# Agreement among Voting Rules under Single-Peaked Preference Distributions

Vincent Mousseau[1], Henri Surugue[1] and Anaëlle Wilczynski[1]
[1] Affiliation: MICS, CentraleSupélec, Université Paris-Saclay
{firstname.lastname}@centralesupelec.fr

**Abstract**

Many different voting rules have been proposed in the literature and they can select very different alternatives. This naturally raises the question of whether this diversity in outcomes often occurs. Previous works have shown that the probability that voting rules agree on the same outcome is generally quite low under impartial culture. In this article, we use a similar probabilistic approach on single-peaked cultures, which are more structured and typically more realistic than impartial culture. We provide conditions for voting rules to agree under standard single-peaked cultures, and show that the probability of agreement between rather large families of voting rules is much higher under such cultures, with fast convergence of this probability with respect to the number of voters. We finally provide some insights on other structured preference distributions, observing that many exhibit similar convergence in agreement, including the Mallows' distribution. Our study reveals a tendency of several well-known voting cultures to bias the outcome of voting rules, which is worth knowing before conducting experiments on synthetic data.

## 1 Introduction

A major topic in voting theory is the design of good voting rules. However, the social choice literature is famous for impossibility theorems, e.g., Arrow's [1] or Gibbard-Satterthwaite [26, 44] theorems, basically stating that no perfect voting rule exists. Many different voting rules have been designed along the years, and a large body of literature is devoted to their axiomatic characterization [2]. In fact, different voting rules can select very different alternatives. However, does this behavior often occur? This question has been raised by many articles [25] which study the probability that different voting rules disagree on their outcome. Indeed, exploring the agreement among voting rules can help understand the similarity between voting rules, in an orthogonal perspective than the axiomatic study.

Most of the works on voting rules' agreement focus on the impartial (anonymous) culture, where each preference order (or score), is uniformly drawn from the whole set of linear orders over candidates. Such study is necessary because the impartial culture can arguably be seen as the most neutral. However, it does not capture real voters' preferences, which are usually far from being uniformly distributed. Moreover, most results on impartial culture highlight that voting rules rarely agree. Therefore, exploring more structured and realistic cultures may provide new insights on differences between voting rules. In this article, we will focus on cultures generating single-peaked preferences [6], which make sense in several contexts such as, e.g., political elections where a left-right axis can structure most voters' preferences. Even though single-peaked cultures are still far from being a perfect match to real data [19], they are much more realistic than impartial culture, so these models can be seen as a better approximation of the reality in some contexts.

In another point of view, studying agreement between voting rules under single-peaked cultures can also improve the understanding of such cultures. A key question in computational social choice, and in particular in voting theory, is how to generate relevant synthetic data for experiments on elections [8]. Conducting an empirical study via computer simulations can indeed be very useful to support or complement theoretical results for many voting problems, e.g., manipulation, winner determination, bribery and control, or the analysis of possible and necessary winners [10]. The ideal solution to perform experiments would be to use real-world data [15, 41, 42], see, e.g., the Preflib platform [35]. However, typically, we only have access to limited and context-dependent real-world data, which makes the experimental results potentially difficult to generalize. In contrast, using synthetic data allows to simulate elections of any size and to control the experiments' parameters. However, for experiments to be meaningful, we also need to simulate realistic elections, raising the question of a compromise between realism and flexibility. A large number of statistical cultures exist for generating elections [45]. Among them, single-peaked distributions are quite often used, as reported by Boehmer et al. [8]. Therefore, exploring voting rules' agreement under single-peaked cultures is relevant to better understand these commonly used cultures and better interpret experimental studies.

Let us illustrate possible issues in the interpretation of experiments. For instance, if one would like to compare how often different rules violate the majority criterion (i.e., a candidate ranked first by half of the voters should be elected), then experiments could be used. However, the conclusions may be very different depending on the voting culture used to generate synthetic data. In particular, using single-peaked cultures may lead to different conclusions compared to impartial culture, especially if the results on voting rules' agreement are very different. In particular, if two voting rules frequently agree under a given culture then the results will be similar because the voting rules are close under that culture, not because of the problem itself. In any case, knowing how the statistical tool works is a prerequisite for a good empirical study.

In this article, we study the probability of agreement of different voting rules under single-peaked cultures. Up to our best knowledge, this question has been surprisingly neglected for cultures more structured than impartial ones. One notable exception is the work of Chatterjee and Storcken [13] on unimodal profiles. We focus our study on two well-known models to generate single-peaked elections: Walsh's [46] and Conitzer's [14] models. They consider different ways of uniformly drawing single-peaked preference orders: either uniformly within the whole single-peaked domain [46], or uniformly with respect to the peak candidate in the order [14].

We particularly examine positional scoring rules (PSRs), which compute scores for the candidates based on their position in the voters' preferences. This family covers many famous voting rules, such as $k$-approval rules like plurality or veto, and the Borda rule. We show that for both Walsh's and Conitzer's distributions, many PSRs tend to elect the median candidate(s) in the single-peaked axis, which turns out to be the asymptotic Condorcet winner, implying that these rules also agree with Condorcet-consistent rules. We also provide a lower bound on the speed of convergence to such a winner, meaning that this result holds for reasonable election sizes. We characterize these rules for both cultures and observe that this set is larger for Walsh's distribution, which is coherent with its definition. Conitzer's distribution seems to be more neutral toward the candidates, in the sense of probability to be elected. We further study this aspect by examining when single-peaked distributions are unbiased, i.e., when they do not favor any candidate with respect to a given voting rule.

Finally, we provide some insights on the agreement among voting rules under two other structured preference distributions: unimodal distributions, which include Mallows' cultures [34], where we complete Theorem 4.1 of Chatterjee and Storcken [13] to prove a rapid convergence to a large probability of agreement; and Pólya-Eggenberger urns [17], where we show that even if the probability of agreement remains high, the convergence toward one is not guaranteed.

Due to space restrictions, some proofs or parts of proofs are deferred to the supplementary material.

## 2 Related Work

The question of agreement among voting rules was initiated by Gehrlein and Fishburn [22, 23] who give an explicit probability of agreement between two positional scoring rules in the case of three candidates under impartial culture. They prove that the probability of all scoring rules to agree in large elections is 0.5346. Many necessary conditions have then been derived to characterize the agreement of all positional scoring rules [39, 40, 43]. In particular, Merlin et al. [39] give the probability (i.e., $0.50116$) under impartial culture that many rules (including positional scoring rules, elimination rules and Condorcet-consistent rules) agree on the same winner in the case of three candidates. This work was complemented via Monte-Carlo simulations by Lepelley et al. [32] for more than three candidates. Similar results with explicit formulas have been found under anonymous impartial culture for three candidates [20]. Most of these works focus on three candidates, sometimes four [30], under the impartial (sometimes anonymous) culture and try to provide explicit formulas. In contrast, we focus on single-peaked distributions with an arbitrary number of candidates and analyze the conditions of convergence toward the same outcome.

In another perspective, many works have studied the Condorcet efficiency of voting rules (see Gehrlein and Lepelley [25] for a survey), i.e., their probability to elect a Condorcet winner, which can be seen as exploring how much these rules agree with Condorcet-consistent rules. This question has also been investigated for structured cultures, such as impartial (anonymous) culture over the single-peaked domain [21, 31, 33], and Pólya-Eggenberger urns [24, 37] but, as far as we know, only for three candidates.

Another close question is the notion of consensus [18, 28], which is essentially setting a distance to find the closest election that satisfies consensus, i.e., the one where the minimum number of voters would disagree. Beyond voting rule agreement, the likelihood of the occurrence of voting paradoxes has been widely investigated [25, 48]. In addition, following the idea of asymptotic results, many studies have been conducted in machine learning, making the link between a voting rule and a maximum likelihood estimator [4, 12, 47]. In the same perspective, a work on the asymptotic probability of ties in elections was proposed [49]. While these directions may sometimes be outside of voting theory, it highlights the importance of our research question.

## 3 The Model

For any positive integer $k$, let $[k]$ denote the set $\{1, \dots, k\}$. Let $N$ be a set of voters where $N = [n]$, and $M$ be a set of $m$ candidates where $M = \{x_1, \dots, x_m\}$. Each voter $i \in N$ has preferences over candidates represented by a linear order $\succ_i$ over $M$; the preference profile is denoted by $\succ = (\succ_i)_{i \in N}$. Let $\Pi^m$ be the set of all possible preference orders for $m$ candidates. For a given preference order $\succ_i \in \Pi^m$, the rank of candidate $x$ in $\succ_i$ is denoted by $r_{\succ_i}(x)$, i.e., $r_{\succ_i}(x) := |\{y \in M : y \succeq_i x\}|$.

We consider a common preference restriction, namely *single-peakedness* [6]. A preference profile $\succ \in (\Pi^m)^n$ is single-peaked if there exists an axis $>$ on $M$ such that, for every voter $i \in N$, and each triple of candidates $x > y > z$, we have $y \succ_i x$ or $y \succ_i z$. All along the article, we consider, w.l.o.g., an axis $>$ on $M$ such that $x_1 > \cdots > x_m$. Let $\Pi^m_>$ be the set of all possible single-peaked preference orders w.r.t. axis $>$ on $M$.

### 3.1 Voting Rules

A voting rule $\mathcal{F} : (\Pi^m)^n \to 2^M \setminus \{\emptyset\}$ selects a non-empty subset of candidates for each preference profile $\succ \in (\Pi^m)^n$. A scoring rule $\mathcal{F}$ is associated with a score function $s^{\mathcal{F}} : M \to \mathbb{R}$ and selects the candidates maximizing this score, i.e., $\mathcal{F}(\succ) \in \arg\max_{x \in M} s^{\mathcal{F}}(x)$ for every preference profile $\succ \in (\Pi^m)^n$.

A *positional scoring rule (PSR)* $\mathcal{F}$ is characterized by a positional score vector $\alpha = (\alpha_1, \ldots, \alpha_m)$ such that $\alpha_1 \geq \cdots \geq \alpha_m$ and $\alpha_1 > \alpha_m$, in such a way that the winner of the election under $\mathcal{F}$ maximizes the sum of the position scores given by each voter according to the position of the candidate in the voter's preferences, i.e., $\mathcal{F}(\succ) \in \arg\max_{x \in M} \sum_{i \in N} \alpha_{r_{\succ_i}(x)}$ for every preference profile $\succ \in (\Pi^m)^n$. The $k$-*approval* voting rule, for $k \in [m-1]$, is a particular case of PSR where $\alpha_j = 1$ for all $j \in [k]$, and $\alpha_j = 0$ for all $k < j \leq m$. The *plurality* rule corresponds to the 1-approval rule and the *veto* rule is the $(m-1)$-approval rule. The *Borda* rule is the PSR characterized by an evenly spaced positional score vector, e.g., $\alpha = (m-1, m-2, \ldots, 1, 0)$.

Instead of evaluating the candidates on their absolute position in the voters' preferences, other voting rules take into account pairwise comparisons of candidates. A candidate $x$ is the *Condorcet winner* in preference profile $\succ \in (\Pi^m)^n$ if it beats all the other candidates in pairwise comparisons, i.e., $|\{i \in N : x \succ_i y\}| > |\{i \in N : y \succ_i x\}|$, for every candidate $y \in M \setminus \{x\}$. A *weak* Condorcet winner $x$ is such that $|\{i \in N : x \succ_i y\}| \geq |\{i \in N : y \succ_i x\}|$, for every candidate $y \in M \setminus \{x\}$. In general, a (weak) Condorcet winner does not always exist. However, a weak Condorcet winner always exists when the preferences are single-peaked as well as a Condorcet winner when, additionally, $m$ is odd [7]. A voting rule which always elects the Condorcet winner, when it exists, is called *Condorcet-consistent*. Note that PSRs are not Condorcet-consistent [16].

### 3.2 Voting Cultures

Let us denote as $C(n, \Pi^m_{sub})$ the probability distribution of drawing $n$ preference orders from $\Pi^m_{sub} \subseteq \Pi^m$ to constitute a preference profile $\succ \in (\Pi^m)^n$. Such a probability distribution $C(n, \Pi^m_{sub})$ is called a *culture*.

When voters' preferences are selected independently and identically distributed, the culture can be defined as drawing $n$ preference orders $\succ_i$ from a given preference distribution $\pi^m : \Pi^m \to [0,1]$ with $\sum_{\succ_i \in \Pi^m} \pi^m(\succ_i) = 1$. The probability for a candidate $x_j$ to be ranked at position $k \in [m]$ under preference distribution $\pi^m$ is given by $\mathbb{P}^m_\pi(j,k) = \sum_{\succ_i \in \Pi^m : r_{\succ_i}(x_j) = k} \pi^m(\succ_i)$. Moreover, the probability for a candidate $x$ to be ranked before a candidate $y$ under preference distribution $\pi^m$ is given by $\mathbb{P}^m_\pi(x \succ_i y) = \sum_{\succ_i \in \Pi^m : x \succ_i y} \pi^m(\succ_i)$. When the context is clear, the superscript $m$ may be omitted.

Let $S^{\mathcal{F}}(x)$ denote the random variable giving the score of a candidate $x \in M$ for a voting rule $\mathcal{F}$. Let $\mathbb{E}_\pi[S^{\mathcal{F}}(x)]$ denote the expected score of candidate $x$ for voting rule $\mathcal{F}$ under distribution $\pi$. For a PSR $\mathcal{F}$ characterized by a positional score vector $\alpha$ and a preference distribution $\pi$, the expected score of each candidate $x$ is given by $\mathbb{E}_\pi[S^{\mathcal{F}}(x)] = \sum_{\succ_i \in \Pi^m} \pi(\succ_i) \cdot \alpha_{r_{\succ_i}(x)}$.

