Anonymous and neutral classification aggregation

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Based on previous results in preference aggregation, we explore the possibility of defining anonymous and neutral aggregators in classification aggregation. We find a necessary condition on the number of individuals, objects, and categories for the existence of anonymous and neutral aggregators and propose such aggregators. We prove that this condition is tight for an equal number of individuals and objects. Experimental evidence suggests this tightness extends to cases with different numbers of individuals and objects, though this remains a conjecture requiring formal proof.

1. Introduction

The classification problem consists of mapping a set of objects into a set of categories. It has been broadly studied, particularly from a machine learning perspective [Kotsiantis et al., 2007, Soofi and Awan, 2017]. We take a collective choice perspective with the goal of aggregating a list of individual classifications into a single one. This model is introduced by Maniquet and Mongin [2016], which follows advances in aggregation of equivalence relations [Mirkin, 1975, Fishburn and Rubinstein, 1986, Dimitrov et al., 2012] and group identification [Kasher and Rubinstein, 1997].

A popular activity that exemplifies classification aggregation is aggregating tier-lists in gaming.[1] The goal is usually to classify some video game characters or items into a predefined set of ranked categories, i.e., tiers. The tiers are generally fixed and a tier-list that leaves a tier empty is usually considered inconsistent, which is in line with the surjectivity condition that is usually imposed over classifications.[2]

The classification aggregation setting has a strong relation to the preference aggregation problem where individual preferences on a set of alternatives are aggregated into a societal preference. This well-known and broadly studied model exhibits a plethora of impossibilities, starting with the seminal work of Arrow [1951], which shows that on an unrestricted domain of preferences, pairwise independence and unanimity are incompatible with non-dictatorial aggregation. Subsequent works extensively studied related axiomatics, such as Wilson [1972], which demonstrates a stronger impossibility under independence and citizen sovereignty. In a similar spirit, Sen [1970] proves that no aggregator can jointly satisfy unanimity and another axiom called liberalism. Later, Moulin [1983] and Moulin [1991] characterize the numbers of individuals and alternatives that allow for

[1] `https://tiermaker.com`

[2] Another example of application, which gained wide public attention in France, is a website called "De droite ou de gauche?" [Delemazure, 2023] that was created to categorize any user-inserted term as right-wing or left-wing. Initially based on GPT, it later collected user opinions to resolve classifications.

anonymous, neutral and resolute social choice rules. This characterization, followed by an analysis of Campbell and Kelly [2015], suggests a major difficulty in defining resolute social choice rules that are both anonymous and neutral. As shown by Doğan and Giritligil [2022] and Bubboloni and Gori [2014], this difficulty extends to social welfare functions that assign a strict ordering to each preference profile.

A considerable part of the literature in classification aggregation studies whether these impossibility results resonate in that setting. Maniquet and Mongin [2016] establishes an *Arrovian* impossibility by showing that independent and unanimous classification aggregation functions are necessarily dictatorial. Alcantud et al. [2019] shows the existence of an equivalent impossibility for fuzzy classification aggregation where objects have a degree of membership in categories. By replacing unanimity with citizen sovereignty, Cailloux et al. [2024] delivers a *Wilsonian* impossibility that generalizes that result of Maniquet and Mongin [2016]. Fioravanti [2024a] proves another impossibility inspired by Sen [1970], replacing the liberalism axiom by variations of an expertise axiom that allows one individual to unilaterally decide on one object's classification.

A natural question is to ask what the axiomatics of Moulin [1983] implies in the context of classification aggregation. We prove an equivalent of the Moulin impossibility for classification aggregation functions. We also propose, for certain sizes of the problem, anonymous and neutral aggregation rules based on aggregators used by Doğan and Giritligil [2022] and Ozkes and Sanver [2024].

Our analysis presents a formal connection between the classification aggregation and preference aggregation frameworks. In particular, we demonstrate that the problem of preference aggregation with an imposed number of equivalence classes is isomorphic to classification aggregation. Dimitrov et al. [2012, p.192] hints at this remark, without a formal explanation, by observing that "The critical difference of [classification aggregation] and the Arrovian framework of individual preference aggregation is that an equivalence relation does not rank the equivalence classes it contains". As a matter of fact, imposing a maximal number of equivalence classes for preference aggregation has been broadly studied in the literature with examples such as approval voting [Brams and Fishburn, 1978], range voting [Smith, 2000, Macé, 2018], and majority judgment [Balinski and Laraki, 2011]. The only work we know that imposes a fixed number of equivalence classes is by Maniquet and Mongin [2015], which focuses on approval voting with at least one approved and one disapproved candidate, thus two equivalence classes.

Section 2 introduces the notation, notions and axioms. Section 3 establishes the relation between classification aggregation and preference aggregation. Section 4 exploits this relation for the special case where the number of objects and categories coincide. Section 5 presents some impossibility theorems for the general case. Section 6 defines anonymous and neutral classification aggregation rules of interest. Section 7 concludes.

2. Preliminaries

We consider a set $N = \{1, \ldots, n\}$ of *individuals* with $n \geq 2$, a set $P = \{p_1, \ldots, p_\rho\}$ of *categories* with $\rho \geq 2$, and a set $X = \{x_1, \ldots, x_m\}$ of *objects* with $m \geq \rho$.

We define a *classification* as a surjective mapping $c : X \to P$ and denote by $\mathcal{C} \subset P^X$ the set of classifications. We write $\boldsymbol{c} = (c_1, \ldots, c_n) \in \mathcal{C}^N$ for a classification profile. Given $\boldsymbol{c} \in \mathcal{C}^N$ and $x \in X$, we write $\boldsymbol{c}_x \in P^N$ for the vector of categories that object x is put into by each individual, thus $\forall i \in N, \boldsymbol{c}_x(i) = c_i(x)$. A *classification aggregation function* (CAF) is a mapping $\alpha : \mathcal{C}^N \to \mathcal{C}$.

Let S_A denote the set of permutations of the set A. Note that a classification profile $\boldsymbol{c}$ can be seen as a function from N to $\mathcal{C}$ or from X to P^N or from $N \times X$ to P. Using this fact, given $\pi \in S_P, \sigma \in S_X$ we will write $\pi \circ \boldsymbol{c}$ to denote the classification profile equal to $(\pi \circ c_i)_{i \in N}$ and $\boldsymbol{c} \circ \sigma$ to denote $(\boldsymbol{c}_{\sigma(x)})_{x \in X}$. Moreover, given $\gamma \in S_N, \boldsymbol{c} \in \mathcal{C}^N$, we let $\boldsymbol{c}^{(\gamma)}$ denote the classification profile

$(c_{\gamma^{-1}(i)})_{i\in N}$.

We give three definitions that are central to our analysis. The first one is the standard equal treatment of voters condition of collective choice theory.

