On Condorcet's Jury Theorem with Abstention

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Abstract

The well-known Condorcet Jury Theorem states that, under majority rule, the better of two alternatives is chosen with probability approaching one as the population grows. We study an asymmetric setting where voters face varying participation costs and share a (possibly heuristic) belief about their pivotality.

In a costly voting setup where voters abstain if their participation cost is greater than their pivotality estimate, we identify a single property of the heuristic belief—weakly vanishing pivotality—that gives rise to multiple stable equilibria in which elections are nearly tied. In contrast, strongly vanishing pivotality (as in the standard Calculus of Voting model) yields a unique, trivial equilibrium where only zero-cost voters participate as the population grows. We then characterize when nontrivial equilibria satisfy a version of the Jury Theorem: below a sharp threshold, the majority-preferred candidate wins with probability approaching one; above it, both candidates either win with equal probability.

1 Introduction

Consider a population of N voters voting over two alternatives A and B, with A being the *better alternative* according to some pre-defined criterion. Consider further that the preference of each individual voter is determined independently by an outcome of a coin toss biased in favour of the better alternative A. That is, each individual voter supports alternative A with probability p and B with the probability $1-p$. Under this setting, the famous Condorcet's Jury Theorem (CJT) states that the majority rule selects candidate A with certainty when population size increases to infinity.

An implicit assumption in Condorcet's theorem is that *everyone votes*, or at least that the decision to vote is independent of one's preference over alternatives. In contrast, in many practical situations such as political elections, or a local or national referendum, abstention is found to be a common and prominent phenomenon. For instance, the voter turnout in US presidential elections has been around 52%-62% over the past 90 years [Martinez and Gill, 2005]. Abstention is also observed to be a significant phenomenon in lab experiments [Blais, 2000; Owen and Grofman, 1984].

From a rational, economic perspective, the surprise is not why some voters abstain, but why anyone votes at all. As Anthony Downs noted in 1957, a rational voter compares the benefit of voting—which occurs only if they are pivotal—against a fixed cost. In large electorates, the probability of being pivotal is so low that the expected benefit rarely outweighs the cost. This leads to the so-called paradox of voting [Downs, 1957], where the only equilibrium is trivial: only voters with zero (or arbitrarily small[1]) costs turn out.

While Condorcet's result holds under the assumption that everyone votes, Down's theory of rational voting predicts meagre participation in large-scale elections. Our goal in this paper is to understand the equilibria that arise from rational voting but with a flexible estimation of one's chances to be pivotal, in an attempt to reconcile the theoretical predictions with the moderate turnout rates we see in practice. We ask: (1) are there equilibria where a significant portion of the population votes? (2) how likely is the better candidate to win, in equilibrium? We are mainly interested in the answer as the size of the population grows to infinity: does the winning probability of the better candidate approach 1? Or does the paradox of voting 'kill' Condorcet's Jury theorem? We now present the standard model of voter turnout from the literature, which formalizes how voters estimate their likelihood of being pivotal.

[1] In large but finite populations, only voters with near-zero voting costs participate.

The Calculus of Voting model: Originally proposed by Downs [1957] and later developed by Riker and Ordeshook [1968], this model of rational voting attributes each voter's decision to abstain to the expected cost-benefit analysis. Let p_i denote the *pivotality* of voter i, $\mathtt{V}_i$ denote the personal benefit she receives if her preferred candidate wins an election, $\mathtt{D}_i$ denote the social benefit she receives by performing a civic duty of voting and $\mathtt{G}_i$ denote the costs of voting she incurs. These costs include the cost of obtaining and processing information and the actual cost of registering and going to polls (see also [Aldrich, 1993] for discussion of voting and rational choice). A voter i votes if and only if

$$p_i \cdot \mathtt{V}_i + \mathtt{D}_i \geq \mathtt{G}_i. \tag{1}$$

The calculus of voting model considers p_i to be the probability that that all voters except i reach a tie. The tie probabilities are derived from the aggregated stochastic votes, and thus the pivot computation and subsequent equilibrium analysis quickly become intractable.

Enter heuristics Several recent models maintain the fundamental game-theoretic approach of equilibrium among strategic voters, but relax the assumption that voters calculate their true pivot probabilities, or engage in probabilistic calculations at all. This includes minmax regret equilibrium [Ferejohn and Fiorina, 1974], sampling equilibrium [Osborne and Rubinstein, 2003], or Local-dominance equilibrium [Meir et al., 2014; Meir, 2015]. Merrill [1981] considers (as we also do later) voters maximizing expected utility but without specifying how they estimate their pivot probability.

Exit rationality Finally, there are models that suggest voting heuristics people may use, without engaging in any equilibrium analysis. A particularly simple example (not related to turnout) is the 'k-pragmatist' heuristic [Reijngoud and Endriss, 2012], and there are many others, see a recent survey in [Meir, 2018]. Some of these essentially model the decision as a function of the (estimated) margin between candidates [Bowman et al., 2014; Fairstein et al., 2019].

Our key takeaway from the long list of existing models, along the entire 'rational-to-heuristic' spectrum, is that the estimated *margin* plays a major role, in addition to the size of the population (that is not always considered). There is also empirical evidence of the connection between (narrow) margin and (high) turnout [Aldrich, 1993], and strong experimental evidence that the decision to abstain is positively correlated with a high margin and with large population [Levine and Palfrey, 2007]. Gerber et al. [2020] conducted a large field experiment, that showed people substantially over-estimate the chance of a small margin (and thus their chances of being pivotal).

It is important to mention a stream of papers that assume voters get a noisy signal of the 'truth', but actually have shared interest (sometimes called 'epistemic voting' [Coleman and Ferejohn, 1986]). In these models a voter may prefer to vote differently from her signal due to Bayesian reasoning, thereby providing additional reasons for failure of the CJT [Austen-Smith and Banks, 1996]. We discuss this in Sec. 7. However we follow the more common assumption that voters (if they vote) always follow their signal.

1.1 Our Contribution

Analyzing every model from the literature separately would be tedious and leave us with an isolated set of narrow results. Instead, we stay within the Downsian framework where voters are rational in the sense of aiming to maximize their expected utility in equilibrium, but allow a wide range of ways in which voters estimate their pivotality. The dashed rectangle in Fig. 1 shows the scope of models in our framework.

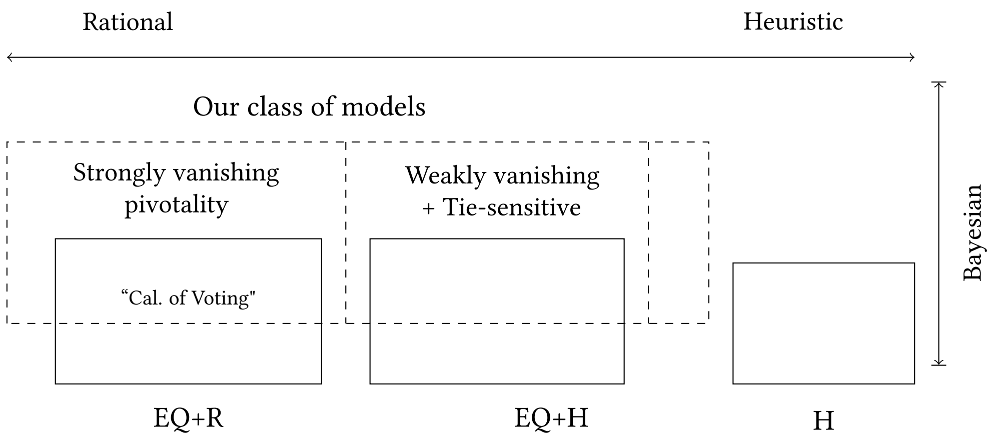

Figure 1: Position of our work within the literature: EQ=Equilibrium; R=Rational; and H=Heuristic.

Our Results: We fully characterize the CJT result for the models considered in the paper. At the heart of our characterization result lie two key elements: (1) the pivot points, which emerge as the limiting points of the sequence of non-trivial equilibria with increasing population size , and (2) the dependence of voter pivotality on two parameters: population size n and margin of victory m. While strongly vanishing pivotality models (Calculus of voting model, for instance) tends to collapse the equilibrium into a unique and trivial outcome, it is the more nuanced, weakly vanishing and tie-sensitive models that gives rise to richer, more complex equilibria. This leads to outcomes that fundamentally diverge from the classical implications of Condorcet's Jury Theorem, revealing new dynamics in collective decision-making under uncertainty. In particular, we show that under weakly vanishing models of pivotality estimation it is possible that a significant fraction of population votes but fails to converge to selecting better alternative even in the limit (hence also for any population size). That is, there is a nonzero probability of a *surprise* in the election irrespective of the population size.

2 Model

We start with the standard Calculus of Voting model, later expanding it to allow heuristic pivotality estimation.

2.1 Calculus of Voting

We study a two-candidate (referred to as A and B) election with N voters. Each voter is either a supporter of A (prefers A, i.e., $A \succ_i B$) or a supporter of B. We adapt the classical two-candidate calculus of voting model as follows. The action of each voter is to vote for one of the two candidates $\{A, B\}$, or abstain ($\perp$). We set the utility of having i's favorite candidate selected as 1, and denote as $c_i := \max\{0, \frac{\mathtt{G}_i - \mathtt{D}_i}{\mathtt{V}_i}\}$ the *effective* cost of voting for voter i (with $\mathtt{G}_i$, $\mathtt{D}_i$ and $\mathtt{V}_i$ as defined in Eq. (1)).

We consider the effective cost of voting and the preference of voter i as an independent sample from a commonly known joint distribution $\mathcal{D}$ over $[0, 1] \times \{A, B\}$ and is denoted by the tuple (c_i, T_i). We assume that $\mathcal{D}$ has no atoms, except possibly at $c = 0$ ("core voters").

Rational equilibrium Note that the only private information a voter has (other than the type distribution, which is common knowledge), is her own type. Moreover, since voting for the less preferred candidate is strictly dominated, we can assume that the only two actions for voter i are $\top$ (vote for T_i) and $\perp$ (abstain). A (pure, ex-ante) strategy profile is therefore a mapping v from the type space $[0, 1] \times \{A, B\}$ to an action $\{\top, \perp\}$. Now, a voting game is composed of a distribution $\mathcal{D}$ and population size N. Every game, together with a strategy profile v, induces a joint distribution on the number of votes for A and B.

The expected (normalized) utility gain from actively voting is thus exactly the probability that the voter is pivotal p^0, minus the voting cost c_i. Since p^0 in turn also depends on v, we define the explicit function $P^0(v) = P^0_{N,\mathcal{D}}(v)$ that represents the probability that a single player is pivotal[2] under strategy profile v in the game $(\mathcal{D}, N)$ (we often omit the game from the definition).