### 3.3 Convergence to the Expected Winners

When voters' preferences are identically and independently drawn w.r.t. distribution $\pi$ and $\mathbb{E}_\pi[S^{\mathcal{F}}(x)]$ is finite for any $x \in M$, by the law of large numbers, the *expected winners* $\mathcal{W}_\pi(\mathcal{F})$ of $\mathcal{F}$ under $\pi$ are $\mathcal{W}_\pi(\mathcal{F}) := \arg\max_{x \in M} \mathbb{E}_\pi[S^{\mathcal{F}}(x)]$. A candidate $x$ is an *asymptotic (weak) Condorcet winner* under distribution $\pi$ if $\mathbb{P}_\pi(x \succ_i y) > \frac{1}{2}$ (resp., $\mathbb{P}_\pi(x \succ_i y) \geq \frac{1}{2}$), for every $y \in M \setminus \{x\}$.

In addition to the guarantee of convergence to the election of expected winners, we provide below a lower bound on the probability that an expected winner actually wins, when we draw voters' preferences independently and identically with respect to a distribution $\pi$.

**Theorem 1.** *Consider a positional scoring rule $\mathcal{F}$ defined by a score vector $\alpha$, and a preference distribution $\pi$ over the set of candidates $M$. When the set of expected winners, defined as $\mathcal{W}_\pi(\mathcal{F}) = \arg\max_{x \in M} \mathbb{E}_\pi[S^{\mathcal{F}}(x)]$, is a singleton, i.e., $\mathcal{W}_\pi(\mathcal{F}) = \{x\}$, the probability that $\mathcal{F}$ elects $x$ satisfies*

$$\mathbb{P}_\pi(x \in \mathcal{F}(\succ)) \geq L_\pi(\mathcal{F}),$$

*where:*

$$L_\pi(\mathcal{F}) := 1 - 2 \cdot \max_{y \in M \setminus \mathcal{W}_\pi(\mathcal{F})} \exp\left( \frac{-2n \cdot \left(\mu_\pi^{\mathcal{F}}(y) - \mathbb{E}_\pi[S^{\mathcal{F}}(y)]\right)^2}{(\max_j \alpha_j - \min_j \alpha_j)^2} \right)$$

$$\textit{and } \mu_\pi^{\mathcal{F}}(y) := \frac{\max_{x \in M} \mathbb{E}_\pi[S^{\mathcal{F}}(x)] + \mathbb{E}_\pi[S^{\mathcal{F}}(y)]}{2}$$

We can thus deduce a lower bound for the speed of convergence for the agreement of several voting rules.

**Corollary 2.** *For two positional scoring rules $\mathcal{F}_1$ and $\mathcal{F}_2$ whose expected winner set under a preference distribution $\pi$ is the same, i.e., $C := \mathcal{W}_\pi(\mathcal{F}_1) = \mathcal{W}_\pi(\mathcal{F}_2)$, the probability of their agreement for electing the same unique candidate from $C$ is such that: $\mathbb{P}_\pi(\mathcal{F}_1(\succ) = \mathcal{F}_2(\succ)) \geqslant \min\{L_\pi(\mathcal{F}_1), L_\pi(\mathcal{F}_2)\}$.*

### 3.4 Single-Peaked Distributions

We particularly consider distributions based on the single-peaked domain. For a given axis $>$ over $M$, a culture $C(n, \Pi^m)$ is single-peaked if $C(n, \Pi^m) = C(n, \Pi^m_>)$.

Let us define the symmetry with respect to the single-peaked axis via the bijection $\tau : [m] \to [m]$ which associates with each candidate $x_j$ its symmetric candidate $x_{\tau(j)}$ where $\tau(j) = m - j + 1$. A single-peaked preference distribution $\pi : \Pi^m_> \to [0,1]$ is said to be *symmetric* if $\mathbb{P}^m_\pi(j, 1) = \mathbb{P}^m_\pi(\tau(j), 1)$, for every candidate $x_j \in M$. Symmetric single-peaked distributions form a rather large family of single-peaked distributions which include, e.g., the distributions $\pi$ such that $\mathbb{P}^m_\pi(x_j \succ_i x_{j+1}) = \mathbb{P}^m_\pi(x_{\tau(j)} \succ_i x_{\tau(j+1)})$ for every $j \in [\lfloor \frac{m}{2} \rfloor]$, but not only. Using symmetric single-peaked distributions turns out to be very natural, in order to derive experiments on the single-peaked domain, without any additional information than the single-peaked axis. In particular, two distributions are commonly used in the literature to sample single-peaked elections: Walsh's [46] and Conitzer's [14] distributions; they are symmetric and capture different types of impartial culture on the single-peaked domain. Roughly, the idea is either to uniformly draw every single-peaked preference order [46], or to uniformly draw every peak candidate and then construct the rest of the preference order by uniformly choosing the next candidate to rank between the closest available candidates on the single-peaked axis [14].

**Definition 1** (Walsh's distribution)**.** *Walsh's distribution $\pi_W : \Pi^m_> \to [0,1]$ is such that $\pi_W(\succ_i) = \frac{1}{2^{m-1}}$, for every $\succ_i \in \Pi^m_>$.*

**Definition 2** (Conitzer's distribution)**.** *Conitzer's distribution $\pi_C : \Pi^m_> \to [0,1]$ is such that $\pi_C(\succ_i) = \frac{1}{m} \cdot \frac{1}{2^{\min\{r_{\succ_i}(x_1), r_{\succ_i}(x_m)\}-1}}$ for every $\succ_i \in \Pi^m_>$.*

This definition adequately translates the algorithm proposed by Conitzer [14]. The peak is selected uniformly at random, corresponding to the $\frac{1}{m}$ term. Once the peak is fixed, the next candidate is chosen uniformly among the two candidates adjacent on the axis $>$, making the process dependent on the relative positions of the two extreme candidates. Specifically, once one of these two extreme candidates is selected, the rest of the ranking is completed by successively adding the remaining candidates on the same side with respect to the axis $>$.

In this article, we aim at understanding the behavior of voting rules under single-peaked distributions. In particular, we analyze the conditions under which PSRs agree, how the expected winners are located with respect to the single-peaked axis and whether they are asymptotic (weak) Condorcet winners.

## 4 The Single-Peaked Domain

Let us start with structural properties of the single-peaked domain. We first recall that $|\Pi^m_>| = 2^{m-1}$. We give below a useful observation on possible candidates' positions in single-peaked orders.

**Observation 3.** *Candidate $x_j$ can never be ranked at a position $k > \max\{j, m-j+1\}$ in a single-peaked order.*

We continue our preliminary remarks on the structure of the single-peaked domain with the next lemma, already stated by Boehmer et al. [9], which will be useful to compute the probability for a candidate to be ranked at a given position.

**Lemma 4** (Boehmer et al. [9]). *The number of single-peaked preference orders in $\Pi^m_>$ in which candidate $x_j$ is ranked at position $k$ is given by the following formula, for each $j, k \in [m]$:*

$$\mathscr{D}_m(j,k) = 2^{k-2}\left(\binom{m-k}{j-1} + \binom{m-k}{j-k}\right).$$[1]

Let $C^*$ denote the set of median candidates in the single-peaked axis, this set is a singleton in case $m$ is odd and is a pair of candidates in case $m$ is even, i.e.,

$$C^* := \begin{cases} \{x_{\lceil \frac{m}{2} \rceil}\} & \text{if } m \text{ is odd} \\ \{x_{\frac{m}{2}}, x_{\frac{m}{2}+1}\} & \text{if } m \text{ is even} \end{cases}.$$

These candidates play an important role in the single-peaked domain. We first show below that more preference orders rank them at good positions compared to the other candidates.

**Lemma 5.** *For every median candidate $x_c \in C^*$ and any other candidate $x_j \in M \setminus C^*$, there exists an index $\gamma_m(j) \in [\max\{j, m-j+1\}]$ such that $\mathscr{D}_m(c,k) \geq \mathscr{D}_m(j,k)$ for every $1 \leq k \leq \gamma_m(j)$ and $\mathscr{D}_m(j,k) > \mathscr{D}_m(c,k)$ for every $\gamma_m(j) < k \leq \max\{j, m-j+1\}$.*

Moreover, we show below that many natural single-peaked distributions favor the median candidates by tending to make them (weak) Condorcet winners.

**Proposition 6.** *Every symmetric single-peaked preference distribution makes the median candidate(s) asymptotic weak Condorcet winner(s). When $m$ is odd, the unique median candidate is the asymptotic Condorcet winner under any symmetric single-peaked distribution $\pi$ which assigns a positive probability to rank the median candidate first, i.e., $\mathbb{P}_\pi(c,1) > 0$ for $x_c \in C^*$.*

*Proof.* Let us consider a symmetric single-peaked distribution $\pi$. Let us compare a median candidate $x_c \in C^*$ and any other candidate $x_j \in M \setminus C^*$ where, w.l.o.g., $c = \lceil \frac{m}{2} \rceil$ and $j < c$. By single-peakedness, a preference order with a candidate $x_\ell$ as a peak candidate must rank $x_c$ before $x_j$ if $\ell \geq c$. It follows that $\mathbb{P}_\pi(x_c \succ_i x_j) \geq \sum_{\ell=c}^m \mathbb{P}_\pi(\ell,1)$. Recall that $\sum_{\ell=1}^m \mathbb{P}_\pi(\ell,1) = 1$.

If $m$ is odd then, by symmetry, we have $\sum_{\ell=1}^{c-1} \mathbb{P}_\pi(\ell,1) = \sum_{\ell=c+1}^m \mathbb{P}_\pi(\ell,1)$, and thus $\mathbb{P}_\pi(x_c \succ_i x_j) \geq \sum_{\ell=c}^m \mathbb{P}_\pi(\ell,1) \geq \frac{1}{2}$. This inequality is strict if $\mathbb{P}_\pi(c,1) > 0$.

If $m$ is even then, by symmetry, we have $\sum_{\ell=1}^{c} \mathbb{P}_\pi(\ell,1) = \sum_{\ell=c+1}^m \mathbb{P}_\pi(\ell,1)$, and thus $\mathbb{P}_\pi(x_c \succ_i x_j) \geq \sum_{\ell=c}^m \mathbb{P}_\pi(\ell,1) \geq \frac{1}{2}$. It remains to compare $x_c$ with the other median candidate $x_{c+1}$. The arguments are similar: a preference order with a candidate $x_\ell$ as a peak candidate must rank $x_c$ before $x_{c+1}$ if $\ell \leq c$. Therefore, $\mathbb{P}_\pi(x_c \succ_i x_{c+1}) \geq \sum_{\ell=1}^c \mathbb{P}_\pi(\ell,1) = \frac{1}{2}$. □

## 5 Walsh's Distribution

We first study Walsh's distribution (Definition 1), which corresponds to impartial culture on the single-peaked domain. The probability that a candidate appears at a given rank then directly follows from Lemma 4.

[1]By convention, $\binom{n}{k} = 0$ when $k > n$ or $k < 0$.

**Observation 7.** *The probability $\mathbb{P}_{\pi_W}(j,k)$ that candidate $x_j$ is ranked at position $k$ under Walsh's distribution, for each $j,k \in [m]$, is equal to $\mathbb{P}_{\pi_W}(j,k) = \frac{\mathscr{D}_m(j,k)}{2^{m-1}}$.*

We first establish that this distribution favors the median candidates since their expected score under every PSR is at least as large as the one of any other candidate.

**Proposition 8.** *For every PSR $\mathcal{F}$, the median candidates always belong to the expected winners of $\mathcal{F}$ under Walsh's distribution, i.e., $C^* \subseteq \mathcal{W}_{\pi_W}(\mathcal{F})$.*

*Sketch of proof.* One can show that the expected score of a median candidate is at least as large as the expected score of any other candidate, no matter the chosen positional score vector for the PSR. When comparing the expected score of a candidate $c \in C^*$ with the one of any other candidate $x_j \in M \setminus C^*$, we can restrict our attention, w.l.o.g., to the median candidate $x_c := x_{\lceil \frac{m}{2} \rceil} \in C^*$ and to any candidate $x_j$ such that $j < \lceil \frac{m}{2} \rceil$ (by symmetry w.r.t. the single-peaked axis). By Observation 3, the expected score of a candidate $x_j$, for Walsh's distribution and a PSR $\mathcal{F}$ characterized by the positional score vector $\alpha$, is given by $\mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x_j)] = \sum_{k=1}^{m-j+1} \frac{\mathscr{D}_m(j,k)}{2^{m-1}} \cdot \alpha_k$ and $\mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x_c)] = \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor+1} \frac{\mathscr{D}_m(j,k)}{2^{m-1}} \cdot \alpha_k$. One can then show to conclude that $\mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x_c)] - \mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x_j)] \geq 0$. □

We now aim to characterize the PSRs for which the median candidates are the only expected winners. We identify them as the PSRs whose associated positional score vector $\alpha$ is such that there exists an index $\ell \in [\lfloor \frac{m}{2} \rfloor + 1]$ with $\alpha_\ell > \alpha_{\ell+1}$. We call them *first-prioritizing* PSRs. Note that all $k$-approval rules for $k \leq \lfloor \frac{m}{2} \rfloor + 1$ are first-prioritizing, as well as the Borda rule.

