

Committee Monotonicity and Proportional Representation for Ranked Preferences

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Abstract

We study committee voting rules under ranked preferences, which map the voters' preference relations to a subset of the candidates of predefined size. In this setting, the compatibility between proportional representation and committee monotonicity is a fundamental open problem that has been mentioned in several works. We address this research question by designing a new committee voting rule called the Solid Coalition Refinement (SCR) rule that simultaneously satisfies committee monotonicity and Dummett's PSC as well as one of its variants called inclusion PSC. This is the first rule known to satisfy both of these properties. Moreover, we show that this is effectively the best that we can hope for as other fairness notions adapted from approval voting are incompatible with committee monotonicity. We also prove that, for truncated preferences, the SCR rule still satisfies PSC and a property called independence of losing voter blocs, thereby refuting a conjecture of Graham-Squire et al. [18]. Finally, we discuss the consequences of our results in the context of rank aggregation.

1 Introduction

The Single Transferable Vote (STV) is a voting rule that allows the selection of multiple winners in an election. Voters provide rankings of the candidates that are running, and STV selects winners following the principles of proportional representation, which ensures that each group of voters has an influence approximately proportional to its size. On a high level, STV is based on a minimum number of voters that a candidate needs to be elected (the so-called quota). A candidate who is placed first by a number of voters exceeding the quota is elected; if not enough such candidates exist, the candidates with the fewest first-place votes are eliminated until additional candidates can be elected. STV in its multi-winner proportional representation version was proposed as early as 1819, and is used for political elections in numerous countries such as Australia, Ireland, New Zealand, and Scotland [34].

Recent literature in computational social choice has noted that proportional representation has applications far beyond parliament elections, such as in budgeting problems [27, 29], recommender systems [24, 31], and multi-criteria decision making [23]. A rule like STV could thus be useful in these applications. However, STV exhibits some undesirable behavior when the number of candidates to be elected is changed: if the number of winners is increased, a candidate that was previously in the winning committee may now be designated as a loser. In other words, STV violates *committee monotonicity*, which requires a voting rule to select a superset of the current winners if the number of winners is increased [15]. Unfortunately, this makes STV unsuitable for many applications of proportionally representative voting, where committee monotonicity is often a requirement because the number of winners is uncertain or might dynamically change. Let us consider some examples.

- A lecturer lets students vote over which topics will be covered in their course. Proportionality is desirable in this context because topics should be chosen so as to keep many students interested. The lecturer expects to be able to cover 6 topics, say, but he needs the flexibility to expand coverage to 7–8 topics in case the class proceeds more quickly than expected. A similar application concerns interactive Q&A systems [19], where the audience submits and votes on questions during a talk or panel discussion. Since it is hard to predict how many of the questions will be asked, committee monotonicity is desirable.

- A multi-criteria recommender system makes recommendations based on modelling the user’s preferences along several dimensions. For example, it might rank hotels based on room size, price, location, etc., with the user specifying their relative importance. The recommender system will recommend 10 hotels, say, by combining these criteria following the importance weight via proportional representation. The users can click a “show more” button that will display 10 more hotels – without thereby wanting to disqualify any of the initial 10 hotels.

Another application of committee monotone rules is *rank aggregation* which is the task of combining several rankings into one. In particular, a committee monotone rule implicitly computes an output ranking of the candidates, whose top- k candidates are the candidates selected by the rule when asked for k winners. Thus, committee monotone voting rules that are proportional can be used to generate *proportional rankings* [32], a task that has received increasing attention [19, 23]. Proportional rankings can be used, for example, to produce university rankings by aggregating rankings based on different criteria such as research output and teaching performance. Standard rank aggregation methods like the Kemeny rule are majoritarian and thus not appropriate when proportionality is desired. The recently-studied Squared Kemeny rule [23] behaves like an average and can thus be seen as providing a proportional outcome. However, its proportionality characteristics apply to the global ranking, and there is no guarantee that the top- k candidates proportionally reflect the voters’ preferences. In many contexts, however, it is natural to place a greater emphasis on the top candidates in a ranking. Committee voting rules that are committee monotone and return proportional committees are a good fit in those cases, as every prefix of their computed ranking corresponds to a proportional committee.

As we mentioned, STV cannot be used for these applications, but STV is not the only proportional voting method. Others have been proposed more recently, such as the Quota Borda System [14] and the Expanding Approvals Rule [2]. However, strikingly, all these rules also fail committee monotonicity. Indeed, as Lederer et al. [23] point out, it is an open question whether there exists any voting rule combining committee monotonicity and strong notions of proportional representation. In this paper, we will address this question by designing the first voting rules for ranked preferences that satisfy both of these desiderata. In more detail, we focus on voters submitting ranked preferences, potentially with indifferences. We will design new *committee voting rules*, which are functions that map the voters’ preferences and a desired number of winners to a subset of the candidates of that size. For proportional representation, we focus on variants of Dummett’s *Proportionality for Solid Coalitions (PSC)* [14]. This is the most common formalization of proportionality for ranked preferences, and has been referred to as “a *sine qua non* for a fair election rule” [36]. It provides representation guarantees for groups of voters who put the same candidates at the top of their rankings, and it has been shown that STV and the other rules we mentioned satisfy PSC. Our formal goal is thus to design committee voting rules that satisfy both PSC and committee monotonicity. A more extensive version of this paper is available at [6].

Contributions. We begin our work by focussing on strict preferences. For this setting, we show that any voting rule satisfying PSC can be bootstrapped into a committee monotone voting rule satisfying PSC by running it in a “reverse sequential” mode ([Theorem 1](#)). In this mode, we repeatedly use the base rule to delete the candidate that is not declared a winner by the base rule when the number of winners is one less than the number of available candidates. Based on this result, we can construct an entire family of committee monotone voting rules that satisfy PSC, for example based on STV or EAR. However, this family of rules is somewhat technical and does not satisfy PSC for weak preferences.

We hence design another committee monotone rule, the Solid Coalition Refinement (SCR) rule, that satisfies PSC while working more directly. This rule is inspired by the D’Hondt apportionment method, and works by identifying virtual parties in the preferences (which correspond to sets of candidates that are frequently ranked on top together). In more detail, analogous to the D’Hondt method of apportionment, the SCR rule repeatedly adds a candidate that represents the most underrepresented party to the winning committee. Since there can be multiple such candidates, the exact candidate is determined through a process of refinement. As we will show, this rule satisfies committee monotonicity

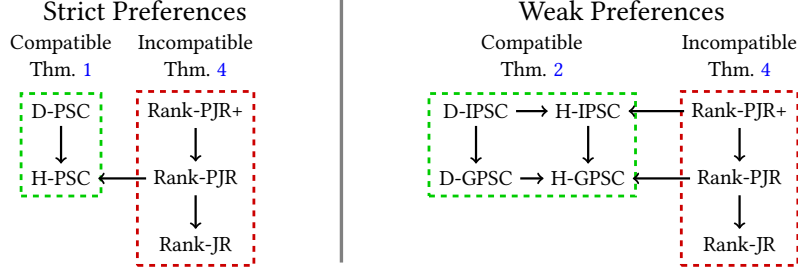


Figure 1: A summary of our results. Axioms in the green boxes are compatible with committee monotonicity, whereas the axioms in the red boxes are incompatible. An arrow from one axiom to another means that the first implies the second. “D-” is short for Droop and “H-” for Hare. Definitions for GPSC (Generalized PSC), Rank-PJR, and Rank-PJR+ as well as proofs of the relationships between the axioms can be found in Brill and Peters [10].

and PSC. Even more, in contrast to reverse sequential rules, the SCR rule is well-defined for weak preferences and satisfies a generalization of PSC called inclusion PSC [3] for this setting (Theorem 2).

The SCR rule also avoids further paradoxes that affect other rules satisfying PSC. In particular, Graham-Squire et al. [18] studied a model with truncated preferences, where voters strictly rank only some of the candidates. They noticed that in this setting, all known PSC methods fail a property they call independence of losing voter blocs, which requires that the outcome of a rule should not change if we delete voters who do not rank any winning candidate. Graham-Squire et al. [18] even hypothesized that PSC and independence of losing voter blocs are incompatible for truncated preferences. However, we show that the SCR rule satisfies both of these properties (Theorem 3), thus disproving their conjecture.

Moreover, we examine the compatibility of committee monotonicity with a family of proportionality notions due to Brill and Peters [10] that adapt fairness axioms from approval-based committee voting to ranked preferences. However, it turns out that committee monotonicity is even incompatible with Rank-JR, the weakest such proportionality notion (Theorem 4). This demonstrates a striking difference between PSC and the new proportionality notions by Brill and Peters, and it suggests that it may be impossible to satisfy stronger proportionality axioms than PSC with committee monotone voting rules.

Finally, we analyze the consequences of our results for rank aggregation, where the outcome is a ranking of the candidates instead of a committee. To this end, we first note that every committee monotone voting rule that satisfies PSC can be seen as a rank aggregation rule that guarantees that each prefix of the chosen ranking satisfies PSC. Moreover, as an alternate concept of proportionality, we investigate the maximum swap distance between an input ranking and a ranking satisfying PSC, as a function of the fraction of voters that report the input ranking. This approach was recently suggested by Lederer et al. [23] to argue in favor of the Squared Kemeny rule, and we show in Theorem 5 that rankings satisfying PSC give comparable guarantees to the Squared Kemeny rule for this proportionality measure.

2 Related Work

We next give a brief overview of prior works on proportional representation and committee monotonicity for committee voting with ranked preferences.