In a (pure) Bayes-Nash equilibrium, each player picks an action that maximizes her expected utility, given the distribution on other players' actions. This distribution, in turn, is induced by the type distribution of the other players (conditional on the player's realized type) and the strategy profile. In our case:

- The type distribution is an i.i.d. sample from $\mathcal{D}$, and the voter's realized type reveals no further information;
- The utility-maximizing action of each voter is to vote ($\top$) iff Eq. (1) holds for her type.

Definition 1 (Threshold profile)**.** *A strategy profile* v *is called a* threshold profile *if there is a threshold* $c = c(v)$ *such that* $\forall c_i \in [0,1], \forall T_i \in \{A, B\}, \quad v(c_i, T_i) = \top$ *iff* $c_i \leq c$.

Observation 1. *A threshold strategy profile* v *is a Bayes-Nash equilibrium of the Calculus of Voting game* $(\mathcal{D}, N)$*, if and only if* $c := c(v) = P^0_{\mathcal{D},N}(v)$.

Proof. Let i be a voter (c_i, T_i) and note that $P^0(v)$ does not depend on her type T_i. As v is a threshold profile, we have $p_i = p^0 = P^0(v)$. The utility-maximizing action of voter is to vote if and only if $c_i \leq P^0(v)$. □

We restrict our attention to threshold profiles $c \in [0,1]$ unless explicitly said otherwise. Note that we did not explicitly say what is the function P^0. However clearly such a function is well-defined, and even without a formal definition it is clear that fixing v, $P^0_{\mathcal{D},N}(v)$ decreases very rapidly with N. This means that only voters with essentially zero cost will vote, regardless of the type distribution. Indeed, this is the well-known paradox of voting, and it is easy to see that it may lead to an almost-certain win of the minority candidate, if happens to be supported by more low-cost voters.

Support functions While $\mathcal{D}$ contains all necessary information regarding the distribution of costs in the population, we would like to present costs in a more intuitive way.

For $T \in \{A, B\}$ let $s_T : [0,1] \to [0,1]$ be a continuous, non-decreasing function, where $s_T(c)$ should be read as the fraction of the distribution that prefers T and has individual cost at most c. We call s_T the *support function* of T, and note that it does not depend on the identity of voter i.

Proposition 2. *Let voter costs and types are sampled i.i.d. from a distribution* $\mathcal{D}$*. Then any distribution* $\mathcal{D}$ *induces a unique pair of support functions* s_A, s_B *with* $s_A(1) + s_B(1) = 1$*, and vice-versa.*

2.2 Perceived Pivotality

The informal statement above regarding the negligible turnout is not only disappointing, but also unrealistic, as in practice a substantial fraction of the population usually votes. We would therefore relax some of the rationality assumptions. These assumptions essentially correspond to the two bullets in the equilibrium characterization in Cor. 1: The second suggests that voters act based on their true

[2]For simplicity we consider p_0 as the probability that both candidates are *exactly* tied with $\frac{N-1}{2}$ votes each, which is indeed the pivot probability when N is even. When N is odd, things get more complicated, but since for large N the differences are negligible, we maintain our simplifying assumption.

probability of being pivotal; and the first means that they maximize their expected utility given this probability and Eq. (1). In what follows, we will maintain the utility maximization assumption but allow voters much more freedom in estimating their pivot probability P.

Pivot functions We highlighted earlier that the two most important factors that determine the probability of various outcomes are (1) the size of voting population, n; and (2) the margin, m. Our simplifying assumption (following e.g., [Myerson and Weber, 1993]) is that voters only consider n and m as expected values. Thus, an *expectation-based Perceived Pivotality Model* (PPM) is specified by a function p, which maps any pair of n and m to $p(n, m) \in (0, 1]$, and is continuous, non-increasing in both parameters, and strictly decreasing when strictly below 1.

PPM quantifies a subjective ex-ante belief of the individual voter about the importance of her vote. The equilibrium analysis crucially depends on the PPM model under consideration. In Sec. 3, we provide several concrete pivotality models, which can either approximate the real pivot probability P^0 or reflect beliefs and other factors affecting utility.

Replacing P^0 with a general PPM $p(n, m)$ allows us to consider a broad set of voters' behaviors. To see why this is useful, we first observe that the expected margin and the number of voters can be easily derived for any threshold profile c. We define the two following functions:

$$n(c, N) := (s_A(c) + s_B(c))N; ; \qquad \text{(expected voters)} \qquad (2)$$

$$m(c) := \frac{|s_A(c) - s_B(c)|}{s_A(c) + s_B(c)}. \qquad \text{(expected margin)} \qquad (3)$$

Observation 3. *Given support functions (i.e. a type distribution)* (s_A, s_B)*, a threshold profile* c*, and population size* N*, the expected number of active voters is* $n(c, N)$ *and the expected margin is* $m(c)$*.*

Election equilibrium We can now broaden our class of games. An *Election Game* is a tuple (s_A, s_B, p, N), where s_A, s_B are the support functions of the type distribution $\mathcal{D}$; p is a PPM; and N is the size of the population. For the special case where $p = P^0_{N,\mathcal{D}}$, we get the Calculus of Voting game. However, in a general election game, a voter votes according to how much she *perceives herself as pivotal.* We extend the equilibrium definition accordingly.

Definition 2. *A strategy profile* v *is an* election equilibrium *of election game* (s_A, s_B, p, N)*, if*

1. v *is a threshold profile; and*
2. $c := c(v) = p(n(c, N), m(c))$.

Proposition 4. *Every election has at least one equilibrium.*

The proof follows from the fact that $g(c) := p(n(c, N), m(c))$ is a continuous function from $[0, 1]$ onto itself and therefore must have a fixed point.

2.3 Issues and Elections

We want to be able to analyze elections as the population size grows. An *issue* is a triple $I = (s_A, s_B, p)$. Thus an issue together with a specific population size N defines an election game (s_A, s_B, p, N) (or just (I, N)) as above. Alternatively, an issue can be thought of as a series of election games, one for every population size N. Denote by $C(I, N) \subseteq [0, 1]$ the set of all equilibrium points of election $E = (I, N)$.

Definition 3 (Issue equilibrium)**.** *An equilibrium of issue* I *is a series of points* $\bar{c} = (c_N)_N$ *s.t.* $\forall N, c_N \in C(I, N)$*, and* $\bar{c}$ *has a limit. We denote the limit by* c^**.*

For an issue equilibrium $\overline{c}$ with limit c^*, if $c_N > c^*$ for all N we say that $\overline{c}$ is a *right equilibrium*. Similarly, if $c_N < c^*$ for all N we say that $\overline{c}$ is a *left equilibrium*.

Trivial equilibria An equilibrium is *trivial* if its limit is 0, meaning only core supporters vote.

3 Perceived Pivotality Models

We first consider models that closely approximate the actual probability that a single voter is pivotal, i.e., the probability of a tie $V_A = V_B$.

3.1 Fully Rational models

We first argue that our model captures the Calculus of Voting as a special case, i.e. that P^0 is also a PPM.

Proposition 5. *For every N, there is a PPM p_N^{CoV} s.t. for every threshold profile c, $P^0_{\mathcal{D},N}(c) = p_N^{CoV}(n(c,N), m(c))$. More precisely,*

$$p_N^{CoV}(n,m) := \mathop{\mathbb{E}}_{n' \sim Bin(N, \frac{n}{N})}\Big[\Pr_{x \sim Bin(n', \frac{1+m}{2})}(x = \lfloor n'/2 \rfloor)\Big].$$

Note that as N grows, n' is highly concentrated around n. We can therefore define an approximate version with a PPM p that does not depend on N:

Example 1 (Binomial PPM)**.**

$$p^{Bin}(n,m) := \Pr_{x \sim Bin\left(n, (1+m)/2\right)}(x = \lfloor n/2 \rfloor). \tag{4}$$

A later model by [Myerson, 1998] suggested drawing the scores of each candidate independently from a Poisson distribution (see Appendix E). Conceptually, the Poisson model is more appropriate in situations where voters can abstain (as the total number of active voters is not fixed), However it behaves very similarly to the Binomial model, and for our purpose they are almost the same. In fact, all three models belong in a much larger class of PPMs, characterized by *strong vanishing pivotality*:

Definition 4 (Vanishing Pivotality)**.** *We say that a PPM p has [strong] vanishing pivotality if $\lim_{n\to\infty} p(n,m) = 0$ for all $m > 0$ [$m \geq 0$].*

As we will later see, issues with vanishing pivotality (v.p.) always admit a trivial equilibrium. Clearly at the trivial equilibrium, Jury theorems are irrelevant: the candidate with more core support always wins with probability that approaches 1 as the population grows, regardless of who is more popular overall. It is not hard to verify (e.g., using Stirling approximation) that in both the Binomial and Poisson PPMs, $p(n,m) = \Theta(\frac{1}{\sqrt{n}})$ for $m = 0$, and decreases exponentially fast in n for any $m > 0$.

3.2 Tie-Sensitive models

We saw that even in the rational models (which have strong vanishing pivotality), the case of $m = 0$ is different, with a substantially higher probability to be pivotal. A simple and perhaps more cognitively plausible assumption is that voters *consider themselves pivotal* if the margin is small enough, regardless of the number of voters.

Definition 5 (Tie-sensitive pivotality)**.** *We say that a PPM is q-*tie-sensitive *if $p(n, 0) \geq q$ for all n.*

That is, if the expected outcome is a tie, everyone thinks they are pivotal at least to some extent, regardless of the number of active voters. By definition, any PPM has either strong v.p. *or* tie-sensitivity. If it is tie-sensitive and has v.p. we say it has *weak* vanishing pivotality. Tie-sensitivity may occur due to various reasons, and we provide and discuss examples in Section 6.

4 Characterizing Equilibrium Limits

We begin by characterizing the trivial equilibrium.

Proposition 6. *Suppose $s_A(0) \neq s_B(0)$. Any issue with vanishing pivotality admits a trivial equilibrium.*

Proposition 7. *Suppose $s_A(0) + s_B(0) > 0$. Any issue with strong vanishing pivotality admits* ***only*** *trivial equilibrium.*

Next, we show that the intersection points of the support functions (where the margin is 0) form the limiting points of equilibria. Recall that by our assumption there is a finite number of such points (but see Appendix C).

Definition 6 (Pivot Points)**.** *For a given pair of support functions s_A, s_B, a* pivot point *is any $c \in (0, 1)$ where $s_A(c) = s_B(c)$.*

For technical reasons we will assume throughout the paper that there is only a finite number of intersection points where $s_A(c) = s_B(c)$, and that all derivatives of s_A, s_B are bounded in some environment of each such point. We explain the more general case in Appendix C.

Theorem 8. *Let I be an issue with a PPM p having weakly vanishing pivotality* and *is q-tie-sensitive. Any pivot point $c^* < q$ has a right- and left-equilibrium with limit c^*. For support functions with finite intersection points, the limit of any equilibrium is either 0 or a pivot point.*

Proof. We start with the existence of the right equilibrium. The proof for the left equilibrium is symmetric. Let $c^* < q$ be some pivot point of I, and let $\delta > 0$. We need to show there is some N_δ and some $c_\delta \in (c^*, c^* + \delta)$ s.t. $c_\delta \in C(I, N_\delta)$.