**Theorem 9.** *The median candidates are the unique expected winners of a PSR $\mathcal{F}$ under Walsh's distribution, i.e., $\mathcal{W}_{\pi_W}(\mathcal{F}) = C^*$, iff $\mathcal{F}$ is first-prioritizing.*

*Sketch of proof.* Consider a PSR $\mathcal{F}$ characterized by a score vector $\alpha$ such that there exists an index $\ell \in [\lfloor \frac{m}{2} \rfloor + 1]$ for which $\alpha_\ell > \alpha_{\ell+1}$. We compare a median candidate $x_c \in C^*$ and another candidate $x_j \in M \setminus C^*$ where, w.l.o.g., $c := \lceil \frac{m}{2} \rceil$ and $j < c$. By Observations 3 and 7 and Lemma 5, one can prove that:

$$
\begin{aligned}
&\mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x_c)] - \mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x_j)] \\
&= \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor+1} \frac{\mathscr{D}_m(c,k)}{2^{m-1}} \cdot \alpha_k - \sum_{k=1}^{m-j+1} \frac{\mathscr{D}_m(j,k)}{2^{m-1}} \cdot \alpha_k \\
&= \frac{1}{2^{m-1}} \Big( \sum_{k=1}^{\gamma_m(j)} (\mathscr{D}_m(c,k) - \mathscr{D}_m(j,k)) \cdot \alpha_k + \\
&\quad \sum_{k=\gamma_m(j)+1}^{m-j+1} (\mathscr{D}_m(c,k) - \mathscr{D}_m(j,k)) \cdot \alpha_k \Big) \\
&> \frac{\alpha_{\gamma_m(j)}}{2^{m-1}} \Big( \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor+1} \mathscr{D}_m(c,k) - \sum_{k=1}^{m-j+1} \mathscr{D}_m(j,k) \Big) = 0
\end{aligned}
$$

Hence, the expected score of a median candidate is always greater than the one of any other candidate $x_j$.

For a PSR $\mathcal{F}$ characterized by a vector $\alpha$ where $\alpha_1 = \cdots = \alpha_\ell$, with $\ell > \lfloor \frac{m}{2} \rfloor + 1$, one can prove that both $x_{\lceil \frac{m}{2} \rceil} \in C^*$ and $x_{\lceil \frac{m}{2} \rceil - 1} \in M \setminus C^*$ are expected winners. □

By Proposition 6 and Theorem 9, the first-prioritizing PSRs tend to elect the (weak) Condorcet winner(s) under Walsh's distribution.

**Corollary 10.** *Under Walsh's distribution, all first-prioritizing PSRs and Condorcet-consistent rules asymptotically agree to elect the median candidates.*

We show a good lower bound for the convergence to the same outcome for a subset of first-prioritizing PSRs, which contains $k$-approval rules and the Borda rule.

**Proposition 11.** *For all $k$-approval voting rules that are first-prioritizing and the Borda rule, under Walsh's distribution, the probability of their agreement for electing one candidate from $C^*$ is lower bounded by $L_\pi(\mathcal{F}_1)$ where $\mathcal{F}_1$ refers to the plurality rule.*

As an illustration, by Proposition 11, for $m = 5$, we have $\mathbb{P}_{\pi_W}(\mathcal{F}_1(\succ) = C^*) \geqslant 1 - 2e^{-\frac{n}{128}}$ and for $n = 600$, the probability of agreement is lower bounded by $0.98$.

## 6 Conitzer's Distribution

We now analyze Conitzer's distribution (Definition 2), which considers a uniform distribution not on the whole single-peaked domain, as Walsh's distribution, but on the peak candidates of the single-peaked orders. It follows that the probability for a given candidate to be ranked at a given rank is a bit less direct, as already stated by Boehmer et al. [9].

**Lemma 12** (Boehmer et al. [9]). *The probability that candidate $x_j$ is ranked at position $k$ under Conitzer's distribution, for each $j, k \in [m]$, is equal to $\mathbb{P}_{\pi_C}(j,k) = Q(j,k) + Q(m-j+1,k)$ where:*

$$Q(j,k) = \begin{cases} \frac{1}{2m} & \text{if } k < j \\ \frac{k}{2m} & \text{if } k = j \\ 0 & \text{otherwise} \end{cases}$$

We first characterize the expected winners of all $k$-approval rules.

**Proposition 13.** *The expected winners of the $k$-approval rule $\mathcal{F}$ under Conitzer's distribution are:*

$$\mathcal{W}_{\pi_C}(\mathcal{F}) = \begin{cases} M \text{ if } k = 1 \\ \{x_k, x_{m-k+1}\} \text{ if } 1 < k \le \lfloor \frac{m}{2} \rfloor + 1 \\ \{x_j \in M : \max\{j, m-j+1\} \le k\} \text{ otherwise} \end{cases}.$$

*Sketch of proof.* We compute the expected score of a candidate $x_j$ where, w.l.o.g., $j \in [\lceil \frac{m}{2} \rceil]$. By Lemma 12,

$$\mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_j)] = \begin{cases} \frac{2k}{2m} & \text{if } k < j \\ \frac{3k-1}{2m} + \frac{k}{2m} \cdot \mathbb{1}_{\{j = \lceil \frac{m}{2} \rceil\}} & \text{if } k = j \\ \frac{2j-1+k}{2m} & \text{if } j < k < m-j+1 \\ 1 & \text{if } k \ge m-j+1 \end{cases}$$

We can then derive the expected winners w.r.t. $k$. □

Hence, the only $k$-approval rule which tends to elect the median candidate(s) as unique expected winner(s) is $\lceil \frac{m}{2} \rceil$-approval (and $\frac{m}{2} + 1$-approval if $m$ is even).

We now characterize more precisely the PSRs which tend to elect the median candidate(s).

**Theorem 14.** *The median candidates are the unique expected winners of a PSR $\mathcal{F}$ under Conitzer's distribution iff the positional score vector $\alpha$ associated with $\mathcal{F}$ satisfies the following inequality, for every $1 \le j < \lceil \frac{m}{2} \rceil$:*

$$\sum_{\ell=j+1}^{\lceil \frac{m}{2} \rceil - 1} \alpha_\ell + \beta(m) + \alpha_{\frac{m}{2}} \mathbb{1}_{\{m \text{ even}\}} > \sum_{\ell=\lceil \frac{m}{2} \rceil + 1}^{m-j} \alpha_\ell + \delta(j,m)$$

*where $\beta(m) := (\lceil \frac{m}{2} \rceil - 1)\alpha_{\lceil \frac{m}{2} \rceil} + (\lfloor \frac{m}{2} \rfloor + 1)\alpha_{\lfloor \frac{m}{2} \rfloor + 1}$ and $\delta(j,m) := (j-1)\alpha_j + (m-j+1)\alpha_{m-j+1}$. A sufficient condition is $\beta(m) > \delta(j,m)$, for every $j < \lceil \frac{m}{2} \rceil$.*

*Sketch of proof.* Consider a PSR $\mathcal{F}$ characterized by a score vector $\alpha$. Let us compare a median candidate $x_c \in C^*$ and another candidate $x_j \in M \setminus C^*$ where, w.l.o.g., $j < c := \lceil \frac{m}{2} \rceil$. The median candidates are unique expected winners iff, for every $j < c$, we have $\mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_c)] - \mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_j)] > 0$. One can prove that this is equivalent to $\sum_{\ell=j+1}^{\lceil \frac{m}{2} \rceil - 1} \alpha_\ell + (\lceil \frac{m}{2} \rceil - 1)\alpha_{\lceil \frac{m}{2} \rceil} + (\lfloor \frac{m}{2} \rfloor + 1)\alpha_{\lfloor \frac{m}{2} \rfloor + 1} + \alpha_{\frac{m}{2}} \mathbb{1}_{\{m \text{ even}\}} > (j-1)\alpha_j + \sum_{\ell=\lceil \frac{m}{2} \rceil + 1}^{m-j} \alpha_\ell + (m-j+1)\alpha_{m-j+1}$.

We always have $\sum_{\ell=j+1}^{\lceil \frac{m}{2} \rceil - 1} \alpha_\ell + \alpha_{\frac{m}{2}} \mathbb{1}_{\{m \text{ even}\}} \geq \sum_{\ell=\lceil \frac{m}{2} \rceil + 1}^{m-j} \alpha_\ell$. Hence, a sufficient condition is $(\lceil \frac{m}{2} \rceil - 1)\alpha_{\lceil \frac{m}{2} \rceil} + (\lfloor \frac{m}{2} \rfloor + 1)\alpha_{\lfloor \frac{m}{2} \rfloor + 1} > (j-1)\alpha_j + (m-j+1)\alpha_{m-j+1}$. □

We observe that the Borda rule satisfies the sufficient condition of Theorem 14, as well as $\lceil \frac{m}{2} \rceil$-approval (and $(\frac{m}{2}+1)$-approval if $m$ is even), proving that these rules eventually elect the median candidates (as already observed in Proposition 13 for the approval rules). While Theorem 14 is not immediately interpretable, the following provides some intuition. Indeed, the underlying intuition is that the characterization corresponds to PSRs associated with a score vector $(\alpha_1, \cdots, \alpha_m)$ such that the first half of the scores is strictly greater than the second half but, for more than 4 candidates, not with too big a gap. More precisely, for m=3 and m=4, we must have $\alpha_2 > \alpha_3$ and $\alpha_2 > \alpha_3$ or $\alpha_3 > \alpha_4$, respectively, and for m=5, we must have $\alpha_2 > \alpha_3$ or $\alpha_3 > \alpha_4$ and $\alpha_2 < 5 \cdot \alpha_3 - 4 \cdot \alpha_4$. Note that, in addition to Borda and $\lceil m/2 \rceil$-approval, this also includes, e.g., all PSRs such that $\alpha_i = 0$ if $i > \lfloor m/2 \rfloor + 1$ and $\alpha_2 < 2 \cdot \alpha_{\lfloor m/2 \rfloor + 1}$.

**Corollary 15.** *The median candidates are the unique expected winners of the Borda rule and the $\lceil \frac{m}{2} \rceil$-approval rule (as well as $(\frac{m}{2}+1)$-approval if $m$ is even) under Conitzer's distribution.*

By Proposition 6 and Corollary 15, the Borda rule, $\lceil \frac{m}{2} \rceil$-approval, as well as all rules identified in Theorem 14 tend to elect the (weak) Condorcet winner(s).

**Corollary 16.** *Under Conitzer's distribution, the Borda rule, $\lceil \frac{m}{2} \rceil$-approval, and Condorcet-consistent rules asymptotically agree to elect the median candidates.*

As an illustration, when we apply Theorem 1 with Borda for $m = 5$, we have $\mathbb{P}_{\pi_C}(\mathcal{F}(\succ) = C^*) \geqslant 1 - 2e^{-\frac{9n}{3200}}$. For example, for $n = 2000$, we have a lower bound of 0.99 for the probability to elect the median candidate.

## 7 Unbiased Distributions

In this section, we aim to identify single-peaked distributions which do not favor any candidate by design, with respect to a given PSR. A preference distribution $\pi : \Pi^m \to [0,1]$ is said to be *unbiased* w.r.t. a given PSR $\mathcal{F}$ if all candidates are expected winners of $\mathcal{F}$ under $\pi$, i.e., $\mathbb{E}_\pi[S^{\mathcal{F}}(x)] = \mathbb{E}_\pi[S^{\mathcal{F}}(y)]$, for every $x, y \in M$. Note that the existence of an unbiased distribution w.r.t. a given PSR can be decided in polynomial time by solving a system of linear equations with real variables.

We first characterize the single-peaked distributions which are unbiased w.r.t. $k$-approval rules.

**Theorem 17.** *There exists an unbiased single-peaked distribution w.r.t. the $k$-approval rule iff $k$ divides $m$.*

*Sketch of proof.* If $k$ divides $m$, then there is an integer $q$ such that $m = k \cdot q$. We partition the candidates $M$ in $q$ groups of size $k$ where $X_j := \{x_{(j-1)k+1}, \ldots, x_{jk}\}$ for each $j \in [q]$, and $M = \bigsqcup_{j \in [q]} X_j$. For each group $X_j$, let $P_j$ denote the non-empty set of single-peaked preference orders where the $k$ candidates in $X_j$ are ranked among the first $k$ candidates, i.e., $P_j := \{\succ_i \in \Pi^m_{\rhd} : r_{\succ_i}(x) \leq k, \forall x \in X_j\}$. One can prove that the distribution $\pi$ such that $\sum_{\succ_i \in P_j} \pi(\succ_i) = \frac{1}{q}$ for each $j \in [q]$, and $\pi(\succ_i) = 0$ for all $\succ_i \in \Pi^m_{\rhd} \setminus \bigcup_{j \in [q]} P_j$ is unbiased w.r.t. $k$-approval. □

From Theorem 17, no single-peaked distribution can be unbiased w.r.t. $k$-approval, for any $k > m/2$ when $m > 2$, which includes the veto rule (i.e., $(m-1)$-approval). Alternatively, there exists a family of single-peaked distributions which are unbiased w.r.t. the plurality rule (i.e., 1-approval), including Conitzer's distribution. In addition, we show that Conitzer's distribution is unbiased only w.r.t. plurality, leading to the following statement.

**Proposition 18.** *Conitzer's distribution is unbiased w.r.t. a positional scoring rule $\mathcal{F}$ iff $\mathcal{F}$ is the plurality rule.*

*Sketch of proof.* Suppose that Conitzer's distribution $\pi_C$ is unbiased w.r.t. some PSR $\mathcal{F}$ defined by the positional score vector $\alpha$. Since all candidates are expected winners of $\mathcal{F}$, we have in particular $\mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_1)] = \mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_2)]$, which leads to $\alpha_m = \frac{2}{m} \cdot \alpha_2 + \frac{m-2}{m} \cdot \alpha_{m-1}$. Because $\alpha_2 \geq \cdots \geq \alpha_m$, it implies that $\alpha_2 = \cdots = \alpha_m$, and $\alpha_1 > \alpha_m$, thus $\mathcal{F}$ corresponds to the plurality rule. $\square$

In contrast, we prove that Walsh's distribution can never be unbiased because, no matter the chosen positional score vector, the expected score of a median candidate will always be strictly greater than the one of an extreme candidate in the single-peaked axis.

