iteration t in (B) candidate $c^{C,t'}$ is already bought up to $\beta(c^{C,t'})$ (otherwise we should buy candidate $c^{C,t'}$ as it has better ρ). Moreover, for every $t'' < t'$ we have that $\rho^{B,t} > \rho^{C,t'} > \rho^{C,t''}$, thus candidate $c^{C,t''}$ is also already bought up to $\beta(c^{C,t''})$ at the start of the iteration t in (B). However, this allows us to bound the utility of v_i in iteration t in (B) as follows

$$x > \sum_{t'' \in [t']} \beta(c^{C,t''}) \cdot u_i(c^{C,t''}) \geqslant \sum_{t'' \in [t']} \beta(c^{C,t''}) \cdot \min_{v_j \in S} u_j(c^{C,t''}) \geqslant x,$$

where the last inequality holds as v_i obtains utility x during iteration t' in (C). But this is a contradiction.

(3) Finally, let t be the first iteration in which the money of voter v_i crossed 0, and let $p \in T$ be a project that was selected in this iteration. Observe that in all iterations up to t, the executions of (A) and (B) coincided. Moreover, unless $b_i^{B,t-1}$ is exactly zero, in iteration t project p was also selected by (A) as its ρ and ρs of all other projects were the same. The difference is that in (A) the smaller fraction of p was bought so as to leave v_i with exactly zero money. Thus, $f_B(x) = f_A(x)$ for x such that $f_B(x) = b/n$. Moreover, if such x would be smaller than $\tilde{\gamma}$, then from the fact that f_B is strictly increasing we would have that $f_B(x) < f_B(\tilde{\gamma}) \leqslant f_C(\tilde{\gamma}) \leqslant b/n$, which is a contradiction. Therefore, the utility of v_i at the end of iteration t in (A) is at least $\tilde{\gamma}$ and at the end of the algorithm it cannot be smaller, which concludes the proof. □

Theorem 2. *BOS runs in polynomial time.*

Proof. It is enough to argue that in each round, the next project can be selected in polynomial time, or equivalently, that we can find parameters ρ and α that minimize ρ/α in polynomial time.

Consider a fixed round of BOS, and assume each voter has a budget b_i. Let S be the set of voters who still have money left. Fix a not-yet-elected candidate c that has supporters in S (otherwise, it is not (α, ρ)-affordable for any finite ρ). If c can be fully funded by voters from S with payments proportional to their utilities, then clearly, the optimal α equals 1, and $\rho = \text{cost}(c)/\sum_{v_i \in S} u_i(c)$.

Assume otherwise. Consider a sequence $(v_{i_1}, \dots, v_{i_\ell})$ of supporters of c who still have money left, sorted in the ascending order by $b_i/u_i(c)$. Note that if we gradually increase the fraction α

of the project that we fund, and for each such value of α minimize the value of ρ, then the first voter to run out of money would be v_{i_1}, the second (or possibly ex-aequo first) would be v_{i_2}, and so on. Let α_j be the fraction of the project for which voter v_{i_j} runs out of money (or $\alpha_j = 1$ if they do not). Assume $\alpha_0 = 0$ and consider buying a fraction $\alpha \in [\alpha_j, \alpha_{j+1}]$ of the project for some $j \in \{0, \dots, \ell - 1\}$. We argue that α equal to α_j or α_{j+1} minimizes the ratio ρ/α on this interval.

For $\alpha \in [\alpha_j, \alpha_{j+1}]$, we know that each voter $v_i \in \{v_{i_1}, \dots, v_{i_j}\}$ would pay b_i, and each voter $v_i \in \{v_{i_{j+1}}, \dots, v_{i_\ell}\}$ would pay $\alpha \cdot u_i(c) \cdot \rho$. Hence, we have $\alpha \cdot \mathrm{cost}(c) = \sum_{h=1}^{j} b_{i_h} + \sum_{h=j+1}^{\ell} \alpha \cdot u_{i_h}(c) \cdot \rho$, and, therefore

$$\rho/\alpha = \frac{\alpha \cdot \mathrm{cost}(c) - \sum_{h=1}^{j} b_{i_h}}{\alpha^2 \cdot \sum_{h=j+1}^{\ell} u_{i_h}(c)}.$$

Taking the derivative with respect to α, we get that

$$\frac{d}{d\alpha}(\rho/\alpha) = \frac{2\sum_{h=1}^{j} b_{i_h} - \alpha \mathrm{cost}(c)}{\alpha^3 \sum_{h=j+1}^{\ell} u_{i_h}(c)}.$$

