Budget Division Via Continuous Thiele's Rules

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Abstract

We introduce the class of *Continuous Thiele's Rules* that generalize the familiar **Thiele's rules** [25] of multi-winner voting to divisible Participatory Budgeting. Each rule in that class maximizes $\sum_i f(\pi^i)$ where π^i is an agent i's satisfaction and f could be any twice differentiable, increasing and concave real function. Based on a single parameter we call the *'Inequality Aversion'* of f (elsewhere known as "Relative Risk Aversion"), we derive bounds on the egalitarian and utilitarian welfare loss, and the approximation of individual and group *Fair Share*. This leads to a quantifiable, continuous presentation of their inevitable trade-offs. In particular, we show that many of these rules satisfy *Individual Fair Share*, and that the Nash Product Rule satisfies *Average Fair Share* in our setting.

1 Introduction

We study the problem of reaching a collective decision concerning the division of a continuous public resource into different channels, e.g. the public budget allocated between various objectives ; time or land allocated between different activities; etc, known as *'divisible Participatory Budgeting'* (PB) [3, 19, 27, 23]. The important defining feature is that we allocate a *public* resource, meaning that all alternatives serve, in principle, everyone, albeit to variable extents according to personal taste or needs. This is in contrast to other branches of Social Choice, e.g. Cake cutting [31] or Fair Division [29], where we allocate resources among agents which enjoy them individually as private goods. The collective decision is reached via some voting *rule* or 'mechanism' that inputs everyone's preference regarding the different possible outcomes and outputs a sole budget division.

1.1 Preference Modeling

General preferences over the set of possible distributions can be quite complex to represent, and impractical for elicitation. In divisible PB, it seems reasonable to ask every agent for her most favorable budget allocation, and nothing further. Thus, as in other Social Choice domains, *"single-peaked"* preferences where an agent's optimal outcome entails the complete description of her preferences over the decision space, are many times favored as a reasonable compromise between the accuracy and applicability [28, 13]. Moreover, such models typically perform better in terms of strategy-proofness. Specifically, the ℓ_1-norm preference where an agent (dis)satisfaction is expressed as the distance $\left\|x^i - x\right\|_1$ between her preferred distribution x^i to the one implemented x, is among the most prevailing choices [19, 24, 23]. In our work we adopt this model, albeit adhering to its equivalent, less common representation that was introduced in [24] of *Overlap* utilities. An agent with preferable allocation x^i enjoys an overlap satisfaction of $|x^i \cap x| = \sum_j \min(x^i_j, x_j)$ in allocation x, where j covers the set of alternatives. This formulation has several advantages. From a practitioner's view [37, 32], PB is about increased engagement and consent over public policies no less than it is a crowd sourcing for better policy making. Thus, a comprehensible and easily justifiable presentation of the goals our mechanism seeks is crucial. The overlap model relates more conveniently to satisfaction rather than dissatisfaction, and intuitively expresses the level to which an outcome matches an individual's proposal. E.g., if $x^i = (0.3, 0.3, 0.4)$ and $x = (0.5, 0, 0.5)$ then $|x^i \cap x| = 0.7$, in other words a 70% match.

Moreover, the Overlap mirrors the canonical *approval* utilities from discrete PB [35, 3] and multi-winner voting [5], where an agent satisfaction equals the number of winning projects/candidates she approved. Thanks to that, *proportional fairness* demands, that have become a gold standard in related fields [7, 8, 5, 33, 9, 27, 36, 30] fit naturally in our setting. However, previous works that assumed ℓ_1 preferences in its conventional form have not applied these straight-forward extensions, and relate to proportional fairness in only a very weak sense [19, 23, 16, 22].

1.2 The three welfares

A well known fact that dates back as far as to Arrow's Theorem [1] is that desired axiomatic properties of social choice rules are often contradictory, which motivates a quantitative investigation of the inevitable trade-offs between them[36, 27, 21, 22]. In this paper we are particularly interested in three different notions of 'social welfare', namely, the ***utilitarian welfare*** $\sum_i \pi^i$, ***egalitarian welfare*** $\min_i \pi^i$, and ***Nash welfare*** $\sum_i \ln(\pi^i)$, where π^i indicates the satisfaction of agent i. The maximizers of these sums — UTIL, EGAL and NASH, respectively — are canonical, massively studied examples of PB rules and Social Choice rules in general [12, 3], each advocating a different view of distributive justice. While justifications for UTIL and EGAL are quite self-explanatory, NASH owes its popularity for typically admitting strong *proportional fairness* demands [7, 14], offering a somewhat middle-ground between utilitarianism and egalitarianism.

Example 1. *Assume a divisible PB instance with two alternatives a and b, where population is divided into two disjoint groups A and B, consisting 70% and 30% of the population, respectively. Voters in each group wish to allocate the full budget to the corresponding alternative a or b. Then, UTIL allocates the full budget to option a in this case, EGAL splits the budget equally between a and b, and NASH divides it proportionally as 0.7 on a and 0.3 on b.*

As illustrated in the above example, each of the rules designed to maximize a different welfare performs quite poorly in terms of the other two. Commonly, such inevitable trade-offs are treated via comparing axiomatic or approximative guarantees of familiar rules, or approximating a certain property conditional on admitting another [19, 22, 36, 27, 7]. However, these are merely isolated points in a much wider range of compromising possibilities.

Example 2. *Consider parliamentary elections with three parties (a, b, c), that won $500, 300,$ and 250 votes, respectively. Many parliamentary democracies apply the NASH rule (up to fractional remainders), which gives the proportional seat allocation of $\sim (0.48, 0.28, 0.24)$. However, we may favor some disproportional bias towards more popular parties, to increase chances of forming a stable coalition. Maximizing $\sum_i \sqrt{\pi^i}$, for example, yields $\sim (0.62, 0.22, 0.16)$. On the contrary, we may want to favor the smaller ones to contract political power inequality, for example if the chosen assembly makes all decisions via a simple majority vote. We may thus choose, e.g., to maximize $\sum_i \frac{-1}{\pi^i}$ which gives $\sim (0.4.0.31, 0.29)$ in this case.*

1.3 contribution

In this work, we offer a non-axiomatic approach for compromising between utilitarianism, egalitarianism and proportional fairness in a divisible Participatory Budgeting context. We introduce the class of *Continuous Thiele's Rules* (CTR), that maximize $\sum_i f(\pi^i)$ where f could be any concave real function. For every CTR we show worst case bounds for utilitarian welfare, egalitarian welfare, and the approximation of the proportionality axiom *Average Fair Share* (**AFS**) [7]. In particular, many of these rules fully satisfy *Individiual Fair Share* [7] and NASH satisfies a slightly stronger demand than **AFS** in our setting. Importantly, these bounds can be derived from a specific characterizing parameter of the function of choice f, that we call the *'Inequality Aversion'* of f (elsewhere known as 'Relative Risk Aversion' [18]). Thus, figuratively speaking, the CTR class forms a continuum of rules that spread 'in between' UTIL, NASH and EGAL, each offering a different concrete balance between the three aforementioned concepts. In relation to existing literature, our work implements proportional fairness in an ℓ_1 preference model, merging into one framework these two important paradigms in the PB research that were almost parallel before.