**Proposition 19.** *No PSR can make Walsh's distribution unbiased.*

*Proof sketch.* One can prove that, no matter the chosen positional score vector, the expected score of a median candidate will always be strictly greater than the one of an extreme candidate in the single-peaked axis. $\square$

We now consider a very degenerate distribution which only puts positive equal probability on the two extreme orders in the single-peaked domain.

**Definition 3** (Polarized distribution)**.** *The polarized single-peaked distribution $\pi : \Pi^m_{>} \to [0,1]$ is defined as:*

$$\pi(\succ_i) = \begin{cases} \frac{1}{2} & \textit{if } x_1 \succ_i \cdots \succ_i x_m \textit{ or } x_m \succ_i \cdots \succ_i x_1 \\ 0 & \textit{otherwise} \end{cases} .$$

Although it is degenerate, the polarized distribution is nevertheless symmetric and is the only single-peaked distribution which is unbiased w.r.t. the Borda rule.

**Theorem 20.** *A single-peaked distribution is unbiased w.r.t. the Borda rule iff it is the polarized distribution.*

*Proof.* The Borda rule is characterized by, e.g., the positional score vector $(m-1, m-2, \ldots, 1, 0)$. Under the polarized distribution, each candidate $x_j$ can be ranked either at position $j$ or at position $m-j+1$, with equal probability. It follows that the expected score of each candidate $x_j$ is equal to $\frac{1}{2}(m-j) + \frac{1}{2}(j-1) = \frac{1}{2}(m-1)$. Therefore, the polarized distribution is unbiased w.r.t. the Borda rule.

Let us now prove that no other distribution is unbiased w.r.t. the Borda rule. Suppose that there exists a single-peaked distribution $\pi$ which is unbiased w.r.t. the Borda rule. Observe that, globally, all the Borda scores that have been distributed to the candidates are equal to $\sum_{\succ_i \in \Pi^m_{>}} \pi(\succ_i) \cdot \sum_{x \in M}(m - r_{\succ_i}(x)) = \sum_{\succ_i \in \Pi^m_{>}} \pi(\succ_i) \cdot \frac{m(m-1)}{2} = \frac{m(m-1)}{2}$. Therefore, since all $m$ candidates must have the same expected score, it must be equal to $\frac{m-1}{2}$. Let us denote by $\Pi^m_{>}(1)$ and $\Pi^m_{>}(m)$ the set of single-peaked orders where candidate $x_1$ and $x_m$ are ranked last, respectively. We have $\Pi^m_{>} = \Pi^m_{>}(1) \sqcup \Pi^m_{>}(m)$. Candidates $x_1$ and $x_m$ get zero points in $\Pi^m_{>}(1)$ and $\Pi^m_{>}(m)$, respectively. Since the maximum number of points to get is $(m-1)$, for $x_1$ and $x_m$ to get an expected score of $\frac{m-1}{2}$, the distribution should be balanced between $\Pi^m_{>}(1)$ and $\Pi^m_{>}(m)$, i.e., we must have $\sum_{\succ_i \in \Pi^m_{>}(1)} \pi(\succ_i) = \sum_{\succ_i \in \Pi^m_{>}(m)} \pi(\succ_i) = \frac{1}{2}$. Moreover, for

$x_1$ and $x_m$ to reach an expected score of exactly $\frac{m-1}{2}$ on only half of the single-peaked orders, they must get $m-1$ points, i.e., be ranked at the first position, in the orders with positive probability in their half. Since both $x_1$ and $x_m$ are ranked first in exactly one single-peaked order, i.e., in the extreme orders $x_1 \succ_i x_2 \succ_i \cdots \succ_i x_m$ and $x_m \succ_i \cdots \succ_i x_2 \succ_i x_1$, respectively, $\pi$ must assign positive equal probability to exactly these two orders, leading to $\pi$ being the polarized distribution. □

# 8 Other Structured Distributions

Finally, we explore structured preference distributions other than single-peaked ones in order to determine whether similar results can be reached. In particular, we study unimodal distributions, including the famous Mallows' distributions [34], introduced in voting theory by Goldsmith et al. [27], and Pólya-Eggenberger urn [17] introduced in voting theory by Berg [5].

## 8.1 Unimodal Distributions

The Kendall tau distance evaluates the similarity between two preference orders by counting the number of pairwise comparisons on which the two orders disagree, i.e., $dist_{KT}(\succ_i, \succ_j) = |\{(x,y) \in M^2 : x \succ_i y \text{ and } y \succ_j x\}|$, for every $\succ_i, \succ_j \in \Pi^m$. The frequency of a preference order $\succ_i \in \Pi^m$ in a preference profile $\succ \in (\Pi^m)^n$ is denoted by $f(\succ_i, \succ)$. A preference profile $\succ \in (\Pi^m)^n$ is *unimodal* [13] if there exists a mode $\succ^* \in \Pi^m$, i.e., a reference preference order, such that $f(\succ_i, \succ) > f(\succ_j, \succ)$ iff $dist_{KT}(\succ^*, \succ_i) < dist_{KT}(\succ^*, \succ_j)$, for every pair of preference orders $\succ_i, \succ_j \in \succ$. *Positively discriminating* rules [13] are social welfare functions which always return the mode as the outcome of the election. Both PSRs and Condorcet-consistent rules are positively discriminating.

We adapt the definition of unimodal profile to distributions. A preference distribution $\pi : \Pi^m \to [0,1]$ is said to be *unimodal* if there exists a mode $\succ^* \in \Pi^m$ such that $\pi(\succ_i) > \pi(\succ_i')$ iff $dist_{KT}(\succ^*, \succ_i) < dist_{KT}(\succ^*, \succ_i')$, for every pair of preference orders $\succ_i, \succ_i' \in \Pi^m$. We consider independent and identical voter preference drawings. By using the Glivenko-Cantelli theorem [11], we deduce that any unimodal distribution will asymptotically generate a unimodal profile, where PSRs and Condorcet-consistent rules agree to select the winner of the mode.

**Corollary 21.** *Under unimodal distributions, all PSRs and Condorcet-consistent rules asymptotically agree to elect the first-ranked candidate of the mode.*

We go further and give a bound for the speed of convergence toward agreement in terms of election size.

**Proposition 22.** *For a unimodal preference distribution $\pi$, the probability that all PSRs and Condorcet-consistent rules agree is lower bounded by $B_\pi := 1 - 2exp(-2n\varepsilon^2)$, for $\varepsilon := \min_{\succ_i, \succ_j \in \Pi^m} |\pi(\succ_i) - \pi(\succ_j)|$.*

A typical example of unimodal distributions are *Mallows' distributions* $\mathcal{M}^{\phi,\sigma}$, for given $\sigma \in \Pi^m$ and $\phi \in [0,1]$, defined by $\mathbb{P}_{\mathcal{M}^{\phi,\sigma}}(\succ_i) = \frac{1}{Z}\phi^{dist_{KT}(\succ_i,\sigma)}$ where $Z = \sum_{\succ_i \in \Pi^m} \phi^{dist_{KT}(\succ_i,\sigma)}$. Mallows' distributions are unimodal when $\phi < 1$. We give below an example of the speed of convergence under Mallows' distributions.

**Example 1.** *Under a Mallows' distribution $\pi^{\phi,\sigma}$, we get $\varepsilon = \phi^k \cdot (1-\phi)$ with $k := \max_{\succ} dist_{KT}(\sigma, \succ_i)$ and thus the bound for agreement is $B_{\pi^{\phi,\sigma}} = 1 - 2exp(-2n(\frac{\phi^k \cdot (1-\phi)}{Z})^2)$.*

*If $\phi = 0.1$, $m = 3$ (then $k = 3$) and $n = 2{,}000{,}000$, we have $B_{\pi^{\phi,\sigma}} = 0.92$. If $\phi = 0.9$, $m = 3$ and $n = 400$, $B_{\pi^{\phi,\sigma}} = 0.97$. When more weight is given to orders close to the mode, voting rules agree faster than when the Mallows' distribution gets closer to impartial culture (i.e., $\phi = 1$).*

### 8.2 Pólya-Eggenberger Urn

In the Pólya-Eggenberger urn model, we consider an urn initially containing $m!$ balls representing the $m!$ different preference orders from $\Pi^m$, i.e., each $\ell^{\text{th}}$ preference order from $\Pi^m$ is initially drawn with probability $\beta_\ell = \frac{1}{m!}$. To draw our preference profile $\succ$ with $n$ voters, for each voter, we draw a ball and assign to the voter the corresponding preference order and put it back into the urn with $R$ additional balls with the same preference order and $R > 0$. We will assume $R = m! \cdot r$, for a given parameter $r$.

The following result generalizes the asymptotic result from Gehrlein [20] for three candidates under impartial anonymous culture (when $R = 1$).

**Proposition 23.** *Under the Pólya-Eggenberger urn culture, the probability that all PSRs asymptotically agree is lower bounded by $\frac{1}{2}$ if $r < \frac{2}{3}$ and $m = 3$, and by $\frac{1}{4}$ if $r < \frac{1}{6}$ and $m = 4$.*

We now analyze the agreement between plurality and Borda rule.

**Proposition 24.** *Under the Pólya-Eggenberger urn culture, the probability that plurality and Borda asymptotically agree is lower bounded by $\frac{3}{4}$ if $r < \frac{2}{3}$ and $m = 3$, and by $\frac{3}{5}$ if $r < \frac{1}{6}$ and $m = 4$.*

To give a comparison, under Walsh's distribution, for the agreement of plurality and the Borda rule to the election of median candidates $C^*$, we have a lower bound given by the plurality rule $\mathcal{F}_1$ (by Proposition 11) which is as follows: if $m = 4$, $\mathbb{P}_{\pi_W}(\mathcal{F}_1(\succ) = C^*) \geqslant 1 - 2e^{-\frac{n}{32}}$ is larger than $\frac{3}{5}$ when $n \geqslant 52$. Therefore, we are able to compare lower bounds and tell that the lower bound of Pólya-Eggenberger urn for $r < \frac{1}{6}$ is reached from $n \geq 52$ for the lower bound of Walsh's distribution.

We finally prove a positive probability of disagreement asymptotically for every pair of PSRs.

**Proposition 25.** *If the election is drawn with a Pólya-Eggenberger urn culture with $R < 4$ then every pair of positional scoring rules $\mathcal{F}_1$ and $\mathcal{F}_2$ asymptotically disagree with a positive probability, i.e., $\lim_{n\to\infty} \mathbb{P}(\mathcal{F}_1(\succ) \neq \mathcal{F}_2(\succ)) > 0$.*

This result means that any pair of positional scoring rules will disagree on a nonempty set asymptotically. Thus, we cannot achieve the same type of convergence results as in single-peaked distributions.

## 9 Conclusion

We have studied the probability of agreement of different voting rules under two single-peaked cultures, classically used for experiments in social choice, namely Walsh's and Conitzer's distributions. These distributions tend to favor the election of median candidate(s) in the single-peaked axis, and these candidates also turn out to be (weak) Condorcet winner(s), implying the agreement of several positional scoring rules (PSRs) with all Condorcet-consistent rules. This (weak) Condorcet efficiency holds in general for all symmetric single-peaked distributions, which are natural distributions for experiments when no additional information other than the single-peaked axis is available. We nevertheless observe that Conitzer's distribution is less biased toward the median candidates because it happens to be unbiased w.r.t. one PSR (namely plurality), contrary to Walsh's distribution. While these single-peaked distributions enable fast convergence to agreement, this is also the case for other structured distributions, such as unimodal ones, where the agreement is very general among voting rules and convergence is rapid. This behavior cannot be extended to Pólya-Eggenberger urns where the probability of disagreement is non-negligible, even if it remains high in some particular cases.

Our findings highlight that particular attention should be taken when using voting cultures for experiments in social choice. Indeed, since we identify cultures in which the agreement of different voting rules rapidly agree as the number of voters increases, conclusions drawn from experiments testing different voting rules for a problem should be interpreted with caution. One could imagine very different conclusions about a problem, not because of the problem itself, but because of the culture used: impartial cultures versus single-peaked cultures, for example. The take-home message of our results is to warn the community to be careful when using such cultures in experiments because some interpretations could be biased by the fact that voting rules mostly agree under these cultures.

Future work could consider bounds on the probability to agree in finite elections with Pólya-Eggenberger urn. The difficulty, however, lies in the dependent structure of this distribution. One idea could also be to consider nearly single-peaked distributions to bridge the gap between impartial and single-peaked cultures and be closer to real political elections. Furthermore, when voting rules asymptotically agree, we might conjecture that the probability of not satisfying certain axioms might also decrease as the election size increases. Finally, the same study could be done in a strategic model where voters can manipulate [38].

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# Technical Appendix

## A The Model

In the case of $m = 2$ candidates, all PSRs coincide and gaining one point for a candidate in a PSR is equivalent for this candidate to be ranked before the other candidate, breaking the gap between absolute and relative evaluation of candidates. In that case, majority voting can appear as the only reasonable voting rule [36]. Therefore, given the focus of our paper, we reasonably assume that $m > 2$.