Proportional Representation. The problem of finding representative committees has a long tradition as STV was already proposed in the 19th century [34]. Since STV is widely used in practice, many works have focused on better understanding this rule [e.g., 9, 13, 15] and it has been shown that that STV satisfies PSC [35, 36]. Moreover, numerous other voting rules have been suggested with the aim of finding representative outcomes. Examples include the Chamberlin-Courant rule [12, 24], Monroe’s rule [26], and variants of positional scoring rules [17]. However, all of these rules fail PSC. Other rules than STV that satisfy this condition include the *Quota Borda System (QBS)* of Dummett [14] and the *Expanding Approvals Rule (EAR)* of Aziz and Lee [2]. Dummett [14] suggests QBS based on the informal

argument that it is less chaotic than STV, while Aziz and Lee [2] motivate EAR by the observation that it satisfies monotonicity conditions that STV fails. Moreover, Brill and Peters [10] show that EAR satisfies a fairness condition called Rank-PJR+ which is violated by STV. Finally, Aziz and Lee [4] characterize committees satisfying PSC in terms of *minimal demand rules*; however, it is not straightforward to derive appealing rules from this characterization. Rules based on pairwise comparisons between committees have also been suggested [30, 34, 35], but it is unclear if they satisfy PSC.

Committee Monotonicity. Just like proportional representation, committee monotonicity was identified early on as a desirable property of committee elections. For instance, in 1880, the chief clerk of the census office of the United States noticed that Alabama would be allocated 8 seats in a 299-seat parliament but only 7 seats in a 300-seat parliament, a paradox which is today known as the “Alabama Paradox” [7]. It is known that committee monotonicity and proportional representation are compatible in the simpler setting of (approval-based) apportionment [7, 11], where voters vote for parties that can be assigned multiple seats. By contrast, the analysis of committee monotonicity for committee elections consists largely of counterexamples showing that specific classes of rules fail this axiom [8, 15, 21, 25, 28, 33]. For instance, Elkind et al. [15] present a counterexample showing that STV fails committee monotonicity. On the other hand, Janson [20] discussed several committee monotone voting rules, such as Phragmén’s Ordered Method and Thiele’s Ordered Method, which, however, fail PSC. Finally, as mentioned before, voting rules that output rankings can be seen as committee monotone committee voting rules. Hence, the study of proportional rank aggregation rules [23, 32] is related to our work, but these papers focus on proportionality notions specific to rankings.

3 Preliminaries

Let $N = \{1, \dots, n\}$ denote a set of n voters and let $C = \{c_1, \dots, c_m\}$ denote a set of m candidates. We assume that every voter $i \in N$ reports a (weak) *preference relation* $\succsim_i$ over the candidates, which is formally a complete and transitive binary relation on C . The notation $c \succsim_i c'$ denotes that voter i weakly prefers candidate c to c' . We write $c \succ_i c'$ if $c \succsim_i c'$ but not $c' \succsim_i c$, which indicates a strict preference. We call a preference relation *strict* if it is antisymmetric, i.e., there is no indifference between any two candidates. The set of all weak preference relations is denoted by $\mathcal{R}$ and the set of all strict preference relations by $\mathcal{L}$. A *preference profile* R is the collection of the voters’ preferences, i.e., it is a function from N to $\mathcal{R}$. A preference profile is *strict* if all voters have strict preference relations. The set of all preference profiles is $\mathcal{R}^N$ and the set of all strict preference profiles is $\mathcal{L}^N$.

Given a preference profile, we aim to select a *committee*, which is formally a subset of the candidates of a given size k . To this end, we use *committee voting rules*, which for every preference profile and committee size return a committee of that size. More formally, a committee voting rule f for strict (resp. weak) preferences maps every profile $R \in \mathcal{L}^N$ (resp. $\mathcal{R}^N$) and target committee size $k \in \{1, \dots, m\}$ to a single winning committee $W = f(R, k)$ with $|W| = k$.

3.1 Committee Monotonicity

The idea of committee monotonicity is that if some candidate is selected for a committee size k , then it should also be selected for every committee size $k' > k$. The formal definition is as follows.

Definition 1 (Committee monotonicity). A committee voting rule f is *committee monotone* if $f(R, k) \subseteq f(R, k+1)$ for all preference profiles R and committee sizes $k \in \{1, \dots, m-1\}$.

Faliszewski et al. [16, Section 2.3.1] deem committee monotonicity a necessity for excellence-based elections, where the goal is to choose the individually best candidates for the considered problem. Similarly, Elkind et al. [15, Section 5] give the example of selecting finalists of a competition where they believe that committee monotonicity is “imperative”. On the other hand, they write that “insisting on a

committee-monotone rule may prevent us from selecting a truly representative committee”. While it is true that committee monotonicity comes at a cost for the representativeness of the chosen committees, we believe that it is a necessary requirement for many applications of proportionally representative voting (see the introduction for examples). Finally, as noted by Elkind et al. [15], a committee monotone rule can be transformed into a rule that returns a ranking of the candidates instead of committees.

3.2 Proportionality for Solid Coalitions

Next, we introduce our central fairness concept called proportionality for solid coalitions (PSC), which is due to Dummett [14]. The idea of this axiom is that, if there is a sufficiently large set of voters $N' \subseteq N$ that all prefer the candidates in a subset $C' \subseteq C$ to the candidates in $C \setminus C'$, then this group should be represented by a number of candidates in C' that is proportional to the size of N' . We will first define this axiom for strict preferences before presenting a variant called inclusion PSC (IPSC) due to Aziz and Lee [3] for weak preferences. Note that we will define these axioms as properties of committees; a committee voting rule satisfies PSC or IPSC if its selected committee always satisfies the given axiom.

To formalize PSC, we define a *solid coalition* for a set of candidates $C' \subseteq C$ as a group of voters $N' \subseteq N$ such that $c' \succ_i c$ for all voters $i \in N'$ and candidates $c' \in C'$, $c \in C \setminus C'$. In this case, we also say that the voters in N' *support* the candidates C' . We emphasize that the voters in N' do not have to agree on the order of the candidates in C' and that a voter can be part of multiple solid coalitions. Now, proportionality for solid coalitions postulates that each solid coalition N' of size $|N'| > \ell \cdot \frac{n}{k+1}$ (for some $\ell \in \mathbb{N}$) is represented by at least ℓ candidates or the set C' if $|C'| < \ell$.

Definition 2 (Proportionality for solid coalitions [14]). A committee W satisfies *proportionality for solid coalitions* (PSC or Droop-PSC) for a preference profile R and a committee size k if for all integers $\ell \in \mathbb{N}$ and solid coalitions N' supporting a set C' with $|N'| > \ell \cdot \frac{n}{k+1}$, it holds that $C' \subseteq W$ or $|W \cap C'| \geq \ell$.

Some authors define the PSC axiom based on the *Hare quota* of $\frac{n}{k}$ instead of the Droop quota of $\frac{n}{k+1}$. This gives rise to a weaker axiom that we call *Hare-PSC*, for which we replace the condition $|N'| > \ell \cdot \frac{n}{k+1}$ by $|N'| \geq \ell \cdot \frac{n}{k}$ in Definition 2. All our positive results will apply to the stronger Droop-PSC, while all our counterexamples will even work for Hare-PSC.

Example 1 (Droop-PSC and Hare-PSC). Consider the profile in Figure 2 with $n = 8$ voters for $k = 2$. Here, the Hare quota $\frac{n}{k}$ is 4 while the Droop quota $\frac{n}{k+1}$ is $\frac{8}{3} < 3$. For this instance, Hare-PSC imposes only one non-trivial inclusion: the voters $\{v_1, \dots, v_4\}$ form a solid coalition for $\{a, b, c\}$, so one of these candidates must be selected. By contrast, Droop-PSC requires that the committee $\{a, e\}$ is chosen as three voters are enough to force a single candidate to be part of the committee.

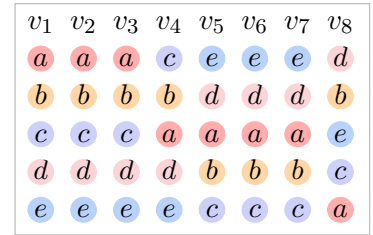


Figure 2: An example for PSC.

We will next present a generalization of PSC to weak preferences due to Aziz and Lee [3]. To this end, we call a set of voters $N' \subseteq N$ a *generalized solid coalition* supporting a set of candidates $C' \subseteq C$ if for all voters $i \in N'$, we have $c' \succsim_i c$ for all candidates $c' \in C'$ and $c \in C \setminus C'$. That is, generalized solid coalitions only have to weakly prefer the candidates in C' to those in $C \setminus C'$. Furthermore, we define the *periphery* $\text{periph}_{N'}(C') = \{c \in C : \text{there exists } i \in N' \text{ and } c' \in C' \text{ such that } c \succsim_i c'\}$ of C' with respect to N' . This is a “closure” of C' , containing C' as well as all candidates that at least one member of N' weakly prefers to a member of C' . Following Aziz and Lee [3], whenever a candidate from the periphery is included in the committee, we will count the candidate as contributing to the satisfaction of N' . Thus, inclusion PSC (IPSC) demands that, for every generalized solid coalition N' supporting a set C' , a number of candidates proportional to $|N'|$ needs to be chosen from the periphery $\text{periph}_{N'}(C')$.

Definition 3 (Inclusion PSC [3]). A committee W satisfies *inclusion PSC* (IPSC) for a profile R and committee size k if for all integers $\ell \in \mathbb{N}$ and generalized solid coalitions N' supporting a set C' with $|N'| > \ell \cdot \frac{n}{k+1}$, it holds that $C' \subseteq W$ or $|W \cap \text{periph}_{N'}(C')| \geq \ell$.

4 PSC and Committee Monotonicity

We will now investigate the compatibility of PSC and committee monotonicity. Based on the profile in Figure 3, we first reason that the combination of these axioms can be tricky to satisfy. For $k = 1$, PSC imposes no constraints in this example, so each singleton set satisfies PSC. However, $\{d\}$ seems to be the best committee since it is a compromise candidate and, e.g., the Condorcet winner. However, for $k = 3$, the unique PSC committee is $\{a, b, c\}$. Therefore, a committee monotone rule satisfying PSC may not choose $\{d\}$ for $k = 1$. However, numerous rules do choose $\{d\}$, including the Expanding Approvals rule [2], the QBS rule [14], and CPO-STV [34, 35]. STV also fails committee monotonicity; an example is given by Elkind et al. [15]. In Appendix A, we construct additional counterexamples showing that all rules in Dummett’s family of *Quota Preference Score* rules fail committee monotonicity.