1.4 Related Work

The original ***Thiele's Voting Rules*** [25] are a well known class of multi-winner voting rules, each characterized by a function f, that choose the winning set W to maximize $\sum_i f\Big(|A_i \cap W|\Big)$. In affinity to the discussion above, this class includes some cannonical multi-winner voting rules, e.g. the welfare maximizing k-approval where f is the identity function, the egalitarian Chamberlin-Courant with $f = \mathbb{1}_{|A_i \cap W| \geq 1}$, and PAV where $f\Big(|A_i \cap W|\Big) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{|A_i \cap W|}$ that satisfies *Extended Justified Representation* (EJR) [6]. When applied to our continuous setting, much resemblance is found between the corresponding rules and their guarantees. However, continuity allows for a much finer characterization of other rules that do not fully satisfy either of the demands. [15] studies the application of Theiele's Rules as *apportionment* methods, which is a particular case of our model where every voter approves a single alternative, as in our above examples.

Compared to its neighboring branches of Social Choice, the literature on divisible PB is not abundant. While Participatory Budgeting has been drawing increased attention in late years, the literature mainly focuses on the discrete form where each project has a fixed cost [30, 4, 35], which is in compliance with the vast majority of PB instances in reality.

Although ℓ_1-norm preferences are common in divisible PB, the approach and technical analysis most similar to this work is actually found in a line of works that assumed binary (*'approval'*) preferences [7, 10, 36, 27], where agents only approve or disapprove each alternative. Conceptually, that model seems less suited for generic PB cases (although certainly applicable in some [7]), where voters could favor a specific allocation that expresses a more complex prioritization over alternatives, and that moreover takes budget limits into account. Technically, the Overlap model does not generalize the binary model, however, all of the results presented here apply to binary preferences too, as we did not explicitly use the assumption that $\{x^i\}_i$ are normalized to 1 in any of the proofs. [7] shows that NASH satisfies **Core Stability** (CS) in the binary model, which does not carry over to our model where it satisfies **AFS** only. [27] and [36] study the trade-off between proportional fairness and utilitarian or egalitarian efficiency, respectively. Each of them introduces several worst-case bounds for approximating one demand under forcing the other, and bounds of different mechanisms. Moreover, [27] also studies the utilitarian/proportionality trade-off within the CTR class ("f-UTIL" mechanisms by their terminology). In that area they show some results quite similar to the corresponding here, however our work does expand on it. Beyond the superior generality of the model mentioned above, we also consider egalitarian loss, and there is our IAV characterization of the results (in fact, their results are only relevant for $IAV \leq 1$).

Fairness/welfare trade-offs, in the sense that we address them here, have not been seriously studied under ℓ_1 preferences prior to our work. The (not too vast) literature in that area mainly focuses on comparing different mechanisms and the axioms they may or may not hold, and only considers very weak proportional-fairness notions [19, 23, 11]. In some works [23, 16, 17], the ℓ_1 distance of an outcome to the average allocation of all votes is taken as a fairness proxy. As for other preference models, [20] presents a mechanism that satisfies **Core stability** for scalar-separable utilities, and, notably, [14] and [11] shows that NASH is moreover strategy-proof if *Minimal Quotient* preferences are assumed.

2 Preliminaries

Let $[n]$ be a set of agents and $[m]$ be a set of alternatives, where $[k] := \{1, \dots, k\}$ for every positive integer k. Subsets of $[n]$ are denoted by lowercase letters e.g. $s \subset [n]$, and in many places we abuse this notation to also represent the subset's size. We use the following notations for vectors in $\mathbb{R}^m_\geq := \{y \in \mathbb{R}^m | y_j \geq 0 \; \forall j \in [m]\}$. For two vectors x, y we define their *overlap* $x \cap y$ as $(x \cap y)_j = \min(x_j, y_j)$.

This is naturally generalized to sets of vectors, i.e., for a set X, $\left(\bigcap_{x\in X}\right)_j = \min_{x\in X} x_j$. For all $x \in \mathbb{R}^m$, we denote by $|x|$ the ℓ_1 norm of x, $|x| := \sum_j |x_j|$. An *allocation* $x \in \Delta^m := \{y \in \mathbb{R}^m_{\geq} || y| = 1\}$ is a distribution of some continuously divisible resource (e.g. money, time) among the m alternatives, where we normalize the overall budget to 1.

The preferences of every agent i are encoded a single allocation $x^i \in \Delta^m$ that she would like to be implemented ideally, and $\vec{x} := (x^1, \dots, x^n)$ is the preferences *profile*. We call a profile $\vec{x}$ **single-minded** [23] if x^i is a unit vector for all i, i.e., every agent allocates the full budget to a single alternative of her choice. $\vec{x}_{-i}$ denotes the partial profile consisting of all agents' ideal allocations, excluding i, and $\vec{x}_{-s}$ is used accordingly for a subset $s \subset [n]$. The *Overlap satisfaction* [24] of agent i in allocation x is $\pi^i_x := |x^i \cap x| = \sum_j \min(x^i_j, x_j)$. Or, just π^i when specifying the allocation is unnecessary.

2.1 Aggregation rules and their properties

Definition 1. *An* aggregation rule *is a function $F : (\Delta^m)^n \to \Delta^m$ that inputs the preferences profile $\vec{x}$ and outputs an allocation $x \in \Delta^m$.*

We survey here several properties that may or may not be satisfied by an allocation x given a profile $\vec{x}$. We say that an aggregation rule F satisfies any of these properties if for all $\vec{x} \in (\Delta^m)^n$, $F(\vec{x})$ satisfies the corresponding property with respect to $\vec{x}$.

- *Efficiency* (**EFF**): An allocation x is *efficient* if no other allocation $y \in \Delta^m$ exists such that $\pi^i_x \geq \pi^i_y \quad \forall i \in [n]$, with at least one agent for which that inequality is strict.
- *Range Respecting* (**RR**) [23]: An allocation x is *Range Respecting* if $\min_i x^i_j \leq x_j \leq \max_i x^i_j \ \forall j \in [m]$.

It is not difficult to see that **RR** is a weaker demand under Overlap preferences.

Proposition 3. ***EFF*** $\implies$ ***RR***.

The proof, **as well as all other missing proofs throughout the paper**, is deferred to the appendix.

In principle, *proportional fairness* [27, 15, 9, 5] means that if a subgroup of agents with shared interests consists a q fraction of population, they are entitled for a proportional influence over collective decisions. In our case, power over at least q of the budget. This translates into many different axioms, including these three that we study in this paper.

- *Individual Fair Share* (**IFS**) [7]: An allocation x satisfies *Individual Fair Share* if $\pi^i_x \geq \frac{1}{n} \ \forall i \in [n]$. From an agents' individual perspective, that means that x allocates at least $1/n$ of the budget according to her desire.
- *Average Fair Share* (**AFS**):[1] A subset of agents $s \subseteq [n]$ is called α-*cohesive* for some $0 < \alpha \leq \frac{|s|}{n}$, if $\left|\bigcap_{i\in s} x^i\right| \geq \alpha$. An allocation x satisfies *Average Fair Share* if for every $\alpha \in (0, 1]$, if $s \subseteq [n]$ is α-cohesive then $\frac{1}{|s|}\sum_{i\in s} \pi^i_x \geq \alpha$.
- *Core Stability* (**CS**)[20]: An allocation x satisfies *Core Stability* if for every $s \subseteq [n]$, no $y \in \mathbb{R}^m_+$ exists such that $|y| = \frac{|s|}{n}$ and $\pi^i_y \geq \pi^i_x \ \forall i \in s$ with strict inequality for at least one member of s.

[1]Under binary preferences [7], **AFS** is formulated as $\frac{1}{|s|}\sum_{i\in s}\pi^i_x \geq \frac{s}{n}$, as cohesiveness does not mean a lot in that model. However, the natural extension from approval voting [34] would be $\frac{1}{|s|}\sum_{i\in s}\pi^i_x \geq \alpha$, as we put it here.