Since $\mathrm{cost}(c) > 0$, this function has only one zero at $\alpha^* = 2\sum_{h=1}^{j} b_{i_h}/\mathrm{cost}(c)$, and it is positive for $\alpha < \alpha^*$ and negative for $\alpha > \alpha^*$. This implies that ρ/α is increasing for $\alpha < \alpha^*$ and decreasing for $\alpha > \alpha^*$. Hence, for any interval, the minimum value is at one of its ends, and so ρ/α is minimal for $\alpha \in \{\alpha_j, \alpha_{j+1}\}$. Note that for the interval $[\alpha_0, \alpha_1]$ all payments are proportional to their utilities, hence ρ is a constant and ρ/α is maximized for $\alpha = \alpha_1$.

This shows that the fraction α that minimizes ρ/α belongs to the set $\{\alpha_1, \dots, \alpha_\ell\} \cup \{1\}$. If $\alpha = \alpha_j < 1$ for some $i_j \in \{1, \dots, \ell\}$, then we get $\rho = b_{i_j}/u_{i_j}(c)$. On the other hand, if $\alpha = 1$, then ρ can be computed as in the Method of Equal Shares. □

Theorem 3. *For cost utilities, BOS satisfies EJR up to* $\frac{n-|S|}{2|S|} \cdot c^*$, *where* $c^* := \max_{c \in C} \mathrm{cost}(c)$.

Proof. Consider a subset of voters $S \subseteq V$ and a subset of candidates $T \subseteq C$ such that S deserves the satisfaction of $\mathrm{cost}(T)$. Towards a contradiction, assume that there exists a candidate $\hat{c} \in T \setminus W$ such that for all voters $v_i \in S$ it holds that $u_i(W) < \mathrm{cost}(T) - \frac{n-|S|}{2|S|} \cdot c^* - \mathrm{cost}(\hat{c})$.

Claim 6. *If there is a candidate* $\hat{c} \in T \setminus W$ *such that for all* $v_i \in S$ *it holds that* $u_i(W) < \mathrm{cost}(T) - \frac{n-|S|}{2|S|} \cdot c^* - \mathrm{cost}(\hat{c})$*, then, in every step of the execution of BOS, it holds* $\frac{\rho}{\alpha} \leqslant \frac{1}{|S|}$.

Proof. Towards a contradiction, suppose that in some round r the rule selected a candidate c with $\rho/\alpha > 1/|S|$. Assume that r was the first round when this happened. Thus, all previous candidates were bought with $\rho/\alpha \leqslant 1/|S|$. Consider a candidate $\hat{c} \in T \setminus W$. At the time c was bought, there was a voter from S, say v_i who had spent strictly more than $b/n - \mathrm{cost}(\hat{c})/|S|$, as otherwise the voters from S would altogether have at least $|S| \cdot (\mathrm{cost}(\hat{c})/|S|)$ money left, and so $\hat{c}$ could have been bought for $\alpha = 1$ and $\rho = 1/|S|$, i.e., for $\rho/\alpha = 1/|S|$. We now compute the satisfaction of voter v_i with respect to W. By the fact that any purchase prior to c was done for $\rho \leqslant 1/|S|$, which is due to the fact that α is upper bounded by 1, we have that the satisfaction of v_i whenever paying p was increasing by at least $|S| \cdot p$. Therefore, the satisfaction of v_i at the considered step was

$$u_i(W) \geqslant |S| \left(\frac{b}{n} - \frac{\mathrm{cost}(\hat{c})}{|S|} \right) \geqslant \mathrm{cost}(T) - \mathrm{cost}(\hat{c}),$$

which contradicts the fact that there are no such voters. □

Claim 7. *Consider a round when some voters overspend. The number of voters that exhaust their budgets in this round is strictly larger than half of the number of all voters who paid for a candidate in this round.*

Proof. Consider a round of the BOS procedure in which a candidate c is selected with certain α and ρ. Let R be the set of all voters that paid for c. The voters from R can be partitioned into two subsets: those who either overspent their budget or spent all of their available funds on c, i.e., they exhausted their budgets in this round, and the remaining voters—those who will still have money to spend in future rounds. Let us denote the first subset by R_- and the second by R_+. Clearly $R = R_- \cup R_+$ and both subsets are disjoint. In what follows, we will show that $|R_-| > |R|/2$.