Since all derivatives are bounded, there is some open interval $(c^*, c^* + t)$ where s_A, s_B differ, and w.l.o.g. $s_A(c) > s_B(c)$ for any $c \in (c^*, c^* + t)$. Let $\underline{\delta} := \min\{t, \delta, q - c^*\}$ and note that by the definition of pivot point, $\varepsilon := m(c^* + \underline{\delta}) > 0$. Also, $n(c^* + \underline{\delta}, N) = (s_A(c^* + \underline{\delta}) + s_B(c^* + \underline{\delta}))N < N$. Thus by weakly vanishing pivotality:

$$p(n(c^* + \underline{\delta}, N), m(c^* + \underline{\delta})) \leq p(N, \varepsilon) \xrightarrow[N \to \infty]{} 0,$$

so there is N_δ for which $p(n(c^* + \underline{\delta}, N_\delta), m(c^* + \underline{\delta})) < \underline{\delta}$. From tie-sensitivity, for any N (including N_δ):

$$p(n(c^*,N), m(c^*)) = p(n(c^*,N), 0) \geq p(N, 0) \geq q > c^* + \underline{\delta}.$$

Let $g(x) := p(n(c^*+x, N_\delta), m(c^*+x)) - (c^*+x)$. Then f is continuous in $x \in [0, \underline{\delta})$ with $g(0) > 0$ and $g(\underline{\delta}) < 0$. From intermediate value theorem there is some x^* where $g(x^*) = 0$ and thus $c_\delta := c^* + x^*$ is an equilibrium of (I, N_δ).

In the other direction, assume towards a contradiction that there is a nontrivial equilibrium with limit $\hat{c}$ that is not a pivot point. Note that by our assumption of finite intersection points, and due to bounded derivatives, $s_A(c) - s_B(c) > \varepsilon > 0$ in some interval $[\hat{c} - \delta, \hat{c} + \delta]$. Thus in any point in this interval the pivotality goes to 0 for sufficiently large N, and in particular is lower than $\hat{c} - \delta$, which means it is not an equilibrium of $(I, \overline{N})$ for any $\overline{N} \geq N$. Note that if q is tight (i.e., the PPM is not q'-tie-sensitive for any $q' > q$), points above q cannot be the limit of any equilibrium, as the pivotality at any c is at most q. □

So we have a rather complete characterization of equilibria, or at least of their limit points, in every issue. Two natural questions are: (a) whether these equilibria are inherently stable; and (b) are these equilibria "good" in the sense of the Condorcet Jury Theorem.

4.1 Stability of Equilibrium Points

Intuitively, stability means that a small perturbation will not cause us to drift from the equilibrium point, but to gravitate back to it [Granovetter, 1978; Palfrey and Rosenthal, 1990].

Definition 7 (Stability (informal))**.** *An equilibrium c_N is* stable*, if there is some $\varepsilon > 0$ such that for any threshold profile c with $|c - c_N| < \varepsilon$, the trend*[3] *at c is towards c_N.*

Theorem 9. *Let $\overline{c}$ be a right-equilibrium of issue I with limit $c^* > 0$. Then for sufficiently large N, c_N is stable.*

We provide here the proof outline. The full proof with the exact definition of stability is given in Appendix D. Note that close to the equilibrium point, both $n(c)$ and $m(c)$ are increasing in c, and thus $p(n(c), m(c))$ is *decreasing* in c. So any perturbation that ends up with fewer active voters will mean higher pivotality, and some voters will join back (and vice versa). Since for sufficiently large N the left- and right-equilibria are the *only* equilibria, and the right ones are stable, the left ones must be unstable.

5 Jury Theorems for Pivot Points

Given an election instance $E = (I, N)$ and a threshold profile $c \in [0, 1]$, a random variable counting the number of active votes for A (and likewise for B) $V_A := \sum_{i \in N} \mathbb{1}[c_i \leq c \wedge T_i = A]$. We further denote the *Winning Probability* of A in profile c of election (I, N) as $\mathbb{WP}_A(I, N, c) := \Pr(V_A > V_B | I, N, c)$. Finally, for any issue I with issue equilibrium $\overline{c}$, we define

$$\mathbb{WP}_A(I, \overline{c}) := \lim_{N \to \infty} \mathbb{WP}_A(I, N, c_N).$$

We emphasize that we, as 'outsiders' to the election, care about the *actual* probability of the event, which is not affected by the perceived pivotality model p, once c is determined.

Our main question regards the non-trivial equilibria, whose limits are the pivot points. For an equilibrium $\overline{c}$ with limit c^*, we define the convergence rate as $cr(\overline{c}) := \lim_{N \to \infty} \sqrt{N} |c_N - c^*|$.

Definition 8. *We say that $\overline{c}$ is converging* fast *if $cr(\overline{c}) = 0$; and* slow *if $cr(\overline{c}) = +\infty$. Otherwise, we say that $\overline{c}$ has* moderate convergence rate *$r(c^*) \in (0, \infty)$.*

Local Jury theorems We say that I admits a *jury theorem* at $c^* \in [0, 1]$, if there is an issue equilibrium $\overline{c}$ with limit c^* s.t. $\mathbb{WP}_A(I, \overline{c}) = 1$. Similarly, I admits a *non-jury theorem* at c^* if $\mathbb{WP}_A(I, \overline{c}) < 1$, meaning that regardless of the size of the population, there is some constant probability that the better candidate A will lose. I admits a *strong non-jury theorem* if $\mathbb{WP}_A(I, \overline{c}) = \frac{1}{2}$.

Theorem 10 (Characterization of Stable Jury Equilibria)**.** *Let I be an issue and let $\overline{c}$ be a nontrivial right equilibrium of I with limit c^*. There are three cases, where $\overline{c}$ admits a*

1. Jury theorem *if $\overline{c}$ converges slowly;*

[3] That is, when starting from c_N, the best response of voters in ε-neighborhood of c_N gets closer to c_N until convergence as we increase N.

PPM Type		Equilibria	Conv. Rate	Jury Thm	Example PPMs
q-tie-sensitive	no v.p.	at q	-	-	Network (fixed κ); Altruist ($f(n) = \exp(\omega(n))$)
	weak vanishing pivotality	at all pivot points below q	slow	Yes	Poly. ($\beta > 2\alpha$); Altruist ($f(n) = \omega(\sqrt{n})$)
			moderate	Weak	Poly. ($\beta = 2\alpha$); Altruist ($f(n) = \Theta(\sqrt{n})$)
			fast	No	Poly. ($\beta < 2\alpha$)
-	strong v.p.	at 0	-	-	CoV; Binomial; Poisson; Altruist ($f(n) = o(\sqrt{n})$)

Table 1: (From L to R) First two columns show the three main types of PPM, as per Def. 4 and 5. The third column summarizes the main results of Sec. 4. Next two columns shows the finer partition of weakly v.p. models, and the winning probability of the leader following the results in Sec. 5. The rightmost column shows the classification of the models in Sections 3 and 6.

2. weak non-Jury *theorem if* $\bar{c}$ *converges moderately; and*

3. strong non-Jury theorem *if* $\bar{c}$ *converges fast.*

Proof Sketch. Let μ and σ denote the mean and standard deviation of the random variable $V_A - V_B$ representing the difference between the votes received by two candidates. We approximate this random variable by a Gaussian distribution. The ratio $\frac{\mu}{\sigma}$ is critical for determining whether the sequence $\bar{c}$ satisfies a specific variant of the Jury theorem, as the winning probability is approaching $\Phi\left(\frac{\mu}{\sigma}\right)$ (here $\Phi(\cdot)$ denotes the CDF of standard normal distribution).

We prove that the margin $m(c_N)$ is essentially proportional to $|c_N - c^*|$. In particular, when the sequence $|c_N - c^*|$ converges at a slow rate, so does the margin, meaning that $m(c_N) = \omega(1/\sqrt{N})$, it can be shown that $\frac{\mu}{\sigma}$ tends to infinity with N, and consequently $\Phi\left(\frac{\mu}{\sigma}\right)$ tends to 1. This result implies that $\bar{c}$ adheres to the Jury theorem since, in this case, the ratio $\frac{\mu}{\sigma}$ grows unbounded as N increases.

On the other hand, under conditions of fast convergence, where $m(c_N) = o(1/\sqrt{N})$, the ratio $\frac{\mu}{\sigma}$ approaches zero as N becomes large. Consequently, $\Phi\left(\frac{\mu}{\sigma}\right) \to 0$, indicating weak non-Jury theorem. This result reflects that the distribution of the voting difference becomes more centered around zero with a diminishing spread, leading to an increasingly uncertain outcome.

For the intermediate case of moderate convergence, where $m(c_N) = \Theta(1/\sqrt{N})$, the ratio $\frac{\mu}{\sigma}$ converges to a finite positive limit as N increases. This behavior implies that $\Phi\left(\frac{\mu}{\sigma}\right)$ remains bounded strictly between 0 and 1. Consequently, $\bar{c}$ does not fully satisfy the Jury theorem but neither does it completely diverge from its principles. A detailed proof is given in Appendix A. □

6 Classification of PPMs

Recall the 'rational' PPMs with strong vanishing pivotality we considered in Sec. 3. As our positive results apply for PPMs that are tie-sensitive, it is important to at least provide some examples of such models, that can also be justified in practice.

We suggest simple models demonstrating how tie-sensitivity may emerge from at least three reasons: limited communication, altruism, and heuristics.

Limited Communication Several authors in the literature considered models in which voters are embedded in an implicit or explicit network, where they only 'see' a limited number of K neighbors [Osborne and Rubinstein, 2003; Michelini et al., 2022], based on which they can assess their pivotality.

Generally, k out of K neighbors will be active in expectation, and k can be a function $k = \kappa(n)$. This yields the following model:

Example 2 (Network PPM). $p^{Network(\kappa)}(n, m) := p^{Bin}(\kappa(n), m)$.

It is not hard to see that if e.g. $k = \kappa(n)$ is a constant, then the model is q_k-tie-sensitive for some $q_k = \Theta(\frac{1}{\sqrt{k}})$ and does not have vanishing pivotality at all. Thus equiliria are not at pivot points.

Otherwise (i.e. k is strictly increasing with n), it has strong vanishing pivotality and thus only the trivial equilibrium.