- *single-minded Proportionality* (**smPR**)[23]: This much weaker demand essentially requires the fulfillment of all the fairness axioms above (**IFS**, **AFS**, **CS**) but only for **single-minded** profiles (under which they all coincide). If $\vec{x}$ is single-minded, satisfying them simply means that x must be the average of proposed allocations, $x = \frac{1}{n}\sum_i x^i$.

The next two properties refer directly to rules rather than allocations.

- *Participation* (**PAR**): Aggregation rule F satisfies *participation* if, for all $\vec{x} \in (\Delta^m)^n$ and $i \in [n]$, $y = F(\vec{x})$ and $z = F(\vec{x}_{-i})$, $\pi^i_y \geq \pi^i_z$. Meaning, agents should always (weakly) prefer voting over abstaining.
- *Strategyproofness* (**SP**): Aggregation rule F is *strategyproof* if for all $\vec{x} \in (\Delta^m)^n$ and $i \in [n]$, $y = F(x^i, \vec{x}_{-i})$ and $z = F(w, \vec{x}_{-i})$ for some $w \neq x^i$, $\pi^i_y \geq \pi^i_z$. Meaning, no agnet i can increase her satisfaction by misreporting some $w \in \Delta^m$ instead of her true preference x^i.

3 Continuous Theile's Rules

In this paper, we study this class of aggregation rules.

Definition 2 (Continuous Thiele's Rules)**.** *For any increasing, twice differentiable and strictly concave function $f : [0,1] \to \mathbb{R}$, we define the corresponding **Continuous Thiele's Rule**, CTR_f, by*

$$CTR_f(\vec{x}) \in \arg\max_{x\in\Delta^m} \sum_{i=1}^{n} f(\pi^i_x) \tag{1}$$

3.1 Axiomatic Properties

Before presenting our main results that concern approximation guarantees of CTRs, we open this section with an overview of their axiomatic properties. The following are immediate by definition.

Proposition 4. *Every rule in the CTR class satisfies **EFF**, **PAR**, and is computed efficiently via convex optimization.*

We dismiss a formal proof of this statement. **EFF** and **PAR** are straight forward since f is increasing. The efficient computation is due to $\sum_{i=1}^n f(\pi^i_x)$ being a concave function of x (note that $\pi^i_x = \sum_j \min(x^i_j, x_j)$ is in itself a sum of concave functions). On the downside, this negative result is also easy to verify.

Proposition 5. *No CTR satisfies **SP**.*

In the proportional fairness area, the only CTR showing any group representation guarantees (for $m > 2$) is NASH where $f = \ln$, that satisfies the relatively strong **AFS**.

Corollary 6. *NASH satisfies **AFS**.*

The proof is dismissed here as a particular case of Theorem 19 (that in fact shows a slightly stronger property). On the contrary, we show that no other CTR satisfies even the much weaker **smPR**.

Proposition 7. *For $m > 2$, NASH is the only CTR that satisfies **smPR**.*

Moreover, NASH itself does not satisfy **CS**. Since **smPR** is weaker than **CS**, we have the following result.

Proposition 8. *No CTR satisfies **CS**.*[2]

[2]Interestingly, the Nash product does output a core solution under the binary model [7]. Whether the core is always non-empty for ℓ_1, and what rule can find core solutions when and if they exist, remains, to the best of our knowledge, an open question.

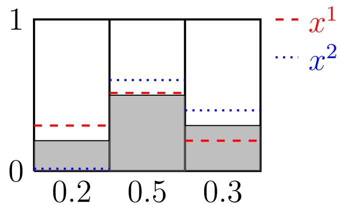


Figure 1: An illustration with $m = 3$ and $n = 2$. $x^1 = (0.3, 0.5, 0.2)$, $x^2 = (0, 0.6, 0.4)$. At $x = (0.2, 0.5, 0.3)$, $\pi_x^1 = 0.2 + 0.5 + 0.2 = 0.9$, $\pi_x^2 = 0 + 0.5 + 0.3 = 0.8$, and: $mc_1^{\uparrow} = mc_1^{\downarrow} = f'(\pi_x^1)$, $mc_2^{\uparrow} = f'(\pi_x^2)$, $mc_2^{\downarrow} = f'(\pi_x^1) + f'(\pi_x^2)$, $mc_3^{\uparrow} = mc_3^{\downarrow} = f'(\pi_x^2)$.

3.2 Optimal Allocations

Thanks to Proposition 4, optimal allocations can be conveniently characterized by KKT conditions. Next, we formalize these constraints that will help us derive much of our later results.

Definition 3. *Let $x \in \Delta^m$. For each $j \in [m]$,*

$$s_j^{\uparrow}(x) := \{i \in [n] | x_j^i > x_j\} \ ; \ s_j^{\downarrow}(x) := \{i \in [n] | x_j^i \geq x_j\}$$

That is, $s_j^{\uparrow}(x)$ consists of all agents $i \in [n]$ for which increasing (or reducing, for $s_j^{\downarrow}(x)$) x_j while keeping all other $\{x_k\}_{k \neq j}$ fixed will increase (reduce) their satisfaction π_x^i. Note that $s_j^{\uparrow}(x) \subseteq s_j^{\downarrow}(x)$. If $x_j^i = x_j$ for some agent i, she suffers from a reduction in x_j while not gaining if we increase it, meaning she is included in $s_j^{\downarrow}(x)$ but not in $s_j^{\uparrow}(x)$. If no such agent exist, the sets coincide and we can just write s_j. We also do that for the sake of abbreviation in many places, where the intention should be clear or the distinction is not crucial. For every agent i and allocation x, we write

$$\sigma_x^{\uparrow(\downarrow)}(i) := \{j \in [m] | i \in s_j^{\uparrow(\downarrow)}(x)\}$$

Next, we want to define the partial derivative of $\sum_{i=1}^n f(\pi_x^i)$ w.r.t. alternative $j \in [m]$.

Definition 4 (marginal contributions)**.** *Let $x \in \Delta^m$. For every $j \in [m]$,*

$$mc_j^{\uparrow(\downarrow)}(x) := \sum_{i \in s_j^{\uparrow(\downarrow)}(x)} f'(\pi_x^i)$$

Note that $mc_j(x)$ depends on x_j only through s_j, however the contribution of each member to the sum is a function of their overall satisfaction $\pi_x^i := \sum_j \min(x_j^i, x_j)$. In case $x_k^i = x_k$ for some $k \in [m]$ (whether $k = j$ or other), $f'(\pi_x^i)$ does not exist, and we take the right or left derivative accordingly. Figure 1 shows an example.

Proposition 9 (Marginal Rate of Substitution (MRS) condition)**.** *For $x \in \arg\max_{y \in \Delta^m} \sum_i f(\pi_y^i)$,*

$$mc_j^{\uparrow}(x) \leq mc_k^{\downarrow}(x) \quad \forall j, k \in [m]$$

Mathematically, these are the KKT conditions for maximizing $\sum_i f(\pi_x^i)$. From an economic perspective, optimal allocations must admit the MRS demand because if $mc_j^{\uparrow}(x) > mc_k^{\downarrow}(x)$ for some j and k, increasing x_j at the expense of x_k would increase $\sum_i f(\pi^i)$, up to the point where the MRS condition is satisfied (and, every local maximum of a concave function is also a global one). When $s_j^{\uparrow} = s_j^{\downarrow}$ and $s_k^{\uparrow} = s_k^{\downarrow}$, marginal contributions also coincide and the two inequalities for j and k are reduced to the equation $mc_j = mc_k$.