If $R_+ = \varnothing$, the thesis holds trivially, thus let us assume that $|R_+| > 0$. Observe that every voter in R_+ pays exactly the same amount in this round (as we assumed cost utilities and their budgets are not finished). Let us denote the amount each of them pays for an α fraction of c by p, i.e., they pay p/α altogether. Since the voters in R_+ pay the highest price, we get $\rho = \frac{p}{\alpha \mathrm{cost(c)}}$ and $\frac{\rho}{\alpha} = \frac{p}{\alpha^2 \mathrm{cost(c)}}$.

Now, consider buying candidate c with alternative $\bar{\alpha}$ and $\bar{\rho}$ which we obtain by making voters in R_+ pay $p + \beta$ instead of p for a fraction of c, where $\beta > 0$ is small enough that each $v_i \in R_+$ has at least $(p + \beta)/\alpha$ money (since after paying p/α they still had a positive amount of money, there is such β). This means that $\bar{\alpha} \geqslant \alpha + \beta \cdot |R_+|/\mathrm{cost(c)}$, as we account for the additional payments. Also, $\bar{\rho} = \frac{p+\beta}{\bar{\alpha}\mathrm{cost(c)}}$. Since BOS has chosen to buy project c with α and ρ, we get

$$\frac{p}{\alpha^2 \mathrm{cost(c)}} = \frac{\rho}{\alpha} \leqslant \frac{\bar{\rho}}{\bar{\alpha}} = \frac{p+\beta}{\bar{\alpha}^2 \mathrm{cost(c)}} \leqslant \frac{p+\beta}{(\alpha + \beta \cdot |R_+|/\mathrm{cost(c)})^2 \mathrm{cost(c)}}.$$

After rearrangement we obtain

$$(\alpha \mathrm{cost(c)} + \beta |R_+|)^2 \leqslant \alpha^2 \mathrm{cost(c)}^2 \frac{(p+\beta)}{p} = \alpha^2 \mathrm{cost(c)}^2 (1 + \beta/p).$$

Observe that the equal payments in R would result in every voter paying $\alpha \mathrm{cost(c)}/|R|$ for the α fraction of c. As voters in R_+ pay the largest share, we get $p \geqslant \alpha \mathrm{cost(c)}/|R|$. Thus, we have that $1 + \frac{\beta}{p} \leqslant \frac{|R|\beta + \alpha \mathrm{cost(c)}}{\alpha \mathrm{cost(c)}}$. Therefore,

$$\alpha^2 \mathrm{cost(c)}^2 + 2\alpha \mathrm{cost(c)} \beta |R_+| + \beta^2 |R_+|^2 = (\alpha \mathrm{cost(c)} + \beta |R_+|)^2 \leqslant \alpha \mathrm{cost(c)} (|R|\beta + \alpha \mathrm{cost(c)}).$$

As $\alpha \mathrm{cost(c)} > 0$ and $\beta > 0$, we divide by $\alpha \mathrm{cost(c)}$, subtract $\alpha \mathrm{cost(c)}$, and divide by β, which yields

$$2|R_+| + \frac{\beta |R_+|^2}{\alpha \mathrm{cost(c)}} \leqslant |R| \Leftrightarrow \frac{\beta |R_+|^2}{\alpha \mathrm{cost(c)}} \leqslant |R| - 2|R_+| = 2|R_-| - |R|.$$

Since the left-hand side is strictly positive, we get that $|R_-| > |R|/2$, which concludes the proof. □

For every voter $v_i \in V$, let us denote the overspending of v_i by $\delta_i = -\min(0, b_i - u_i(c^*) \cdot \rho^*)$, where c^* is the candidate chosen in the last round in which v_i was paying, ρ^* is ρ in that round, and b_i is the remaining budget of v_i at the beginning of that round.

Claim 8. *If there is a candidate $\hat{c} \in T \setminus W$ such that for all $v_i \in S$ it holds that $u_i(W) < \mathrm{cost}(T) - \frac{n-|S|}{2|S|} \cdot c^* - \mathrm{cost}(\hat{c})$, then, the total overspending is bounded by $\sum_{v_j \in V} \delta_j \leqslant \frac{n-|S|}{2|S|} \cdot c^*$.*

Proof. First, observe that the voters from S will never overspend, i.e., $\delta_i = 0$, for every $v_i \in S$. Indeed, from Claim 6 we know that $\rho \leqslant 1/|S|$ in every round, which combined with the fact that $u_i(W) < \mathrm{cost}(T)$, bounds the total amount of money that voter v_i spends by

$$u_i(W)\rho < \mathrm{cost}(T)\rho \leqslant \mathrm{cost}(T) \cdot (1/|S|) \leqslant b/n.$$