Definition 1. A CAF α is *anonymous* if $\forall \pi \in S_N, \forall \boldsymbol{c} \in \mathcal{C}^N, \alpha(\boldsymbol{c}^{(\pi)}) = \alpha(\boldsymbol{c})$.

Next, we impose the standard "equal treatment of alternatives" condition of collective choice theory to objects and categories separately.

Definition 2. A CAF α is *object neutral* if $\forall \pi \in S_X, \boldsymbol{c} \in \mathcal{C}^N, \alpha(\boldsymbol{c} \circ \pi) = \alpha(\boldsymbol{c}) \circ \pi$.

Definition 3. A CAF α is *category neutral* if $\forall \pi \in S_P, \forall \boldsymbol{c} \in \mathcal{C}^N, \alpha(\pi \circ \boldsymbol{c}) = \pi \circ \alpha(\boldsymbol{c})$.

3. Relation between ordinal preferences and classifications

3.1. Preference aggregation

We consider a set N of individuals and a set X of objects, with $m = |X| \geq 2$. We let $\mathcal{L}(X)$ be the set of linear orders, i.e., complete, transitive and asymmetric binary relations on X. Given a individual $i \in N$, we write $P_i \in \mathcal{L}(X)$ for the preference of individual i on X. An n-tuple P_N in $\mathcal{L}(X)^N$ is called a preference profile.

A *social welfare function* (SWF) is a mapping from $\mathcal{L}(X)^N$ to $\mathcal{L}(X)$. We now write the standard anonymity and neutrality conditions introduced in Moulin [1980], naming these *welfare anonymity* and *welfare neutrality* so as not to create confusion with the axioms we introduced earlier for classification aggregation. Given a preference profile $P_N \in \mathcal{L}(X)^N$ and a permutation $\gamma \in S_N$, we write $\gamma(P_N) = (P_{\gamma^{-1}(i)})_{i\in N}$ for the profile induced by the permutation of individuals γ. A SWF f is said to satisfy *welfare anonymity* if $\forall \gamma \in S_N, \forall P_N \in \mathcal{L}(X)^N, f(\gamma(P_N)) = f(P_N)$.

By abuse of notation, given an individual preference $P_i \in \mathcal{L}(X)$ and a permutation $\pi \in S_X$, let $\pi(P_i)$ be such that $\pi(x)\ \pi(P_i)\ \pi(y) \iff x\ P_i\ y$ for all $x, y \in A$. Given a preference profile $P_N \in \mathcal{L}(X)^N$ and a permutation $\pi \in S_X$, we write $\pi(P_N) = (\pi(P_i))_{i\in N}$ for the profile induced by a permutation of objects π in P_N. A SWF f is said to satisfy *welfare neutrality* if $\forall \pi \in S_X, \forall P_N \in \mathcal{L}(X)^N, f(\pi(P_N)) = \pi(f(P_N))$.

We introduce $\mathcal{W}_\rho(X)$ to be the set of weak orders of X with exactly ρ equivalence classes. A ρ-SWF is a mapping from $\mathcal{W}_\rho(X)^N$ to $\mathcal{W}_\rho(X)$.

3.2. An isomorphism to classification aggregation

We build a bijection from $\mathcal{C}$ to $\mathcal{W}_\rho(X)$, which induces a bijection from the set of CAFs to the set of SWFs with a fixed number of equivalence classes.

Given $P = \{p_1, \ldots, p_m\}$ and an arbitrary strict ordering $\succ$ of P, we can define $\theta : \mathcal{C} \to \mathcal{W}_\rho(X)$ as $\forall c \in \mathcal{C}, \forall x, y \in X, x\ \theta(c)\ y$ *iff* $c(x) \succ c(y)$. One can check that $\theta(c) \in \mathcal{W}_\rho(X)$ and that θ is a bijection. Note that for $m = \rho$, we have $\mathcal{L}(X) = \mathcal{W}_\rho(X)$, so there exists a bijection from $\mathcal{C}$ to $\mathcal{L}(X)$. In what follows, we let θ be any bijection from $\mathcal{C}$ to $\mathcal{W}_\rho(X)$ and φ any bijection from $\mathcal{W}_\rho(X)$ to $\mathcal{C}$.

Note that the bijection θ from $\mathcal{C}$ to $\mathcal{W}_\rho(X)$ induces a bijection from $\mathcal{C}^N$ to $\mathcal{W}_\rho(X)^N$ and therefore from the set of CAFs to the set of SWFs with a fixed number of equivalence classes. By abuse of notation, given a CAF α, we let $\theta(\alpha)$ denote the SWF defined as $\forall \boldsymbol{c} = (c_i)_{i\in N} \in \mathcal{C}^N, \theta(\alpha)((\theta(c_i))_{i\in N}) = \theta(\alpha(\boldsymbol{c}))$. Likewise, given an SWF f, we define $\varphi(f)$ as $\forall P_N \in \mathcal{L}(X), \varphi(f)((\varphi(P_i))_{i\in N}) = \varphi(f(P_N))$.

4. Equal number of objects and categories

We exploit the relation between preference aggregation and classification aggregation discussed in section 3 to derive results for the case $m = \rho$. By using the bijection between preference and classification profiles, we can extend Doğan and Giritligil's [2022] and Bubboloni and Gori's [2014] results to classification aggregation: we prove that an anonymous and object neutral CAF exists iff 1 is the only divisor of n that is smaller than m. To show this, we start by establishing the equivalence between the anonymity of a CAF and the welfare anonymity of the SWF to which this CAF is isomorphic. In this section, we let θ be a bijection from $\mathcal{C}$ to $\mathcal{L}(X)$ and φ be a bijection from $\mathcal{L}(X)$ to $\mathcal{C}$.

Proposition 1. *A CAF α satisfies anonymity iff $\theta(\alpha)$ satisfies welfare anonymity.*

Proof. Let α be a CAF.

$\Rightarrow$ Suppose α satisfies anonymity, and let $\gamma \in S_N$ be a permutation of N. We have $\forall \boldsymbol{c} = (c_i)_{i\in N} \in \mathcal{C}^N, \alpha((c_{\gamma^{-1}(i)})_{i\in N}) = \alpha(\boldsymbol{c})$. Let $P_N \in \mathcal{L}(X)$ be any preference profile, then $\exists \boldsymbol{c} \in \mathcal{C}, \theta(\boldsymbol{c}) = P_N$ as θ is a bijection. Therefore, $\theta(\alpha)(\gamma(P_N)) = \theta(\alpha)((\theta(c_{\gamma^{-1}(i)}))_{i\in N}) = \theta(\alpha(\boldsymbol{c}^{(\gamma)})) = \theta(\alpha(\boldsymbol{c})) = \theta(\alpha)((\theta(c_i))_{i\in N}) = \theta(\alpha)(P_N)$ as α satisfies anonymity, so $\alpha(\boldsymbol{c}^{(\gamma)}) = \alpha(\boldsymbol{c})$. Then, $\theta(\alpha)$ satisfies welfare anonymity.