Altruist voters Another possibility is that voters correctly estimate their pivotality (say, using the Binomial or Poisson model above), but that their *value* $\mathtt{V}_i$ scales with the size of the population as $\mathtt{V}_i \cdot f(N)$. That is, voters consider large elections as 'more important'. We note that this argument is sometimes used as a possible explanation for the paradox of voting [Downs, 1957]. Note that

$$\frac{\mathtt{G}_i - \mathtt{D}_i}{\mathtt{V}_i \cdot f(N)} < p \iff c_i = \frac{\mathtt{G}_i - \mathtt{D}_i}{\mathtt{V}_i} < p \cdot f(N).$$

We therefore get another class of PPMs (considering that n and N scale roughly at the same rate):

Example 3. $p^{alt(q,f)}(n, m) := \min\{q, p^{Bin}(n, m) \cdot f(n)\}$.

It is not hard to see that the Altruist PPM is weakly vanishing for any sub-exponential function f, and that it is q-tie-sensitive whenever $f = \Omega(\sqrt{n})$. In fact, we can classify all the regimes of altruist PPMs:

Proposition 11. *Let I be an issue with an Altruist PPM with function f, and let $\bar{c}$ be a non-trivial equilibrium. Then:*

1. *if $f = e^{\omega(n)}$ then $\bar{c}$ is a fixed equilibrium at q; else*
2. *if $f = \omega(\sqrt{n})$ then $\bar{c}$ converges slowly; else*
3. *if $f = \Theta(\sqrt{n})$ then $\bar{c}$ converges moderately; else*
4. *$\bar{c}$ is trivial.*

Interestingly, there is no f for which there is a non-trivial equilibrium with fast convergence (See also Table 1).

Polynomial heuristics Another PPM is induced by defining the dependency on n and m directly:

Example 4 (Polynomial PPM). *For $q, \alpha, \beta > 0, p^{Poly(q,\alpha,\beta)}(n, m) = \min\{q,\ m^{-\alpha} \cdot n^{-\beta}\}$.*

It is immediate from the definition that the Polynomial PPM is both q-tie-sensitive and has a weakly vanishing pivotality, so it remains to classify the models according to rate of convergence.

Suppose that the support functions have different derivatives at c^* (see Appendix C).

Lemma 12. *Let I be an issue with a Polynomial PPM and let $\bar{c} = (c_N)_N$ be a non-trivial equilibrium of I with limit c^*. Then $c_N = c^* + \Theta(N^{-\frac{\beta}{\alpha}})$.*

From the lemma (and Theorem 10), we conclude that the critical threshold for the Jury theorem is at $\alpha = 2\beta$: If $\alpha > 2\beta$ then convergence is fast and there is a strong non-jury theorem; and if $\alpha < 2\beta$ then convergence is slow enough to guarantee a local jury theorem. See Table 1.

The threshold includes the special case where $\beta = \frac{1}{2}, \alpha = 1$. This is case is interesting and natural because the dependency on the number of active voters is $\frac{1}{\sqrt{n}}$—just as in the fully rational models—whereas the dependency on the margin is linear.

Moreover, in the case of moderate convergence rate where $\alpha = 2\beta$, A wins w.p. $\Phi((c^*)^{-\frac{1}{\alpha}})$. Conveniently, most terms cancel out and we get that $\mathbb{WP}_A(I, \bar{c}) = \Phi((c^*)^{-\frac{1}{\alpha}}) < 1$. Interestingly, the probability does not depend on the shape of the support functions at all (except their intersection point), and neither does it depend on β.

6.1 Sensitivity to model parameters

Our theoretical results provide a sharp threshold between 'positive' and 'negative' results, which depends on the PPM or its parameters. However, there results hold *in the limit* as N grows. We next study via an example how A's winning probability changes as population size increases and/or when we vary the parameters. For simplicity and easy computation we use the Polynomial PPM with $\beta = \frac{1}{2}$ [4].

Example 5. *Suppose $s_A(c) = 0.1 + c/2$ and $s_B(c) = 0.4$. That is, B has 40% overall support, all of them core supporters and A has 60% overall support and a fraction of voters are distributed over cost range* $[0, 1]$. *We assume the Tie-sensitive PPM with $\alpha = 1$.*

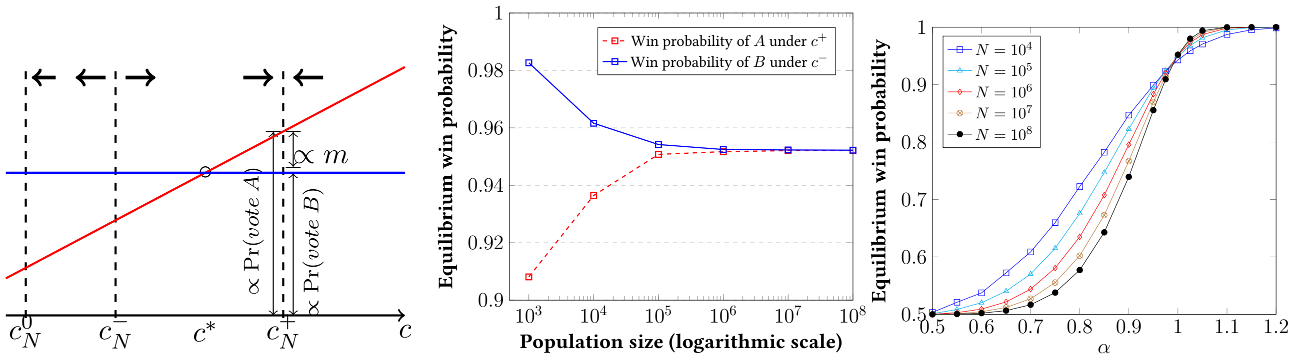

Figure 2: (L) demonstrates election instance from Section 6.1. For large value N, the pivot point c^*, two non-trivial equilibria c_N^+ and c_N^-, and the trivial equilibrium c^0 are shown. For c_N^+, the probability of a random voter to vote A is proportional to $s_A(c_N^+)$. The $m(c_N^+)$ is proportional to the margin of victory. The bold arrows indicate that c_N^+, c_N^0 are stable equilibria whereas c_N^- is not stable. (C) Win probability for different values of N under respective induced equilibria. (R) Win probability of A for $\beta = 0.5$ and different values of α in polynomial PPM model for different values of N. The trend reversal can be observed at $\alpha = 1$.

As noted earlier in the main paper, the unique pivot point is 0.6, so we expect the winning probability to approach $\Phi(0.6^{-1}) \cong 0.952$ in both c_N^+ and c_N^- as N increases.

Figure 2 (C) demonstrates the Non-Jury Theorem, showing that the winning probability of A in the stable equilibrium c^+ is bounded away from 1, even as the total population size N is increasing. Interestingly, the winning probability of B behaves differently and is *decreasing* towards the c^- equilibrium point. Hence B would always prefer a smaller fraction of the population to vote whereas the popular candidate prefers a large population to vote. Showing that the equilibrium c^- is not stable.

We observe that the win probabilities under c^+ and c^- for respective winning candidates are significantly different. While B wins with a high probability under c^-, A wins under c^+ with a high probability for

[4] The code used for the simulation results is available at https://anonymous.4open.science/r/NJT-E778/.

the same population size, for a fixed population size, the win probabilities of these candidates under their favoured equilibria are not the same.

Figure 2 (R) shows the win probability of candidate A under c^+ and for different values of α. We observe that with higher values of N equilibrium win probability of at c^+ approaches a step function around $\alpha = 1 = 2\beta$. The larger value of α pushes the equilibrium point c^+ towards the right, increasing the win probability of the popular candidate. Yet this occurs slowly and even for high N there is a gradual increase of the win probability of A with α.

7 Discussion

The starting point of this paper was the Paradox of Voting, which entails the Condorcet Jury Theorem (CJT) would not hold in a population with heterogeneous voting costs, due to abstention and bias. We showed that when voters are sufficiently responsive to the expected margin, the (locally) more popular candidate wins with a probability approaching one, aligning with the classical CJT.

There is of course ample literature on the CJT analyzing theoretical and practical conditions where it may fail (see [McCannon, 2015]). In the introduction, we already mentioned *epistemic voting*, that is supposedly outside the scope of models we consider: e.g., in epistemic voting minority voters are unaware of this and everyone gains from high turnout. However, a recent paper by Michellini et al. [Michelini et al., 2022] suggest a way to re-obtain a CJT when voters are only exposed to few neighbors. This can be thought of as a special case of heuristics that increases their perceived pivotality, so perhaps our results could be extended to cover 'heuristic' epistemic voting as well, with some modifications.

Strategic behavior in the epistemic voting scenario is more problematic though, leading to the 'swing voter curse' [Feddersen and Pesendorfer, 1996; Austen-Smith and Banks, 1996]. It is less likely that considering self as more pivotal would help, as the problem is with the voter's belief about her *competence*.

Other factors affecting vote decisions like external pressure [Bolle, 2022], bandwagon effect [Morton et al., 2015], and information cascades [Golub and Jackson, 2010; Acemoglu et al., 2011], which provide alternative reasons for the failure of group wisdom, are outside the scope of our paper.

Future directions In light of the above discussion, we believe that relaxing the rigid rationality assumption in favor of more general pivot probability models is an important step in understanding the boundaries of positive and negative results in the social choice literature (in our case, CJT and paradox fo voting, respectively).

While this paper focused on the limit case, studying the winning probabilities in small-population settings is also important, following our preliminary empirical example. Another avenue is to explore heterogeneity not only in participation costs but also in how voters estimate their pivotality. While it is relatively straightforward to show that an equilibrium still exists in this extended model (Appendix F), characterizing these equilibria and understanding how this diversity affects the CJT remains an open and intriguing question.

References

Daron Acemoglu, Munther A. Dahleh, Ilan Lobel, and Asuman Ozdaglar. Bayesian learning in social networks. *The Review of Economic Studies*, 78(4):1201–1236, 03 2011. ISSN 0034-6527. doi: 10.1093/restud/rdr004. URL https://doi.org/10.1093/restud/rdr004.

John H. Aldrich. Rational choice and turnout. *American Journal of Political Science*, 37(1):246–278, 1993. ISSN 00925853, 15405907. URL http://www.jstor.org/stable/2111531.

David Austen-Smith and Jeffrey S Banks. Information aggregation, rationality, and the condorcet jury theorem. *American political science review*, 90(1):34–45, 1996.

André Blais. *To Vote or Not to Vote: The Merits and Limits of Rational Choice Theory*. University of Pittsburgh Press, 2000. ISBN 9780822941293. URL http://www.jstor.org/stable/j.ctt5hjrrf.

Friedel Bolle. Voting with abstention. *Journal of Public Economic Theory*, 24(1):30–57, 2022.

Clark Bowman, Jonathan K Hodge, and Ada Yu. The potential of iterative voting to solve the separability problem in referendum elections. *Theory and decision*, 77:111–124, 2014.

Jules Coleman and John Ferejohn. Democracy and social choice. *Ethics*, 97(1):6–25, 1986.

Anthony Downs. An economic theory of political action in a democracy. *Journal of Political Economy*, 65(2):135–150, 1957. ISSN 00223808, 1537534X. URL http://www.jstor.org/stable/1827369.