Now, consider a subset of voters, $R \subseteq V$, that pay for candidate c selected with α and ρ in a certain round of BOS. In what follows, we will bound the overspending in this round. From the proof of Claim 7 and using the notation from there we know that $\frac{\rho}{\alpha} = \frac{p}{\alpha^2 \mathrm{cost(c)}}$ and $p \geqslant \frac{\alpha \mathrm{cost(c)}}{|R|}$. Then, Claim 6 yields $|S| \leqslant \frac{\alpha}{\rho} = \frac{\alpha^2 \mathrm{cost(c)}}{p} \leqslant \alpha|R|$. Hence, $\alpha \geqslant |S|/|R|$. On the other hand, observe that the overspending of voters in $R_{\_}$ (so all voters that exhaust their funds in this round) cannot be greater than what remains after buying α fraction of c, i.e.,

$$\sum_{v_i \in R_{\_}} \delta_i \leqslant (1-\alpha)\mathrm{cost(c)} \leqslant c^*(1 - |S|/|R|).$$

Then, by Claim 7, the average overspending of voters in $R_{\_}$ is at most

$$\sum_{v_i \in R_{\_}} \frac{\delta_i}{|R_{\_}|} \leqslant \frac{c^*(|R|-|S|)/|R|}{|R|/2} = \frac{2c^*(|R|-|S|)}{|R|^2} \leqslant \frac{c^*}{2|S|},$$

where the last inequality follows from the fact that $|R|^2 - 4|S| \cdot |R| + 4|S|^2 = (|R| - 2|S|)^2 \geqslant 0$. Thus, the average overspending in $V \setminus S$ is also bounded by $c^*/(2|S|)$, which concludes the proof. $\square$

Recall that $\hat{c} \in T \setminus W$ is such that $u_i(W) < \mathrm{cost}(T) - \frac{n-|S|}{2|S|} \cdot c^* - \mathrm{cost}(\hat{c})$, for every $v_i \in S$. Observe that the voters in $V \setminus S$ can spend in total at most $(n - |S|) \cdot b/n$ from their initial budgets and, by Claim 8, $\frac{n-|S|}{2|S|} c^*$ from overspending. Thus, voters from S must have spent at least

$$b - \left(\frac{n-|S|}{n} b + \frac{n-|S|}{2|S|} c^*\right) - \mathrm{cost}(\hat{c}) = \frac{|S|}{n} b - \frac{n-|S|}{2|S|} c^* - \mathrm{cost}(\hat{c}) \geqslant \mathrm{cost}(T) - \frac{(n-|S|)c^*}{2|S|} - \mathrm{cost}(\hat{c})$$

as otherwise they could afford to buy $\hat{c}$.

From Claim 6, we know that any payment of p for a voter in S results in a satisfaction of at least $|S| \cdot p$. Thus, there must exist a voter in S with a satisfaction of at least

$$\frac{1}{|S|} \cdot \left(\mathrm{cost}(T) - \frac{n-|S|}{2|S|} c^* - \mathrm{cost}(\hat{c})\right) \cdot \frac{1}{\rho} \geqslant \mathrm{cost}(T) - \frac{n-|S|}{2|S|} c^* - \mathrm{cost}(\hat{c}).$$

This gives a contradiction and concludes the proof of the theorem. $\square$

Proposition 4. *For each ℓ there exists an approval-based committee election where a group of voters deserves $\left\lceil \frac{k-\ell}{4\ell} \right\rceil$ candidates, but they all get no representatives under BOS.*

Proof. Fix $\ell \geqslant 1$ and consider the following instance of approval-based committee elections with $|V| = 4\ell^2 + \ell$ voters and $|C| = 4\ell^2 + 2\ell$ candidates; the available budget is equal to $b = 4\ell^2 + \ell$. The preferences of the voters are as follows:

- The first ℓ voters, referred to as group V_1, approve all ℓ candidates from $C_1 = \{c_1, c_2, \ldots, c_\ell\}$.
- The remaining $4\ell^2$ voters, referred to as group V_2, all approve $4\ell^2 - \ell$ common candidates from $C_2 = \{c_{\ell+1}, c_{\ell+2}, \ldots, c_{4\ell^2}\}$. These $4\ell^2$ voters are further divided into 2ℓ subgroups, $V_2^{(1)}, V_2^{(2)}, \ldots, V_2^{(2\ell)}$, each consisting of 2ℓ voters who each approve one of the remaining 2ℓ candidates. Let us denote the set of these candidates as $C_3 = \{c_{4\ell^2+1}, c_{4\ell^2+2}, \ldots, c_{4\ell^2+2\ell}\}$.