3.3 Inequality Aversion

CTRs are defined by the choice of the function f. In this part, we develop a characterization for such functions we call *Inequality Aversion*. All of our upcoming results will depend on this sole factor. That is, the approximative guarantees we will provide for different CTRs with respect to utilitarian welfare, egalitarian welfare, Individual Fare Share and Avergae Fair Share, are all expressed in terms of the Inequality Aversion of the corresponding function f.

Definition 5 (Inequality Aversion[3]). *For any increasing, strictly concave function $f : [0,1] \to \mathbb{R}$, the Inequality Aversion of f, $IAV_f : (0,1] \to \mathbb{R}$ is $IAV_f(t) := -\frac{tf'(t)}{f''(t)}$.*

Here are some examples of functions f and their corresponding IAV_f.

$$\begin{array}{rccccc} f(t): & -e^{-t} & -t^{-p}, {\scriptstyle p>0} & \ln(t) & t^p, {\scriptstyle 1>p>0} & t(2-t) \\ IAV_f(t): & t & 1+p & 1 & 1-p & t(t-1) \end{array}$$

Putting it roughly, IAV_f "measures the concavity" of f, meaning, the extent to which $f\left(\frac{1}{n}\sum_i \pi_x^i\right)$ exceeds $\frac{1}{n}\sum_i f(\pi_x^i)$. The higher it is, the less dispersed is the optimal distribution of $\{\pi^i\}_i$. For our needs, however, the following implication is what's important.

Lemma 10. *For any twice differentiable concave function $f : [0,1] \to \mathbb{R}$, $IAV_f \overset{(\leq)}{\geq} \lambda$ for some $\lambda > 0$ iff $\forall \alpha > 1$, $\frac{f'(t)}{f'(\alpha t)} \overset{(\leq)}{\geq} \alpha^\lambda$.*

This interpretation of the IAV will come handy in our later analysis. The example below demonstrates the basic idea.

Example 11. *Let $m = 2$. Consider a single-minded profile where the second alternative has higher support, i.e., $s_2 > s_1$ in the notations of Definition 3. Note that in single minded profiles, $s_j^\uparrow = s_j^\downarrow\ \forall j$ and we thus only write s_j and mc_j. Also, for $i \in s_j$, $\pi_x^i = x_j$. Thus,*

$$mc_1 = \sum_{i \in s_1} f'(\pi_x^i) = s_1 f'(x_1) \quad ; \quad mc_2 = \sum_{i \in s_2} f'(\pi_x^i) = s_2 f'(x_2)$$

For any rule CTR_f we may apply. At the optimum $x = CTR_f(\vec{x})$, the MRS conditions give $mc_1 = mc_2 \implies \frac{f'(x_1)}{f'(x_2)} = \frac{s_2}{s_1}$. Now if λ bounds IAV_f, it also bounds the extent to which CTR_f favors the majority s_2. If $IAV_f \geq \lambda$, $\left(\frac{x_2}{x_1}\right)^\lambda \leq \frac{f'(x_1)}{f'(x_2)} = \frac{s_2}{s_1} \implies \frac{x_2}{x_1} \leq \left(\frac{s_2}{s_1}\right)^{1/\lambda}$. In particular, for $\lambda \to \infty$ we get the egalitarian maxmin allocation $x_1 = x_2$. For $IAV_f \leq \lambda$ we will have the inequalities reversed: $\left(\frac{x_2}{x_1}\right)^\lambda \geq \frac{f'(x_1)}{f'(x_2)} = \frac{s_2}{s_1} \implies \frac{x_2}{x_1} \geq \left(\frac{s_2}{s_1}\right)^{1/\lambda}$, so that at the limit $\lambda \to 0$ the welfare maximizing allocation $x = (0,1)$ is approached. Finally, if $IAV = 1$ (meaning $f = \ln$) the budget is allocated proportionally as $\frac{x_2}{x_1} = \frac{f'(x_1)}{f'(x_2)} = \frac{s_2}{s_1}$.

Presented in the next section, our main results build on the idea demonstrated in **Example 11** to provide egalitarian, utilitarian and proportionality guarantees within the CTR class.

We can figuratively map this class to $(0,\infty)$ based on the IAV parameter, as demonstrated in Figure 2. UTIL marks the left boundary where $IAV \to 0$, at $IAV = 1$ we have NASH, and at the right limit $IAV \to \infty$ there is EGAL. Every CTR thus represents a different compromise between these three

[3] More commonly, $-\frac{tf'(t)}{f''(t)}$ is known as the "Relative Risk Aversion" of f, a major concept in decision making under uncertainty [18]. While the two contexts are unrelated, the mathematical similarity is comprehensible [26, 2].

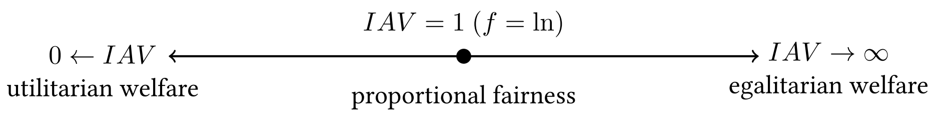


Figure 2: The CTR class continuum.

contractors and the objectives they represent. In general, $\max_{t\in(0,1)} IAV_f(t)$ determines worst case utilitarian loss of CTR_f (Theorem 12), $\min_{t\in(0,1)} IAV_f(t)$ determines its egalitarian loss (Theorem 17), and $\max_{t\in(0,1)} |IAV_f(t) - 1|$ the proportionality approximation (Theorem 19). In particular, if f has a homogeneous derivative, e.g. $t^p, -t^{-p}, \ln(t)$, then IAV_f is constant, and CTR_f is represented in a single point. Moreover, for every point $p \in (0, \infty)$ there exits such corresponding CTR, meaning that the full range of possibilities is available through these elementary functions.

4 Trade-Offs In The CTR Class

4.1 Utilitarian Welfare Loss

We will now show how utilitarian welfare loss is diminished with IAV. Formally, the **Utilitarian Welfare** in allocation x is $UW(x) := \sum_i \pi_x^i$, and:

Definition 6. *The **utilitarian loss** in allocation x is* $\mathbf{UL}(x) := 1 - \frac{UW(x)}{UW(y)}$, *where* $y = UTIL(\vec{x})$.

Theorem 12 bounds welfare loss, as a function of the IAV, for $m \geq 3$. Following it, Theorem 13 (of which the proof is deferred to the appendix) gives the complementary result for $m = 2$.