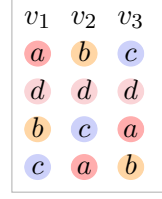


Figure 3: A difficulty with satisfying both PSC and committee monotonicity.

Since none of the known voting rules satisfy both PSC and committee monotonicity, we will design new rules to achieve both axioms simultaneously. To this end, we introduce the *reverse sequential rule* f^{RS} of a committee voting rule f . Roughly, these reverse sequential rules compute the winning committee by repeatedly using the original rule f to identify candidates that are to be removed from the winning committee. To make this more formal, we let $R|_X$ denote the restriction of a profile R to the set $X \subseteq C$, i.e., we derive $R|_X$ by deleting the candidates $C \setminus X$ from R . Then, the reverse sequential rule f^{RS} of a committee voting rule f is defined recursively by $f^{RS}(R, m) = C$ and for all $k \in \{m - 1, \dots, 1\}$, $f^{RS}(R, k) = f(R|_X, k)$ where $X = f^{RS}(R, k + 1)$. We show next that for strict preferences, the reverse sequential rule f^{RS} satisfies committee monotonicity and PSC if the original rule f satisfies PSC. This means that, e.g., reverse sequential STV satisfies both of our desiderata.

Theorem 1. *For strict preferences, for every committee voting rule f that satisfies PSC, the reverse sequential rule f^{RS} satisfies PSC and is committee monotone.*

Proof. The reverse sequential rule f^{RS} of a committee voting rule f satisfies committee monotonicity because $f^{RS}(R, k) = f(R|_{f^{RS}(R, k+1)}, k) \subseteq f^{RS}(R, k + 1)$ for all $k \in \{1, \dots, m - 1\}$ and $R \in \mathcal{L}^N$. Next, suppose that f satisfies PSC. Fix a profile $R \in \mathcal{L}^N$ and let $W_k = f^{RS}(R, k)$ for all $k \in \{1, \dots, m\}$. Further, we fix some k , and we will show that W_k satisfies PSC. For this, let $N' \subseteq N$ be a solid coalition supporting a set of candidates $C' \subseteq C$ such that $|N'| > \ell \frac{n}{k+1}$ for some $\ell \in \mathbb{N}$. To establish PSC, we need to show that $|W_k \cap C'| \geq \min(|C'|, \ell)$. We prove by a backwards induction that $|W_t \cap C'| \geq \min(|C'|, \ell)$ for all $t \in \{m, m - 1, \dots, k\}$. When $t = m$, this holds because $W_m = C$ and thus $W_m \cap C' = C'$. So, suppose that $|W_t \cap C'| \geq \min(|C'|, \ell)$ for some $t \in \{m, \dots, k + 1\}$. If even $|W_t \cap C'| > \min(|C'|, \ell)$, we are done since W_{t-1} is obtained from W_t by deleting only one candidate, so $|W_{t-1} \cap C'| \geq \min(|C'|, \ell)$. Next, assume that $|W_t \cap C'| = \min(|C'|, \ell)$. We note that N' is a solid coalition with $|N'| > \ell \frac{n}{k+1} \geq \ell \frac{n}{t}$ supporting the set $W_t \cap C'$ in the profile $R|_{W_t}$. Since f satisfies PSC, at least $\min(|W_t \cap C'|, \ell)$ candidates from $W_t \cap C'$ are chosen by f for $R|_{W_t}$ and the committee size $t - 1$. Because $|W_t \cap C'| = \min(|C'|, \ell)$, it holds that $\min(|W_t \cap C'|, \ell) = \min(|C'|, \ell)$ and thus $W_t \cap C' \subseteq f(R|_{W_t}, t - 1) = W_{t-1}$. It follows that $|W_{t-1} \cap C'| \geq \min(|C'|, \ell)$, establishing the induction step. $\square$

Unfortunately, Theorem 1 does not extend to weak preferences and IPSC. For example, in the profile shown in Figure 4, the committee $\{a, b, c\}$ satisfies IPSC, and the committee $\{a, b\}$ satisfies IPSC after restricting the profile to $\{a, b, c\}$. However, $\{a, b\}$ violates IPSC because the last two voters form a generalized solid coalition supporting $\{d\}$ while no candidate from the periphery $\{c, d\}$ is selected.

For another example, consider the proportional approval voting (PAV) rule which is defined for approval preferences, a special case of weak preferences, and satisfies IPSC [3]. If Theorem 1 held for weak preferences, then the reverse sequential version of PAV should also satisfy IPSC, and thus also the weaker axioms PJR and JR [3]. However, it is known that reverse sequential PAV fails JR [1].

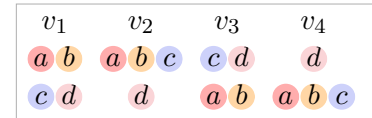


Figure 4: For weak preferences, reverse sequential rules may fail IPSC.

5 The Solid Coalition Refinement Rule

A drawback of committee voting rules obtained by the reverse sequential transformation is that they are rather technical and fail IPSC for weak preferences. In addition, these rules can be computationally intensive, since we need to run $m - k$ rounds of the base rule to determine k winners. This is a concern especially when computing small committees in the presence of a large number of candidates. These drawbacks motivate the search for a more direct rule that combines PSC and committee monotonicity. We thus present next the *Solid Coalition Refinement rule* (or SCR for short), which satisfies both of our desiderata. Moreover, this rule generalizes to weak preferences and satisfies IPSC in this case. Due to space restrictions, we defer all proofs of this section to the appendix.

Since the SCR rule is inspired by the D'Hondt method of apportionment, we first give a description of this method. In apportionment, a set of p parties with their respective vote counts $v_1, \dots, v_p$ need to be assigned a total of h seats. The D'Hondt method (also known as Jefferson method) intuitively tries to assign these seats such that *each seat represents as many voters as possible*. For example, if a party with 100 votes is assigned 2 seats, each seat represents 50 voters. Now, the D'Hondt method proceeds sequentially, in each step assigning the next seat to a party that optimizes the per-seat representation. Formally, this means selecting a party i that maximizes $v_i/(s_i + 1)$, where s_i is the number of seats assigned to i thus far. Note that $v_i/(s_i + 1)$ corresponds to the per-seat representation should the next seat go to party i . This procedure is repeated until all h seats are filled.

In committee voting, there are only individual candidates instead of parties. However, we can view sets of candidates as “virtual parties” that receive votes from the solid coalitions supporting them. Thus, each subset $C' \subseteq C$ forms a party supported by $v_{C'} = |\{i \in N : \forall c' \in C', c \in C \setminus C' : c' \succ_i c\}|$ voters. Now, the SCR rule sequentially builds up a committee W based on same principle as the D'Hondt method: at each step, we view the party C' as having been assigned $s_{C'} = |C' \cap W|$ seats thus far and we give the next seat to the party C' maximizing $v_{C'}/(s_{C'} + 1)$. In selecting this party, the method ignores parties that are already completely contained in the committee. Moreover, after identifying the most-deserving party C' , we still need to decide which candidate from $C' \setminus W$ to add to the committee. If $|C' \setminus W| = 1$, it is clear that we add the sole candidate in $C' \setminus W$. On the other hand, if $|C' \setminus W| > 1$, we choose the candidate based on the same philosophy of maximizing the per-seat representation. Thus, we will look for a strict sub-party C'' of C' (i.e., $C'' \subsetneq C'$) with $C'' \setminus W \neq \emptyset$ that maximizes $v_{C''}/(s_{C''} + 1)$. We repeat this refinement step until we find a party contained in C' that includes only a single unselected candidate, which is then added to the committee. This procedure is repeated until all k seats are filled.

Let us further generalize the SCR rule to weak preferences and IPSC. In this setting, a virtual party will not just refer to a set of candidates but to a pair (N', C') where N' is a generalized solid coalition supporting the set C' of candidates. The number of voters for this virtual party is $|N'|$, and the number of seats assigned to the party by a committee W is $|W \cap \text{periph}_{N'}(C')|$. Thus, let us define $\rho(W, N', C') = \frac{|N'|}{|W \cap \text{periph}_{N'}(C')| + 1}$ as the *underrepresentation value* of the party (N', C') under the committee W . Intuitively, a large underrepresentation value means that the generalized solid coalition (or party) is far from being proportionally represented. Indeed, we can establish a direct relationship between the underrepresentation value and the IPSC property.

Proposition 1. *A committee W satisfies IPSC for a profile R and a committee size k if and only if $\rho(W, N', C') \leq \frac{n}{k+1}$ for every generalized solid coalition N' that supports a set C' with $C' \not\subseteq W$.*

Following again the philosophy of the D'Hondt method, the generalized solid coalition with the highest underrepresentation value has the strongest claim to decide the next seat. As a final piece of notation, we denote by $\Phi(R, W, D)$ the set of pairs (N', C') where N' is a generalized solid coalition supporting the set C' such that $C' \not\subseteq W$ (the coalition is not fully satisfied) and $C' \subsetneq D$ (it refines a given set D). Then, the SCR rule works as follows for weak preferences: in each round, we identify the party (N', C') that maximizes the underrepresentation value $\rho(W, N', C')$ among all parties in $\Phi(R, W, C)$.