According to the EJR, the voters in V_1 are entitled to ℓ candidates, since they all approve the same set of ℓ candidates and $|V_1| \geqslant \ell \cdot {}^{|V|}/_{b} = \ell$. Further, $\ell = \left\lceil \frac{b-\ell}{4\ell} \right\rceil$. However, we will demonstrate that the outcome of BOS will not include any candidates from C_1.

Each voter starts with a unit budget. A candidate from C_1 can be purchased with $\rho = 1/\ell$ and $\alpha = 1$. Similarly, a candidate from C_2 can be bought with $\rho = 1/4\ell^2$ and $\alpha = 1$, and a candidate from C_3 with $\rho = 1/2\ell$ and $\alpha = 1$. Selecting all $4\ell^2 - \ell$ candidates from C_2 leaves each voter in V_2 with a remaining budget of:

$$1 - (4\ell^2 - \ell) \cdot \frac{1}{4\ell^2} = \frac{1}{4\ell}.$$

At this stage, a candidate from C_3 can be purchased with $\rho = 1/2\ell$ and $\alpha = 1/2$, which results in $\rho/\alpha = 1/\ell$. Therefore, BOS may choose any candidate from C_1 or C_3. If we break ties in favor of C_3, then ultimately, BOS will select all candidates from C_2 and C_3. □

Theorem 5. *For cost utilities, BOS+ satisfies EJR up to* $\frac{n-|S|}{2|S|} \cdot c^*$, *where* $c^* := \max_{c \in C} \mathrm{cost}(c)$.

Proof. The proof follows a similar strategy to that of Theorem 3. Note that both BOS and BOS+ initially identify the best-affordable project c in the same way. The key difference is that BOS+ includes an additional step in which the budget is temporarily increased. As a result, the part of the proof concerning the identification of the best-affordable project c can be directly reused. In particular, this applies to Claims 6 and 7. Additionally, a significant portion of the proof of Claim 8 remains applicable. Specifically, as in the proof of Claim 8, we obtain that

$$\alpha \geqslant \frac{|S|}{|R|}.$$

Now, we can bound the temporary increase in the voters' budgets, as

$$\Delta b = \frac{\mathrm{cost}(c) - \alpha \mathrm{cost}(c)}{|\{v_i \in V \colon \rho u_i(c) \geqslant b_i > 0\}|} = \frac{\mathrm{cost}(c)(1-\alpha)}{|R_-|}.$$

By Claim 7 we further get that $|R_-| > |R|/2$ and thus

$$\Delta b \leqslant \frac{2\mathrm{cost}(c)(1-\alpha)}{|R|} \leqslant \frac{2\mathrm{cost}(c)(1-\frac{|S|}{|R|})}{|R|} = \frac{2\mathrm{cost}(c)(|R|-|S|)}{|R|^2} \leqslant \frac{c^*}{2|S|},$$

where the inequality follows simply by the fact that $|R|^2 - 4|S| \cdot |R| + 4|S|^2 \geqslant 0$.

As with BOS, we came to the conclusion that the overall overspending of the voters is upper-bounded by $\frac{n-|S|}{2|S|} c^*$. From there the proof follows the same way as the proof of Theorem 3.
□

Appendix D. Additional Empirical Results

In Table 2, we report the aggregated statistics for each of our rules based on their performance in the Pabulib real-world instances. Additionally, we present the values of these statistics when limited to the instances with a particular type of ballots used for voting. Pabulib distinguishes four such ballot types: *approval ballots*, where a voter indicates a subset of project he or she approves, *choose-1 ballots*, in which a voter must select exactly one project, *cumulative ballots*—a voter distributes a number of points between the projects, and *ordinal ballots*, where a voter provides an ordering of (a subset of) projects—we use *Borda scores* to transform them to utilities.

In Table 3, we report the mean values of statistics presented in the first three plots of Figure 1 in Section 6 together with standard deviations and five quantiles. In the same table, we also present the p-values for the significance of the differences between the average values of statistics for each pair of rules. We note that they are very small, rarely exceeding 0.05, which means that almost all the observed differences between the behavior of the rules are statistically significant. In particular, we note that BOS has a lower average exclusion ratio than the original Method of Equal Shares for every instance size and the difference is significant at the 0.01 level.