Theorem 12. *Let $x = CTR_f(\vec{x})$ such that $IAV_f \leq \lambda$. Then, for $m \geq 3$,*

$$\mathbf{UL}(x) \leq 1 - \left(\frac{1}{m}\right)^{\lambda}$$

We just need the following notation before proceeding with the proof. Given two allocations x and y, we write:

$$J_{x,y} := \{j \in [m] | x_j \geq y_j\} \;;\; J_{y,x} := \{j \in [m] | x_j < y_j\}$$
$$\delta_j := |x_j - y_j| \; \forall j \in [m] \;;\; \text{and, } \delta := \sum_{j\in J_{x,y}} \delta_j = \sum_{j \in J_{y,x}} \delta_j$$

Proof. Let $y = UTIL(\vec{x})$. When shifting from x to y, the increase in welfare cannot exceed

$$\begin{aligned}
\sum_{k\in J_{y,x}} s_k^{\uparrow}(x)(y_k - x_k) - \sum_{j\in J_{x,y}} s_j^{\downarrow}(x)(x_j - y_j) \quad &\leq s_{\max}^{\uparrow} \sum_{k\in J_{y,x}} (y_k - x_k) - \sum_{j \in J_{x,y}} s_j^{\downarrow}(x)(x_j - y_j) \\
= \sum_{j\in J_{x,y}} (s_{\max}^{\uparrow} - s_j^{\downarrow}(x))(x_j - y_j) \quad &\leq \sum_{j\in J_{x,y}} (s_{\max}^{\uparrow} - s_j^{\downarrow}(x))x_j
\end{aligned}$$

where $s_{\max}^{\uparrow} := \max_{k\in J_{y,x}} s_k^{\uparrow}$ and we used $\sum_{k\in J_{y,x}}(y_k - x_k) = \sum_{j\in J_{x,y}}(x_j - y_j)$. Now, by MRS and $IAV_f \leq \lambda$ we have for all $j, k \in [m]$:

$$s_k^{\uparrow} f'(1) \leq mc_k^{\uparrow} \leq mc_j^{\downarrow} \leq s_j^{\downarrow} f'(x_j) \implies s_k^{\uparrow} \leq s_j^{\downarrow} \frac{f'(x_j)}{f'(1)} \leq s_j^{\downarrow} \left(\frac{1}{x_j}\right)^{\lambda}$$

and therefore $\sum_{j\in J_{x,y}}(s^{\uparrow}_{\max} - s^{\downarrow}_j(x))x_j \le s^{\uparrow}_{\max}\sum_{j\in J_{x,y}}(1-x_j^{\lambda})x_j$. On the other hand, let $k\in[m]$ such that $s_k = s^{\uparrow}_{\max}$, and then

$$\begin{aligned} UW(x) &= \sum_i \pi^i_x \ge \sum_i \sum_{j\in\sigma(i)} x_j = \sum_j s^{\downarrow}_j x_j \ge s^{\uparrow}_{\max}x_k + \sum_{j\neq k} s^{\downarrow}_j x_j \ge s^{\uparrow}_{\max}\Big(x_k + \sum_{j\neq k} x_j^{\lambda}\cdot x_j\Big) \\ &= s^{\uparrow}_{\max}\Big(1 - \sum_{j\neq k} x_j + \sum_{j\neq k} x_j^{\lambda}\cdot x_j\Big) = s^{\uparrow}_{\max}\Big(1-\sum_{j\neq k}(1-x_j^{\lambda})x_j\Big) \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{UW(y)-UW(x)}{UW(x)} \le \frac{s^{\uparrow}_{\max}\sum_{j\neq k}(1-x_j^{\lambda})x_j}{s^{\uparrow}_{\max}\Big(1-\sum_{j\neq k}(1-x_j^{\lambda})x_j\Big)} \\ \implies &\frac{UW(y)-UW(x)}{UW(y)} \le \sum_{j\neq k}(1-x_j^{\lambda})x_j \le \left(1-\left(\frac{1-x_k}{m-1}\right)^{\lambda}\right)(1-x_k) \end{aligned}$$

Due to the concavity of $(1-x_j^{\lambda})x_j$. For $m\ge 4$, the function $g_m(t) = (1-(\frac{t}{m-1})^{\lambda})t$ is increasing in [0,1], therefore

$$\mathbf{UL}(x) \le g_m(1) = 1-\left(\frac{1}{m-1}\right)^{\lambda},$$

as required. For $m=3$ just take $g_4(t)\ge g_3(t)$ instead to bound $\mathbf{UL}(x)$, yielding the stated result. □

Theorem 13. *Let $m=2$, and $x = CTR_f(\vec{x})$ such that $IAV_f \le \lambda$. Then $\mathbf{UL}(x) \le \lambda(\lambda+1)^{-\frac{\lambda+1}{\lambda}}$.*

4.1.1 Lower Bound

Theorem 14. *Let CTR_f be a CTR such that $IAV_f = \lambda$. Then for every $m\ge 2$ there exists a profile $\vec{x}$ such that*

$$\mathbf{UL}\Big(CTR_f(\vec{x})\Big) = \max_{x_k\in(0,1)}\left(1-\left(\frac{1-x_k}{x_k(m-1)}\right)^{\lambda}\right)(1-x_k)$$

As it turns out, the lower bound we found is also an upper bound for welfare loss *in single-minded profiles*. (A short explanation of this corollary is given in the appendix as well).

Corollary 15 (Tight bound for single-minded profiles)**.** *In any single minded profile $\vec{x}$,*

$$\mathbf{UL}\Big(CTR_f(\vec{x})\Big) \le \max_{x_k\in(0,1)}\left(1-\left(\frac{1-x_k}{x_k(m-1)}\right)^{\lambda}\right)(1-x_k)$$

and this bound is tight.

4.2 Individual Fairness and Egalitarian Loss

We now want to explore the egalitarian welfare guarantees of CTR's. The less demanding interpretation of that would be looking at $\min_i \pi^i_x$ a rule can guarantee.

Theorem 16. *Let $x = CTR_f(\vec{x})$ such that $\lambda \le IAV_f$. Then,*

$$\pi^{\min}_x \ge \begin{cases} \frac{1}{\sqrt[\lambda]{n}} & \lambda\le 1 \\ \max\left(\frac{1}{n}, \frac{1}{m\sqrt[\lambda]{n}}\right) & \lambda > 1 \end{cases}$$

*In particular, x satisfies **IFS** whenever $IAV_f \ge 1$.*

For $\lambda \to \infty$, Theorem 16 gives $1/m$, which is indeed the guarantee of EGAL. However, this is only its worst case guarantee. The following definition formalizes egalitarian welfare approximation by its full meaning.

Definition 7. *The* **egalitarian loss** *of allocation x is $\mathbf{EL}(x) := 1 - \frac{\min_i \pi_x^i}{\min_i \pi_y^i}$, where $y = EGAL(\vec{x})$.*

Theorem 16 alone provides a trivial bound of $1 - \pi_x^{\min}$ for both utilitarian and egalitarian losses, which *is* an improvement on Theorems 12 and 17 (that will follow shortly), but only in the 'very bad' cases, meaning utilitarian loss for high IAV and egaitarian loss for small IAV. Bounding $\mathbf{EL}(x)$ in general, is, however, much more challenging. The proof of Theorem 17 can be roughly sketched as follows. While the egalitarian allocation enables no improvement on *all* argmin agents, CTR's exhibit a relaxation of that impossibility, that exacerbates as the IAV increases. As the proof will show, an improvement for the $\arg\min$ set implies the existence of an agent that suffers from the change and has satisfaction *close to* minimal. Formalizing this requires the following definition.

Definition 8 (Directional Derivative). *For every agent i an two allocations $x, y \in \Delta^m$, define the derivative of π^i towards y at x as*

$$\frac{\partial \pi_x^i}{\partial (y-x)} := \frac{\mathrm{d}}{\mathrm{d}\alpha} \pi^i(\alpha y + (1-\alpha)x)\Big|_{\alpha=0}$$

Note that

$$\frac{\partial \pi_x^i}{\partial (y-x)} = \frac{\mathrm{d}}{\mathrm{d}\alpha}\Big[\sum_l \min(\alpha y_l + (1-\alpha)x_l, x_l^i)\Big] = \sum_{k \in J_{y,x} \cap \sigma_x^{\uparrow}(i)} \delta_k - \sum_{j \in J_{x,y} \cap \sigma_x^{\downarrow}(i)} \delta_j \geq \pi_y^i - \pi_x^i$$

The inequality stems from the fact that agents might lose some of δ_j for $j \in J_{x,y}$ even if $j \notin \sigma_x^{\downarrow}(i)$, that is if $y_j < x_j^i < x_j$, and similarly, gain less then the full δ_k gap when $x_k < x_k^i < y_k$. Now we ready for this part's main result.