**Lemma 26** (Hoeffding [29]). *Let $X_k$ be some independent real random variables, and $(a_k)_{k\in[n]}$ and $(b_k)_{k\in[n]}$ two real sequences such that for every $k \in [n]$, we have $a_k < b_k$ and $\mathbb{P}(a_k \leqslant X_k \leqslant b_k) = 1$. Then, for every $t > 0$, $\mathbb{P}(S_n - \mathbb{E}(S_n) \geqslant t) \leqslant e^{\frac{-2t^2}{\sum_{k=1}^n (b_k - a_k)^2}}$, where $S_n = \sum_{k=1}^n X_k$.*

**Lemma 27** (Hoeffding [29]). *Let $X_k$ be some independent real random variables, and $(a_k)_{k\in[n]}$ and $(b_k)_{k\in[n]}$ two real sequences such that for every $k \in [n]$, we have $a_k < b_k$ and $\mathbb{P}(a_k \leqslant X_k \leqslant b_k) = 1$. Then, for every $t > 0$, $\mathbb{P}(S_n - \mathbb{E}(S_n) \leqslant -t) \leqslant e^{\frac{-2t^2}{\sum_{k=1}^n (b_k - a_k)^2}}$, where $S_n = \sum_{k=1}^n X_k$.*

**Theorem 1.** *Consider a positional scoring rule $\mathcal{F}$ defined by a score vector $\alpha$, and a preference distribution $\pi$ over the set of candidates $M$. When the set of expected winners, defined as $\mathcal{W}_\pi(\mathcal{F}) = \arg\max_{x\in M} \mathbb{E}_\pi[S^{\mathcal{F}}(x)]$, is a singleton, i.e., $\mathcal{W}_\pi(\mathcal{F}) = \{x\}$, the probability that $\mathcal{F}$ elects $x$ satisfies*

$$\mathbb{P}_\pi(x \in \mathcal{F}(\succ)) \geq L_\pi(\mathcal{F}),$$

*where:*

$$L_\pi(\mathcal{F}) := 1 - 2 \cdot \max_{y\in M\setminus\mathcal{W}_\pi(\mathcal{F})} \exp\left(\frac{-2n \cdot \left(\mu_\pi^{\mathcal{F}}(y) - \mathbb{E}_\pi[S^{\mathcal{F}}(y)]\right)^2}{\left(\max_j \alpha_j - \min_j \alpha_j\right)^2}\right)$$

$$\textit{and } \mu_\pi^{\mathcal{F}}(y) := \frac{\max_{x\in M} \mathbb{E}_\pi[S^{\mathcal{F}}(x)] + \mathbb{E}_\pi[S^{\mathcal{F}}(y)]}{2}$$

*Proof.* Let $\mathbb{E}_\pi[S^{\mathcal{F}}(y)]_i$ be the expected score of candidate $y$ with rule $\mathcal{F}$ for voter $i$. Let $\mathcal{W}_\pi(\mathcal{F}) = \{x\}$, we have:

$$\mathbb{P}_\pi(x \in \mathcal{F}(\succ)) = \mathbb{P}_\pi[\forall y \neq x, \sum_{i=1}^n S^{\mathcal{F}}(x)_i > \sum_{i=1}^n S^{\mathcal{F}}(y)_i]$$

Using Bonferroni's inequality we get:

$$\mathbb{P}_\pi[\forall y \neq x, \sum_{i=1}^n S^{\mathcal{F}}(x)_i > \sum_{i=1}^n S^{\mathcal{F}}(y)_i]$$

$$\geqslant \sum_{x\neq y} \mathbb{P}_\pi[\sum_{i=1}^n S^{\mathcal{F}}(x)_i > \sum_{i=1}^n S^{\mathcal{F}}(y)_i] - (m-2)$$

$$\geqslant (m-1) \cdot \min_{y\neq x} \mathbb{P}_\pi[\sum_{i=1}^n S^{\mathcal{F}}(x)_i > \sum_{i=1}^n S^{\mathcal{F}}(y)_i] - (m-2)$$

Let us now compute a lower bound for $\mathbb{P}_\pi[\sum_{i=1}^n S^{\mathcal{F}}(x)_i > \sum_{i=1}^n S^{\mathcal{F}}(y)_i]$. Using again Bonferroni's inequality we have:

$$\mathbb{P}_\pi[\sum_{i=1}^n S^{\mathcal{F}}(x)_i > \sum_{i=1}^n S^{\mathcal{F}}(y)_i]$$

$$\geqslant \mathbb{P}_\pi[\sum_{i=1}^{n} S^{\mathcal{F}}(x)_i < n \cdot \mu_\pi^{\mathcal{F}}(y)] + \mathbb{P}_\pi[\sum_{i=1}^{n} S^{\mathcal{F}}(x)_i > n \cdot \mu_\pi^{\mathcal{F}}(y)] - 1$$

Now, we work on each term separately,

$$\mathbb{P}_\pi[\sum_{i=1}^{n} S^{\mathcal{F}}(x)_i < n \cdot \mu_\pi^{\mathcal{F}}(y)]$$

$$= 1 - \mathbb{P}_\pi[\sum_{i=1}^{n} S^{\mathcal{F}}(x)_i \geqslant n \cdot \mu_\pi^{\mathcal{F}}(y)]$$

Using the first Hoeffding's inequality (Lemma 26) with $a_i = \min_y \alpha_y$ and $b_i = \max_y \alpha_y$,

$$\mathbb{P}_\pi[\sum_{i=1}^{n} S^{\mathcal{F}}(x)_i \geqslant n \cdot \mu_\pi^{\mathcal{F}}(y)]$$

$$= \mathbb{P}_\pi[\sum_{i=1}^{n} S^{\mathcal{F}}(x)_i - n \cdot \mathbb{E}_\pi[S^{\mathcal{F}}(y)]_i \geqslant n \cdot \mu_\pi^{\mathcal{F}}(y) - n \cdot \mathbb{E}_\pi[S^{\mathcal{F}}(y)]_i]$$

$$\leqslant e^{\frac{-2n(\mu_\pi^{\mathcal{F}}(y) - \mathbb{E}_\pi[S^{\mathcal{F}}(y)]_i)^2}{(\max_y \alpha_y - \min_y \alpha_y)^2}}$$

We reproduce the exact same reasoning for the second term $\mathbb{P}_\pi[\sum_{i=1}^{n} S^{\mathcal{F}}(x)_i > n \cdot \mu_\pi^{\mathcal{F}}(y)]$ but we use the second Hoeffding's inequality (Lemma 27). We summarize and find:

$$\mathbb{P}_\pi[\sum_{i=1}^{n} S^{\mathcal{F}}(x)_i > \sum_{i=1}^{n} S^{\mathcal{F}}(y)_i]$$

$$\leqslant 1 - e^{\frac{-2n(\mu_\pi^{\mathcal{F}}(y) - \mathbb{E}_\pi[S^{\mathcal{F}}(y)]_i)^2}{(\max_y \alpha_y - \min_y \alpha_y)^2}} - e^{\frac{-2n(\mu_\pi^{\mathcal{F}}(y) - \mathbb{E}_\pi[S^{\mathcal{F}}(x)]_i)^2}{(\max_y \alpha_y - \min_y \alpha_y)^2}}$$

Finally, we get:

$$\mathbb{P}_\pi(x \in \mathcal{F}(\succ))$$

$$\geqslant 1 - 2 \cdot \max_{y \neq x} e^{\frac{-2n \cdot (\mu_\pi^{\mathcal{F}}(y) - \mathbb{E}_\pi[S^{\mathcal{F}}(y)])^2}{(\max_y \alpha_y - \min_y \alpha_y)^2}}$$

□

**Corollary 2.** *For two positional scoring rules $\mathcal{F}_1$ and $\mathcal{F}_2$ whose expected winner set under a preference distribution $\pi$ is the same, i.e., $C := \mathcal{W}_\pi(\mathcal{F}_1) = \mathcal{W}_\pi(\mathcal{F}_2)$, the probability of their agreement for electing the same unique candidate from $C$ is such that: $\mathbb{P}_\pi(\mathcal{F}_1(\succ) = \mathcal{F}_2(\succ)) \geqslant \min\{L_\pi(\mathcal{F}_1), L_\pi(\mathcal{F}_2)\}$.*

*Proof.* We apply twice Theorem 1 and deduce that both rules $\mathcal{F}_1$ and $\mathcal{F}_2$ have to agree on the same outcome with a probability higher than the minimum of both lower bounds. □

## B The Single-Peaked Domain

**Observation 3.** *Candidate $x_j$ can never be ranked at a position $k > \max\{j, m-j+1\}$ in a single-peaked order.*

*Proof.* If $k > m - j + 1$ (resp., $k > j$), then it means that there are not enough positions between position $k$ and position $m$ to place at least all candidates $x_1, \ldots, x_{j-1}$ (resp., $x_{j+1}, \ldots, x_m$), which is necessary in order to rank $x_j$ at position $k$, by single-peakedness. It follows that, under such a condition, no single-peaked preference order can rank $x_j$ at position $k$. □

**Lemma 5.** *For every median candidate $x_c \in C^*$ and any other candidate $x_j \in M \setminus C^*$, there exists an index $\gamma_m(j) \in [\max\{j, m-j+1\}]$ such that $\mathscr{D}_m(c,k) \geq \mathscr{D}_m(j,k)$ for every $1 \leq k \leq \gamma_m(j)$ and $\mathscr{D}_m(j,k) > \mathscr{D}_m(c,k)$ for every $\gamma_m(j) < k \leq \max\{j, m-j+1\}$.*

*proof.* Let us compare a median candidate $x_c \in C^*$ and another candidate $x_j \in M \setminus C^*$ where, w.l.o.g., $c := \lceil \frac{m}{2} \rceil$ and $j < c$. Our goal is to compare $\mathscr{D}_m(c,k)$ and $\mathscr{D}_m(j,k)$ for a given position $k \in [m-j+1]$, and thus, by Lemma 4, to compare $\binom{m-k}{c-1} + \binom{m-k}{c-k}$ and $\binom{m-k}{j-1} + \binom{m-k}{j-k}$. Observe that $\binom{m-k}{c-k} = \binom{m-k}{m-c} = \binom{m-k}{\lfloor \frac{m}{2} \rfloor}$, and thus $\binom{m-k}{c-k} = \binom{m-k}{c-\mathbb{1}_{\{m \text{ odd}\}}}$, implying that $\binom{m-k}{c-k} = \binom{m-k}{c-1}$ when $m$ is odd.

Let us recall that, when $n$ is fixed, the binomial coefficient $\binom{n}{\ell}$ is strictly increasing from $\ell = 0$ to $\ell = \frac{n}{2}$ and then strictly decreasing from $\ell = \frac{n}{2}$ to $\ell = n$ (in case $n$ is odd, the two maximal values are taken for $\ell = \lfloor \frac{n}{2} \rfloor$ and $\ell = \lceil \frac{n}{2} \rceil$, so it is fine to simply consider that the closest $\ell$ is to $\frac{n}{2}$, the biggest the value $\binom{n}{\ell}$. In our case, we have $n = m-k$, therefore the maximal value of $\binom{m-k}{\ell}$ is taken for $\ell$ the closest to $\frac{m-k}{2}$. Observe that $\frac{m-k}{2} \geq c-k$. It follows that $\binom{m-k}{c-k} \geq \binom{m-k}{j-k}$, since we are in the increasing part. Moreover, the maximal value $\ell = \frac{m-k}{2}$ is always closer to $c-1$ than to $j-k$: if $\frac{m-k}{2} \geq c-1$ it is obvious and if $\frac{m-k}{2} < c-1$, then supposing $j-k$ is closer would imply $\frac{m-k}{2} - j + k < c - 1 - \frac{m-k}{2}$ and thus, because $j < c = \lceil \frac{m}{2} \rceil$, we would have $c > m-j+1 \geq \lfloor \frac{m}{2} \rfloor + 2$, a contradiction. It follows that $\binom{m-k}{c-1} \geq \binom{m-k}{j-k}$.

First observe that if $k \leq c-j+1$, then we have $j-1 \leq c-k$ and thus $j-k \leq j-1 \leq c-k \leq c-1$. Since $c-k \leq \frac{m-k}{2}$, it follows that $\binom{m-k}{c-k} \geq \binom{m-k}{j-1}$, and thus, since $\binom{m-k}{c-1} \geq \binom{m-k}{j-k}$, we have that $\mathscr{D}_m(c,k) \geq \mathscr{D}_m(j,k)$.

By Observation 3, we know that $\mathscr{D}_m(j,k) = 0$ iff $k > \max\{j, m-j+1\}$, therefore $\mathscr{D}_m(c,k) = 0$ when $k > \lfloor \frac{m}{2} \rfloor + 1$ and $\mathscr{D}_m(j,k) = 0$ when $j > m-j+1$. It follows that $\mathscr{D}_m(j,k) > \mathscr{D}_m(c,k)$ for all $\lfloor \frac{m}{2} \rfloor + 1 < k \leq m-j+1$.

To summarize, now we know that when $k \leq c-j+1$, we have $\mathscr{D}_m(c,k) \geq \mathscr{D}_m(j,k)$ and when $k > \lfloor \frac{m}{2} \rfloor + 1$, we have $\mathscr{D}_m(j,k) > \mathscr{D}_m(c,k)$. It means that there exists an index $k$ such that $\mathscr{D}_m(c,k) \geq \mathscr{D}_m(j,k)$ and $\mathscr{D}_m(j,k+1) > \mathscr{D}_m(c,k+1)$. Let us consider the greatest such index $k_0$ as our base case and suppose, by induction, that $\mathscr{D}_m(c,k') \geq \mathscr{D}_m(j,k')$ for all indices $k'$ such that $k \leq k' \leq k_0$ for a given index $k \leq k_0$. We will prove that if $\mathscr{D}_m(c,k) \geq \mathscr{D}_m(j,k)$ holds, then $\mathscr{D}_m(c,k-1) \geq \mathscr{D}_m(j,k-1)$ also holds, which will be sufficient to prove our statement about the existence of a unique threshold $\gamma_m(j)$ to distinguish when $\mathscr{D}_m(c,k) \geq \mathscr{D}_m(j,k)$ and when $\mathscr{D}_m(j,k) > \mathscr{D}_m(c,k)$.