Finally, in Figure 3, we present the full results of the experiment on the Euclidean elections. Recall that we have defined the utility of voter v_i for candidate c as $u_i(c) = (\mathrm{dist}(v_i, c) + \lambda)^{-1}$, where $\mathrm{dist}(v_i, c)$ is the Euclidean distance between them, and the denominator is shifted by the constant of λ to bound the maximal utility. In the main body of the paper, we presented results for $\lambda = 1$. Here, we also include results for $\lambda = 2$ and $\lambda = 0.5$. Note that taking $\lambda = 2$ instead of $\lambda = 1$ leads to exactly the same outcomes as multiplying all distances by 0.5 since all of the considered rules are invariant towards scaling all the utilities by a constant. Similarly, taking $\lambda = 0.5$ instead of $\lambda = 1$ is equivalent to doubling the distances.

Metric	Util.	Eq. Shares	BOS	BOS+	FrES
Avg. rel. cost satisfaction	1.000	0.836	0.903	0.909	0.921
instances with approval ballots	1.000	0.846	0.890	0.900	0.960
instances with choose-1 ballots	1.000	0.908	0.984	0.984	0.825
instances with cumulative ballots	1.000	0.751	0.890	0.887	0.885
instances with ordinal ballots	1.000	0.821	0.896	0.902	0.885
Avg. rel. score satisfaction	1.000	1.201	1.160	1.169	1.029
instances with approval ballots	1.000	1.286	1.201	1.215	1.052
instances with choose-1 ballots	1.000	0.964	1.004	1.004	0.855
instances with cumulative ballots	1.000	1.099	1.138	1.140	1.070
instances with ordinal ballots	1.000	1.187	1.164	1.171	1.060
Avg. exclusion ratio	19.85%	17.62%	16.16%	16.50%	0.00%
instances with approval ballots	13.92%	11.86%	10.36%	10.88%	0.00%
instances with choose-1 ballots	41.36%	43.37%	41.19%	41.19%	0.00%
instances with cumulative ballots	29.57%	27.12%	24.90%	24.89%	0.00%
instances with ordinal ballots	12.38%	5.10%	5.54%	5.85%	0.00%
Avg. running time in sec.	0.001	6.822	0.086	0.263	2.151
instances with approval ballots	0.001	10.585	0.120	0.382	3.296
instances with choose-1 ballots	0.000	1.087	0.044	0.116	0.028
instances with cumulative ballots	0.000	0.751	0.014	0.031	0.060
instances with ordinal ballots	0.000	2.914	0.060	0.151	1.676
Avg. EJR+ violations	0.953	0.000	0.061	0.060	0.000
instances with approval ballots	1.138	0.000	0.059	0.058	0.000
instances with choose-1 ballots	0.141	0.000	0.071	0.071	0.000
EJR+ violation instances	26.20%	0.00%	4.59%	4.48%	0.00%
instances with approval ballots	29.62%	0.00%	4.29%	4.16%	0.00%
instances with choose-1 ballots	11.18%	0.00%	5.88%	5.88%	0.00%
Avg. budget spending	96.47%	44.84%†	93.98%	93.95%	93.40%†
instances with approval ballots	96.09%	53.86%†	93.20%	93.16%	92.51%†
instances with choose-1 ballots	97.40%	14.98%†	96.35%	96.35%	95.94%†
instances with cumulative ballots	95.79%	27.08%†	92.77%	92.77%	90.90%†
instances with ordinal ballots	98.16%	56.85%†	96.64%	96.59%	98.02%†
Exhausted budgets	100.00%	6.99%†	99.84%	99.69%	24.65%†
instances with approval ballots	100.00%	8.31%†	99.73%	99.60%	17.29%†
instances with choose-1 ballots	100.00%	4.12%†	100.00%	100.00%	65.88%†
instances with cumulative ballots	100.00%	4.00%†	100.00%	99.50%	16.00%†
instances with ordinal ballots	100.00%	7.59%†	100.00%	100.00%	25.95%†

Table 2: Aggregated statistics from running our rules on instances from Pabulib. The values for Equal Shares assume Add1U completion, except for the average budget spending and exhausted budgets. Similarly, FrES is completed in a utilitarian fashion except for these two cases. The usage of a completion method is denoted by †.