Roy Fairstein, Adam Lauz, Reshef Meir, and Kobi Gal. Modeling people's voting behavior with poll information. In *Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems*, pages 1422–1430, 2019.

Timothy J Feddersen and Wolfgang Pesendorfer. The swing voter's curse. *The American economic review*, pages 408–424, 1996.

John A. Ferejohn and Morris P. Fiorina. The paradox of not voting: A decision theoretic analysis*. *American Political Science Review*, 68(2):525–536, 1974. URL https://EconPapers.repec.org/RePEc:cup:apsrev:v:68:y:1974:i:02:p:525-536_11.

Alan Gerber, Mitchell Hoffman, John Morgan, and Collin Raymond. One in a million: Field experiments on perceived closeness of the election and voter turnout. *American Economic Journal: Applied Economics*, 12(3):287–325, 2020.

Benjamin Golub and Matthew O. Jackson. Naïve learning in social networks and the wisdom of crowds. *American Economic Journal: Microeconomics*, 2(1):112–49, February 2010. doi: 10.1257/mic.2.1.112. URL https://www.aeaweb.org/articles?id=10.1257/mic.2.1.112.

Mark Granovetter. Threshold models of collective behavior. *American journal of sociology*, 83(6):1420–1443, 1978.

David K Levine and Thomas R Palfrey. The paradox of voter participation? a laboratory study. *American political science Review*, 101(1):143–158, 2007.

Michael D Martinez and Jeff Gill. The effects of turnout on partisan outcomes in us presidential elections 1960–2000. *The Journal of Politics*, 67(4):1248–1274, 2005.

Bryan C McCannon. Condorcet jury theorems. In *Handbook of social choice and voting*, pages 140–160. Edward Elgar Publishing, 2015.

Reshef Meir. Plurality voting under uncertainty. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 29, 2015.

Reshef Meir. *Strategic Voting*. Synthesis Lectures on Artificial Intelligence and Machine Learning. Morgan & Claypool Publishers, 2018. doi: 10.2200/S00849ED1V01Y201804AIM038. URL https://doi.org/10.2200/S00849ED1V01Y201804AIM038.

Reshef Meir, Omer Lev, and Jeffrey S. Rosenschein. A local-dominance theory of voting equilibria. In *Proceedings of the Fifteenth ACM Conference on Economics and Computation*, EC '14, page 313–330, New York, NY, USA, 2014. Association for Computing Machinery. ISBN 9781450325653. doi: 10.1145/2600057.2602860. URL https://doi.org/10.1145/2600057.2602860.

Samuel Merrill. Strategic decisions under one-stage multi-candidate voting systems. *Public Choice*, 36(1):115–134, Jan 1981. ISSN 1573-7101. doi: 10.1007/BF00163774. URL https://doi.org/10.1007/BF00163774.

Matteo Michelini, Adrian Haret, and Davide Grossi. Group wisdom at a price: Jury theorems with costly information. In *31st International Joint Conference on Artificial Intelligence, IJCAI 2022*, pages 419–425. IJCAI, 2022.

Rebecca B Morton, Daniel Muller, Lionel Page, and Benno Torgler. Exit polls, turnout, and bandwagon voting: Evidence from a natural experiment. *European Economic Review*, 77:65–81, 2015.

Roger B. Myerson. Population uncertainty and poisson games. *International Journal of Game Theory*, 27(3):375–392, Nov 1998. ISSN 1432-1270. doi: 10.1007/s001820050079. URL https://doi.org/10.1007/s001820050079.

Roger B. Myerson and Robert J. Weber. A theory of voting equilibria. *The American Political Science Review*, 87(1):102–114, 1993. ISSN 00030554, 15375943. URL http://www.jstor.org/stable/2938959.

Martin J. Osborne and Ariel Rubinstein. Sampling equilibrium, with an application to strategic voting. *Games and Economic Behavior*, 45(2):434–441, 2003. ISSN 0899-8256. doi: https://doi.org/10.1016/S0899-8256(03)00147-7. URL https://www.sciencedirect.com/science/article/pii/S0899825603001477. Special Issue in Honor of Robert W. Rosenthal.

Guillermo Owen and Bernard Grofman. To vote or not to vote: The paradox of nonvoting. *Public Choice*, 42(3):311–325, Jan 1984. ISSN 1573-7101. doi: 10.1007/BF00124949. URL https://doi.org/10.1007/BF00124949.

Thomas R Palfrey and Howard Rosenthal. Testing game-theoretic models of free riding: New evidence on probability bias and learning. 1990.

Annemieke Reijngoud and Ulle Endriss. Voter response to iterated poll information. In *Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systems-Volume 2*, pages 635–644, 2012.

William H. Riker and Peter C. Ordeshook. A theory of the calculus of voting. *The American Political Science Review*, 62(1):25–42, 1968. ISSN 00030554, 15375943. URL http://www.jstor.org/stable/1953324.

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A Missing Proofs

Proposition 2. *Let voter costs and types are sampled i.i.d. from a distribution $\mathcal{D}$. Then any distribution $\mathcal{D}$ induces a unique pair of support functions s_A, s_B with $s_A(1) + s_B(1) = 1$, and vice-versa.*

Proof. Suppose there are N voters sampled from $\mathcal{D}$. Given a cost threshold c, let the random variable $X_A^{(i)}(c) := \frac{1}{N-1}\sum_{j\neq i} \mathbb{1}[c_j \leq c \text{ and } A \succ_j B]$ denote fraction of supporters—other than voter i—of candidate A having the voting cost at-most c. Similarly, define $X_B^{(i)}(c) := \frac{1}{N-1}\sum_{j\neq i}^{N-1} \mathbb{1}[c_j \leq c \text{ and } B \succ_j A]$. Note that, as $X_T^{(i)}(c)$ does not depend on her type T_i and her private cost c_i. Since every agent is exposed to the same knowledge, we have, $X_T(c) := X_T^{(1)}(c) = X_T^{(2)}(c) = \cdots = X_T^{(N)}(c)$ for all $T \in \{A, B\}$. Finally, let $s_T(c) := \mathbb{E}[X_T(c)]$, and note that it depends neither on i or N.

To see that $s_T(c)$ is continuous, let s^+, s^- the right and left limits of $s_T(c)$ at $c^* \in (0,1)$. If $s^- < s^+$ then $Pr_{\mathcal{D}}[(c^*, T)] = s^+ - s^- > 0$, in contradiction to our assumption that $\mathcal{D}$ has no atoms in $(0,1)$.

To show the direction, we define $Pr_{\mathcal{D}}[T_i = A] := s_A(1)$, then by assumption $Pr_{\mathcal{D}}[T_i = B] := 1 - s_A(1) = s_B(1)$. For each $T \in \{A, B\}$, the function $s_T(c)/s_T(1)$ is the CDF of $Pr_{\mathcal{D}}[c_i|T_i]$. □

Proposition 4. *Every election has at least one equilibrium.*

Proof. For any $c \in [0,1]$, define $f(c) := p(n(c), m(c))$. Since the support functions are continuous, $n(c)$ and $m(c)$ are continuous (by Eqs. (2),(3)). By continuity of p, the function f is also continuous. Finally, every continuous function from $[0,1]$ to itself has a fixed point, due to intermediate value theorem applied to $f(x) - x$. □

n size).

Lemma 13. *The Altruist PPM has v.p. iff $f = \exp(o(n))$ and is q-tie-sensitive for some $q > 0$ iff $f = \Omega(\sqrt{n})$.*

Proof. For any $m > 0$, we have

$$p^{Bin}(n,m) \cdot f(n) = \exp(-\Theta(n)) f(n),$$

which goes to 0 if $f(n)$ is subexponential, and goes to infinity otherwise.

Similarly, at $m = 0$, we have

$$p^{Bin}(n,m) \cdot f(n) = \Theta(\frac{1}{\sqrt{n}}) f(n),$$

which goes to 0 if $f(n) = o(\sqrt{n})$ (in which case the model has strong vanishing pivotality); goes to infinity if $f(n) = \omega(n)$; and approaches some positive constant q' if $f(n) = \Theta(\sqrt{n})$, in which case we define $q := \min\{1, q'\}$. □

Proposition 7. *Suppose $s_A(0) + s_B(0) > 0$. Any issue with strong vanishing pivotality admits **only** trivial equilibrium.*

Proof. Assume towards a contradiction that there is a nontrivial equilibrium $\bar{c} = (c_N)_N$ with limit $c^* > \delta$ for some $\delta > 0$. This means that $c_N > q$ for all sufficiently large $N > N'$.

Now, since $\lim_{n\to\infty} p(n,0) = 0$, there is some n^* such that $p(n,0) < q$ for all $n \geq n^*$. Let $N^* := \frac{n^*}{s_A(c_{N^*})+s_B(c_{N^*})}$, so that $n(c_{N^*}, N^*) = n^*$. Also w.l.o.g. $N^* > N'$. We then have that

$$\begin{aligned} q < c_{N^*} &= p(n(c_{N^*}, N^*), m(c_{N^*})) \\ &\leq p(n(c_{N^*}, N^*), 0) \\ &= p(n^*, 0) < q, \end{aligned}$$

i.e. a contradiction. □

Proposition 6. *Suppose $s_A(0) \neq s_B(0)$. Any issue with vanishing pivotality admits a trivial equilibrium.*

Proof. For any $\delta > 0$, we should show that there is some N_δ and $c_\delta < \delta$ such that $c_\delta \in C(I, N_\delta)$.

W.l.o.g. $s_B(0) > s_A(0)$, and by the bounded derivatives, the margin $m(c)$ is at least some $\varepsilon > 0$ in some neighborhood $c \in [0, q]$. Let $\underline{\delta} := \min\{q, \delta\}$.

By weakly vanishing pivotality,

$$p(n(\underline{\delta}), m(\underline{\delta})) \leq p(n(\underline{\delta}), \varepsilon) = p(s_B(0)N, \varepsilon) \to 0$$

as N grows. In particular there is some N_δ for which $p(s_B(0)N_\delta, \varepsilon) < \underline{\delta}$. We argue that there must be an equilibrium in the range $[0, \underline{\delta}]$.

Indeed, fix N_δ and let $f(x) := p(n(x), m(x)) - x$. We know that f is continuous, that $f(0) \geq 0$ and that $f(\underline{\delta}) = p(n(\underline{\delta}), m(\underline{\delta})) - \underline{\delta} < \underline{\delta} - \underline{\delta} = 0$. From the intermediate value theorem, there must be some $c_\delta \in [0, \underline{\delta})$ for which $f(c_\delta) = 0$, meaning $c_\delta = p(n(c_\delta), m(c_\delta))$. Thus $c_\delta \in C(I, N_\delta)$. □

Lemma 14. $m(c_N) = |c_N - c^*|(m^* + o(1))$.

Proof. Since the first and second derivatives of s_A, s_B are bounded, so are those of the functions $n(c, N)$ and $m(c)$ (as per Eq. (2),(3)).