Suppose that $\mathscr{D}_m(c,k) \geq \mathscr{D}_m(j,k)$ holds for a given position $k$. By Lemma 4, it means that $\binom{m-k}{c-1} + \binom{m-k}{c-\mathbb{1}_{\{m \text{ odd}\}}} \geq \binom{m-k}{j-1} + \binom{m-k}{j-k}$. By Pascal's identity, we thus have $\binom{m-k+1}{c-1} - \binom{m-k}{c-2} + \binom{m-k+1}{c-\mathbb{1}_{\{m \text{ odd}\}}} - \binom{m-k}{c-1-\mathbb{1}_{\{m \text{ odd}\}}} \geq \binom{m-k+1}{j-1} - \binom{m-k}{j-2} + \binom{m-k+1}{j-k+1} - \binom{m-k}{j-k+1}$, which is equivalent to $\binom{m-k+1}{c-1} + \binom{m-k+1}{c-\mathbb{1}_{\{m \text{ odd}\}}} \geq \binom{m-k+1}{j-1} + \binom{m-k+1}{j-k+1} + \binom{m-k}{c-2} + \binom{m-k}{c-1-\mathbb{1}_{\{m \text{ odd}\}}} - \binom{m-k}{j-2} - \binom{m-k}{j-k+1}$. If $\binom{m-k}{c-2} + \binom{m-k}{c-1-\mathbb{1}_{\{m \text{ odd}\}}} - \binom{m-k}{j-2} - \binom{m-k}{j-k+1} \geq 0$ holds, then we have $\binom{m-k+1}{c-1} + \binom{m-k+1}{c-\mathbb{1}_{\{m \text{ odd}\}}} \geq \binom{m-k+1}{j-1} + \binom{m-k+1}{j-k+1}$ and our claim follows, i.e., we have $\mathscr{D}_m(c,k-1) \geq \mathscr{D}_m(j,k-1)$. Let us thus assume, for the sake of contradiction, that $\binom{m-k}{c-2} + \binom{m-k}{c-1-\mathbb{1}_{\{m \text{ odd}\}}} - \binom{m-k}{j-2} - \binom{m-k}{j-k+1} < 0$.

$$\begin{aligned}
&\binom{m-k}{c-2} + \binom{m-k}{c-1-\mathbb{1}_{\{m \text{ odd}\}}} < \binom{m-k}{j-2} + \binom{m-k}{j-k+1} \\
\Leftrightarrow &\binom{m-k}{c-1} \cdot \frac{c-1}{m-k-c+2} + \binom{m-k}{c-\mathbb{1}_{\{m \text{ odd}\}}} \cdot \frac{c-\mathbb{1}_{\{m \text{ odd}\}}}{m-k-c+1+\mathbb{1}_{\{m \text{ odd}\}}} \\
&< \binom{m-k}{j-1} \cdot \frac{j-1}{m-k-j+2} + \binom{m-k}{j-k} \cdot \frac{m-j}{j-k+1}
\end{aligned}$$

$$\Leftrightarrow \frac{c-1}{m-k-c+2} \cdot \left( \binom{m-k}{c-1} + \binom{m-k}{c - \mathbb{1}_{\{m \text{ odd}\}}} \right) +$$
$$\binom{m-k}{c} \cdot \frac{(m-k+1) \cdot \mathbb{1}_{\{m \text{ even}\}}}{(m-k-c+2)(m-k-c+1)} <$$
$$\binom{m-k}{j-1} \cdot \frac{j-1}{m-k-j+2} + \binom{m-k}{j-k} \cdot \frac{m-j}{j-k+1}$$

Since we have assumed $\mathscr{D}_m(c,k) \geq \mathscr{D}_m(j,k)$, it follows that:

$$\frac{c-1}{m-k-c+2} \cdot \left( \binom{m-k}{j-1} + \binom{m-k}{j-k} \right) +$$
$$\binom{m-k}{c} \cdot \frac{(m-k+1) \cdot \mathbb{1}_{\{m \text{ even}\}}}{(m-k-c+2)(m-k-c+1)} <$$
$$\binom{m-k}{j-1} \cdot \frac{j-1}{m-k-j+2} + \binom{m-k}{j-k} \cdot \frac{m-j}{j-k+1}$$
$$\Leftrightarrow \binom{m-k}{j-1} \cdot \left( \frac{c-1}{m-k-c+2} - \frac{j-1}{m-k-j+2} \right) +$$
$$\binom{m-k}{c} \cdot \frac{(m-k+1) \cdot \mathbb{1}_{\{m \text{ even}\}}}{(m-k-c+2)(m-k-c+1)} <$$
$$\binom{m-k}{j-k} \cdot \left( \frac{m-j}{j-k+1} - \frac{c-1}{m-k-c+2} \right)$$
$$\Leftrightarrow \binom{m-k}{j-1} \cdot \left( \frac{c-1}{m-k-c+2} - \frac{j-1}{m-k-j+2} \right) +$$
$$\binom{m-k}{j-1} \cdot \frac{\prod_{p=1}^{c-j+1}(m-k-c+p)}{\prod_{p=1}^{c-j+1}(j-1+p)} \cdot \frac{(m-k+1) \cdot \mathbb{1}_{\{m \text{ even}\}}}{(m-k-c+2)(m-k-c+1)} <$$
$$\binom{m-k}{j-1} \cdot \frac{\prod_{p=1}^{k-1}(j-k+p)}{\prod_{p=1}^{k-1}(m-k-j+1+p)} \cdot \left( \frac{m-j}{j-k+1} - \frac{c-1}{m-k-c+2} \right)$$
$$\Leftrightarrow \left( \frac{c-1}{m-k-c+2} - \frac{j-1}{m-k-j+2} \right) +$$
$$\frac{\prod_{p=1}^{c-j}(m-k-c+1+p)}{\prod_{p=1}^{c-j+1}(j-1+p)} \cdot \frac{(m-k+1) \cdot \mathbb{1}_{\{m \text{ even}\}}}{(m-k-c+2)} <$$
$$\frac{\prod_{p=1}^{k-1}(j-k+p)}{\prod_{p=1}^{k-1}(m-k-j+1+p)} \cdot \left( \frac{m-j}{j-k+1} - \frac{c-1}{m-k-c+2} \right)$$
$$\Leftrightarrow \frac{(c-j)(m-k+1)}{(m-k-c+2)(m-k-j+2)} +$$
$$\frac{\prod_{p=1}^{c-j}(m-k-c+1+p)}{\prod_{p=1}^{c-j+1}(j-1+p)} \cdot \frac{(m-k+1) \cdot \mathbb{1}_{\{m \text{ even}\}}}{(m-k-c+2)} <$$
$$\frac{\prod_{p=1}^{k-1}(j-k+p)}{\prod_{p=1}^{k-1}(m-k-j+1+p)} \cdot \frac{(m-k+1)(m-c-j+1)}{(j-k+1)(m-k-c+2)}$$
$$\Leftrightarrow \frac{(c-j)}{(m-k-j+2)} + \frac{\prod_{p=1}^{c-j}(m-k-c+1+p)}{\prod_{p=1}^{c-j+1}(j-1+p)} \cdot \mathbb{1}_{\{m \text{ even}\}} <$$

$$
\begin{aligned}
&\frac{\prod_{p=1}^{k-1}(j-k+p)}{\prod_{p=1}^{k-1}(m-k-j+1+p)} \cdot \frac{(m-c-j+1)}{(j-k+1)} \\
\Leftrightarrow &(c-j) + \frac{\prod_{p=1}^{c-j+1}(m-k-c+1+p)}{\prod_{p=1}^{c-j+1}(j-1+p)} \cdot \mathbb{1}_{\{m \text{ even}\}} < \\
&\frac{\prod_{p=2}^{k-1}(j-k+p)}{\prod_{p=2}^{k-1}(m-k-j+1+p)} \cdot (m-c-j+1) \\
\Leftrightarrow &(c-j) \cdot \frac{\prod_{p=2}^{k-1}(m-k-j+1+p)}{\prod_{p=2}^{k-1}(j-k+p)} + \\
&\frac{\prod_{p=1}^{c-j+1}(m-k-c+1+p)}{\prod_{p=1}^{c-j+1}(j-1+p)} \cdot \frac{\prod_{p=2}^{k-1}(m-k-j+1+p)}{\prod_{p=2}^{k-1}(j-k+p)} \cdot \mathbb{1}_{\{m \text{ even}\}} \\
&< (m-c-j+1) \\
\Leftrightarrow &(c-j) \cdot \frac{\prod_{p=2}^{k-1}(m-k-j+1+p)}{\prod_{p=2}^{k-1}(j-k+p)} + \\
&\frac{\prod_{p=1}^{k+c-j-1}(m-k-c+1+p)}{\prod_{p=1}^{k+c-j-1}(j-k+1+p)} \cdot \mathbb{1}_{\{m \text{ even}\}} < (m-c-j+1)
\end{aligned}
$$

Since $j < c = \lceil \frac{m}{2} \rceil$, we have $m-j+1 > j$ and $m-c \geq j$, therefore $\frac{\prod_{p=2}^{k-1}(m-k-j+1+p)}{\prod_{p=2}^{k-1}(j-k+p)} > 1$ and $\frac{\prod_{p=1}^{k+c-j-1}(m-k-c+1+p)}{\prod_{p=1}^{k+c-j-1}(j-k+1+p)} \geq 1$. It follows that $(c-j) + \mathbb{1}_{\{m \text{ even}\}} < (m-c-j+1)$, whereas $m-c+1 = \lfloor \frac{m}{2} \rfloor + 1 = c + \mathbb{1}_{\{m \text{ even}\}}$, a contradiction. $\square$

## C The Walsh's Distribution

**Proposition 8.** *For every PSR $\mathcal{F}$, the median candidates always belong to the expected winners of $\mathcal{F}$ under Walsh's distribution, i.e., $C^* \subseteq \mathcal{W}_{\pi_W}(\mathcal{F})$.*

*Proof.* When comparing the expected score of a candidate $c \in C^*$ with the one of any other candidate $x_j \in M \setminus C^*$, we can restrict our attention, w.l.o.g., to the median candidate $x_c := x_{\lceil \frac{m}{2} \rceil} \in C^*$ and to any candidate $x_j$ such that $j < \lceil \frac{m}{2} \rceil$ (by symmetry w.r.t. the single-peaked axis). The expected score of a candidate $x_j$, for the Walsh's distribution and a PSR $\mathcal{F}$ characterized by the positional score vector $\alpha$, is given by $\mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x_j)] = \sum_{k=1}^{m} \mathbb{P}_{\pi_W}(j,k) \cdot \alpha_k = \sum_{k=1}^{m} \frac{\mathscr{D}_m(j,k)}{2^{m-1}} \cdot \alpha_k$. By the fact that $j < c = \lceil \frac{m}{2} \rceil$, we have $\max\{j, m-j+1\} = m-j+1$ and $\max\{c, m-c+1\} = \lfloor \frac{m}{2} \rfloor + 1$. And thus, by Observation 3, $\mathscr{D}_m(c,k) = 0$ for every $k > \lfloor \frac{m}{2} \rfloor + 1$ and $\mathscr{D}_m(j,k) = 0$ for every $k > m-j+1 > \lfloor \frac{m}{2} \rfloor + 1$. Therefore, we have $\mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x_j)] = \sum_{k=1}^{m-j+1} \frac{\mathscr{D}_m(j,k)}{2^{m-1}} \cdot \alpha_k$ and $\mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x_c)] = \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor + 1} \frac{\mathscr{D}_m(j,k)}{2^{m-1}} \cdot \alpha_k$ Let us compare the expected scores of both candidates:

$$
\begin{aligned}
&\mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x_c)] - \mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x_j)] \\
=&\sum_{k=1}^{\lfloor \frac{m}{2} \rfloor+1} \frac{\mathscr{D}_m(c,k)}{2^{m-1}} \cdot \alpha_k - \sum_{k=1}^{m-j+1} \frac{\mathscr{D}_m(j,k)}{2^{m-1}} \cdot \alpha_k \\
=&\frac{1}{2^{m-1}} \left( \sum_{k=1}^{\gamma_m(j)} (\mathscr{D}_m(c,k) - \mathscr{D}_m(j,k)) \cdot \alpha_k + \right. \\
&\left. \sum_{k=\gamma_m(j)+1}^{m-j+1} (\mathscr{D}_m(c,k) - \mathscr{D}_m(j,k)) \cdot \alpha_k \right) \\
\geq&\frac{\alpha_{\gamma_m(j)}}{2^{m-1}} \left( \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor+1} \mathscr{D}_m(c,k) - \sum_{k=1}^{m-j+1} \mathscr{D}_m(j,k) \right) \\
=&0
\end{aligned}
$$

The inequality comes from the fact that, by Lemma 5, $\mathscr{D}_m(c,k) \geq \mathscr{D}_m(j,k)$, for every $k \in [\gamma_m(j)]$ and $\mathscr{D}_m(c,k) < \mathscr{D}_m(j,k)$ for every $\gamma_m(j) < k \leq m-j+1$, and that $\alpha_1 \geq \cdots \geq \alpha_\gamma \geq \cdots \geq \alpha_m$. The last equality to 0 is because $\sum_{k=1}^{\lfloor \frac{m}{2} \rfloor+1} \mathscr{D}_m(c,k) = \sum_{k=1}^{m-j+1} \mathscr{D}_m(j,k) = 2^{m-1}$.