$\Leftarrow$ Suppose $\theta(\alpha)$ satisfies welfare anonymity, and let $\boldsymbol{c} = (c_i)_{i\in N} \in \mathcal{C}^N$ be a classification profile and $\gamma \in S_N$ be a permutation. As $\theta(\alpha)$ satisfies welfare anonymity, we have $\theta(\alpha)(\gamma((\theta(c_i))_{i\in N})) = \theta(\alpha)((\theta(c_i))_{i\in N})$.

Then $\theta(\alpha(\boldsymbol{c}^{(\gamma)})) = \theta(\alpha)(\gamma((\theta(c_i))_{i\in N})) = \theta(\alpha)((\theta(c_i))_{i\in N}) = \theta(\alpha(\boldsymbol{c}))$. As θ is a bijection, $\theta(\alpha(\boldsymbol{c}^{(\gamma)})) = \theta(\alpha(\boldsymbol{c}))$ implies $\alpha(\boldsymbol{c}^{(\gamma)}) = \alpha(\boldsymbol{c})$, so α satisfies anonymity. ☐

Corollary 1. *A SWF f satisfies welfare anonymity iff $\varphi(f)$ satisfies anonymity.*

In a similar vein, we establish the equivalence between the object neutrality of a CAF and the welfare neutrality of the SWF to which this CAF is isomorphic.

Proposition 2. *A CAF α satisfies object neutrality iff $\theta(\alpha)$ satisfies welfare neutrality.*

Proof. Suppose α satisfies object neutrality, and let $\pi \in S_X$ be a permutation of X.

We have $\forall \boldsymbol{c} \in \mathcal{C}^N, \alpha((c_i \circ \pi)_{i\in N}) = \alpha(\boldsymbol{c}) \circ \pi$. Now take $P_N = (P_i)_{i\in N} \in \mathcal{L}(X)^N$, as θ is a bijection, $\exists \boldsymbol{c} = (c_i)_{i\in N} \in \mathcal{C}^N, \forall i \in N, \theta(c_i) = P_i$.

Now, $\theta(\alpha)(\pi(P_N)) = \theta(\alpha)(\pi((\theta(c_i))_{i\in N})) = \theta(\alpha)((\theta(c_i \circ \pi))_{i\in N}) = \theta(\alpha((c_i \circ \pi)_{i\in N})) = \theta(\alpha(\boldsymbol{c}) \circ \pi) = \pi(\theta(\alpha)(P_N))$. Then, $\theta(\alpha)$ satisfies welfare neutrality.

Suppose $\theta(\alpha)$ satisfies welfare neutrality, and let π be a permutation of X.

We have $\forall P_N \in \mathcal{L}(X)^N, \theta(\alpha)(\pi(P_N)) = \pi(\theta(\alpha)(P_N))$. Now, let $\boldsymbol{c} = (c_i)_{i\in N} \in \mathcal{C}^N$, and consider $P_N = (\theta(c_i))_{i\in N} \in \mathcal{L}(X)^N$.

We have $\theta(\alpha((c_i \circ \pi)_{i\in N})) = \theta(\alpha)((\theta(c_i \circ \pi)_{i\in N}) = \theta(\alpha)((\pi(\theta(c_i)))_{i\in N}) = \theta(\alpha)(\pi(P_N)) = \pi(\theta(\alpha)(P_N)) = \theta(\pi(\alpha(\boldsymbol{c})))$. Then, as θ is a bijection, $\alpha((c_i \circ \pi)_{i\in N}) = \alpha(\boldsymbol{c}) \circ \pi$, so α satisfies object neutrality. ☐

Corollary 2. *A SWF f satisfies welfare neutrality iff $\varphi(f)$ satisfies object neutrality.*

We now quote the result of Bubboloni and Gori [2014] for preference aggregation.

Theorem 1 (Bubboloni and Gori [2014])**.** *Given $n, m \geq 2$, there exists a welfare anonymous and welfare neutral SWF iff all prime divisors of n exceed m.*

The following result is the counterpart of Bubboloni and Gori [2014] for classification aggregation.

Theorem 2. *Given $n \geq 2, m = \rho \geq 2$, there exists an anonymous and object neutral CAF iff all prime divisors of n exceed m.*

Proof. Let α be a CAF and θ be a bijection from $\mathcal{C}$ to $\mathcal{L}(X)$. proposition 1 states that α is anonymous iff $\theta(\alpha)$ is welfare anonymous while by proposition 2, α is object neutral iff $\theta(\alpha)$ is welfare neutral.

Then there exist an anonymous and object neutral CAF iff there exist a welfare anonymous and welfare neutral SWF. Thus, applying theorem 1, there exists an anonymous and object neutral CAF iff all prime divisors of n exceed m.

□

Doğan and Giritligil [2022] introduced a SWF in their Theorem 3.1 that they proved to be well-defined and to satisfy welfare anonymity and welfare neutrality whenever all prime divisors of n exceed m. We refer to it as the greedy SWF.

Algorithm 1 The greedy SWF

```
while ∃P_N ∈ L(X)^N s.t. f(P_N) is not defined do
    Pick P_N ∈ L(X)^N s.t. f(P_N) is not defined, and l ∈ L(X)
    Set f(P_N) = l
    for P̃_N ∈ L(X)^N s.t. ∃π ∈ S_X, γ ∈ S_N, ∀i ∈ N, P̃_i = π(P_{γ^{-1}(i)}) do
        f(P̃_N) = π(l)
    end for
end while
```

We now introduce the greedy anonymous and neutral CAF which is isomorphic to the greedy SWF. TAs the greedy SWF is welfare anonymous and welfare neutral, by corollary 1 and corollary 2, we get proposition 3.

Algorithm 2 The greedy CAF

```
while ∃c ∈ C^N s.t. α(c) is not defined do
    Pick c ∈ C^N s.t. α(c) is not defined, and c̃ ∈ C
    Set α(c) = c̃
    for c' ∈ C^N s.t. ∃π ∈ S_X, γ ∈ S_N, ∀i ∈ N, c'_i = c_{γ^{-1}(i)} ∘ π do
        α(c') = α(c) ∘ π
    end for
end while
```

Proposition 3. *The greedy anonymous and neutral CAF satisfies anonymity and object neutrality iff all prime divisors of n exceed m.*

5. More objects than categories

5.1. Impossibility results

We let $\mathbb{N}$ denote the set of natural numbers and given $i, j \in \mathbb{N}$ write $i \mid j$ when i divides j (with $1 \mid i$ and $i \mid i$). Given $j, k \in \mathbb{N}$, let $[\![j, k]\!] = [j, k] \cap \mathbb{N}$ denote the set of integers from j to k (with $[\![j, k]\!] = \emptyset$ if $k < j$). Given $k \in \mathbb{N}$, we denote by $D(k) = \{l \in \mathbb{N} : l \mid k\}$ the set of divisors of k. We now state our first theorem, which resonates with Moulin [1983].