Algorithm 1: The Solid Coalition Refinement Rule

Input : A preference profile R and committee size k **Output** : A committee of k candidates

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1  $W \leftarrow \emptyset$ 
2 for  $i \in \{1, \dots, k\}$  do
3    $D \leftarrow C$ 
4   while  $|D \setminus W| > 1$  do
5      $\Phi(R, W, D) \leftarrow \{(N', C') : N' \text{ is a generalized solid coalition supporting } C' \subsetneq D \text{ with } C' \not\subseteq W\}$ 
6      $(N^*, C^*) \leftarrow \arg \max_{(N', C') \in \Phi(R, W, D)} \frac{|N'|}{|W \cap \text{periph}_{N'}(C')| + 1}$ 
7      $D \leftarrow C^*$ 
8    $W \leftarrow W \cup D$ 
9 return  $W$ 
```

Moreover, to decide which candidate from $C' \setminus W$ to choose, we again identify the party (N'', C'') that maximizes $\rho(W, N'', C'')$ among all parties in $\Phi(R, W, C')$. By repeating this refinement step, we eventually arrive at a generalized solid coalition N^* supporting a set C^* such that $|C^* \setminus W| = 1$, and we add the single candidate in $C^* \setminus W$ to W . This process is again repeated until k candidates are selected. A pseudocode description of the SCR rule is given in [Algorithm 1](#).

We note that multiple generalized solid coalitions in $\Phi(R, W, D)$ may have the same maximal underrepresentation value in some steps; in such cases, we assume that ties are broken by an arbitrary but fixed ranking $\triangleright$ over the sets of candidates $C' \subseteq C$. That is, if two generalized solid coalitions N' and N'' with candidate sets C' and C'' have the same maximal underrepresentation value in some step of the SCR rule, we choose (N', C') if $C' \triangleright C''$ and (N'', C'') otherwise.

Example 2. Consider the preference profile in [Figure 5](#) with $n = 5$ voters and $m = 4$ candidates. With a committee size $k = 3$, the SCR rule runs as follows.

v_1	v_2	v_3	v_4	v_5
a	a	c	d	d
b	c	b	a	a
c	b	a	b	b
d	d	d	c	c

Figure 5: An example for the SCR rule.

1) Initially, $D = \{a, b, c, d\}$ and $W = \emptyset$. Then:

- The solid coalition $N' = \{v_1, v_2, v_3\}$ supporting $C' = \{a, b, c\}$ has $\rho(W, N', C') = 3$, which is maximal among all solid coalitions in $\Phi(R, W, D)$. SCR sets $D = \{a, b, c\}$.
- The solid coalition $N' = \{v_1, v_2\}$ supporting $C' = \{a\}$ has $\rho(W, N', C') = 2$, which is maximal among all solid coalitions in $\Phi(R, W, D)$. SCR sets $D = \{a\}$. Hence, a is selected and $W = \{a\}$.

2) D is reset to $\{a, b, c, d\}$. Then:

- The solid coalition $N' = \{v_4, v_5\}$ supporting $C' = \{d\}$ has $\rho(W, N', C') = 2$, which is maximal among all solid coalitions in $\Phi(R, W, D)$. SCR sets $D = \{d\}$. Hence, d is selected and $W = \{a, d\}$.

3) D is reset to $\{a, b, c, d\}$. Then:

- The solid coalition $N' = \{v_1, v_2, v_3\}$ supporting $C' = \{a, b, c\}$ has $\rho(W, N', C') = \frac{3}{2}$, which is maximal among all solid coalitions in $\Phi(R, W, D)$. SCR sets $D = \{a, b, c\}$.
- The solid coalition $N' = \{v_3\}$ supporting $C' = \{c\}$ has $\rho(W, N', C') = 1$, which is maximal among all solid coalitions in $\Phi(R, W, D)$. SCR sets $D = \{c\}$. This means that c is selected and the final committee is $W = \{a, d, c\}$.

We next show that SCR is well-defined and runs in polynomial time if the voters have strict preferences.

Proposition 2. *The SCR rule always terminates and produces a committee of the target size k . Furthermore, for strict preferences, it can be implemented to run in polynomial time.*

We leave it open whether the SCR rule can also be computed in polynomial time for weak preferences as it is not clear how to efficiently compute [line 6](#) in this case. Indeed, if we could solve this maximization problem for every profile R and committee W , we could also decide whether there is a generalized solid coalition N' supporting a set C' such that $C' \not\subseteq W$ and $\rho(W, N', C') > \frac{n}{k+1}$. However, this is equivalent to deciding whether a committee satisfies IPSC ([Proposition 1](#)), which is coNP-complete [[10](#)].

Finally, we will show that the SCR rule satisfies committee monotonicity and IPSC.

Theorem 2. *Even for weak preferences, the SCR rule is committee monotone and satisfies IPSC.*

Proof Sketch. Since the SCR rule selects candidates sequentially and independently of the target committee size, it clearly satisfies committee monotonicity. Next, for IPSC, fix a profile $R \in \mathcal{R}^N$ and a committee size k and let $W = \text{SCR}(R, k)$. For our proof, we assume that each voter i is given a virtual budget of 1, which is used to buy the candidates for a price of $\frac{n}{k+1}$. We then show that there is a payment scheme for the committee W such that (i) no voter pays more than his budget and (ii) if a voter i is part of a generalized solid coalition N' supporting a candidate set $C' \not\subseteq W$ such that $\rho(W, N', C') > \frac{n}{k+1}$, voter i only spends his budget on candidates in $\text{periph}_{N'}(C')$. The existence of this payment scheme implies that W satisfies IPSC. Indeed, if this was not the case, there is a generalized solid coalition N' supporting a set $C' \not\subseteq W$ and an integer ℓ such that $|N'| > \ell \cdot \frac{n}{k+1}$ and $|W \cap \text{periph}_{N'}(C')| < \ell$. This implies that $\rho(W, N', C') \geq \frac{|N'|}{\ell} > \frac{n}{k+1}$. By condition (ii) of our payment scheme, the voters in N' have only paid for candidates in $\text{periph}_{C'}(N')$, so they spent a budget of at most $(\ell - 1) \frac{n}{k+1}$. Since $|N'| > \ell \cdot \frac{n}{k+1}$, the remaining budget of these voters is strictly larger than $\frac{n}{k+1}$. However, the total remaining budget after buying k candidates is $\frac{n}{k+1}$. Hence, a voter $i \notin N'$ must have spent more than his budget, which violates condition (i). This contradiction shows that SCR satisfies IPSC. $\square$

6 PSC and Irrelevant Voter Blocks

We next use the SCR rule to answer an open question of Graham-Squire et al. [[18](#)] by showing that there is a rule that satisfies both independence of losing voter blocs and PSC. In more detail, these authors study the setting of *truncated* preferences, i.e., voters have strict preferences but it is no longer necessary to rank all candidates. The notion of solid coalitions, and thus also PSC as well as the SCR rule, can be easily extended to this setting: solid coalitions are defined just as before and, in particular, only form for sets of voters who rank all candidates in the supported set of candidates. Then, PSC and the SCR rule can be adapted to truncated preferences by using this new definition of solid coalitions.

Furthermore, Graham-Squire et al. [[18](#)] suggest two consistency notions regarding the behavior of committee voting rules when some voters are removed from the election. One of these notions, independence of losing voter blocs, requires that the outcome should not change when we remove voters who only rank unchosen candidates. To formalize this notion, we define by $X(\succsim_i)$ the set of candidates that are not ranked in the truncated preference relation $\succsim_i$. Moreover, $R_{-N'}$ denotes the profile derived from another profile R by deleting the voters in $N' \subseteq N$ from R .

Definition 4 (Independence of losing voter blocs). A committee voting rule f satisfies *independence of losing voter blocs* if $f(R, k) = f(R_{-N'}, k)$ for all truncated preference profiles R , committee sizes k , and sets of voters $N' \subsetneq N$ such that $f(R, k) \subseteq X(\succsim_i)$ for all $i \in N'$.

Graham-Squire et al. [[18](#)] show that no classical proportional rules such as STV and EAR satisfy this property. In fact, they even conjecture that no voting rule simultaneously satisfies PSC and independence of losing voter blocs. We show that this conjecture is false as SCR satisfies both conditions.

Theorem 3. *For truncated preferences, the SCR rule satisfies PSC and independence of losing voter blocs.*

Proof. First, an analogous argument as for [Theorem 2](#) shows that SCR satisfies PSC for truncated preferences. Hence, we focus on independence of losing voter blocs. Fix a profile R , a committee size

k , and a set of voters N' such that $W \subseteq X(\succsim_i)$ for all $i \in N'$, and let $W = \text{SCR}(R, k)$. We need to show that SCR also chooses the committee W for the profile $R_{-N'}$ and the committee size k . For this, let W' denote the intermediate committee and D the current set of candidates of Algorithm 1 during some step of the execution of SCR for R such that $|D \setminus W'| > 1$. Moreover, let (N^*, C^*) be the solid coalition and the set of candidates that is chosen next at line 6. Since $W \subseteq X(\succsim_i)$ for all $i \in N'$, there must be for every voter $i \in N'$ a candidate $x \in C^* \cap X(\succsim_i)$. Otherwise, voter i ranks the candidate that will be selected next because $C^* \subseteq C \setminus X(\succsim_i)$, which contradicts that $W \subseteq X(\succsim_i)$. Since solid coalitions only form over sets of voters that rank all candidates in the given set, this implies that $(N^*, C^*) \in \Phi(R_{-N'}, W', D)$. Further, it holds that $\Phi(R_{-N'}, W', D) \subseteq \Phi(R, W', D)$ as introducing new voters can only create more solid coalitions. Thus, if SCR agrees on W' and D for both R and $R_{-N'}$ in the current step, it will still agree on these sets for the next step. Since it initially always holds that $W' = \emptyset$ and $D = C$, it follows that SCR chooses W for both R and $R_{-N'}$. $\square$