Hence, the expected score of a median candidate $x_c$ is always at least as good as the expected score of any other candidate, which completes the proof. □

**Theorem 9.** *The median candidates are the unique expected winners of a PSR $\mathcal{F}$ under Walsh's distribution, i.e., $\mathcal{W}_{\pi_W}(\mathcal{F}) = C^*$, iff $\mathcal{F}$ is first-prioritizing.*

*Proof.* Consider first a PSR $\mathcal{F}$ characterized by a positional score vector $\alpha$ such that there exists an index $\ell \in [\lfloor \frac{m}{2} \rfloor + 1]$ for which $\alpha_\ell > \alpha_{\ell+1}$. Let us compare, w.l.o.g., the median candidate $x_c$ with $c := \lceil \frac{m}{2} \rceil$ and a candidate $x_j$ such that $1 \leq j < c$ where, by definition, $x_j \in M \setminus C^*$. By the fact that $j < c = \lceil \frac{m}{2} \rceil$, we have $\max\{j, m-j+1\} = m-j+1$ and $\max\{c, m-c+1\} = \lfloor \frac{m}{2} \rfloor + 1$. And thus, by Observation 3, $\mathscr{D}_m(c,k) = 0$ for every $k > \lfloor \frac{m}{2} \rfloor + 1$ and $\mathscr{D}_m(j,k) = 0$ for every $k > m-j+1 > \lfloor \frac{m}{2} \rfloor + 1$. Let us compare the expected scores of both candidates:

$$
\begin{aligned}
&\mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x_c)] - \mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x_j)] \\
=&\sum_{k=1}^{\lfloor \frac{m}{2} \rfloor+1} \frac{\mathscr{D}_m(c,k)}{2^{m-1}} \cdot \alpha_k - \sum_{k=1}^{m-j+1} \frac{\mathscr{D}_m(j,k)}{2^{m-1}} \cdot \alpha_k \\
=&\frac{1}{2^{m-1}} \left( \sum_{k=1}^{\gamma_m(j)} (\mathscr{D}_m(c,k) - \mathscr{D}_m(j,k)) \cdot \alpha_k + \right. \\
&\left. \sum_{k=\gamma_m(j)+1}^{m-j+1} (\mathscr{D}_m(c,k) - \mathscr{D}_m(j,k)) \cdot \alpha_k \right) \\
>&\frac{\alpha_{\gamma_m(j)}}{2^{m-1}} \left( \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor+1} \mathscr{D}_m(c,k) - \sum_{k=1}^{m-j+1} \mathscr{D}_m(j,k) \right) \\
=&0
\end{aligned}
$$

The inequality comes from the fact that, by Lemma 5, $\mathscr{D}_m(c,k) \geq \mathscr{D}_m(j,k)$, for every $k \in [\gamma_m(j)]$ and $\mathscr{D}_m(c,k) < \mathscr{D}_m(j,k)$ for every $\gamma_m(j) < k \leq m-j+1$, and that $\alpha_1 \geq \cdots \geq \alpha_{\gamma_m(j)} \geq \cdots \geq \alpha_m$. This inequality is strict because there exists an index $\ell$ such that $1 \leq \ell \leq \lfloor \frac{m}{2} \rfloor + 1 < m-j+1$ for which $\alpha_\ell > \alpha_{\ell+1}$. Hence, the expected score of a median candidate is always greater than the expected score of any other candidate $x_j$, and thus the median candidates are the only expected winners.

Consider now a PSR $\mathcal{F}$ characterized by a positional score vector $\alpha$ where $\alpha_1 = \cdots = \alpha_\ell$ for a given $\ell > \lfloor \frac{m}{2} \rfloor + 1$. Let us compare, w.l.o.g., the median candidate $x_c$ with $c := \lceil \frac{m}{2} \rceil$ and the candidate $x_{c-1}$ (which must exist since $m > 2$) where, by definition, $x_{c-1} \in M \setminus C^*$. By the fact that $c-1 < c = \lceil \frac{m}{2} \rceil$, we have $\max\{c, m-c+1\} = \lfloor \frac{m}{2} \rfloor + 1$ and $\max\{c-1, m-(c-1)+1\} = \lfloor \frac{m}{2} \rfloor + 2$. And thus, by Observation 3, $\mathscr{D}_m(c,k) = 0$ for every $k > \lfloor \frac{m}{2} \rfloor + 1$ and $\mathscr{D}_m(c-1,k) = 0$ for every $k > \lfloor \frac{m}{2} \rfloor + 2$. Note that, by assumption, we have $\ell > \lfloor \frac{m}{2} \rfloor + 1$, and thus $\alpha_1 = \cdots = \alpha_{\lfloor \frac{m}{2} \rfloor + 1} = \alpha_{\lfloor \frac{m}{2} \rfloor + 2}$. Let us compare the expected scores of both candidates:

$$
\begin{aligned}
&\mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x_c)] - \mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x_{c-1}))] \\
=&\sum_{k=1}^{\lfloor \frac{m}{2} \rfloor + 1} \frac{\mathscr{D}_m(c,k)}{2^{m-1}} \cdot \alpha_k - \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor + 2} \frac{\mathscr{D}_m(c-1,k)}{2^{m-1}} \cdot \alpha_k \\
=&\frac{\alpha_1}{2^{m-1}} \left( \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor + 1} \mathscr{D}_m(c,k) - \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor + 2} \mathscr{D}_m(c-1,k) \right) \\
=&0
\end{aligned}
$$

Hence, the expected scores of $x_c$ and $x_{c-1}$ are equal, whereas $x_{c-1}$ is not a median candidate. Therefore, the median candidates are not the only expected winners. □

**Proposition 11.** *For all $k$-approval voting rules that are first-prioritizing and the Borda rule, under Walsh's distribution, the probability of their agreement for electing one candidate from $C^*$ is lower bounded by $L_\pi(\mathcal{F}_1)$ where $\mathcal{F}_1$ refers to the plurality rule.*

*Proof.* Let $\mathcal{A}$ be the set of all $k$-approval rules that are first-prioritizing. Thanks to Corollary 2, it is enough to look at $\min_{\mathcal{F} \in \mathcal{A}}\{L_\pi(\mathcal{F})\}$. Using the expression of $L$ in Theorem 1, we can greatly simplify our question to the finding of $\mathcal{F}$ such that $\max_{x \in M} \mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x)] - \mathbb{E}_{\pi_W}[S^{\mathcal{F}}(y)]$ is minimal, where $\max_{x \in M} \mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x)] = \mathbb{E}_{\pi_W}[S^{\mathcal{F}}(c)]$ if $c \in C^*$ and $y \in M \setminus C^*$. However, $\max_{x \in M} \mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x)] - \mathbb{E}_{\pi_W}[S^{\mathcal{F}}(y)] = \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor + 1} \frac{\mathscr{D}_m(c,k)}{2^{m-1}} \cdot \alpha_k - \sum_{k=1}^{m-j+1} \frac{\mathscr{D}_m(y,k)}{2^{m-1}} \cdot \alpha_k$. This can again be reduced to the following minimization: $\sum_{k=1}^{\lfloor \frac{m}{2} \rfloor + 1} \mathscr{D}_m(c,k) - \sum_{k=1}^{m-j+1} \mathscr{D}_m(y,k)$. Thanks to Lemma 5 and the fact that $\sum_{k=1}^{\lfloor \frac{m}{2} \rfloor + 1} \mathscr{D}_m(c,k) = 2^{m-1}$, we can conclude that the plurality rule minimizes this quantity. It remains to show that the Borda rule always reaches a higher bound $L$. In fact, it is enough to show that $\max_{x \in M} \mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x)] - \mathbb{E}_{\pi_W}[S^{\mathcal{F}}(y)]$ is at least $m-1$ times bigger than for plurality since we will divide by $(\max_j \alpha_j - \min_j \alpha_j) = m-1$. Nevertheless, the Borda score for the candidate ranked first is $m-1$, so that this quantity has to be larger for Borda. It is even strictly larger thanks to the next Borda scores. □

## D The Conitzer's Distribution

**Proposition 13.** *The expected winners of the $k$-approval rule $\mathcal{F}$ under Conitzer's distribution are:*

$$
\mathcal{W}_{\pi_C}(\mathcal{F}) = \begin{cases} M \text{ if } k = 1 \\ \{x_k, x_{m-k+1}\} \text{ if } 1 < k \leq \lfloor \frac{m}{2} \rfloor + 1 \\ \{x_j \in M : \max\{j, m-j+1\} \leq k\} \text{ otherwise} \end{cases}.
$$

*Proof.* Since the Conitzer's distribution is symmetric, we restrict our analysis, w.l.o.g., to the case of a candidate $x_j$ where $j \in [\lceil \frac{m}{2} \rceil]$. By Lemma 12, we have the following expected score for $x_j$:

$$\mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_j)] = \sum_{\ell=1}^{k} \mathbb{P}_{\pi_C}(j,k)$$
$$= \sum_{\ell=1}^{\min\{k,j\}} Q(j,\ell) + \sum_{\ell=1}^{\min\{k,m-j+1\}} Q(m-j+1,\ell)$$
$$= \begin{cases} \frac{2k}{2m} & \text{if } k < j \\ \frac{3k-1}{2m} + \frac{k}{2m} \cdot \mathbb{1}_{\{j=\lceil \frac{m}{2} \rceil\}} & \text{if } k = j \\ \frac{2j-1+k}{2m} & \text{if } j < k < m-j+1 \\ 1 & \text{if } k \geq m-j+1 \end{cases}$$

If $k \geq \lfloor \frac{m}{2} \rfloor + 1$, then there exist candidates $x_j$ such that $k \geq m-j+1$, and all of them get the maximal expected score of 1, thus they are all expected winners. If $k = 1$, then $\mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_1)] = \frac{3k-1}{2m} = \frac{1}{m}$ and $\mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_j)] = \frac{2k}{2m} = \frac{1}{m}$ for all other candidates $x_j$. It follows that all candidates are expected winners. Finally, if $1 < k \leq \lfloor \frac{m}{2} \rfloor + 1$, then the expected winners are those corresponding to the case where $k = j$ because $3k - 1 > 2k$ when $k > 1$ and $2j - 1 + k < 3k - 1$ when $j < k$. □

**Theorem 14.** *The median candidates are the unique expected winners of a PSR $\mathcal{F}$ under Conitzer's distribution iff the positional score vector $\alpha$ associated with $\mathcal{F}$ satisfies the following inequality, for every $1 \leq j < \lceil \frac{m}{2} \rceil$:*

$$\sum_{\ell=j+1}^{\lceil \frac{m}{2} \rceil - 1} \alpha_\ell + \beta(m) + \alpha_{\frac{m}{2}} \mathbb{1}_{\{m \text{ even}\}} > \sum_{\ell=\lceil \frac{m}{2} \rceil+1}^{m-j} \alpha_\ell + \delta(j,m)$$

*where $\beta(m) := (\lceil \frac{m}{2} \rceil - 1)\alpha_{\lceil \frac{m}{2} \rceil} + (\lfloor \frac{m}{2} \rfloor + 1)\alpha_{\lfloor \frac{m}{2} \rfloor + 1}$ and $\delta(j,m) := (j-1)\alpha_j + (m-j+1)\alpha_{m-j+1}$. A sufficient condition is $\beta(m) > \delta(j,m)$, for every $j < \lceil \frac{m}{2} \rceil$.*

*Proof.* Consider a PSR $\mathcal{F}$ characterized by a positional score vector $\alpha$. Let us compare a median candidate $x_c \in C^*$ and another candidate $x_j \in M \setminus C^*$ where, w.l.o.g., $j < c := \lceil \frac{m}{2} \rceil$. By Lemma 12, the expected score of candidate $x_c$ is given by: $\mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_c)] = \frac{1}{m} \sum_{\ell=1}^{\lceil \frac{m}{2} \rceil - 1} \alpha_\ell + \frac{\lceil \frac{m}{2} \rceil}{2m} \cdot \alpha_{\lceil \frac{m}{2} \rceil} + \frac{\lfloor \frac{m}{2} \rfloor + 1}{2m} \cdot \alpha_{\lfloor \frac{m}{2} \rfloor + 1} + \frac{1}{2m} \cdot \alpha_{\frac{m}{2}} \cdot \mathbb{1}_{\{m \text{ even}\}}$.

Moreover, the expected score of candidate $x_j$ is given by: $\mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_j)] = \frac{1}{m} \sum_{\ell=1}^{j-1} \alpha_\ell + \frac{j+1}{2m}\alpha_j + \frac{1}{2m} \sum_{\ell=j+1}^{m-j} \alpha_\ell + \frac{m-j+1}{2m} \alpha_{m-j+1}$.