Theorem 3. *For $m > \rho \geq 3$, if $D(n) \cap [\![2, m]\!] \neq \emptyset$, there is no CAF that is anonymous and object neutral.*

We prove this theorem by combining lemma 1 and lemma 2.

Lemma 1. *For $m > \rho \geq 2$, if $D(n) \cap [\![2, \rho]\!] \neq \emptyset$, there is no CAF that is anonymous and object neutral.*

Proof. Let α be an anonymous and object neutral CAF. Suppose for a contradiction that some number $k \in [\![2, \rho]\!]$ divides n. We partition N with subsets $(N_1, \dots N_k)$ of equal size. Let π be the permutation of $[\![1, k]\!]$ such that $\forall i \in [\![1, k]\!], \pi(i) = (i \bmod k) + 1$. For $j \in \mathbb{N}^*$ we will use $\pi^j = \pi \circ \pi^{j-1}$ defined by induction with $\pi^0 = id$. One can see that $\pi^k = id$.

We define a classification profile $\boldsymbol{c}$ as follows. In this profile, each individual in a given set N_j, with $1 \leq j \leq k$, has the same classification. We denote this classification with c_{N_j}. Construct profile $\boldsymbol{c}$ as follows:

object	$c_{N_j}, 1 \leq j \leq k$
$x_i, 1 \leq i \leq k$	$p_{\pi^{i-1}(j)}$
$x_i, k+1 \leq i \leq \rho$	p_i
$x_i, \rho+1 \leq i \leq m$	p_ρ

Formally, $\forall i, j \in [\![1, k]\!], \forall l \in N_j, c_l(x_i) = p_{\pi^{i-1}(j)}, \forall i \in [\![k+1, \rho]\!], \boldsymbol{c}_{x_i} = (p_i, \dots p_i)$, and $\forall i \in [\![\rho+1, m]\!], \boldsymbol{c}_{x_i} = (p_\rho, \dots, p_\rho)$.

Let $\gamma \in S_X$ be the permutation defined as $\forall i \in [\![1, k]\!], \gamma(x_i) = x_{\pi(i)}, \forall i \in [\![k+1, m]\!], \gamma(x_i) = x_i$. We define $\boldsymbol{c} \circ \gamma$ as the classification profile induced from applying γ to $\boldsymbol{c}$ thus, for $i \in [\![1, m]\!], \boldsymbol{c}^{(\gamma)}(x_i) = \boldsymbol{c}(\gamma(x_i))$. It follows that $(c \circ \gamma)_{N_j}(x_i) = c_{N_j}(x_{\pi(i)})$.

object	$(c \circ \gamma)_{N_j}, 1 \leq j \leq k$
$x_i, 1 \leq i \leq k$	$c_{N_j}(x_{\pi(i)}) = p_{\pi^{i-1}(\pi(j))}$
$x_i, k+1 \leq i \leq \rho$	p_i
$x_i, \rho+1 \leq i \leq m$	p_ρ

For $1 \leq i \leq k-1$, we have $c_{N_j}(x_{\pi(i)}) = c_{N_j}(x_{i+1}) = p_{\pi^i(j)}$. And for $i = k$, we have $c_{N_j}(x_{\pi(i)}) = c_{N_j}(x_{\pi(k)}) = c_{N_j}(x_1) = p_{\pi^0(j)} = p_{\pi^k(j)} = p_{\pi^i(j)}$ as π is a cycle of size k. Also, $p_{\pi^i(j)} = p_{\pi^{i-1}(\pi(j))}$.

It follows that $\forall j \in [\![1, k]\!], (c \circ \gamma)_{N_j} = c_{N_{\pi(j)}}$, thus, $\boldsymbol{c} \circ \gamma$ can be obtained by a permutation of the individuals in $\boldsymbol{c}$.

One can check that $\boldsymbol{c}, \boldsymbol{c} \circ \gamma \in \mathcal{C}^N$. By anonymity, we must have $\alpha(\boldsymbol{c}) = \alpha(\boldsymbol{c} \circ \gamma)$. Now, for $z \in X \setminus \{x_1, \dots, x_\rho\}, \boldsymbol{c}_z = \boldsymbol{c}_{x_\rho}$, thus if we swap object x_ρ and object z in $\boldsymbol{c}$, the classification profile remains the same. Therefore object neutrality imposes, $\forall z \in X \setminus \{x_1, \dots, x_\rho\}, \alpha(\boldsymbol{c})(z) = \alpha(\boldsymbol{c})(x_\rho)$. Again, by object neutrality and by definition of $\boldsymbol{c} \circ \gamma, \forall j \in [\![1, k]\!], \alpha(\boldsymbol{c} \circ \gamma)(x_j) = \alpha(\boldsymbol{c})(x_{\pi(j)})$.

All together, we can see that $\forall j \in [\![1, k]\!], \alpha(\boldsymbol{c})(x_j) = \alpha(\boldsymbol{c} \circ \gamma)(x_j) = \alpha(\boldsymbol{c})(x_{\pi(j)}) = \ldots = \alpha(\boldsymbol{c})(x_1)$. Therefore, $|\{\alpha(\boldsymbol{c})(x), x \in X\}| \leq \rho - k + 1 < \rho$ as $k > 1$, so $\alpha(\boldsymbol{c})$ is not surjective, which is a contradiction. $\square$

Lemma 2. *For $m > \rho \geq 3$, if $D(n) \cap [\![\rho+1, m]\!] \neq \emptyset$, there is no CAF that is anonymous and object neutral.*

Proof. Suppose for a contradiction that $[\![\rho+1, m]\!]$ contains a divisor k of n and there exists a CAF α that is anonymous and object neutral. Let $(N_1, \dots N_k)$ be any partition of N with subsets of equal size.

Let π be the permutation of $[\![1, k]\!]$ such that $\forall i \in [\![1, k]\!], \pi(i) = (i \bmod k) + 1$. Also, for $j \in \mathbb{N}$, $\pi^j = \pi \circ \pi^{j-1}$ with $\pi^0 = id$.

Construct $\boldsymbol{c}$ as follows.

object	$c_{N_j}, 1 \leq j \leq k$
$x_i, 1 \leq i \leq k$	$p_{\min(\pi^{i-1}(j),\rho)}$
$x_i, k+1 \leq i \leq m$	p_1

Formally, $\forall i, j \in [\![1, k]\!], \forall l \in N_j, c_l(x_i) = p_{\min(\pi^{i-1}(j),\rho)}$, and $\forall \rho < i \leq m, \boldsymbol{c}_{x_i} = (p_1, \ldots, p_1)$.