7 Committee Monotonicity and Rank-JR

In light of our positive results so far, it is a natural follow-up question whether the SCR rule—or, more generally, any committee monotone voting rule—also satisfies other forms of proportionality. Unfortunately, we will give a negative answer to this question by showing that no committee monotone voting rule satisfies a proportionality notion called Rank-JR due to Brill and Peters [10].

In more detail, Brill and Peters suggested a family of proportionality notions inspired by fairness axioms from approval-based committee elections. To explain these axioms, we define the *rank* of a candidate x in a preference relation $\succsim_i$ as $\text{rank}(\succsim_i, x) = |\{y \in A : y \succ_i x\}| + 1$. Thus, the most-preferred candidate in a preference relation has rank 1, while the least-preferred candidate has rank m . Now, Brill and Peters observe that for each $r \in \{1, \dots, m\}$, we can transform a preference profile R into an approval profile $A(R, r)$ by letting each voter approve the candidates with a rank of at most r . Based on this insight, proportionality notions for approval-based committee elections can be transferred to ranked preferences by requiring that a committee satisfies the given proportionality axiom for the approval profiles $A(R, r)$ for all $r \in \{1, \dots, m\}$. Applying this approach to justified representation (JR) [5], one of the weakest fairness notions in approval-based committee voting, results in the following axiom.

Definition 5 (Rank-JR). A committee W satisfies Rank-JR for a preference profile R and committee size k if for all ranks $r \in \{1, \dots, m\}$ and groups of voters $N' \subseteq N$ such that $|N'| \geq \frac{n}{k}$ and $\bigcap_{i \in N'} \{x \in C : \text{rank}(\succsim_i, x) \leq r\} \neq \emptyset$, it holds that $W \cap \bigcup_{i \in N'} \{x \in C : \text{rank}(\succsim_i, x) \leq r\} \neq \emptyset$.

Brill and Peters [10] also adopt several other approval-based fairness axioms to ranked preferences (see Figure 1). However, because Rank-JR is their weakest fairness notion for ranked preferences, the incompatibility of Rank-JR and committee monotonicity shown in the next theorem implies that no committee monotone voting rule satisfies any of the proportionality conditions of Brill and Peters.

Theorem 4. *Even for strict preferences, no committee voting rule satisfies both committee monotonicity and Rank-JR if $n \geq 4$ and $m \geq \frac{n}{q} + 1$, where $q \in \mathbb{N}$ satisfies that $\frac{n}{q} \geq 4$ and $\frac{n}{q} \in \mathbb{N}$.*

Proof. Fix some n , let $q \in \mathbb{N}$ denote a divisor of n such that $\frac{n}{q} \geq 4$, and define $\ell = \frac{n}{q}$. Consider a set of at least $\ell + 1$ candidates, the first $\ell + 1$ of which we label as $c_1, \dots, c_\ell$ and d , and let f be a committee voting rule that satisfies Rank-JR. Lastly, consider the profile R where, for every $j \in \{1, \dots, \ell\}$, there are q voters who top-rank c_j , second-rank d , and order the remaining alternative arbitrarily. We will analyze the committees selected by f for $k = 2$ and $k = \ell$. We first claim that $d \in f(R, 2)$. If this was not the case, at most two of the groups in our profile receive their top-ranked candidate as $k = 2$. Since $\ell \geq 4$, there are at least $\lceil \frac{n}{2} \rceil$ voters who do not receive their top-ranked candidate. However, Rank-JR for $r = 2$ requires that one of the top-ranked candidates of these voters is chosen, as all of them rank d second but $d \notin f(R, 2)$, a contradiction. On the other hand, when $k = \ell$, Rank-JR for $r = 1$ requires that the top-ranked candidate of each group of voters is selected, so $f(R, \ell) = \{c_1, \dots, c_\ell\}$. This implies that f fails committee monotonicity since $d \in f(R, 2)$ but $d \notin f(R, \ell)$. $\square$

8 Application to Rank Aggregation

As our last contribution, we will discuss the consequences of our results for rank aggregation. Like in the committee voting setting, the rank aggregation problem takes as input a preference profile R . However, the goal is now to compute a winning ranking instead of a winning committee. More formally, rank aggregation is done using *social welfare functions (SWFs)*, which are functions that map every strict preference profile R to a single strict winning ranking $f(R) = \succ \in \mathcal{L}$.

Interestingly, there is a 1-to-1 correspondence between SWFs and committee monotone committee voting rules. To formalize this, we recall that the rank of an candidate x in a preference relation $\succ$ is $\text{rank}(\succ, x) = |\{y \in C : y \succ x\}| + 1$. Then, an SWF f can be converted into a committee monotone voting rule g by setting $g(R, k) = \{x \in C : \text{rank}(f(R), x) \leq k\}$, i.e., g chooses the top- k candidates with respect to $f(R)$. Similarly, a committee monotone committee voting rule g can be turned into an SWF by ranking the element in $g(R, k+1) \setminus g(R, k)$ in the $k+1$ -st position. Furthermore, PSC naturally extends to a rank-based proportionality notion for SWFs by requiring that for each $k \in \{1, \dots, m\}$, the top- k candidates with respect to the chosen ranking satisfies PSC. We define this condition again as a property of rankings and an SWF satisfies PSC if its chosen ranking always satisfies this condition.

Definition 6 (Proportionality for solid coalitions for rankings). Given a preference profile R , a ranking $\succ$ satisfies Droop-PSC (Hare-PSC) if for every $k \in \{1, \dots, m\}$, the set $\{x \in C : \text{rank}(\succ, x) \leq k\}$ satisfies Droop-PSC (Hare-PSC) for the profile R and the committee size k .

It is easy to see that all our positive results carry over to SWFs, e.g., the SCR rule also satisfies PSC when viewed as an SWF. However, in the context of rank aggregation, there are alternate approaches for measuring proportionality. In particular, Lederer et al. [23] propose to analyze the worst-case swap distance of an input ranking to the output ranking, as a function of the fraction of voters that report the given ranking. To make this more formal, we define the swap distance between two ranking $\succ$ and $\succ'$ by $\text{swap}(\succ, \succ') = |\{(x, y) \in C^2 : x \succ y \text{ and } y \succ' x\}|$, i.e., the number of pairs the rankings disagree on. The central idea of Lederer et al. is that, if an α fraction of the voters report a ranking $\succ$, then its swap distance to the output ranking should ideally be $(1 - \alpha) \binom{m}{2}$ since $\binom{m}{2}$ is the maximal swap distance between two rankings. In other words, in the ideal case, every α fraction of the voters that report the same ranking would agree with the output ranking on at least $\alpha \binom{m}{2}$ pairs. With this goal in mind, Lederer et al. suggest the Squared Kemeny rule and show that it approximately satisfies this condition: if a ranking $\succ$ is reported by at least $\alpha \cdot n$ voters, the maximum swap distance between $\succ$ and a ranking $\succ'$ chosen by the Squared Kemeny rule is at most $\text{swap}(\succ, \succ') \leq \sqrt{(1 - \alpha)/\alpha} \cdot \binom{m}{2}$.

We next aim to give similar bounds for rankings that satisfy PSC. Specifically, we will show that, up to a small constant factor, the swap distance of a ranking that is reported by an α fraction of the voters and a ranking satisfying PSC is linear in α . For instance, this means that the SCR rule gives strong fairness guarantees simultaneously in terms of the swap distance and in terms of PSC. We defer the full proof of this result to the appendix and discuss a short proof sketch instead.

Theorem 5. Let R be a ranking profile in which at least $\alpha \cdot n$ voters report $\succ$. Further, let $\succ_H$ be a ranking satisfying Hare-PSC for R and $\succ_D$ a ranking satisfying Droop-PSC for R . It holds that

- (i) $\text{swap}(\succ, \succ_H) \leq (1 - \alpha + \frac{1+\alpha}{m}) \binom{m}{2}$ and
- (ii) $\text{swap}(\succ, \succ_D) \leq (1 - \alpha) \binom{m}{m-1} \binom{m}{2}$.

Proof Sketch. We will focus here on Claim (i) as Claim (ii) follows from analogously reasoning. Fix a profile R and let $\succ_H$ denote a ranking satisfying Hare-PSC. Moreover, let N' denote a group of voters reporting a ranking $\succ = c_1 \succ c_2 \succ \dots \succ c_m$ in R and let $\alpha \in [0, 1]$ such that $|N'| \geq \alpha \cdot n$. For every rank $r \in \{1, \dots, m\}$, the group N' forms a solid coalition for $\{c_1, \dots, c_r\}$. By Hare-PSC, all candidates in $\{c_1, \dots, c_r\}$ must be selected when the committee size k satisfies $\alpha \geq \frac{r}{k}$, so they