Theorem 17. *Let $x = CTR_f(\vec{x})$ such that $IAV_f \geq \lambda$. Then*

$$m(n-1) \geq \left(\frac{1}{1 - \mathbf{EL}(x)}\right)^{\lambda} \cdot \mathbf{EL}(x)$$

Note that this means $\mathbf{EL} \xrightarrow[\lambda \to \infty]{} 0$.

Proof. Let $y = EGAL(\vec{x})$. Since x is an optimum of $\sum_i f(\pi^i)$,

$$0 \geq \frac{\partial}{\partial (y-x)}\left[\sum_i f(\pi^i)\right] = \sum_i f'(\pi^i) \frac{\partial \pi_x^i}{\partial (y-x)}$$

Let $\alpha := \min_{\{i : \pi_x^i \leq \pi_y^i\}} \frac{\pi_x^i}{\pi_x^{\min}}$. Note that all agents $j \in \arg\min_i \pi_x^i$ must have higher satisfaction in y, thus $\alpha > 1$. Moreover, $\frac{\partial \pi_x^i}{\partial (y-x)} \leq |y - x| \leq 1$ for all agents. Lastly, denote $\mu := \frac{\partial \pi_x^i}{\partial (y-x)}$ for some agent i in the $\arg\min$ set. Thus,

$$0 \geq \frac{\partial}{\partial (y-x)}\left[\sum_i f(\pi^i)\right] \geq f'(\pi_x^{\min}) \cdot \mu - (n-1) f'(\alpha \pi_x^{\min})$$
$$\implies n - 1 \geq \mu \frac{f'(\pi_x^{\min})}{f'(\alpha \pi_x^{\min})} \geq \mu \alpha^{\lambda}$$

Since we have an agent for which $\pi_y^i \leq \pi_x^i = \alpha\pi_x^{\min}$, $\mathbf{EL}(x) = 1 - \frac{\pi_x^{\min}}{\pi_y^{\min}} \leq 1 - \frac{1}{\alpha}$. And, as there exists an agent for which $\mu \geq \pi_y^i - \pi_x^i = \pi_y^i - \pi_x^{\min} \geq \pi_y^{\min} - \pi_x^{\min}$, and $\pi_y^{\min} \geq \frac{1}{m}$, we also have $\mathbf{EL} \leq \frac{\mu}{\pi_y^{\min}} \leq m\mu$. Thus,

$$m(n-1) \geq \left(\frac{1}{1 - \mathbf{EL}(x)}\right)^{\lambda} \cdot \mathbf{EL}(x)$$

□

4.2.1 Lower Bound

Here too, we provide lower bound via looking at the worst single minded profile, as they are more convenient to deal with. But unlike in section 4.1 where we discussed utilitarian loss, calculating the egalitarian loss upper bound in single minded profiles is straight forward. We thus combine the two results.

Theorem 18. *Let $\vec{x}$ be single-minded profile $x = CTR_f(\vec{x})$ such that $IAV_f \leq \lambda$. Then,*

$$\mathbf{EL}\Big(CTR_f(\vec{x})\Big) \leq 1 - \frac{2}{1 + (n-1)^{\frac{1}{\lambda}}}$$

In particular, this bound is tight when $IAV_f = \lambda$.

4.3 Utilitarian Vs. Egalitarian Welfare

Figure 3 provides the meaning of our so far results in terms of trading possibilities. It shows upper and lower bounds on utilitarian and egalitarian loss for different settings of m and n, as a function of the IAV. Note that deriving a welfare loss bound requires an upper bound on the IAV, while for egalitarian loss we need an IAV lower bound, thus a concrete choice of f may correspond to an interval rather than a single value of λ. However, functions with homogeneous derivatives such as logarithmic and power functions have a constant IAV, and for every $\lambda \in (0, \infty)$ we have the corresponding function that guarantees the exact bounds we see on the y-axis in each plot. Moreover, note that our results only relate to the worst case with respect to either welfare, which are generically not the same preferences profiles, meaning that the bounds for a given λ certainly do not mean that achieving both of them in some instances is possible. For example, consider NASH executed on the single minded profile with $m = 2, s_1 = 1, s_2 = n - 1$. This profile implements the tight egalitarian bound for single-minded profiles from Theorem 18, of $\mathbf{EL} = 1 - \frac{2}{1+(n-1)^{\frac{1}{\lambda}}}$, in this case $1 - \frac{2}{n}$. As NASH outputs $(\frac{1}{n}, \frac{n-1}{n})$, the utilitarian loss is

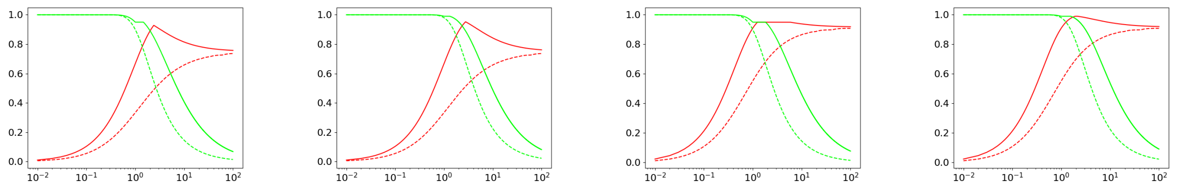
$$\mathbf{UL} = 1 - \frac{\frac{1}{n} + (n-1)\frac{n-1}{n}}{n-1} = \frac{n-2}{n(n-1)} \approx \frac{1}{n},$$

which is far below $1/4$, the upper bound by Theorem 13.

4.4 Group Representation

Lastly, we proceed now to explore proportional fairness. With the IAV characterization, the results presented in this section highlight the fact that we may interpret proportional fairness as the 'middle-ground' between utilitarian and egalitarian welfare. The next and final result shows the proportionality approximation for IAV in the neighboorhood of 1.

Theorem 19 (**λ-AFS**)**.** *Let $x = CTR_f(\vec{x})$ such that $IAV_f \in [\lambda_1, \lambda_2]$ for some $\lambda_1 \leq 1 \leq \lambda_2$. Then, for every α-cohesive set $s \in [n]$, either $\pi_x^i \geq \alpha\ \forall i \in s$, or $\frac{1}{|s|}\sum_{i \in s} \pi_x^i \geq \frac{\sqrt[\lambda_2]{s}}{\sqrt[\lambda_1]{n}}$.*



(a) $m = 4, n = 20$ (b) $m = 4, n = 100$ (c) $m = 12, n = 20$ (d) $m = 12, n = 100$

Figure 3: Utilitarian and egalitarian loss bounds as a function of the IAV. Each graph shows upper and lower bounds as the IAV goes from 10^{-2} to 10^2, for different values of m and n. Red lines show welfare loss, green showing egalitarian loss. Solid lines are for upper bounds, dashed for lower bounds.

For NASH, this means a slightly stronger guarantee then **AFS**.