We next introduce $\gamma \in S_N$ as $\forall j \in [\![1, m]\!], \gamma^{-1}(N_j) = N_{\pi(j)}$ and consider the classification profile $\boldsymbol{c}^{(\gamma)}$.

object	$c^{(\gamma)}_{N_j}, 1 \leq j \leq k$
$x_i, 1 \leq i \leq k$	$p_{\min(\pi^{i}(j),\rho)}$
$x_i, k+1 \leq i \leq m$	p_1

We have $\forall i \in [\![1, k]\!], \boldsymbol{c}^{(\gamma)}_{x_i} = \boldsymbol{c}_{x_{\pi(i)}}$. In fact, $\forall i, \in [\![1, k]\!], c^{(\gamma)}_{N_j}(x_i) = c_{N_{\pi(j)}}(x_i) = p_{\min(\pi^{i-1}(\pi(j)),\rho)} = c_{N_j}(x_{\pi(i)})$. Then, $\boldsymbol{c}^{(\gamma)} = \sigma \circ \boldsymbol{c}$ with $\sigma \in S_X$ defined as $\forall i \in [\![1, m]\!], \sigma(x_i) = x_{\pi(i)}$.

One can check that both $\boldsymbol{c}$ and $\boldsymbol{c}^{(\gamma)}$ are surjective. By applying anonymity, we get $\alpha(\boldsymbol{c}) = \alpha(\boldsymbol{c}^{(\gamma)})$. Moreover, object neutrality entails that $\forall z \in X \setminus \{x_1, \ldots, x_k\}, \alpha(\boldsymbol{c})(x_z) = \alpha(\boldsymbol{c})(x_{k+1})$ and $\forall i \in [\![1, k]\!], \alpha(\boldsymbol{c}^{(\gamma)})(x_i) = \alpha(\boldsymbol{c})(x_{\pi(i)})$.

Altogether, these results imply that $\forall i \in [\![1, k]\!], \alpha(\boldsymbol{c})(x_i) = \alpha(\boldsymbol{c})(x_1)$. Moreover, object neutrality imposes that $\forall z \in X \setminus \{x_1, \ldots, x_k\}, \alpha(\boldsymbol{c})(z) = \alpha(\boldsymbol{c})(x_{k+1})$. Therefore, $|\{\alpha(\boldsymbol{c}(x), x \in X\}| \leq 2 < \rho$, so $\alpha(\boldsymbol{c})$ is not surjective, a contradiction. $\square$

Remark 1. Note that this lemma also holds for $m > \rho = 2$ if m divides n. Indeed, in the profile from the previous proof we have that all objects $x_1, \ldots, x_k$ should be mapped to the same category. Thus $m = k$ imposes the classification to fail surjectivity. $\triangle$

Theorem 4. *For $m > \rho \geq 3$, if $D(n) \cap [\![2, \rho - 1]\!] \neq \emptyset$, there is no CAF that is anonymous and category neutral.*

Proof. We prove this result through a contradiction. Suppose $[\![2, \rho - 1]\!]$ contains a divisor k of n and there exists a CAF α that is anonymous and object neutral. Let $(N_1, \ldots N_k)$ be any partition of N with subsets of equal size. Let π be the permutation of $[\![1, k]\!]$ such that $\forall i \in [\![1, k]\!], \pi(i) = (i \bmod k) + 1$.

We define a classification profile $\boldsymbol{c}$ as follows, $\forall i, j \in [\![1, k]\!], \forall l \in N_j, c_l(x_i) = p_{\pi^{i-1}(j)}, \forall i \in [\![k + 1, \rho]\!], \boldsymbol{c}_{x_i} = (p_i, \ldots p_i)$, and $\forall \rho < i \leq m, \boldsymbol{c}_{x_i} = (p_\rho, \ldots, p_\rho)$. Note that for $j \in [\![1, k]\!]$ all $(c_l)_{l \in N_j}$ are the same, we denote this classification with c_{N_j}.

Now, let $\gamma \in S_P$ s.t. $\forall i > k, \gamma(p_i) = p_i$ and $\forall i \in [\![1, k]\!], \gamma(p_i) = p_{\pi(i)}$ and consider the profile $\gamma \circ \boldsymbol{c}$.

object	$(\gamma \circ c)_{N_j}, 1 \leq j \leq k$
$x_i, 1 \leq i \leq k$	$p_{\pi^i(j)}$
$x_i, k+1 \leq i \leq \rho$	$\gamma(p_i) = p_i$
$x_i, \rho+1 \leq i \leq m$	p_ρ

One can check that $\boldsymbol{c}, \gamma \circ \boldsymbol{c} \in \mathcal{C}^N$ and that $\forall i \in [\![1, k]\!], (\gamma \circ c)_{N_i} = c_{N_{\pi(i)}}$. By anonymity, we must have $\alpha(\boldsymbol{c}) = \alpha(\gamma \circ \boldsymbol{c})$. Also, by category neutrality, we have $\forall x \in X, \alpha(\gamma \circ \boldsymbol{c})(x) = \gamma(\alpha(\boldsymbol{c})(x))$.

All together, we can see that $\forall x \in X, \alpha(\boldsymbol{c})(x) = \gamma(\alpha(\boldsymbol{c})(x))$, so $\alpha(\boldsymbol{c})(x) \notin \{p_1, \ldots, p_k\}$. Therefore, $\alpha(\boldsymbol{c})$ is not surjective, a contradiction. $\square$

6. Defining anonymous and neutral CAFs

We proved the non-existence of anonymous and (category or object) neutral CAFs when n admits a divisor that is smaller than m but exceeds 1. When 1 is the only divisor of n that is smaller than m, the greedy algorithm in section 4 defines CAFs that are anonymous and object neutral. As these CAFs are rather artificial and only defined for $m = \rho$, we now define one that is more natural and works when $m > \rho$.

Given $m, \rho \in \mathbb{N}, m > \rho$ we write $S_{m,\rho} = \sum_{i=0}^{\rho} \binom{\rho}{i} \times (-1)^{\rho-i} \times i^m$ to denote the Stirling number of the second kind associated to (m, ρ). For work on Stirling numbers, we refer to Rennie and Dobson [1969]. What is important to note here is that $\rho! \times S_{m,\rho}$ is the number of surjective mappings from a set of size m to a set of size ρ. Thus, $|\mathcal{C}| = \rho! \times S_{m,\rho}$.