Statistics		Score satisfaction				Cost satisfaction				Exclusion ratio			
No. projects		1–8	9–16	17–28	29+	1–8	9–16	17–28	29+	1–8	9–16	17–28	29+
Util.	mean	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.206	0.209	0.178	0.199
	std	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.201	0.201	0.185	0.166
	q10	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.000	0.013	0.006	0.050
	q25	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.025	0.051	0.040	0.086
	q50	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.148	0.145	0.121	0.147
	q75	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.324	0.304	0.239	0.252
	q90	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.526	0.523	0.457	0.414
MES	mean	1.061	1.148	1.265	1.350	0.820	0.845	0.834	0.848	0.229	0.200	0.134	0.135
	std	0.463	0.620	0.625	0.635	0.302	0.230	0.177	0.123	0.227	0.204	0.155	0.155
	q10	0.860	0.955	0.996	1.038	0.264	0.429	0.575	0.671	0.000	0.005	0.002	0.019
	q25	1.000	1.000	1.000	1.114	0.770	0.777	0.735	0.816	0.022	0.047	0.019	0.041
	q50	1.000	1.000	1.115	1.257	1.000	0.970	0.890	0.886	0.164	0.136	0.072	0.081
	q75	1.000	1.075	1.281	1.475	1.000	1.000	0.970	0.927	0.395	0.291	0.190	0.159
	q90	1.114	1.324	1.573	1.682	1.000	1.000	1.000	0.960	0.573	0.500	0.398	0.302
BOS	mean	1.024	1.085	1.240	1.314	0.941	0.924	0.878	0.866	0.201	0.186	0.126	0.127
	std	0.148	0.300	0.579	0.632	0.162	0.146	0.145	0.103	0.200	0.191	0.149	0.155
	q10	0.996	0.987	0.999	1.005	0.793	0.739	0.678	0.755	0.000	0.005	0.002	0.018
	q25	1.000	1.000	1.000	1.086	1.000	0.908	0.830	0.813	0.018	0.046	0.020	0.035
	q50	1.000	1.000	1.095	1.224	1.000	1.000	0.920	0.883	0.147	0.122	0.072	0.069
	q75	1.000	1.033	1.261	1.411	1.000	1.000	0.992	0.941	0.319	0.267	0.171	0.150
	q90	1.007	1.265	1.538	1.612	1.000	1.000	1.000	0.978	0.525	0.476	0.366	0.283
BOS+	mean	1.025	1.088	1.249	1.337	0.941	0.926	0.881	0.885	0.201	0.187	0.129	0.137
	std	0.148	0.300	0.590	0.634	0.163	0.146	0.147	0.103	0.199	0.191	0.149	0.153
	q10	1.000	0.999	1.000	1.023	0.793	0.731	0.678	0.781	0.000	0.007	0.002	0.022
	q25	1.000	1.000	1.000	1.101	1.000	0.912	0.831	0.849	0.021	0.047	0.022	0.042
	q50	1.000	1.000	1.104	1.239	1.000	1.000	0.923	0.905	0.147	0.124	0.073	0.081
	q75	1.000	1.038	1.268	1.433	1.000	1.000	1.000	0.956	0.319	0.267	0.174	0.164
	q90	1.007	1.265	1.549	1.667	1.000	1.000	1.000	0.985	0.525	0.476	0.372	0.304
FrES	mean	1.037	0.996	1.048	1.038	0.973	0.943	0.904	0.857	0.000	0.000	0.000	0.000
	std	0.255	0.272	0.442	0.378	0.150	0.162	0.144	0.104	0.000	0.000	0.000	0.000
	q10	0.812	0.790	0.802	0.772	0.781	0.731	0.749	0.713	0.000	0.000	0.000	0.000
	q25	0.943	0.886	0.894	0.902	0.910	0.875	0.837	0.814	0.000	0.000	0.000	0.000
	q50	1.000	0.970	0.977	1.014	0.990	0.966	0.920	0.881	0.000	0.000	0.000	0.000
	q75	1.074	1.027	1.084	1.126	1.006	1.011	0.982	0.933	0.000	0.000	0.000	0.000
	q90	1.236	1.222	1.282	1.296	1.127	1.071	1.016	0.960	0.000	0.000	0.000	0.000
Util. vs. MES		0.008	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.017	0.000	0.000
Util. vs. BOS		0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.015	0.000	0.000	0.000
Util. vs. BOS+		0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.028	0.000	0.000	0.000
Util. vs. FrES		0.004	0.385	0.029	0.041	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
MES vs. BOS		0.064	0.019	0.024	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.000	0.000
MES vs. BOS+		0.066	0.024	0.107	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.072
MES vs. FrES		0.133	0.000	0.000	0.000	0.000	0.000	0.000	0.055	0.000	0.000	0.000	0.000
BOS vs. BOS+		0.224	0.000	0.000	0.000	0.441	0.076	0.010	0.000	0.032	0.000	0.000	0.000
BOS vs. FrES		0.139	0.000	0.000	0.000	0.004	0.036	0.003	0.115	0.000	0.000	0.000	0.000
BOS+ vs. FrES		0.148	0.000	0.000	0.000	0.004	0.053	0.008	0.000	0.000	0.000	0.000	0.000

Table 3: Detailed values of statistics from running our rules on instances from Pabulib. In the first 35 rows, for each rule, statistic, and size range, we provide the average, standard deviation, median, 1st and 3rd quartile, as well as 10th and 90th centile of the observed values of the statistic. In the last 10 rows, we present p-values for the significance of the difference between the average values for a given pair of rules.