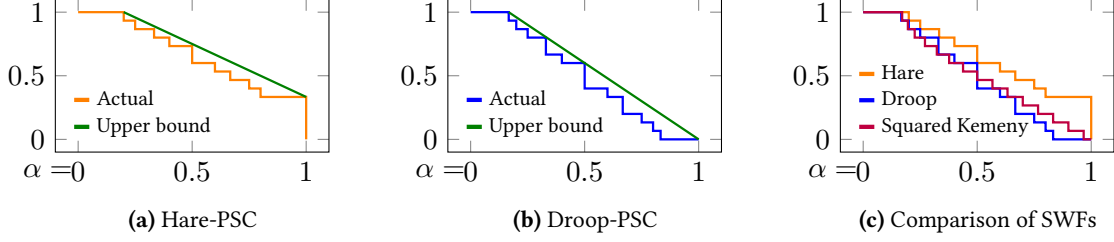


Figure 6: The maximum (normalized) swap distance between the output rankings for various rules to an input ranking, as a function of the fraction of the voters that report the input ranking. The swap distance is normalized to be in $[0, 1]$. The left figure shows the maximum swap distance between an input ranking of weight α and a ranking that satisfies Hare-PSC. Specifically, we display both our theoretical upper bound (green) and the actual upper bound computed for $m = 6$ (orange). In Figure (b), we show an analogous plot for rankings satisfying Droop-PSC, with green corresponding to our theoretical upper bound and blue to the actual upper bound for $m = 6$. Finally, on the right figure, we compare the maximum swap distance to an input ranking of weight α for the Squared Kemeny rule (purple) and rankings satisfying Droop-PSC (blue) and Hare-PSC (orange).

are among the top- $\lceil \frac{r}{\alpha} \rceil$ candidates of $\mathcal{R}_H$. Hence, for every $r \in \{0, \dots, m-1\}$ and candidate x with $\lceil \frac{r}{\alpha} \rceil + 1 \leq \text{rank}(\mathcal{R}_H, x) \leq \lceil \frac{r+1}{\alpha} \rceil$, it holds that $x = c_i$ for some $i > r$. Therefore, candidate x can have at most $\text{rank}(\mathcal{R}_H, x) - 1 - r$ inversions to higher-ranked candidates. This implies that $\text{swap}(\mathcal{Z}, \mathcal{R}_H) \leq \sum_{r=1}^{r'} \sum_{i=\lceil (r-1)/\alpha \rceil + 1}^{\lceil r/\alpha \rceil} (i-r) + \sum_{i=\lceil r'/\alpha \rceil + 1}^m (i-r'-1)$, where r' is the maximal integer with $\lceil \frac{r'}{\alpha} \rceil \leq m-1$. From here, our bound follows from algebraic transformations and inequalities. $\square$

To gain a better intuition for Theorem 5, we plot in Figure 6 our theoretical upper bounds as well as the actual worst-case bound for $m = 6$ candidates for Hare-PSC and Droop-PSC. As we can see, Droop-PSC gives significantly better bounds. For example, when α is close to 1, Hare-PSC allows even the bottom-ranked candidate in the input ranking to be top-ranked in the output ranking, whereas Droop-PSC precludes this behavior. Furthermore, in Figure 6(c), we compare the worst-case rankings that satisfy Hare-PSC and Droop-PSC to the worst-case ranking that can be chosen by the Squared Kemeny rule. As we can see, rankings that satisfy Droop-PSC perform similarly well as the ones chosen by the Squared Kemeny rule. Since the latter rule is NP-hard to compute [23] while rules like SCR run in polynomial time for strict preferences, this provides further motivation for using our methods.

9 Conclusion

In this paper, we present the first committee voting rules for ranked preferences that satisfy committee monotonicity and proportionality for solid coalitions (PSC). Specifically, we first show that such rules can be defined based on known rules that satisfy PSC by running these rules in a reverse sequential mode. However, this approach results in rather technical rules, so we further design a new committee voting rule called the Solid Coalition Refinement rule that simultaneously satisfies committee monotonicity and PSC. This rule can even be generalized to weak preferences, for which it satisfies a strengthening of PSC called inclusion PSC. Using this rule, we also disprove a conjecture by Graham-Squire et al. [18] regarding the compatibility of PSC and a notion called independence of losing voter blocs for truncated preferences. Additionally, we show that committee monotonicity is not compatible with a recently suggested family of fairness concepts by Brill and Peters [10]. Finally, we re-examine our results in the context of rank aggregation and provide bounds on the maximum swap distance between a given input ranking and a PSC ranking depending on the fraction of voters that report the input ranking.

Our results also give other directions for future work. In particular, we leave it open whether there is a polynomial-time computable rule for weak preferences that satisfies the axiomatic properties of SCR. Moreover, much remains unknown about the compatibility of committee monotonicity and proportionality under approval preferences [see also 22].

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A Committee Monotonicity of Quota Preference Score Rules

QBS was introduced by Dummett [14] and later generalized into a class called *Minimal Demand* (MD) rules by Aziz and Lee [4]. For each rank r , starting at 1 and increasing one by one, these rules perform two steps:

1. Partition the voters into solid coalitions, where two voters are in the same solid coalition if their top r ranked candidates are the same, regardless of the ordering within the top r .
2. Consider each solid coalition N' supporting candidate set C' in the partition. If N' is entitled to more representation under PSC, then additional candidates from C' are elected until this entitlement is met.

The family of rules differ by the tie-breaking used in the second step. Dummett [14] suggests that the Borda score¹ be used. Aziz and Lee [4] showed that PSC is satisfied regardless of the tie-breaking method.

We show that MD rules fail committee monotonicity whenever positional scoring is used for tie-breaking. A positional scoring rule consists of a score vector $(s_1, s_2, \dots, s_m)$, where $s_1 \geq s_2 \geq \dots \geq s_m \geq 0$ and $s_1 > s_m$. A candidate earns s_r points if a voter ranks them in position r . The tie-breaking in step 2 selects the candidate with the highest total score across all n voters. Further tie-breaking may be needed if there are ties in the positional scoring, but our impossibility result holds regardless of the additional tie-breaking method.

Proposition 3. *Minimal Demand (MD) rules that break ties using positional scoring do not satisfy committee monotonicity, even for strict preferences.*

Proof. Consider an MD rule with a positional scoring vector $(s_1, s_2, \dots, s_m)$. We assume $s_m = 0$: if it wasn't, then we could use a new vector $(s_1 - s_m, s_2 - s_m, \dots, 0)$ without changing any tie-breaking decisions. We consider two cases depending on the scoring vector.

Case 1: $s_1 < 2s_2$. For this case, we construct a profile R with $n = 2$ voters and $m = 3$ candidates. To increase n , the entire profile can be duplicated multiple times. To increase m , extra candidates can be added to the preference relations of the voters as shown below.

Voter 1: $c_1 \succ c_3 \succ \text{any extra candidates} \succ c_2$
Voter 2: $c_2 \succ c_3 \succ \text{any extra candidates} \succ c_1$

For $k = 1$, a solid coalition must include both voters to be entitled to representation and so no candidates are elected until $r = m$. The tie-breaking will select candidate c_3 since its score of $2s_2$ exceeds the scores of the other candidates. For $k = 2$, voters 1 and 2 both separately form solid coalitions when $r = 1$, so candidates c_1 and c_2 are selected. This violates committee monotonicity because c_3 is elected when $k = 1$ but not when $k = 2$.

Case 2: $s_1 \geq 2s_2$. For this case, we construct a profile R with $n = 11$ voters and $m = 12$ candidates. To increase n , the entire profile can be duplicated ℓ times. To increase m , extra candidates can be added to the end of preference relations of each voter.

Voter 1: $c_1 \succ c_2 \succ c_3 \succ c_4 \succ c_9 \succ \text{other candidates}$

¹For each voter, their lowest-ranked candidate is given 0 points, their second lowest candidate 1 point, their third lowest 2 points, etc. The candidate with the highest total score is chosen.

Voter 2: $c_2 \succ c_3 \succ c_4 \succ c_1 \succ c_9 \succ$ other candidates
 Voter 3: $c_3 \succ c_4 \succ c_1 \succ c_2 \succ c_9 \succ$ other candidates
 Voter 4: $c_4 \succ c_1 \succ c_2 \succ c_3 \succ c_9 \succ$ other candidates
 Voter 5: $c_5 \succ c_6 \succ c_7 \succ c_8 \succ c_9 \succ$ other candidates
 Voter 6: $c_6 \succ c_7 \succ c_8 \succ c_5 \succ c_9 \succ$ other candidates
 Voter 7: $c_7 \succ c_8 \succ c_5 \succ c_6 \succ c_9 \succ$ other candidates
 Voter 8: $c_8 \succ c_5 \succ c_6 \succ c_7 \succ c_9 \succ$ other candidates
 Voters 9 to 11: $c_9 \succ c_{10} \succ c_{11} \succ c_{12} \succ$ other candidates

Note that voters 1 to 4 form a solid coalition supporting $\{c_1, c_2, c_3, c_4\}$ and voters 5 to 8 form a solid coalition supporting $\{c_5, c_6, c_7, c_8\}$. For $k = 2$, the committee will consist of one candidate from $\{c_1, c_2, c_3, c_4\}$ and another from $\{c_5, c_6, c_7, c_8\}$. Now consider $k = 1$. A solid coalition needs at least 6 voters to be entitled to representation, and any such solid coalition will include c_9 in its supported candidates. We will show that c_9 has a higher score than all of c_1 through c_8 , meaning that committee monotonicity will be violated:

- Candidates c_1 through c_8 are ranked first, second, third, and fourth by exactly one voter each. Therefore their scores are upper bounded by $s_1 + s_2 + s_3 + s_4 + 7s_5 \leq s_1 + 3s_2 + 7s_5$.
- Candidate c_9 has a score of $3s_1 + 8s_5$.

If $s_2 = 0$, then $s_i = 0$ for all $i \geq 2$ and c_9 's score of $3s_1$ is higher than c_1 through c_8 's score of s_1 . Otherwise, assume that $s_2 > 0$:

$$\begin{aligned}
 3s_1 + 8s_5 &\geq s_1 + 4s_2 + 8s_5 && \text{since } s_1 \geq 2s_2 \\
 &> s_1 + 3s_2 + 8s_5 && \text{since } s_2 > 0 \\
 &\geq s_1 + 3s_2 + 7s_5,
 \end{aligned}$$

and so c_9 has a higher score than c_1 through c_8 . □

Since the Borda rule is a positional scoring rule, we have the following corollary.

Corollary 1. *QBS does not satisfy committee monotonicity.*

B Proof of Proposition 1

Proposition 1. *A committee W satisfies IPSC for a profile R and a committee size k if and only if $\rho(W, N', C') \leq \frac{n}{k+1}$ for every generalized solid coalition N' that supports a set C' with $C' \not\subseteq W$.*

Proof. Fix a profile R and a committee size k . First, suppose that the committee W satisfies IPSC for R and k . Let N' be a generalized solid coalition supporting a candidate set C' with $C' \not\subseteq W$ and let ℓ^* denote the maximal integer such that $|N'| > \ell^* \frac{n}{k+1}$, which means that $|N'| \leq (\ell^* + 1) \frac{n}{k+1}$. By IPSC, we have that $|W \cap \text{periph}_{N'}(C')| \geq \ell^*$, so

$$\rho(W, N', C') = \frac{|N'|}{|W \cap \text{periph}_{N'}(C')| + 1} \leq \frac{|N'|}{\ell^* + 1} \leq \frac{n}{k+1}.$$

Conversely, suppose that the committee W fails IPSC for R and k . Thus, there is a generalized solid coalition N' supporting a set C' and an integer $\ell \in \mathbb{N}$ such that $|N'| > \frac{\ell n}{k+1}$, $C' \not\subseteq W$, and

$|W \cap \text{periph}_{N'}(C')| < \ell$. The underrepresentation value for (N', C') is

$$\rho(W, N', C') = \frac{|N'|}{|W \cap \text{periph}_{N'}(C')| + 1} \geq \frac{|N'|}{\ell} > \frac{n}{k+1},$$

which proves the direction from right to left of our proposition. $\square$

C Proof of Proposition 2

Proposition 2. *The SCR rule always terminates and produces a committee of the target size k . Furthermore, for strict preferences, it can be implemented to run in polynomial time.*

Proof. First, we will show that if the SCR rule terminates, it produces a committee of size k . To this end, we note that each iteration of the outer loop (line 2) adds exactly one candidate to W . The reason for this is that the set $\Phi(R, W, D)$ only contains pairs (N', C') such that $C' \not\subseteq W$, which means that D will always contain at least one candidate not in W . On the other hand, we only reach line 8 after the end of the while-loop, which requires that $|D \setminus W| \leq 1$. In combination, this means that $|D \setminus W| = 1$, so each iteration of the outer loop adds a single candidate.

Next, we will show that SCR always terminates. To this end, we note that if $\Phi(R, W, D)$ is always non-empty during the execution of SCR, then each iteration of the while-loop (line 4) reduces the size of D by at least 1. This holds because C' is a strict subset of D for all $(N', C') \in \Phi(R, W, D)$. Now, consider the set D in some round during the execution of the SCR rule and suppose that $|D \setminus W| > 1$. We first note that D is supported by a generalized solid coalition N'' . In more detail, if $D = C$, this holds as C is supported by the set of all voters N . On the other hand, if $D \neq C$, then D was chosen as the set of candidates supported by a generalized solid coalition N'' in the previous iteration of the while-loop. Next, let $i \in N''$, which means that $d \succsim_i c$ for all $d \in D, c \in C \setminus D$. Moreover, let d^* denote one of voter i 's least favorite candidates in $D \setminus W$ and define $D' = \{c \in D \setminus \{d^*\} : c \succsim_i d^*\}$. By the definition of this set and d^* , it holds that $D' \subsetneq D$, $D' \not\subseteq W$, and that $\{i\}$ is a generalized solid coalition supporting D' . Hence, $(\{i\}, D') \in \Phi(R, W, D)$, which proves that $\Phi(R, W, D) \neq \emptyset$. Thus, the SCR rule is indeed well-defined.

Finally, we show that the SCR rule can be computed in polynomial time for strict preference profiles. Since line 4 is repeated at most mk times, we only need to show that the procedure in the while-loop can be done in polynomial time. For this, we note that for strict preferences, IPSC coincides with PSC and a generalized solid coalition is equivalent to a solid coalition. Furthermore, there are only nm candidate sets C' such that (N', C') can be in $\Phi(R, W, D)$: for each voter i and each rank r , we can obtain one such set by considering the r most preferred candidates of voter i . Finally, we can compute line 6 by evaluating $\rho(W, N', C')$ for all such sets C' and the maximal solid coalitions N' supporting C' . This works because for any two solid coalitions N', N'' supporting C' , it holds that $\text{periph}_{N'}(C') = \text{periph}_{N''}(C')$ and thus $\rho(W, N', C') \geq \rho(W, N'', C')$ if $|N'| \geq |N''|$. Moreover, we can compute the maximal solid coalition supporting a set C' by simply checking for each voter whether he supports C' . Since all of these steps can be done in polynomial time, it follows that the SCR rule can be computed in polynomial time for strict preferences. $\square$

D Proof of Theorem 2

Theorem 2. *Even for weak preferences, the SCR rule is committee monotone and satisfies IPSC.*

Proof. Since the SCR rule selects candidates sequentially and independently of the target committee size, it follows immediately that it satisfies committee monotonicity.

Next, to show that SCR satisfies IPSC, we fix a profile R and a committee size k . Moreover, for all $t \in \{1, \dots, k\}$, we let c_t denote the t -th candidate that SCR adds to the winning committee for R , and we define $W^t = \{c_1, \dots, c_t\}$ for all $t \in \{1, \dots, k\}$ and $W^0 = \emptyset$. For our proof, we assume that each voter $i \in N$ has a virtual budget $b_i = 1$ and that the candidates will be bought using these budgets for a price of $\frac{n}{k+1}$. We will next show that, for every $t \in \{1, \dots, k\}$, there is a payment scheme for W^t such that

- (i) the budgets of all voters are always non-negative, and
- (ii) if a voter i is part of a generalized solid coalition N' supporting a set C' such that $C' \not\subseteq W^t$ and $\rho(W^t, N', C') > \frac{n}{k+1}$, then voter i only spends a non-zero amount on candidates in $\text{periph}_{N'}(C')$.

To derive such a scheme, we fix $t \in \{1, \dots, k\}$ and inductively assume that there is a payment scheme for the committee W^{t-1} that satisfies conditions (i) and (ii). If $t = 1$, such a scheme exist for W^{t-1} as $W^0 = \emptyset$ and no money has been spent. We next explain how to extend the payment scheme for W^{t-1} to W^t by reasoning how to pay for the candidate c_t .

For this, we proceed with a case distinction and first assume that $\rho(W^{t-1}, N', C') \leq \frac{n}{k+1}$ for all generalized solid coalitions N' that support a set C' with $C' \not\subseteq W^{t-1}$. In this case, we deduct the money for candidate c_t arbitrarily from the budgets of the voters while ensuring that no budget becomes negative. This is possible as the voters' total budget exceeds the necessary budget to pay for k candidates, i.e., $n > k \frac{n}{k+1}$. Moreover, it holds that $\rho(W^t, N', C') \leq \rho(W^{t-1}, N', C') \leq \frac{n}{k+1}$ for all generalized solid coalitions N' with set $C' \not\subseteq W^t$, so condition (ii) holds trivially.

For the second case, suppose that there is a generalized solid coalition N' supporting a set C' such that $C' \not\subseteq W^{t-1}$ and $\rho(W^{t-1}, N', C') > \frac{n}{k+1}$. In this case, let N^* and C^* denote the last generalized solid coalition and the corresponding set of candidates in the execution of the while-loop of SCR (line 4) that satisfies these conditions. By condition (ii), the voters in N^* have only spent money on the candidates in $\text{periph}_{N^*}(C^*)$ so far. Hence, these voter have spent a total budget of at most $|W^{t-1} \cap \text{periph}_{N^*}(C^*)| \frac{n}{k+1}$. By rearranging the assumption that $\rho(W^{t-1}, N^*, C^*) = \frac{|N^*|}{|W^{t-1} \cap \text{periph}_{N^*}(C^*)| + 1} > \frac{n}{k+1}$, we infer that $|N^*| > (|W^{t-1} \cap \text{periph}_{N^*}(C^*)| + 1) \frac{n}{k+1}$. Therefore, these voters have a total remaining budget of at least $\frac{n}{k+1}$, so they can pay for the candidate c_t without violating condition (i).