We say that a tuple (m, n, ρ) satisfies condition μ if $\nexists(\lambda_k)_{0\leq k\leq m}$ s.t.

$$
\begin{gathered}
n = \sum_{k=0}^{m} k\lambda_k \\
\textstyle\sum_{k=0}^{m} \lambda_k = \rho! \times S_{m,\rho} = |\mathcal{C}| \\
\forall k \in [\![0, m]\!], \lambda_k \neq 1
\end{gathered}
$$

Given a profile $\boldsymbol{c} \in \mathcal{C}^N$, we define a ranking of $\mathcal{C}$ as $\forall c' \in \mathcal{C}, \mathit{rank}(c') = |\{i \in N \mid c_i = c'\}|$. We define the equivalence relation $\sim$: $\forall c, c' \in \mathcal{C}, c \sim c'$ iff $\mathit{rank}(c) = \mathit{rank}(c')$. Given $c \in \mathcal{C}$, we set $K_c = \{c' \in \mathcal{C}, c \sim c'\}$ as the equivalence class of c according to $\sim$. Let $L = \arg\min_{c\in\mathcal{C}} |K_c|$. We define the rule $\alpha^\star$ that outputs $\arg\max_{c\in L} \mathit{rank}(c)$.

Theorem 5. *For $m > \rho \geq 3$, the rule $\alpha^\star$ is a CAF iff (m, n, ρ) satisfies condition μ.*

Note that if $D(n) \cap [\![2, m]\!] \neq \emptyset$, there exists a divisor k of n in $[\![2, m]\!]$ and we can set $\lambda_k = q$ where $n = kq, \lambda_0 = \rho! \times S_{m,\rho} - q$ and $\forall i \in [\![1, m]\!] \setminus \{k\}, \lambda_i = 0$. Then $\alpha^\star$ is not a CAF in the cases covered by theorem 3.

Proof. Let $\boldsymbol{c} \in \mathcal{C}^N$, and suppose $|\alpha^\star(\boldsymbol{c})| > 1$, so that $|\{c \in L : \forall c' \in L,\ \mathit{rank}(c) \geq \mathit{rank}(c')\}| > 1$. Then, $\forall c \in L, |K_c| > 1$. Given $k \in [\![0, m]\!]$, we set $\lambda_k = |\{c' \in \mathcal{C}, \mathit{rank}(c') = k\}|$. Then, we have $\forall k \in [\![0, m]\!], \lambda_k \neq 1$, also $\sum_{k=0}^{m} \lambda_k = |\mathcal{C}| = \rho! \times S_{m,\rho}$ and $\sum_{k=0}^{m} k\lambda_k = \sum_{c'\in\mathcal{C}} \mathit{rank}(c') = n$. Therefore, μ holds only if $\alpha^\star$ is a CAF.

If μ holds, as $\sum_{k=0}^{m} \lambda_k = |\mathcal{C}|$, and $\sum_{k=0}^{m} k\lambda_k = n$, we can define a partition $\mathcal{C}_0, \ldots, \mathcal{C}_m$ of $\mathcal{C}$ s.t. $\forall k \in [\![0, m]\!], |\mathcal{C}_k| = \lambda_k$. Then if we consider $\boldsymbol{c}$ s.t. $\forall k \in [\![0, k]\!], \forall c \in \mathcal{C}_k, \mathrm{rank}(c) = k$ then $|\alpha^\star(\boldsymbol{c})| > 1$ as $\forall k \in [\![0, k]\!], \lambda_k \neq 1$. Thus, $\alpha^\star$ is a CAF only if μ holds. □

Theorem 6. *Under condition $\mu, \alpha^\star$ satisfies anonymity, unanimity, object neutrality and category neutrality.*

Proof. Let $\boldsymbol{c} \in \mathcal{C}^N$ be a classification profile and $\boldsymbol{c}' \in \mathcal{C}^N$ be any profile obtained from a permutation of the individuals in $\boldsymbol{c}$. Then $\{c_i, i \in N\} = \{c'_i, i \in N\}$, so L is the same in both cases. Then, $\alpha^\star(\boldsymbol{c}) = \alpha^\star(\boldsymbol{c}')$.

Let $c \in \mathcal{C}$, and $\boldsymbol{c} = (c, \ldots, c)$ be the profile where all individuals pick c. Then $L = \{c\}$, so $\alpha^\star(\boldsymbol{c}) = c$.

Let $\boldsymbol{c}, \boldsymbol{c}' \in \mathcal{C}^N$ s.t. $\exists \pi \in S_X, \boldsymbol{c}' = \boldsymbol{c} \circ \pi$. We denote by $L^{(\boldsymbol{c})}$ and $L^{(\boldsymbol{c}')}$ the sets of equivalence classes of smallest sizes for $\boldsymbol{c}$ and $\boldsymbol{c}'$. Let $c \in \mathcal{C}$, we must have $\mathit{rank}_{\boldsymbol{c}}(c) = \mathit{rank}_{\boldsymbol{c}'}((c_{\pi(x)})_{x\in X})$. Then, $c \in L^{(\boldsymbol{c})}$ *iff* $(c_{\pi(x)})_{x\in X} \in L^{(\boldsymbol{c}')}$.

Let $\boldsymbol{c}, \boldsymbol{c'} \in \mathcal{C}^N$ s.t. $\exists \pi \in S_P, \boldsymbol{c'} = \pi \circ \boldsymbol{c}$. We denote by $L^{(c)}$ and $L^{(c')}$ the sets of equivalence classes of smallest sizes for $\boldsymbol{c}$ and $\boldsymbol{c'}$. Let $c \in \mathcal{C}$, we must have $rank_{\boldsymbol{c}}(c) = rank_{\boldsymbol{c'}}(\pi \circ c)$. Then, $c \in L^{(c)}$ *iff* $\pi \circ c \in L^{(c')}$. □

Condition μ is weaker than requiring 1 to be the only divisor of n that is smaller than m. Thus, instances where Condition μ holds but n admits a divisor smaller than m and different than 1 present a gap over which $\alpha^\star$ is not a CAF and theorem 6 does not hold. Tables 1 and 2 give a picture of the gap for small sizes of n, m and ρ. In Table 1, the information of the first row comes from theorem 2 and the information of the columns where n is even comes from lemma 1. In Table 2, the information of the first row comes from theorem 2 and the remaining information comes from Theorems 3 and 4.