We denote $m'(c) := \frac{\partial m(c)}{\partial c}$ and $n'(c) := \frac{1}{N}\frac{\partial n(c,N)}{\partial c}$, so neither function depends on N.

Also note that $n'(c) \geq 0$ everywhere and $m^* = m'(c^*) > 0, n'(c^*) > 0$ by the definition of pivot point. We also denote $s^* := s_A(c^*) + s_B(c^*)$ and note that $s^* \cdot N = n(c^*)$.

Due to bounded derivatives, there must be some interval $[c^* - \delta, c^* + \delta]$ and constants $\overline{m}, \underline{m}, \overline{n}, \underline{n} > 0$ such that $m'(c) \in [\underline{m}, \overline{m}]$ and $n'(c) \in [\underline{n}, \overline{n}]$ for all $c \in [c^*, c^* + \delta]$. Moreover, $\underline{m}, \overline{m}$ can be arbitrarily close to m^* and likewise for n' (the functions are nearly-linear near c^*). We consider only N's large enough so that c_N is in this interval, and so that $p(n(c_N), m(c_N)) < 1$.

For any $c \in [c^*, c^* + \delta]$, we have that

$$\begin{aligned} m(c) &\in [m(c^*) + |c - c^*|\underline{m}, m(c^*) + |c - c^*|\overline{m}] \\ &= |c - c^*| \cdot [\underline{m}, \overline{m}] = |c - c^*|(m^* + o(1)), \end{aligned}$$

where the last equality is due to the bounded second derivatives of the support functions, and in particular of their difference. □

Theorem 10 (Characterization of Stable Jury Equilibria)**.** *Let I be an issue and let $\overline{c}$ be a nontrivial right equilibrium of I with limit c^*. There are three cases, where $\overline{c}$ admits a*

1. Jury theorem *if $\overline{c}$ converges slowly;*
2. weak non-Jury *theorem if $\overline{c}$ converges moderately; and*

3. strong non-Jury theorem *if $\overline{c}$ converges fast.*

Proof. Let $\overline{c} = (c_N)_N$ be a nontrivial equilibrium with limit at pivot point $c^* < q$.

Denote $p_A := s_A(c_N); p_B := s_B(c_N)$ the probabilities that a single random voter is voting actively for A or B, respectively with $p_A > p_B$ and let $s^* := p_A + p_B$.

The variables V_A, V_B are coming from a single multinomial distribution with parameters N and $(p_A, p_B, 1-(p_A+p_B))$. In the limit of N, these are two *correlated* Normal variables, with mean $\mu_A = N \cdot p_A, \mu_B = N \cdot p_B$ and variance $\sigma_A^2 = Np_A(1-p_A), \sigma_B^2 = Np_B(a-p_B)$. Their difference $V_A - V_B$ is a Normal variable with expectation

$$\mu = \mu_A - \mu_B = N(p_A - p_B) = N \cdot m(c_N)s^*$$

and with variance

$$\begin{aligned}
\sigma^2 &= \sigma_A^2 + \sigma_B^2 - 2COV(V_A, V_B) \\
&= N(p_A(1-p_A) + p_B(1-p_B)) + 2N(p_A p_B) \\
&= N(p_A + p_B - (p_A^2 - 2p_A p_B + p_B^2)) \\
&= N(p_A + p_B - (p_A - p_B)^2) \\
&= N(p_A + p_B - (m(c_N)(p_A + p_B))^2) \\
&= N((p_A + p_B)(1 - m(c_N)^2(p_A + p_B))) \\
&= n(c_N, N) - O(N \cdot m(c_N)^2) = N \cdot s^* - o(N),
\end{aligned}$$

where the last equality is since $m(c_N)$ is diminishing.

From the calculations above,

$$\frac{\mu}{\sigma} = \frac{N \cdot s^* \cdot m(c_N)}{\sqrt{s^* \cdot N - o(N)}} \cong \sqrt{N}\sqrt{s^*} \cdot m(c_N), \tag{5}$$

where $x \cong y$ is a shorthand for $\lim_{N\to\infty} |x - y| = 0$. This is the main fact we need for all three cases.

Slow convergence Suppose that $\overline{c}$ converges slowly to c^*. Then by Lemma 14, $m(c_N) = \Theta(1)|c_N - c^*| = \omega(1/\sqrt{N})$, and from Eq. (5),

$$\frac{\mu}{\sigma} = \sqrt{N}\sqrt{s^*} \cdot \omega(1/\sqrt{N}) = \omega(1),$$

i.e., $\frac{\mu}{\sigma}$ goes to infinity as N grows. As a result, using Normal approximation of $V_A - V_B$, we have

$$\begin{aligned}
&\mathbb{WP}_A(I, N, c_N) \cong \Pr_{x\sim\mathcal{N}(\mu,\sigma^2)}(x > 0) = \Pr_{x\sim\mathcal{N}(0,1)}(x < \frac{\mu}{\sigma}) \Rightarrow \\
&\mathbb{WP}_A(I, N, c_N) \cong \Phi(\frac{\mu}{\sigma}) \xrightarrow[N\to\infty]{} 1,
\end{aligned}$$

where Φ is the CDF of the standard Normal distribution function.

Fast convergence This case is very similar, except now $m(c_N) = \Theta(1)|c_N - c^*| = o(1/\sqrt{N})$, which by Eq. (5) means $\frac{\mu}{\sigma} = \sqrt{N}\sqrt{s^*} \cdot o(1/\sqrt{N}) = o(1)$, i.e. in this case $\frac{\mu}{\sigma}$ goes to 0 as N grows, and thus

$$\mathbb{WP}_A(I, N, c_N) \cong \Phi(\frac{\mu}{\sigma}) \xrightarrow[N\to\infty]{} 0.5.$$

Moderate convergence In the knife-edge case where $cr(\overline{c}) = r(c^*)$ is a positive constant, we have by Lemma 14 that $m(c_N) = \Theta(1/\sqrt{N})$, and more precisely, that

$$m(c_N) = |c_N - c^*|(m^* + o(1)) = \frac{r(c^*) \cdot m^*}{\sqrt{N}}.$$

This means that the ratio $\frac{\mu}{\sigma}$ also has a finite positive limit, specifically

$$\frac{\mu}{\sigma} = \sqrt{N}\sqrt{s^*} \cdot m(c_N) = \sqrt{s^*} \cdot r(c^*)(m^* + o(1)),$$

and by continuity of Φ, we can define the constant $\phi(c^*) := \lim_{N\to\infty} \Phi(\frac{\mu}{\sigma}) = \Phi(\sqrt{s^*} \cdot r(c^*)m^*)$, and it holds that

$$\mathbb{WP}_A(I, N, c_N) \cong \Phi(\frac{\mu}{\sigma}) \xrightarrow[N\to\infty]{} \phi(c^*) < 1,$$

as required. □

B Parameters of Specific PPMs

B.1 Polynomial PPM

Lemma 12. *Let I be an issue with a Polynomial PPM and let $\overline{c} = (c_N)_N$ be a non-trivial equilibrium of I with limit c^*. Then $c_N = c^* + \Theta(N^{-\frac{\beta}{\alpha}})$.*

That is, the point c_N must be at distance that decreases proportionally to $N^{-\frac{\beta}{\alpha}}$: not closer neither farther away.

Proof. We will prove for $c_N > c^*$. The proof for $c_N < c^*$ is symmetric.

By Theorem 8, c^* is a pivot point.

Since the first and second derivatives of s_A, s_B are bounded, so are those of the functions $n(c)$ and $m(c)$ (as per Eq. (2),(3)).

We denote $m'(c) := \frac{\partial m(c)}{\partial c}$ and $n'(c) := \frac{1}{N}\frac{\partial n(c,N)}{\partial c}$, so neither function depends on N.

Also note that $n'(c) \geq 0$ everywhere and $m'(c^*) > 0, n'(c^*) > 0$ by the definition of pivot point. We also denote $s^* := s_A(c^*) + s_B(c^*)$ and note that $s^* \cdot N = n(c^*)$.

Therefore there must be some interval $[c^*, c^*+\delta]$ and constants $\overline{m}, \underline{m}, \overline{n}, \underline{n} > 0$ such that $m'(c) \in [\underline{m}, \overline{m}]$ and $n'(c) \in [\underline{n}, \overline{n}]$ for all $c \in [c^*, c^* + \delta]$. Moreover, $\underline{m}, \overline{m}$ can be arbitrarily close to $m'(c^*)$ and likewise for n' (the functions are nearly-linear near c^*). We consider only N's large enough so that c_N is in this interval, and so that $p(n(c_N), m(c_N)) < 1$.

Now, let $\varepsilon := c_N - c^*$, and w.l.o.g. $\varepsilon < \min\{0.1, 0.1/\overline{n}\}$. We want to show $\varepsilon = \Theta(\frac{1}{N^{\frac{\beta}{\alpha}}})$. From the definition of equilibrium,

$$\begin{aligned} c^* + \varepsilon = c_N = p(n(c_N), m(c_N)) &= \frac{1}{m(c_N)^{\alpha} n(c_N)^{\beta}} \\ &= \frac{1}{m(c^* + \varepsilon)^{\alpha} n(c^* + \varepsilon)^{\beta}} \Rightarrow \end{aligned}$$

$$m(c^* + \varepsilon) = \frac{1}{(c^* + \varepsilon)^{\frac{1}{\alpha}} n(c^* + \varepsilon)^{\frac{\beta}{\alpha}}}. \tag{6}$$

Now, by the constant bounds on the derivatives,

$$
\begin{aligned}
m(c^*+\varepsilon) &\in [m(c^*)+\underline{m}\varepsilon, m(c^*)+\overline{m}\varepsilon] &(7)\\
&= [\underline{m}\varepsilon, \overline{m}\varepsilon], &(8)\\
n(c^*+\varepsilon) &\in [n(c^*)+\underline{n}\varepsilon\cdot N, n(c^*)+\overline{n}\varepsilon\cdot N]\\
&= [(s^*+\underline{n}\varepsilon)N, (s^*+\overline{n}\varepsilon)N]\\
&\subset [s^*N, (s^*+0.1)N] &(9)
\end{aligned}
$$

For the upper bound, we get from Eqs.(6),(8),(9)

$$
\begin{aligned}
\underline{m}\varepsilon \le m(c^*+\varepsilon) &\le \frac{1}{(c^*)^{\frac{1}{\alpha}}(s^*N)^{\frac{\beta}{\alpha}}} \Rightarrow\\
\varepsilon \le \frac{1}{\underline{m}\cdot(c^*)^{\frac{1}{\alpha}}(s^*)^{\frac{\beta}{\alpha}}N^{\frac{\beta}{\alpha}}} &= O(\frac{1}{N^{\frac{\beta}{\alpha}}}),
\end{aligned}
$$

since $\underline{m}$, c^* and s^* are all constants. Likewise, for the lower bound,

$$
\begin{aligned}
\overline{m}\varepsilon \ge m(c^*+\varepsilon) \ge \frac{1}{(c^*+0.1)^{\frac{1}{\alpha}}((s^*+0.1)N)^{\frac{\beta}{\alpha}}} \Rightarrow\\
\varepsilon = \Omega(\frac{1}{N^{\frac{\beta}{\alpha}}}) \text{ as required.}
\end{aligned}
$$

□

Clearly the term "0.1" used in the proof is arbitrary and could be replaced by a smaller constant. Thus the lemma shows

$$c^* + m(c^*)^{-1}\cdot \underline{t}\cdot N^{-\frac{\beta}{\alpha}} < c_N < c^* + m(c^*)^{-1}\cdot \overline{t}\cdot N^{-\frac{\beta}{\alpha}},$$

where m' is the derivative of $m(c)$ at c^*. Moreover, from Eq. (7),

$$\underline{t}\cdot N^{-\frac{\beta}{\alpha}} < m(c_N) < \overline{t}\cdot N^{-\frac{\beta}{\alpha}},$$

which we use in the proof of Theorem 10. In the limit as N grows, both $\overline{t}, \underline{t}$ converge to

$$t^* = (c^*)^{-\frac{1}{\alpha}}(s^*)^{-\frac{\beta}{\alpha}}, \tag{10}$$

and thus $r^* = t^*\cdot\sqrt{s^*} = (c^*)^{-\frac{1}{\alpha}}(s^*)^{0.5-\frac{\beta}{\alpha}}$.