Corollary 20. *Let $x = NASH(\vec{x})$. Then, for every α-cohesive set $s \in [n]$, either $\pi_x^i \geq \alpha\ \forall i \in s$, or $\frac{1}{|s|}\sum_{i\in s}\pi_x^i \geq \frac{s}{n}$.*

Proof. Let $s \subset [n]$ an α-cohesive group and let $\pi^s := \sum_{i\in s}\pi_x^i$. Since $|\bigcap_{i\in s} x^i| \geq \alpha$, if $\bigcap_{i\in s} x^i$ is covered by x we are done. Otherwise, there exists $j \in [m]$ such that $\min_{i\in s} x_j^i > x_j$ and, since f' is convex,

$$mc_j^{\uparrow} = \sum_{i\in s_j^{\uparrow}} f'(\pi_x^i) \geq \sum_{i\in s} f'(\pi_x^i) \geq sf'(\pi^s)$$

Thus, $sf'(\pi^s) \leq \min_k mc_k^{\downarrow} \leq \sum_i \pi^i f'(\pi^i)$. (See proof of Theorem 16). Now,

$$\begin{aligned} s &\leq \sum_i \pi^i \frac{f'(\pi^i)}{f'(\pi^s)} = \sum_i \pi^i \frac{f'(\pi^i)}{f'(\pi^{\min})}\frac{f'(\pi^{\min})}{f'(\pi^s)} \leq \sum_i (\pi^i)^{1-\lambda_1}(\pi_x^{\min})^{\lambda_1-\lambda_2}(\pi^s)^{\lambda_2} \\ &\leq n(\pi_x^{\min})^{\lambda_1-\lambda_2}(\pi^s)^{\lambda_2} \end{aligned}$$

and now using Theorem 16, $s \leq n\left(\frac{1}{n}\right)^{\frac{\lambda_1-\lambda_2}{\lambda_1}}(\pi^s)^{\lambda_2} \implies \pi^s \geq \frac{s^{\frac{1}{\lambda_2}}}{n^{\frac{1}{\lambda_1}}}$, as required.

□

5 Concluding Remarks

We presented the class of Continious Thiele's Rules (CTR) for distribution aggregation problems. Each rule in this class is defined by a real function f, which are characterized in turn by their *'Inequality Aversion'* (IAV) measure. We gave some positive and negative bounds on the approximations of different rules for utilitarian optimality, Average Fair Share, Individual Fare Share and egalitarian optimality, all depending on the IAV. In general, smaller IAV corresponds to favoring utilitarian welfare, large IAV to egalitarian welfare, and IAV around 1 to proportional fairness. From a practical point of view, such presentation of the range of possibilities might be appealing and convenient to work with for social planners.

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Appendix: Missing proofs

Proposition 3. ***EFF*** $\implies$ ***RR***.

Proof. Let $x \in \Delta^m$ that violates *Range-Respecting*. We show that x must fail Pareto-efficiency (**EFF**). If $x_j > \max_i x_j^i$, reducing x_j down to $\max_i x_j^i$ affects the satisfaction of no agent, and the excess $x_j - \max_i x_j^i$ can surely be distributed to benefit at least one agent. If $x_j < \min_i x_j^i$, then increasing x_j up to $\min_i x_j^i$ benefits all agents, while the necessary reduction in other alternatives can surely be done so that at least one agent is not harmed by it. ☐

Proposition 5. *No CTR satisfies* ***SP***.

Proof. Consider the following counter example. Let $n = m = 2$, $x^1 = (.5, .5)$ and $x^2 = (0, 1)$. Let $x = CTR_f(\vec{x})$. Since CTR_f satisfies **RR**, we know that $x = (y, 1-y)$ for some $y \in [0, 0.5]$, so that $s_1 = 1$, $s_2 = 2$, $\pi_x^1 = y + 0.5$ and $\pi_x^2 = 1 - y$. Thus,

$$f'(y + 0.5) = mc_1 = mc_2 = f'(1-y) \implies y = .25$$

(f' is strictly decreasing, thus injective). Now assume agent 1 misreports $\hat{x}^1 = (1, 0)$ instead, and call the new outcome $(z, 1-z)$. Then

$$f'(z) = mc_1 = mc_2 = f'(1-z) \implies z = .5$$

and agent 1's satisfaction has increased from 0.75 to 1. ☐

Proposition 7. *For $m > 2$, The logarithmic CTR $CTR_{\ln}$ is the only CTR that satisfies* ***PROP***.

Proof. Let $\vec{x}$ be a single-minded preferences vector, and $T_f(\vec{x}) = x$. By MRS, $s_j f'(x_j) = s_k f'(x_k) \forall j, k$ (note that under single-minded profiles $mc_j^{\uparrow} = mc_j^{\downarrow}$ for all j such that $0 < x_j < 1$). If x satisfies **PROP** then $x_j = \frac{s_j}{n}$, yielding

$$s_j f'(\frac{s_j}{n}) = s_k f'(\frac{s_k}{n}) \implies \frac{s_j}{n} f'(\frac{s_j}{n}) = \frac{s_k}{n} f'(\frac{s_k}{n})$$

Satisfying this for general n, $1 \le s_j, s_k \le n$ means that $xf'(x)$ is a constant function, thus $f = \ln$. ☐

Proposition 8. *No CTR satisfies* ***CS***.

Proof. We give an example of the Nash rule $CTR_{\ln}$ violating **CS**. Since **CS** implies **PROP**, the above Lemma completes the proof. Let $m = 3$ and $[n]$ consisting of 3 disjoint sets of sizes n_1, n_2, n_3, such that $n_1 = n_2 = 0.3n$, $n_3 = 0.4n$. Each of the subsets is homogeneous with preferences $x^1 = (1, 0, 0), x^2 = (.5, .5, 0), x^3 = (0, 0, 1)$ respectively. Then $CTR_{\ln}(\vec{x}) = x = (.5, 0, .5)$. However, with $y = (0.55, 0.05)$, $|y| = \frac{n_1 + n_2}{n}$, every agent $i \in n_1 \cup n_2$ has $\pi_y^i = 0.55 > \pi_x^i = 0.5$. ☐

Lemma 10. *The following are equivalent for any twice differentiable concave function $f : [0, 1] \to \mathbb{R}$:*

- $IAV_f \overset{(\le)}{\ge} \lambda$ *for some* $\lambda > 0$.

- $\forall \alpha > 1,\ \frac{f'(t)}{f'(\alpha t)} \overset{(\leq)}{\geq} \alpha^\lambda$

Proof. Essentially, $-\frac{tf''(t)}{f'(t)} \overset{(\leq)}{\geq} \lambda$ means that $f'(t)$ decreases faster (slower) than $t^{-\lambda}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[t^\lambda f'(t)\Big] = t^{\lambda-1}\big(\lambda f'(t) + tf''(t)\big) \leq 0 \iff -\frac{tf''(t)}{f'(t)} \geq \lambda$$

And if $t^\lambda f'(t)$ is non-increasing (non-decreasing), then

$$\forall \alpha > 1,\ \ t^\lambda f'(t) \overset{(\leq)}{\geq} (\alpha t)^\lambda f'(\alpha t) \iff \frac{f'(t)}{f'(\alpha t)} \overset{(\leq)}{\geq} \alpha^\lambda$$

□

Theorem 13. *Let $m = 2$, and $x = CTR_f(\vec{x})$ such that $IAV_f \leq \lambda$. Then $\mathbf{UL}(x) \leq \lambda(\lambda+1)^{-\frac{\lambda+1}{\lambda}}$.*

Proof. The proof is almost identical to that of Theorem 12. The difference comes near the end when we bound $g_m(t) = (1 - (\frac{t}{m-1})^\lambda)t$. $g_2(t)$ has global maximum at $t^* = (\lambda+1)^{\frac{-1}{\lambda}} < 1$ and thus

$$\mathbf{UL}(x) \leq g_2(t^*) = \lambda(\lambda+1)^{-\frac{\lambda+1}{\lambda}}.$$