Table 1: Possibility of having anonymous and object neutral aggregators for $\rho = 2$ (n in columns, m in lines)

	2	3	4	5	6	7	8	9	10
2	no	yes	no	yes	no	yes	no	yes	no
3	no	no[2]	no	yes[1]	no	yes[1]	no	no[2]	no
4	no	?	no	?	no	?	no	?	no
5	no	?	no	?	no	?	no	?	no

[1] From theorem 6
[2] From remark 1

Table 2: Possibility of having anonymous and object neutral aggregators for $\rho = 3$ (n in columns, m in lines)

	2	3	4	5	6	7	8	9	10
3	no	no	no	yes[1]	no	yes[1]	no	no	no
4	no	no	no	?	no	?	no	no	no
5	no	no	no	no	no	?	no	no	no

[1] From theorem 2
[2] From theorem 6

We present two different approaches to fill the uninformed cells. In appendix A, we describe a prospective experiment using an algorithm based on Doğan and Giritligil [2022] that greedily builds an anonymous and (object and category) neutral CAF. As the number of profiles to be checked ($|\mathcal{C}|^n$) is exponential, we used a trick introduced by Nardi [2021]: instead of mapping profiles one by one, we let one profile represent each the equivalence classes under anonymity, therefore shrinking the number of profiles to be explored. This allowed us to explore problem sizes that were not possible to take on by brute force. This experiment was quite promising as we were able to find anonymous and neutral CAFs for all problem sizes that we tried for which 1 is the only prime divisor of n that is smaller than m. In appendix B, we build a CAF that is anonymous, object neutral and category neutral for $m = 4, \rho = 2, n = 5$, which is the smallest open case left after the experiment.

7. Conclusion

We proved that depending on the number of individuals, objects and categories, an anonymous and neutral (object neutral or category neutral) CAF might not exist, obtaining a result similar to

Moulin [1983]. We were able to fully characterize this impossibility when the number of objects and categories are the same by using a bijection between preference and classification profiles and using Bubboloni and Gori's [2014] results. When there are more objects than categories, we were only able to find an anonymous and neutral aggregator for a smaller domain than the complementary of our impossibility domain. Filling this gap would be an interesting question and it appears that some anonymous and neutral aggregators could still exist within this gap as illustrated in appendices A and B.

The anonymous and neutral aggregator that we point to is not Pareto optimal and it seems impossible to define an aggregator that would be Pareto optimal, anonymous and neutral, as Moulin's [1983] result hints. Even more, any such aggregator might select classifications that are not even supported by any individual, making a similarity with Campbell and Kelly's [2015] claim that in the context on preference aggregation, anonymous and neutral aggregators might select bottom-ranked alternatives.

Exploring escape routes to this and other impossibility results in classification aggregation is an interesting direction some researchers have already pursued. Fioravanti [2024b] relaxed the problem of fuzzy classifications and found an independent, unanimous, and anonymous aggregator that we claim is also object neutral. Craven [2023] used the approach of domain restriction to build a non-dictatorial aggregation function that is independent and unanimous, and Craven [2024] explored the problem of classification aggregation with ranked categories and objects, which makes it possible to have an independent, unanimous and anonymous aggregators. One could also adapt the idea of Ozkes and Sanver [2021] to classification aggregation, namely replacing object neutrality by a weaker notion of consequential object neutrality in order to avoid impossibilities. Other possible directions for positive results include allowing for ties and dropping the surjectivity constraint on classifications.

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A. Greedy search for Anonymous and Neutral CAFs

We aim to identify an anonymous, object neutral and category neutral CAF using an algorithm similar to algorithm 2 which is a brute force algorithm. We follow Nardi [2021]; for each class of profiles Γ that are equivalent under anonymity ($\forall \boldsymbol{c}, \boldsymbol{c}' \in \Gamma, \exists \gamma \in S_N$ s.t. $\boldsymbol{c}' = \boldsymbol{c}^{(\gamma)}$), we use only the profile that is minimal under lexicographic order. Thanks to this, we reduce the number of profiles to explore, and hence the computation time of our algorithm.

Table 3: Possibility of having anonymous and neutral aggregators for $\rho = 2$ (n in columns, m in lines)

	2	3	4	5	6	7	8	9	10
2	no	yes[1]	no	yes[1]	no	yes[1]	no	yes[1]	no
3	no	no[2]	no	yes[3]	no	yes[3]	no	no[2]	no
4	no	yes[5]	no	yes[4]	no	yes[5]	no	?	no
5	no	yes[5]	no	no	no	?	no	?	no

[1] From theorem 2
[2] From remark 1
[3] From theorem 6
[4] From appendix B
[5] From experiments

Table 4: Possibility of having anonymous and neutral aggregators for $\rho = 3$ (n in columns, m in lines)

	2	3	4	5	6	7	8	9	10
3	no	no	no	yes[1]	no	yes[1]	no	no	no
4	no	no	no	yes[4]	no	?	no	no	no
5	no	no	no	no	no	?	no	no	no

[1] From theorem 2
[2] From theorem 6
[4] From experiments

Unlike in algorithm 2, as we work with more objects than categories, it is possible that some but not all classifications do not suit a given profile. For example, if all individuals classified two objects on the same category, object neutrality imposes that these objects are mapped in the same category. For any $\boldsymbol{c} \in \mathcal{C}^N$, we let $\mathrm{lexmin}(\boldsymbol{c})$ denote the profile $\mathrm{lexmin}(\boldsymbol{c}) = \arg\min\{\boldsymbol{c}^{(\gamma)}, \gamma \in S_N\}$. The new version of our algorithm is written in algorithm 3 below.

Algorithm 3 The greedy CAF

while $\exists \boldsymbol{c}' \in \mathcal{C}^N$ s.t. $\alpha(\boldsymbol{c}')$ is not defined **do**
  Pick $\boldsymbol{c} = \text{lexmin}(\boldsymbol{c}') \in \mathcal{C}^N, \alpha(\boldsymbol{c})$ is not defined
  $E \leftarrow \{\}$
  $b \leftarrow$ False
  while (not b) and $|E| < |\mathcal{C}|$ **do**
    $b \leftarrow$ True
    $\alpha' \leftarrow \alpha$
    $\alpha'(\boldsymbol{c}) \leftarrow \tilde{c}$ for $\tilde{c} \in \mathcal{C} \setminus E$
    for $\boldsymbol{c}' \in \mathcal{C}^N$ s.t. $\exists \pi \in S_X, \boldsymbol{c}' = \boldsymbol{c} \circ \pi$ **do**
      $\boldsymbol{c}^{(1)} \leftarrow \text{lexmin}(\boldsymbol{c}')$
      if $\alpha'(\boldsymbol{c}^{(1)}) \neq \alpha'(\boldsymbol{c}) \circ \pi$ **then**
        $b \leftarrow$ False
      else
        $\alpha'(\boldsymbol{c}') \leftarrow \alpha'(\boldsymbol{c}) \circ \pi$
      end if
    end for
  end while
  if b **then**
    $\alpha \leftarrow \alpha \cup \alpha'$
  else
    return False
  end if
end while
return True

For all sizes of the problem where theorem 3 is not defined that we have tested, it was possible to define an anonymous, object neutral and category neutral CAF. There are still some sizes we have not tested because of the running time, but we conjecture that an equivalent of theorem 1 should hold for the classification aggregation problem: Given $n, m \geq 2$, there exists an anonymous and object neutral CAF iff all prime divisors of n exceed m. Future work could try and see if this conjecture holds.