B.2 Altruist PPM

Proposition 11. *Let I be an issue with an Altruist PPM with function f, and let $\overline{c}$ be a non-trivial equilibrium. Then:*

1. *if $f = e^{\omega(n)}$ then $\overline{c}$ is a fixed equilibrium at q; else*
2. *if $f = \omega(\sqrt{n})$ then $\overline{c}$ converges slowly; else*
3. *if $f = \Theta(\sqrt{n})$ then $\overline{c}$ converges moderately; else*
4. *$\overline{c}$ is trivial.*

Note that there is no f for which there is a non-trivial equilibrium with fast convergence.

Proof. Case 1. To see that $c = q$ is an equilibrium, note that the margin at $m(q)$ is strictly less than 1 (say, $> 1 - \delta$) and this $p^{Bin}(n, m(q)) > \delta^n$. Also, $f(n) > e^{Z \cdot n}$ for any constant Z and in particular for $Z = -\log \delta$.

Thus

$$\begin{aligned} p^{alt(q,f)}(n, m(q)) &= \min\{q, f(n)\delta^n\} > \min\{q, e^{-\log \delta \cdot n}\delta^n\} \\ &= \min\{q, \delta^{-n}\delta^n\} = \min\{q, 1\} = q. \end{aligned}$$

That is, all voters with cost $c \leq q$ consider themselves pivotal.

For Cases 2 and 3, we know from Lemma 13 and Theorem 8 that $\bar{c}$ converges to a pivot point $c^* \in (0, 1)$.

For Case 2, assume towards a contradiction that convergence is (at least) moderate, i.e., there is a constant $X > 0$ s.t. $|c_N - c^*| < X/\sqrt{N}$ for all N.

Now, we have that in equilibrium $c = c_N$ with $n := n(c_N, N)$ and

$$\varepsilon := m(c_N) \leq \sup_c m'(c) \cdot |c_N - c^*| \leq Y/\sqrt{N},$$

for some constant Y, due to bounded derivatives.

Set $Z := e^{Y^2}$. Since $f(n) = \omega(\sqrt{n})$, for any sufficiently large n, we have $f(n) > 4Z\sqrt{n}$. Note that the number of active voters at the pivot point c^* is $s^* \cdot N$, and that $n(c_N) = s_N \cdot N \cong s^* N$.

Note that for all n,

$$\begin{aligned} p^{Bin}(2n+1, 2\varepsilon) &= \binom{2n}{n}(\frac{1}{2} + \varepsilon)^n(\frac{1}{2} - \varepsilon)^n \\ &> \frac{4^n}{2\sqrt{\pi n}}(\frac{1}{4} - \varepsilon^2)^n \\ &> \frac{1}{\sqrt{n}}(1 - \varepsilon^2)^n > \frac{1}{\sqrt{n}}(1 - (Y/\sqrt{N})^2)^n \\ &= \frac{1}{\sqrt{n}}(1 - \cdot Y^2/N)^n \geq \frac{1}{\sqrt{n}}(1 - \frac{Y^2}{n})^n \\ &> \frac{1}{2\sqrt{n}}e^{-Y^2} \end{aligned}$$

for sufficiently large n.

Finally, pick N large enough so that $n = n(c_N)$ respects the conditions above, then

$$\begin{aligned} 1 > q \geq c_N &= p^{alt(q,f)}(n, \varepsilon) = p^{Bin}(n, \varepsilon)f(n) \\ &> p^{Bin}(n, \varepsilon)Z\sqrt{n} \\ &> \frac{1}{2\sqrt{n/2}}e^{-Y^2} \cdot 4Z\sqrt{n} \\ &> e^{-Y^2} \cdot Z = 1, \end{aligned}$$

which is a contradiction.

Case 3. We start from $p^{Bin}(n, \varepsilon) = O(1)\frac{1}{\sqrt{n}}(1 - \varepsilon^2)^n$, which means $p^{alt(f)}(n, \varepsilon) = O(1)(1 - \varepsilon^2)^n$. Since this converges to some pivot point c^*, we have that

$$(1 - \varepsilon^2)^n \to Z,$$

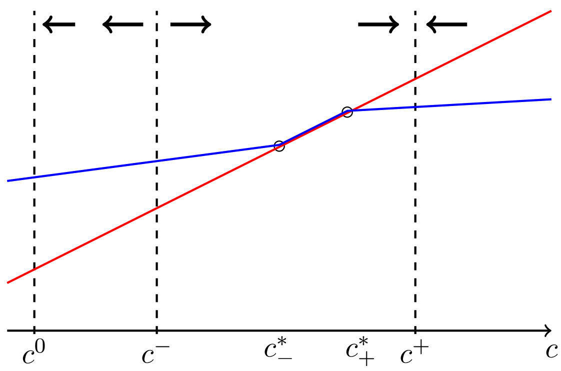

Figure 3: Pivot points and equilibria when the support functions are partially overlapping.

for some constant $Z < 1$. Thus the margin at equilibrium holds

$$\varepsilon = \varepsilon(C_N) \cong \sqrt{1 - Z^{1/n}} = \sqrt{\frac{1}{n} \log \frac{1}{Z}} = \Theta(\sqrt{1/N}),$$

where we use the fact that $\lim_{n\to\infty}(1 - Z^{1/n})n = \log \frac{1}{Z}$. Finally, the gap $|c_N - c^*|$ is linear in the margin ε due to bounded derivatives.

Case 4 follows immediately from Lemma 13 and Prop. 7, as the PPM is strongly vanishing. $\square$

C Partially-coinciding Support Functions

The assumption of a finite number of pivot points for s_A, s_B is somewhat restrictive. Suppose there are segments of the cost distribution where voters are equally likely to prefer A or B: this would translate to intervals $[c^*_-, c^*_+]$ where $s_A(c) = s_B(c)$ for all $c \in [c^*_-, c^*_+]$. See Fig. 3. By our assumption on the introduction the support functions should be differentiable near the pivot points (so the figure is not allowed), but we actually only need them to have bounded derivative in some *open right environment* of c^*_+, and likewise for the left pivot point. In other words, the left- and right- derivative at c^* could be different but has to exist.

The effect on our result in Theorem 8 is that the right equilibrium will converge to c^*_+ (from the right), and the left equilibrium will converge to c^*_- from the left. The proof itself remains unchanged.

Can points on the interval $[c^*_-, c^*_+]$ be an equilibrium? note that since $m(c) = 0$ for all $c \in [c^*_-, c^*_+]$, we have $p(n, m(c)) = p(n, 0) > q$ by q-tie-sensitivity, and for sufficiently large N we should get $p(n(c, N), 0) = q$. We conclude that if the interval contains q then q will also be an equilibrium. Also note that this equilibrium will be stable, as with any small perturbation all voters still think they are q-pivotal.

An assumption we cannot relax is that the margin at the [right side of the right-] pivot point must be strictly increasing. If the support functions have the same asymptote at c^* then equilibrium will still exist but we cannot say anything about the rate of convergence of c_N to c^*, or on the winning probability.

D Stability

Recall that every agent has two (undominated) actions: vote/abstain. A Profile is a mapping from the nonatomic set of agents to a subset of active voters. So for a given election, we can think of every profile

P as a subset of agents. Every agent in profile P has a best response (to P), which is either to keep her action or switch to the other action. We thus denote by $BR(P)$ the profile obtained from P if all voters simultaneously switch to their best-response. Note that the Bayes-Nash equilibria are those profiles where $BR(P) = P$.

Definition 9. *Let P, Q be two profiles.*

1. *Write $d(P, Q)$ to denote their* difference*, i.e. the fraction of agents playing differently.*
2. *The ε*-neighborhood *of a profile P is the set of profiles with difference at most ε.*

Recall that P is a *threshold profile* if there is c such that exactly all voters with $c_i \leq c$ vote in P. In such a case we denote the threshold by $c(P)$.

An easy observation is that for any P, $BR(P)$ is a threshold profile. This is since in P there is some expected number of active voters n_P and some expected margin m_P and thus the threshold of $BR(P)$ would be $c(BR(P)) = p(n_P, m_P)$.

In particular, $BR(P)$ for some threshold profile with $c(P) = a$ is also a threshold profile with some $c(BR(P)) = b$. We then denote $b = BR(a)$.

Definition 10. *We say that a threshold profile P is* moving towards *a threshold q if both q and $c(BR(P))$ are on the same side of $c(P)$.*

Definition 11 (Stable equilibrium)**.** *An election equilibrium Q with threshold c_N is* stable*, if there is some $\varepsilon > 0$ such that for any profile P in a ε-neighborhood of Q:*

- *If P is a threshold profile then it is moving towards c_N;*
- *Otherwise, $BR(P)$ (which is a threshold profile) is moving towards c_N.*

This means that when starting from profile P, a best response by few agents will get closer to Q, so it will still be in the ε-neighborhood of Q and will continue to get closer in every step until convergence.

Lemma 15. *There is some interval $Z = [c^*, c^* + z]$ where for every N, $p(n(c, N), m(c))$ is monotonically decreasing in $c \in Z$. Moreover, for every $\varepsilon > 0$ there is sufficiently large N_ε such that*

1. *$p(n(c, N_\varepsilon), m(c))$ is strictly decreasing in c in the range $c \in [c^* + \varepsilon, c^* + z]$;*
2. *$c_{N_\varepsilon} \in (c^* + \varepsilon, c^* + z/2]$.*

Proof. Note that $n(c, N)$ is non-decreasing in c everywhere. In addition, $m(c^*) = 0$, and by our assumption that s_A, s_B have different derivatives at c^*, we get $m'(c^*) > 0$. So from our bounded derivative assumption m must be strictly increasing in some range $[c^*, c^* + z]$ that is independent of N. Since $p(n, m)$ is monotonically non-increasing in both parameters, p is non-increasing in c in the entire range $[c^*, c^* + z]$, and this holds for every N.