□

Theorem 14. *Let CTR_f be a CTR such that $IAV_f = \lambda$. Then for every $m \geq 2$ there exists a profile $\vec{x}$ such that*

$$\mathbf{UL}\Big(CTR_f(\vec{x})\Big) = \max_{x_k \in (0,1)} \left(1 - \left(\frac{1-x_k}{x_k(m-1)}\right)^\lambda\right)(1-x_k)$$

Proof. Let $\vec{x}$ be a single minded profile such that $s_k = \max_{j\in[m]} s_j$ and $s_j = s_\ell\ \forall l, j \neq k$. In single-minded profiles, the utilitarian allocation puts all the budget on the alternatives with maximal support, i.e. $\max_{y\in\Delta^m} UW(y) = s_k$ in our case. As it must be that $x_j = x_\ell$ for all $j, \ell \neq k$, the welfare loss in x is

$$\mathbf{UL}(x) = \frac{s_k - \Big((m-1)s_j \cdot \frac{1-x_k}{m-1} + s_k \cdot x_k\Big)}{s_k} = \frac{(s_k - s_j)(1-x_k)}{s_k} = \Big(1 - \frac{s_j}{s_k}\Big)(1-x_k)$$

where $j \neq k$. By MRS condition,

$$\frac{s_j}{s_k} = \frac{f'(x_k)}{f'(x_j)} = \left(\frac{x_j}{x_k}\right)^\lambda \implies 1 - x_k = \sum_{j\neq k} x_j = (m-1)\left(\frac{s_j}{s_k}\right)^{\frac{1}{\lambda}} x_k$$
$$\implies \frac{s_j}{s_k} = \left(\frac{1-x_k}{x_k(m-1)}\right)^\lambda$$

which yields the result. □

Corollary 15. *In any single minded profile $\vec{x}$,*

$$\mathbf{UL}\Big(CTR_f(\vec{x})\Big) \le \max_{x_k \in (0,1)} \left(1 - \left(\frac{1-x_k}{x_k(m-1)}\right)^{\lambda}\right)(1-x_k)$$

and this bound is tight.

To see why, we need to look back at the proof of Theorem 12. When we bounded $\frac{s_j^{\downarrow}}{s_k^{\uparrow}}$ using the MRS (second equation in the proof), for single-minded profiles we can replace $f'(1)$ with $f'(x_k)$, as we know that $\pi_x^i = x_k$ for all $i \in s_k$. Thus we will have

$$s_k f'(x_k) \le mc_k \le mc_j \le s_j f'(x_j) \implies \frac{s_j}{s_k} \ge \left(\frac{x_j}{x_k}\right)^{\lambda}$$

This $\frac{1}{x_k}$ factor would drag along as the proof proceeds, ultimately resulting in

$$\frac{UW(y) - UW(x)}{UW(y)} \le \sum_{j \ne k}(1 - (x_j/x_k)^{\lambda})x_j \le \left(1 - \left(\frac{1-x_k}{x_k(m-1)}\right)^{\lambda}\right)(1-x_k)$$

Theorem 16. *Let $x = CTR_f(\vec{x})$ such that $\lambda \le IAV_f$. Then,*

$$\pi_x^{\min} \ge \begin{cases} \frac{1}{\sqrt[\lambda]{n}} & \lambda \le 1 \\ \max\left(\frac{1}{n}, \frac{1}{m\sqrt[\lambda]{n}}\right) & \lambda > 1 \end{cases}$$

*In particular, x satisfies **IFS** whenever $IAV_f \ge 1$.*

Proof. Let $j \in [m]$ such that the voter with minimal π_x^i is in $mc_j^{\uparrow}$, so that $mc_j^{\uparrow} \ge f'(\pi_x^{\min})$ (if no such j exists then $\pi_x^{\min} = 1$). Since $mc_k^{\downarrow} \ge mc_j^{\uparrow}\ \forall k$, we now want to bound $\min_k mc_k^{\downarrow}$. Indeed,

$$\sum_k mc_k \cdot x_k = \sum_k x_k \sum_{i \in s_k} f'(\pi_x^i) = \sum_i \sum_{k \in \sigma(i)} x_k f'(\pi^i) \le \sum_i \pi^i f'(\pi^i)$$

Thus, since $\sum_k x_k = 1$ we have that $f'(\pi_x^{\min}) \le \min_k mc_k^{\downarrow} \le \sum_i \pi^i f'(\pi^i)$. Hence,

$$1 \le \sum_i \pi^i \frac{f'(\pi^i)}{f'(\pi_x^{\min})} \le \sum_i \pi^i \left(\frac{\pi_x^{\min}}{\pi_x^i}\right)^{\lambda} = (\pi_x^{\min})^{\lambda} \sum_i (\pi_x^i)^{1-\lambda}$$

Now if $\lambda \le 1$ then $(\pi_x^i)^{1-\lambda} \le 1\ \forall i$, and thus $1 \le (\pi_x^{\min})^{\lambda} \cdot n$, as needed. If $\lambda > 1$,= then $(\pi_x^i)^{1-\lambda} \le (\pi_x^{\min})^{1-\lambda}\ \forall i$, and thus $1 \le (\pi_x^{\min})^{\lambda} \cdot n(\pi_x^{\min})^{1-\lambda} = n\pi_x^{\min}$. Moreover, $mc_k^{\downarrow} \le nf'(x_k)\ \forall k$, and since there exist x_k such that $x_k \ge 1/m$, we will have $f'(\pi_x^{\min}) \le nf'(1/m) \implies \pi_x^{\min} \ge \frac{1}{m\sqrt[\lambda]{n}}$ (note that if $x_j \ge 1/m$ then $\pi_x^{\min} \ge 1/m$). □

Theorem 18. *Let $\vec{x}$ be single-minded profile and $x = CTR_f(\vec{x})$ such that $IAV_f \ge \lambda$. Then,*

$$\mathbf{EL}(x) \le 1 - \frac{2}{1 + (n-1)^{\frac{1}{\lambda}}}$$

In particular, this bound is tight when $IAV_f = \lambda$.

Proof. Let $k = |\{j \in [m] : s_j > 0\}|$ be the number of projects that has non-empty support in $\vec{x}$. Then $\max_{y \in \Delta^m} \min_i \pi_y^i = \frac{1}{k}$, making the egalitarian loss in x $\mathbf{EL}(x) = 1 - k \min_{x_j}$. Assume w.l.o.g. that $s_1 = \min_{j: s_j > 0} s_j$. It is not difficult (See for example the proof of Theorem 14) to calculate that, given that $IAV_f \geq \lambda$,

$$x_1 \geq \frac{s_1^{\frac{1}{\lambda}}}{\sum_j s_j^{\frac{1}{\lambda}}} \geq \frac{1}{1 + \sum_{j>1} s_j^{\frac{1}{\lambda}}}$$

which gives

$$\text{for } \lambda \geq 1, \quad kx_1 \geq \frac{k}{1 + (k-1)\left(\frac{n-1}{k-1}\right)^{\frac{1}{\lambda}}} \geq \frac{2}{1 + (n-1)^{\frac{1}{\lambda}}}$$

$$\text{for } \lambda < 1, \quad kx_1 \geq \frac{k}{1 + (n-1)^{\frac{1}{\lambda}}} \geq \frac{2}{1 + (n-1)^{\frac{1}{\lambda}}}$$

and thus

$$\mathbf{EL}(x) \leq 1 - \frac{2}{1 + (n-1)^{\frac{1}{\lambda}}}$$

And indeed, if $IAV_f = \lambda$ and $s_1 = 1, s_2 = n - 1$ we get that exact result. $\square$