B. An anonymous and neutral CAF for 4 objects, 2 categories and 5 individuals.

We consider the case $m = 4, \rho = 2$ and $n = 5$. This is the smallest open case for the existence of anonymous and neutral CAFs after using our prospective algorithm in appendix A. In order to prove existence of anonymous and neutral CAFs in this case, we exhibit one. Our CAF is based on the majority rule (MAJ) defined in Craven [2023] as the function that associates each profile $\mathcal{C}^N$ to the mapping $\text{MAJ}(\boldsymbol{c}) = (\arg\max_{p \in P} |\{i \in N, c_i(x) = p\}|)_{x \in X}$. Note that MAJ is not a CAF because the mapping $\text{MAJ}(\boldsymbol{c})$ is not always a classification as it is not necessarily surjective.

Given $p \in P$ and $c \in P^X$, we write $\|c\|_p = |\{x \in X : c_x = p\}|$. For any profile $\boldsymbol{c} \in \mathcal{C}^n$, we associate the "$p$-vector" $(\|c_i\|_p)_{i \in N}$ for $p \in P$, which identifies how many times p is assigned by each individual.

We build $\hat{\alpha}$ by considering 4 disjoint cases on the classification profile $\boldsymbol{c} \in \mathcal{C}^N$:

- If $\text{MAJ}(\boldsymbol{c})$ is surjective, we define $\hat{\alpha}(\boldsymbol{c}) = \text{MAJ}(\boldsymbol{c})$.

- Otherwise, $\exists q \in P$ s.t. $||\text{MAJ}(\boldsymbol{c})||_q = 0$. Therefore $\sum\limits_{i \in N} ||c_i||_q \leq 8$: as MAJ classifies all objects in p, for each object at most 2 individuals classified it in q. We then have only 3 possibilities for the q-vector:
  - If $\exists! i \in N, \|c_i\|_q = k$ for some $k \in [\![2,3]\!]$, we define $\hat{\alpha}(\boldsymbol{c}) = \arg\max_{c_i, i \in N}\{k, \exists! i \in N, \|c_i\|_q = k\}$.
  - If the q-vector is a permutation of (1,1,2,2,2), we apply MAJ on the profile restricted to the individuals $\boldsymbol{c}' = \{i \in N, \|c_i\|_q = 2\}$, in this case, each category received 6 votes in total so $\text{MAJ}(\boldsymbol{c}')$ is surjective and we define $\hat{\alpha}(\boldsymbol{c}) = \text{MAJ}(\boldsymbol{c}')$.
  - If the q-vector is a permutation of (1,1,1,2,2) or (1,1,1,1,1), there are 7 or 5 votes for q in total. We can apply the following rule: given $x \in X$, if $|\{i \in N, c_i(x) = q\}| \geq 2$ we define $\hat{\alpha}(\boldsymbol{c})(x) = q$ and $\hat{\alpha}(\boldsymbol{c})(x) = p$ otherwise.

Proposition 4. *$\hat{\alpha}$ is anonymous, object neutral and category neutral.*

Proof. We will prove this for each possible case, let $\boldsymbol{c} \in \mathcal{C}^N, \pi \in S_X, \sigma \in S_P$ and $\gamma \in S_N$.

- If $\text{MAJ}(\boldsymbol{c})$ is surjective, $\text{MAJ}(\boldsymbol{c}^{(\gamma)})$, $\text{MAJ}(\boldsymbol{c} \circ \pi)$ and $\text{MAJ}(\sigma \circ \boldsymbol{c})$ will be surjective as well. As MAJ is anonymous, object neutral and category neutral, and $\hat{\alpha}$ copies MAJ, we will have $\hat{\alpha}(\boldsymbol{c}^{(\gamma)}) = \hat{\alpha}(\boldsymbol{c}), \hat{\alpha}(\boldsymbol{c} \circ \pi) = \hat{\alpha}(\boldsymbol{c}) \circ \pi$ and $\hat{\alpha}(\sigma \circ \boldsymbol{c}) = \sigma \circ \hat{\alpha}(\boldsymbol{c})$.

- Otherwise, $\exists q \in P$ s.t. $||\text{MAJ}(\boldsymbol{c})||_q = 0$.
  - $\exists! i \in N, \|c_i\|_q = k$ for some $k \in [\![2,3]\!]$:

    Any permutation on individuals, object or categories will not change this property (q is defined as the category that is not represented in MAJ), then this also holds for $\boldsymbol{c}^{(\gamma)}, \boldsymbol{c} \circ \pi$ and $\sigma \circ \boldsymbol{c}$.

    It is clear that selecting $\arg\max_{c_i, i \in N}\{k, \exists! i \in N, \|c_i\|_q = k\}$ is anonymous and object neutral, and $\arg\max_{c_i, i \in N}\{k, \exists! i \in N, \|c_i\|_{\sigma(q)} = k\} = \arg\max_{c_i, i \in N}\{k, \exists! i \in N, \|c_i\|_q = k\}$.
  - The q-vector is a permutation of (1,1,2,2,2): Any permutation on individuals, object or categories will not change this property (q is defined as the category that is not represented in MAJ), then this also holds for $\boldsymbol{c}^{(\gamma)}, \boldsymbol{c} \circ \pi$ and $\sigma \circ \boldsymbol{c}$. The set of voters' classifications having 2 objects in the least represented category does not change with swaps of voters, objects or categories. Therefore, as we apply MAJ on this set of classifications, we remain anonymous, object neutral and category neutral.
  - The q-vector is a permutation of (1,1,1,2,2) or (1,1,1,1,1): Any permutation on individuals, object or categories will not change this property (q is defined as the category that is not represented in MAJ), then this also holds for $\boldsymbol{c}^{(\gamma)}, \boldsymbol{c} \circ \pi$ and $\sigma \circ \boldsymbol{c}$.

    Now, given $x \in X$, $|\{i \in N, (c \circ \pi)_i(x) = q\}| \geq 2$ iff $|\{i \in N, c_i(\pi(x)) = q\}| \geq 2$, $|\{i \in N, (c^{(\gamma)})_i(x) = q\}| \geq 2$ iff $|\{i \in N, c_i(x) = q\}| \geq 2$ and $|\{i \in N, (\sigma \circ c)_i(x) = q\}| \geq 2$ iff $|\{i \in N, c_i(x) = \sigma(q)\}| \geq 2$. Then we have $\hat{\alpha}(\boldsymbol{c}^{(\gamma)}) = \hat{\alpha}(\boldsymbol{c}), \hat{\alpha}(\boldsymbol{c} \circ \pi) = \hat{\alpha}(\boldsymbol{c}) \circ \pi$ and $\hat{\alpha}(\sigma \circ \boldsymbol{c}) = \sigma \circ \hat{\alpha}(\boldsymbol{c})$.

□