Showing that it strictly decreasing is not immediate because in principle it could equal the maximum value in the entire range. Indeed this is the case for the Polynomial PPM if N is low. However, from weak vanishing pivotality, we know that $p(n(c^* + \varepsilon, N), m(c^* + \varepsilon))$ goes to 0 with N, and thus drops below the maximal value for sufficiently large N_ε. From our assumption on $p(n, m)$, it is strictly decreasing in both parameters below its maximal value, and thus strictly decreasing in c in the range $(c^* + \varepsilon, c^* + z]$ whenever $N \geq N_\varepsilon$.

For the minimal ε for which this holds, we have $p((n(c^* + \varepsilon, N_\varepsilon), m(c^* + \varepsilon)) = 1 > c_N$ and so c_N must be more to the right, where the pivotality decreases.

We increase N further as needed until c_N is in the range $(c^* + \varepsilon, c^* + z/2)$ (must occur eventually as the limit of $(c_N)_N$ is c^*). $\square$

The next definition generalizes Def. 7:

Definition 12 (Stable equilibrium)**.** *An election equilibrium Q with threshold c_N is stable, if there is some $\varepsilon > 0$ such that for any profile P in a ε-neighborhood of Q:*

- *If P is a threshold profile then it is moving towards c_N ;*
- *Otherwise, $BR(P)$ (which is a threshold profile) is moving towards c_N.*

Theorem 9. *Let $\overline{c}$ be a right-equilibrium of issue I with limit $c^* > 0$. Then for sufficiently large N, c_N is stable.*

Proof. We first explain how to select N. Suppose that N is sufficiently large so that by Lemma 15, p is strictly decreasing in c in some range $(c^* + \varepsilon', c^* + z]$ and $c_N \in (c^* + \varepsilon', c^* + z/2]$.

Consider a profile P in the ε-neighborhood of c_N, where ε is determined below.

Threshold profile: Suppose first that P is a threshold profile with threshold $\hat{c}$. Then choose ε small enough so that $|c_N - \hat{c}| < \min\{c_N - (c^* + \varepsilon'), z/2\}$. This guarantees that:

$$\hat{c} > c_N - (c_N - (c^* + \varepsilon') = c^* + \varepsilon',$$
$$\hat{c} < c_N + z/2 < c^* + z.$$

Since both $\hat{c}, c_N$ are in the interval $[c^* + \varepsilon', c^* + z]$ where p is strictly decreasing, when $\hat{c} < c_N$:

$$BR(\hat{c}) = p(n(\hat{c}), m(\hat{c})) > p(n(c_N), m(c_N)) = c_N > \hat{c},$$

so both $c_N, BR(\hat{c})$ are above $\hat{c}$. Similarly, when $\hat{c} > c_N$ then both would be below.

Non-threshold profile: Next, suppose that P is *not* a threshold profile. Then we only need to show that $\hat{c} = c(BR(P))$ is also in the interval $[c^* + \varepsilon', c^* + z]$.

Denote $\varepsilon := (c_N - (c^* + \varepsilon'))/t$, where t will be determined later.

Denote by $\hat{n}, \hat{m}$ the expected number of voters and expected margin in the current profile P. Note that

$$\hat{n} \in (1 - \varepsilon)(s_A(c_N) + s_B(c_N))N \pm \varepsilon \cdot N \subseteq n(c_N, N) \pm \varepsilon$$

and similarly

$$\hat{m} \in \frac{|s_A(c_N) - s_B(c_N)| \pm \varepsilon}{s_A(c_N) + s_B(c_N) \pm \varepsilon} \subseteq m(c_N) \pm 2\varepsilon \cdot s',$$

where $s' = s_A(c_N) + s_B(c_N)$ is a constant.

Now set $\hat{c} := p(\hat{n}, \hat{m})$ and note that since p has bounded derivatives,

$$\hat{c} = p(\hat{n}, \hat{m}) \in p(n(c_N, N), m(c_N)) \pm O(\varepsilon) = c_N \pm h \cdot \varepsilon,$$

for some constant h. We now set $t > h$, so that

$$\hat{c} > c_N - h \cdot \varepsilon > c_N - t \cdot \varepsilon = c_N - (c_N - (c^* + \varepsilon')) = c^* + \varepsilon',$$

and

$$\begin{aligned} \hat{c} &< c_N + h \cdot \varepsilon \\ &< c_N + (c_N - c^*) \\ &= c^* + 2(c_N - c^*) \\ &< c^* + 2(z/2) \\ &= c^* + z. \end{aligned}$$

Thus $\hat{c}$ (which is the threshold of the best-response to profile P) is still on the monotone interval and the previous part of the proof shows it is moving towards c_N.

□

E Calculus of Voting PPM

Example 6 (CoV PPM). *The Calculus of Voting PPM (note it also depends on the size of the entire population N)*

$$\begin{aligned} p^{CoV}(n,m,N) &:= E_{n'\sim Bin(N,n/N)}[p^{Bin}(n',m)] \\ &= E_{n'\sim Bin(N,n/N)}[\Pr_{x\sim Bin\left(n',(1+m)/2\right)}(x = \lfloor n'/2 \rfloor)]. \end{aligned} \tag{11}$$

Note that from Equations (2),(3) we can also derive the expected fraction of A and B supporters from n, m and N, as

$$s_A, s_B = \frac{n^2 \pm mN^2}{2n^2} = \frac{1}{2} \pm (\frac{N}{n})^2 m.$$

Proposition 16. *The following three terms coincide:*

1. *The probability of a tie, i.e. $Pr(V_A = V_B | I, N, c)$;*
2. *$p^{CoV}(n,m,N)$;*
3. *$Pr_{(V_A,V_B,V_0)\sim Mult(N,(s_A,s_B,1-s_A-s_B))}[V_A = V_B]$.*

Proof. There is a tie if there is exactly the same number of active A voters and B voters. One way to compute this is to first sample n' active voters by flipping a coin for each of the N voters. Then for each of the n' active voters decide (w.p. s_A vs. s_B) if she is an A or B supporter. There is a tie if there are exactly $n'/2$ A supporters, which happens w.p. $p^{Bin}(n',m)$.

Alternatively, we could just decide for each of the N voters whether she is an active A supporter (which occurs w.p. $s_A = Pr_{(c_i,T_i)\sim\mathcal{D}}[i \text{ is active and supports } A]$), an active B supporter, or inactive. Then we check if the two (correlated) multinomial variables V_A and V_B are the same. □

Proposition 17. *The CoV PPM has strongly vanishing pivotality. I.e., for every fixed c, $\lim_{N\to\infty} Pr(V_A = V_B | I, N, c) = 0$. Moreover,*

- *$p^{CoV}(n,m,N) = \Theta(\frac{1}{\sqrt{n}})$ if $m = 0$; and*
- *$p^{CoV}(n,m,N) = exp(-\Theta(n))$ if $m > 0$.*

(assuming the ratio n/N remains fixed)

Proof. As N goes to infinity, w.h.p we have $n' \in [0.99n, 1.01n]$. Also note that m remains fixed. Thus

$$Pr(V_A = V_B | I, N, c) \le p^{Bin}(0.99n, m) + exp(\Theta(-N)),$$

and

$$Pr(V_A = V_B | I, N, c) \ge p^{Bin}(1.01n, m) - exp(\Theta(-N)).$$

Now we have $p^{Bin}(n,m) = \Theta(\frac{1}{\sqrt{n}})$ for $m = 0$ and $p^{Bin}(n,m) = exp(-\Theta(n))$ for $m > 0$, and changing n by a constant fraction does not change the asymptotic result. □

E.1 Poisson PPM

Example 7 (Poisson PPM)**.** *The Poisson PPM considers the perceived pivotality as the probability that an equal number of supporters are drawn from Poisson distributions with parameters $N \cdot s_A$ and $N \cdot s_B$:*

$$p^{Poi}(n, m) := \Pr_{\substack{X_A \sim \mathit{Poisson}((1+m)n/2) \\ X_B \sim \mathit{Poisson}((1-m)n/2)}} (X_A = X_B) \tag{12}$$

Note that while s_A, s_B and N cannot be inferred from n, m, the Poisson parameters can be inferred as $s_A \cdot N = \frac{s_A}{s_A+s_B} n = \frac{1+m}{2} n$, and $s_B \cdot N = \frac{1-m}{2} n$.

F Diverse PPM

Suppose that each type includes not just voting cost c_i and preferred candidate T_i, but also i's own pivotality estimation function $p_i(n, m)$, which we still assume to be continuous and decreasing in both parameters.

We fix an election (I, N), where I is a distribution over types.

Proposition 18. *Every election (I, N) has a pure Bayes-Nash equilibrium.*

Proof. The functions $s_A(c)$ and $s_B(c)$ become useless, since there is no meaningful cost threshold that applies to all agents.

Also let a, b be the fraction of core supporters of A and B, respectively. However, we will maintain the notation s_A, s_B for the fraction of all voters who actively vote for A and B, respectively.

Note that

$$(s_A, s_B) \in \Delta := \{(x, y) : (x \geq a) \wedge (y \geq b) \wedge (x + y \leq 1)\}.$$

Given a pair of numbers s_A, s_B, we can still compute the expected number of active voters and expected margin as in Eq. (2),(3) (recall that N is fixed):

$$n(s_A, s_B) := (s_A + s_B)N; \qquad \text{(active voters)} \tag{13}$$

$$m(s_A, s_B) := \frac{|s_A - s_B|}{s_A + s_B}. \qquad \text{(expected margin)} \tag{14}$$

We now define functions S_A, S_B which map n and m to $(s_A, s_B) \in \Delta$, by integrating over the distribution I:

$$S_A(n, m) := Pr_{i \sim I}[c_i \leq p_i(n, m) \wedge T_i = A];$$
$$S_B(n, m) := Pr_{i \sim I}[c_i \leq p_i(n, m) \wedge T_i = B];$$

Finally we get a function $F : \Delta \to \Delta$ defined as

$$F(s_A, s_B) := (S_A(n(s_A, s_B), m(s_A, s_B)), S_B(n(s_A, s_B), m(s_A, s_B)))\,.$$

Since I is atomless, p_i are continuous, and $n(), m()$ are continuous, so are $S_A()$ and $S_B()$. Thus F is a continuous function from a compact and convex set Δ onto itself. From Brouwer's fixed point theorem, F has a fixed point. This point is a pure Nash equilibrium of (I, N). ☐