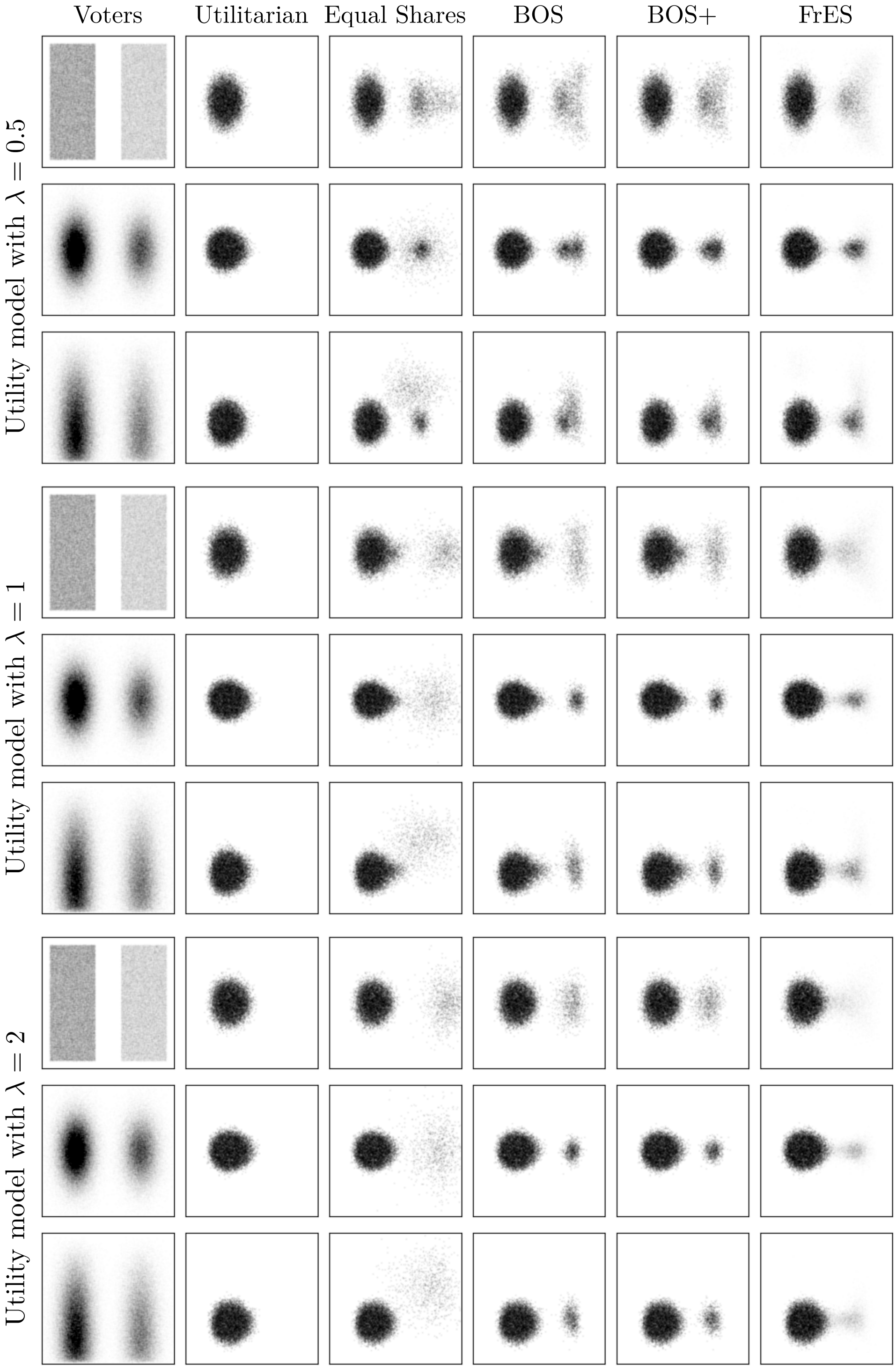

Figure 3: Results of the experiment on Euclidean elections. The first column presents the superimposed positions of voters in all 1000 generated elections. The following columns show positions of candidates selected by respective rules (for FrES, the opacity of each point is proportional to the selected fraction of the respective candidate).