It remains to show that this payment scheme for W^t satisfies condition (ii). For this, let N'' denote a generalized solid coalition supporting a set C'' such that $C'' \not\subseteq W^t$ and $\rho(W^t, N'', C'') > \frac{n}{k+1}$. We assume for contradiction that there is a voter $i \in N''$ who spent money on candidates outside of $\text{periph}_{N''}(C'')$. Since $\rho(W^{t-1}, N'', C'') \geq \rho(W^t, N'', C'') > \frac{n}{k+1}$, we infer from the induction hypothesis that voter i has not spent money on candidates outside of $\text{periph}_{N''}(C'')$ during the first $t - 1$ steps. Hence, voter i spent money on candidate c_t and $c_t \notin \text{periph}_{N''}(C'')$. This means that $i \in N^*$, so C^* forms a prefix of voter i 's preference relation. Further, because $c_t \notin \text{periph}_{N''}(C'')$, it follows that $C'' \subsetneq C^*$. However, this means that after SCR selected the solid coalition N^* with set C^* in the while-loop, there is another iteration of the while-loop such that

$$\frac{|N'|}{|W^{t-1} \cap \text{periph}_{N'}(C')| + 1} \geq \frac{|N'|}{|W^t \cap \text{periph}_{N'}(C')| + 1} > \frac{n}{k+1}.$$

This violates the definition of N^* and C^* , so the assumption that voter i spent some money on c_t is wrong and condition (ii) holds.

Finally, we will show that SCR satisfies IPSC. Assume for contradiction that the committee W^k fails IPSC for R and k . Thus, there is a generalized solid coalition N' supporting a set C' and an integer $\ell \in \mathbb{N}$ such that $|N'| > \ell \frac{n}{k+1}$, $C' \not\subseteq W^k$, and $|W^k \cap \text{periph}_{N'}(C')| < \ell$. This implies that

$$\rho(W^k, N', C') = \frac{|N'|}{|W^k \cap \text{periph}_{N'}(C')| + 1} \geq \frac{|N'|}{\ell} > \frac{n}{k+1}.$$

In turn, condition (ii) of the payment scheme shows that the voters in N' only paid for candidates in $\text{periph}_{N'}(C')$. Thus, they spent a budget of at most $(\ell - 1)\frac{n}{k+1}$. Since the total initial budget of these voters is $|N'| > \ell\frac{n}{k+1}$, their remaining budget is strictly larger than $\frac{n}{k+1}$. However, the total remaining budget of all voters after k candidates have been bought is $n - \frac{kn}{k+1} = \frac{n}{k+1}$. Hence, there must be a voter with a negative budget, which violates condition (i). So, SCR satisfies IPSC. $\square$

E Proof of Theorem 5

Theorem 5. *Let R be a ranking profile in which at least $\alpha \cdot n$ voters report $\succsim$. Further, let $\succsim_H$ be a ranking satisfying Hare-PSC for R and $\succsim_D$ a ranking satisfying Droop-PSC for R . It holds that*

- (i) $\text{swap}(\succsim, \succsim_H) \leq (1 - \alpha + \frac{1+\alpha}{m})\binom{m}{2}$ and
- (ii) $\text{swap}(\succsim, \succsim_D) \leq (1 - \alpha)\binom{m}{m-1}\binom{m}{2}$.

Proof. We prove both claims of the theorem separately.

Proof of Claim (i): Let $\succsim = c_1 \succ c_2 \succ \dots \succ c_m$ and fix a profile R . Further, let N' denote the group of voters reporting $\succsim$ and suppose that $|N'| \geq \alpha \cdot n$ for some $\alpha \in [0, 1]$. For every rank $r \in [m]$, the set N' forms a solid coalition supporting the set $\{c_1, \dots, c_r\}$. Because $|N'| \geq \alpha \cdot n$, Hare-PSC requires that all r of these candidates are selected when the committee size k satisfies that $\alpha \geq \frac{r}{k}$. Next, let $\succsim_H$ denote a ranking that satisfies Hare-PSC for R . From the previous observation, we obtain that all of $c_1, \dots, c_r$ are all ranked among the first $\lceil \frac{r}{\alpha} \rceil$ candidates in $\succsim_H$. Now, let x be a candidate ranked by $\succsim_H$ at rank t with $\lceil \frac{r}{\alpha} \rceil + 1 \leq t \leq \lceil \frac{r+1}{\alpha} \rceil$ for some $r \in \{0, \dots, m\}$ such that $\lceil \frac{r}{\alpha} \rceil \leq m - 1$. It holds that $x = c_i$ for some candidate i with $i > r$ as otherwise Hare-PSC is violated. Consequently, candidate x can have at most $t - r - 1$ inversions to candidates ranked before it in $\succsim_H$, as none of the candidates between c_1 to c_r incur an inversion. Let r' be the last rank for which $\lceil r'/\alpha \rceil \leq m - 1$, i.e., $r' = \lfloor \alpha(m - 1) \rfloor$. By our previous observation, we can upper bound the number of inversions of $\succsim_H$ by

$$\begin{aligned} \text{swap}(\succsim, \succsim_H) &\leq \sum_{i=1}^{\lceil 1/\alpha \rceil} (i - 1) + \sum_{i=\lceil 1/\alpha \rceil + 1}^{\lceil 2/\alpha \rceil} (i - 2) + \dots + \sum_{i=\lceil r'/\alpha \rceil + 1}^m (i - 1 - r') \\ &= \sum_{i=0}^{\lceil 1/\alpha \rceil - 1} (i) + \sum_{i=\lceil 1/\alpha \rceil}^{\lceil 2/\alpha \rceil - 1} (i - 1) + \dots + \sum_{i=\lceil r'/\alpha \rceil}^{m-1} (i - r') \end{aligned}$$

In this sum, the term $\lceil i/\alpha \rceil - i$ for $i \in \{1, \dots, r'\}$ appears twice, namely as the last term of the i -th sum and as the first term of the $(i + 1)$ -th sum. By contrast, each other term appears exactly once. Therefore, we get that

$$\begin{aligned} \text{swap}(\succsim, \succsim_H) &\leq \sum_{i=0}^{m-r'-1} (i) + \sum_{i=1}^{r'} (\lceil i/\alpha \rceil - i) \\ &\leq \sum_{i=0}^{m-r'-1} (i) + (1/\alpha - 1) \sum_{i=1}^{r'} (i) + r' \\ &= (m - r')(m - r' - 1)/2 + (1/\alpha - 1)(r' + 1)(r')/2 + r' \\ &= (r')^2/(2\alpha) + (1 + \frac{1}{2\alpha} - m)r' + m^2/2 - m/2. \end{aligned}$$

Next, let $f(x) = x^2/(2\alpha) + (1 + \frac{1}{2\alpha} - m)x + m^2/2 - m/2$. The derivative of this function is $\frac{x}{\alpha} + (1 + \frac{1}{2\alpha} - m)$, so we deduce that f has a minimum point at $x' = \alpha(m - 1) - \frac{1}{2}$. Since f is a quadratic function and thus symmetric around x' , we know that $f(x' - \varepsilon) < f(x' + \varepsilon)$ for all

$\varepsilon > \varepsilon' > 0$. Now, if $r' = \lfloor \alpha(m-1) \rfloor \geq (\alpha(m-1)) - \frac{1}{2}$, we directly obtain that $f(r') \leq f(\alpha(m-1))$ since $r' \leq \alpha(m-1)$. On the other hand, if $r' < (\alpha(m-1)) - \frac{1}{2}$, we define $\delta = (\alpha(m-1)) - \frac{1}{2} - r' < \frac{1}{2}$. By our previous observation, we also get that $f(r') = f(x' - \delta) < f(x' + \frac{1}{2}) = f(\alpha(m-1))$. Hence, in both cases, it holds that $f(r') \leq f(\alpha(m-1))$, so we get that

$$\begin{aligned} \text{swap}(\succsim, \mathbb{D}_H) &\leq f(r') \leq f(\alpha(m-1)) \\ &= \alpha^2(m-1)^2/(2\alpha) + (1 + \frac{1}{2\alpha} - m)\alpha(m-1) + m^2/2 - m/2 \\ &= \left(1 - \alpha + \frac{1 + \alpha}{m}\right) \binom{m}{2}. \end{aligned}$$

This proves Claim (i) of our theorem.

Proof of Claim (ii): We fix again a profile R , a group of voters N' that all report the same preference relation $\succsim$ in R , and we let $\alpha \in [0, 1]$ such that $|N'| \geq \alpha \cdot n$. Since N' forms a solid coalition for every set $\{c_1, \dots, c_r\}$ with $r \in \{1, \dots, m\}$, Droop-PSC implies that all candidates $c_1, \dots, c_r$ are chosen when the committee size k satisfies that $\alpha > \frac{r}{k+1}$. Next, let $\mathbb{D}_D$ denote a ranking that satisfies Droop-PSC for R . Solving our previous inequality for k shows that, for all r , the candidates $\{c_1, \dots, c_r\}$ are among the top- ℓ in every ranking that satisfies Droop-PSC, where ℓ is the smallest rank larger than $\frac{r}{\alpha} - 1$. Now, let $\varepsilon > 0$ be fixed. Then, based on our previous considerations, we know that the candidates $c_1, \dots, c_r$ are all ranked in the first $\lceil \frac{r}{\alpha} + \varepsilon \rceil - 1$ candidates. Let x be a candidate ranked by $\mathbb{D}_D$ at rank t with $\lceil \frac{r}{\alpha} + \varepsilon \rceil \leq t \leq \lceil \frac{r+1}{\alpha} + \varepsilon \rceil - 1$. By Droop-PSC, it must be that $x = c_i$ for some $i > r$, so candidate x can have at most $t - r - 1$ inversions to candidates ranked before it in $\mathbb{D}_D$. The last rank r' with $\lceil \frac{r'}{\alpha} + \varepsilon \rceil - 1 \leq m - 1$ is now $r' = \lfloor \alpha(m - \varepsilon) \rfloor$. Using this and the same technique as for the Hare quota, we derive the following inequality:

$$\begin{aligned} \text{swap}(\succsim, \mathbb{D}_D) &\leq \sum_{i=0}^{\lceil 1/\alpha + \varepsilon \rceil - 2} (i) + \sum_{i=\lceil 1/\alpha + \varepsilon \rceil - 1}^{\lceil 2/\alpha + \varepsilon \rceil - 2} (i-1) + \dots + \sum_{i=\lceil r'/\alpha + \varepsilon \rceil - 1}^{m-1} (i - r') \\ &\leq \sum_{i=0}^{m-r'-1} (i) + \sum_{i=1}^{r'} (\lceil i/\alpha + \varepsilon \rceil - i - 1) \\ &\leq (m - r')(m - r' - 1)/2 + \sum_{i=1}^{r'} (i/\alpha - i + \varepsilon) \\ &= (m - r')(m - r' - 1)/2 + (1/\alpha - 1)(r' + 1)(r')/2 + \varepsilon r'. \end{aligned}$$

Just as in the previous case, this is a quadratic function in terms of r' . This quadratic function $((r')^2/(2\alpha) + (\varepsilon + \frac{2}{\alpha} - m)r' + m^2/2 - m/2)$ has a minimum point at $\alpha(m - \varepsilon) - \frac{1}{2}$ which again allows us to upper bound the swap-distance by substituting r' with $\alpha(m - \varepsilon)$, which gives us

$$\begin{aligned} \text{swap}(\succsim, \mathbb{D}_D) &\leq (m - \alpha(m - \varepsilon))(m - \alpha(m - \varepsilon) - 1)/2 \\ &\quad + (1/\alpha - 1)(\alpha(m - \varepsilon) + 1)(\alpha(m - \varepsilon))/2 + \varepsilon\alpha(m - \varepsilon) \\ &= \frac{\alpha\varepsilon^2}{2} - \frac{\varepsilon}{2} - \frac{\alpha m^2}{2} + \frac{m^2}{2} + \varepsilon\alpha(m - \varepsilon). \end{aligned}$$

For $\varepsilon \rightarrow 0$, this term converges to the desired bound of $((1 - \alpha)\frac{m}{m-1}) \cdot \binom{m}{2}$. $\square$