It follows that the median candidates are unique expected winners iff, for every $j < c$, we have:

$$\mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_c)] - \mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_j)] > 0$$

$$\Leftrightarrow$$

$$\frac{1}{m} \sum_{\ell=1}^{\lceil \frac{m}{2} \rceil - 1} \alpha_\ell + \frac{\lceil \frac{m}{2} \rceil}{2m} \alpha_{\lceil \frac{m}{2} \rceil} + \frac{\lfloor \frac{m}{2} \rfloor + 1}{2m} \alpha_{\lfloor \frac{m}{2} \rfloor + 1} + \frac{1}{2m} \alpha_{\frac{m}{2}} \mathbb{1}_{\{m \text{ even}\}}$$
$$- \frac{1}{m} \sum_{\ell=1}^{j-1} \alpha_\ell - \frac{j+1}{2m} \alpha_j - \frac{1}{2m} \sum_{\ell=j+1}^{m-j} \alpha_\ell - \frac{m-j+1}{2m} \alpha_{m-j+1} > 0$$

$$\Leftrightarrow$$

$$\frac{1}{2m} \sum_{\ell=j+1}^{\lceil \frac{m}{2} \rceil - 1} \alpha_\ell + \frac{\lceil \frac{m}{2} \rceil - 1}{2m} \alpha_{\lceil \frac{m}{2} \rceil} + \frac{\lfloor \frac{m}{2} \rfloor + 1}{2m} \alpha_{\lfloor \frac{m}{2} \rfloor + 1} + \frac{1}{2m} \alpha_{\frac{m}{2}} \mathbb{1}_{\{m \text{ even}\}}$$
$$> \frac{j-1}{2m} \alpha_j + \frac{1}{2m} \sum_{\ell=\lceil \frac{m}{2} \rceil + 1}^{m-j} \alpha_\ell + \frac{m-j+1}{2m} \alpha_{m-j+1}$$

$$\Leftrightarrow$$

$$\sum_{\ell=j+1}^{\lceil\frac{m}{2}\rceil-1}\alpha_\ell+(\lceil\frac{m}{2}\rceil-1)\alpha_{\lceil\frac{m}{2}\rceil}+(\lfloor\frac{m}{2}\rfloor+1)\alpha_{\lfloor\frac{m}{2}\rfloor+1}+\alpha_{\frac{m}{2}}\mathbb{1}_{\{m\text{ even}\}}$$
$$>(j-1)\alpha_j+\sum_{\ell=\lceil\frac{m}{2}\rceil+1}^{m-j}\alpha_\ell+(m-j+1)\alpha_{m-j+1}$$

We always have $\sum_{\ell=j+1}^{\lceil\frac{m}{2}\rceil-1}\alpha_\ell+\alpha_{\frac{m}{2}}\mathbb{1}_{\{m\text{ even}\}}\geq\sum_{\ell=\lceil\frac{m}{2}\rceil+1}^{m-j}\alpha_\ell$. It follows that a sufficient condition to get $\mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_c)]-\mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_j)]>0$ is $(\lceil\frac{m}{2}\rceil-1)\alpha_{\lceil\frac{m}{2}\rceil}+(\lfloor\frac{m}{2}\rfloor+1)\alpha_{\lfloor\frac{m}{2}\rfloor+1}>(j-1)\alpha_j+(m-j+1)\alpha_{m-j+1}$, for every $1\leq j<\lceil\frac{m}{2}\rceil$. □

**Corollary 15.** *The median candidates are the unique expected winners of the Borda rule and the $\lceil\frac{m}{2}\rceil$-approval rule (as well as $(\frac{m}{2}+1)$-approval if $m$ is even) under Conitzer's distribution.*

*Proof.* We simply show that these rules satisfy the sufficient condition of Theorem 14.

The Borda rule is characterized by the positional score vector $\alpha=(m-1,\dots,0)$, therefore we have $\alpha_j=m-j$, for every $j\in[m]$. Thus, for every $1\leq j<\lceil\frac{m}{2}\rceil$, we have:

$$\begin{aligned}
&(\lceil\frac{m}{2}\rceil-1)\alpha_{\lceil\frac{m}{2}\rceil}+(\lfloor\frac{m}{2}\rfloor+1)\alpha_{\lfloor\frac{m}{2}\rfloor+1}\\
&-(j-1)\alpha_j-(m-j+1)\alpha_{m-j+1}\\
=&(\lceil\frac{m}{2}\rceil-1)\lfloor\frac{m}{2}\rfloor+(\lfloor\frac{m}{2}\rfloor+1)(\lceil\frac{m}{2}\rceil-1)\\
&-(j-1)(m-j)-(m-j+1)(j-1)\\
=&(\lceil\frac{m}{2}\rceil-1)(2\lfloor\frac{m}{2}\rfloor+1)-(j-1)(2m-2j+1)
\end{aligned}$$

The previous quantity is decreasing w.r.t. $j$, therefore it takes its minimum value for $j=\lceil\frac{m}{2}\rceil-1$, where this quantity is equal to:

$$\begin{aligned}
&(\lceil\frac{m}{2}\rceil-1)(2\lfloor\frac{m}{2}\rfloor+1)-(\lceil\frac{m}{2}\rceil-2)(2m-2\lceil\frac{m}{2}\rceil+3)\\
=&(\lceil\frac{m}{2}\rceil-2)(2\lfloor\frac{m}{2}\rfloor+1-2m+2\lceil\frac{m}{2}\rceil-3)+(2\lfloor\frac{m}{2}\rfloor+1)\\
=&(\lceil\frac{m}{2}\rceil-2)(-2)+(2\lfloor\frac{m}{2}\rfloor+1)\\
=&-2\lceil\frac{m}{2}\rceil+4+2\lfloor\frac{m}{2}\rfloor+1\\
=&-\mathbb{1}_{\{m\text{ odd}\}}+4-\mathbb{1}_{\{m\text{ odd}\}}+1\\
=&-2\cdot\mathbb{1}_{\{m\text{ odd}\}}+5\\
>&0
\end{aligned}$$

Hence, the Borda rule satisfies the sufficient condition of Theorem 14.

Under the $\lceil\frac{m}{2}\rceil$-approval rule, $\alpha_j=1$ for all $1\leq j\leq\lceil\frac{m}{2}\rceil$ and $\alpha_j=0$ for all $j>\lceil\frac{m}{2}\rceil$. Therefore, for every $1\leq j<\lceil\frac{m}{2}\rceil$, we have $(\lceil\frac{m}{2}\rceil-1)\alpha_{\lceil\frac{m}{2}\rceil}+(\lfloor\frac{m}{2}\rfloor+1)\alpha_{\lfloor\frac{m}{2}\rfloor+1}-(j-1)\alpha_j-(m-j+1)\alpha_{m-j+1}=\lceil\frac{m}{2}\rceil-1+(\lfloor\frac{m}{2}\rfloor+1)\mathbb{1}_{\{m\text{ odd}\}}-(j-1)>0$, because $j<\lceil\frac{m}{2}\rceil$. Hence, $\lceil\frac{m}{2}\rceil$-approval satisfies the sufficient condition of Theorem 14. If $m$ is even, then $(\frac{m}{2}+1)$-approval also satisfies the sufficient condition of Theorem 14 because $\alpha_{\lfloor\frac{m}{2}\rfloor+1}=1$ and thus we have $\lceil\frac{m}{2}\rceil-1+\lfloor\frac{m}{2}\rfloor+1-(j-1)>0$. □

## E Unbiased Distributions

**Theorem 17.** *There exists an unbiased single-peaked distribution w.r.t. the $k$-approval rule iff $k$ divides $m$.*

*Proof.* Let us assume that $k$ divides $m$, i.e., there exists an integer $q$ such that $m = k \cdot q$. Let us partition the set of candidates $M$ in $q$ groups of size $k$ as follows: $\{x_1, x_2, \dots, x_k\}, \{x_{k+1}, \dots, x_{2k}\}, \dots, \{x_{(q-1)k+1}, \dots, x_{qk}\}$ where $X_j$ denotes the group $\{x_{(j-1)k+1}, \dots, x_{jk}\}$ for each $j \in [q]$ and $M = \bigsqcup_{j\in[q]} X_j$. For each group $X_j$, let us denote by $P_j$ the set of single-peaked preference orders where the $k$ candidates in $X_j$ are ranked among the first $k$ candidates, i.e., $P_j : \{\succ_i \in \Pi^m_> : r_{\succ_i}(x) \leq k, \forall x \in X_j\}$. Observe that $P_j$ is necessarily non-empty for every $j \in [q]$ because, e.g., the following single-peaked order $\succ_i$ belongs to $P_j$: $x_{(j-1)k+1} \succ_i \cdots \succ_i x_{jk} \succ_i x_{(j-1)k} \succ_i \cdots \succ_i x_1 \succ_i x_{jk+1} \succ_i \cdots \succ_i x_m$. We consider the single-peaked preference distribution $\pi : \Pi^m_> \to [0, 1]$ such that $\sum_{\succ_i \in P_j} \pi(\succ_i) = \frac{k}{m} = \frac{1}{q}$ for each $j \in [q]$, and $\pi(\succ_i) = 0$ for all $\succ_i \in \Pi^m_> \setminus \bigcup_{j\in[q]} P_j$. We can check that $\pi$ is a valid distribution because $\sum_{\succ_i \in \Pi^m_>} \pi(\succ_i) = \sum_{j\in[q]} \sum_{\succ_i \in P_j} \pi(\succ_i) = q \cdot \frac{k}{m} = 1$.

In the $k$-approval rule, each candidate gains one point per preference order where it is ranked among the first $k$ candidates. Under the described preference distribution $\pi$, it occurs for candidate $x_\ell$ with a positive probability only in preference orders in $P_j$ with the unique $j$ such that $x_\ell \in X_j$. It follows that the expected score of each candidate $x_\ell$ is equal to $\sum_{\succ_i \in P_j : x_\ell \in X_j} \pi(\succ_i) \cdot 1 = \frac{k}{m}$.

Let us now assume that $k$ does not divide $m$. Let us denote by $q$ and $r$ the unique integers such that $m = k \cdot q + r$ with $0 < r < k$. Suppose, for the sake of contradiction, that there exists a single-peaked distribution $\pi$ unbiased with respect to the $k$-approval rule. We will prove by induction that a preference order ranking candidate $x_{(j-1)k+\ell}$ among the first $k$ candidates, for $\ell \in [k]$, can be assigned a positive probability in $\pi$ only if all the $k$ candidates $x_{(j-1)k+1}, \dots, x_{jk}$ are ranked among the first $k$ candidates in this preference order, for every $j \in [q]$. For the base case, candidate $x_1$ gets one point under the $k$-approval rule iff it is ranked among the first $k$ candidates. However, if $x_1$ is ranked among the first $k$ candidates then, by single-peakedness, it must also be the case of all candidates $x_j$ for $1 < j \leq k$. Since the expected score of $x_1$ must be the same as the one of all candidates $x_j$ for $1 < j \leq k$, then no positive probability can be assigned to other preference orders where some candidate $x_j$, for $1 < j \leq k$, is ranked among the first $k$ candidates. We now assume that a preference order ranking candidate $x_{(j'-1)k+\ell}$ among the first $k$ candidates, for $\ell \in [k]$, can be assigned a positive probability in $\pi$ only if all the $k$ candidates $x_{(j'-1)k+1}, \dots, x_{j'k}$ are ranked among the first $k$ candidates in this preference order, for every $1 \leq j' < j$, for a given $j \in [q]$. It follows that candidate $x_{(j-1)q+1}$ cannot be ranked within the top $k$ of a preference order with positive probability where some candidate $x_{\ell'}$, for $\ell' < (j-1)q + 1$, is also ranked within the top $k$. Therefore, if $x_{(j-1)q+1}$ is ranked within the top $k$ of a preference order with positive probability, then it must also be the case of all the candidates $x_{(j-1)k+2}, \dots, x_{jk}$. Since the expected score of $x_{(j-1)q+1}$ must be the same as the one of all candidates $x_{(j-1)k+2}, \dots, x_{jk}$, then no positive probability can be assigned to other preference orders where some candidate among $x_{(j-1)k+2}, \dots, x_{jk}$, is ranked among the first $k$ candidates, proving the claim.

Now, let us analyze the case of candidate $x_m$. If $x_m$ is ranked among the first $k$ candidates then, by single-peakedness, it must also be the case of all the $k-1$ candidates $x_j$, for $m - k + 1 \leq j < m$. Since $k$ does not divide $m$, there exist integers $j \in [q]$ and $\ell \in [k]$ such that $m - k + 1 = (j-1)k + \ell$ and thus candidate $x_{m-k+1}$ is approved in single-peaked orders approving candidates $(j-1)k + \ell'$, for $\ell' \in [k]$, and in the disjoint ones approving candidate $x_m$, therefore its expected score would be equal to the sum of the expected score of $x_m$ and the expected score of $x_{m-k}$, contradicting the fact that $\pi$ is unbiased. ☐

**Proposition 18.** *Conitzer's distribution is unbiased w.r.t. a positional scoring rule $\mathcal{F}$ iff $\mathcal{F}$ is the plurality rule.*

*Proof.* By Proposition 13, all candidates are expected winners of the 1-approval rule (i.e., plurality) under the Conitzer's distribution. Therefore, the Conitzer's distribution is unbiased w.r.t. plurality.

Suppose that the Conitzer's distribution is unbiased w.r.t. some positional scoring rule $\mathcal{F}$ defined by the positional score vector $\alpha = (\alpha_1, \ldots, \alpha_m)$ such that, by definition, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m$ and $\alpha_1 > \alpha_m$. It follows that all candidates are expected winners of $\mathcal{F}$, i.e., $\mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_i)] = \mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_j)]$, for every $i, j \in [m]$. By Lemma 12, the expected score of a candidate $x_j$ is the following:

$$
\begin{aligned}
\mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_j)] &= \sum_{k=1}^{m} \mathbb{P}_{\pi_C}(j,k) \cdot \alpha_k \\
&= \sum_{k=1}^{m} (Q(j,k) + Q(m-j+1,k)) \cdot \alpha_k \\
&= \sum_{k=1}^{j-1} \frac{1}{2m} \cdot \alpha_k + \frac{j}{2m} \cdot \alpha_j + \\
&\quad \sum_{k=1}^{m-j} \frac{1}{2m} \cdot \alpha_k + \frac{m-j+1}{2m} \cdot \alpha_{m-j+1}
\end{aligned}
$$

By considering, in particular, candidates $x_1$ and $x_2$, we have $\mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_1)] = \frac{1}{2m}\alpha_1 + \frac{1}{2m}\sum_{k=1}^{m-1}\alpha_k + \frac{1}{2}\alpha_m$ and $\mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_2)] = \frac{1}{2m}\alpha_1 + \frac{2}{2m} \cdot \alpha_2 + \frac{1}{2m}\sum_{k=1}^{m-2}\alpha_k + \frac{m-1}{2m} \cdot \alpha_{m-1}$. For candidates $x_1$ and $x_2$ to be both expected winners, they need to have the same expected score. It follows that:

$$
\begin{aligned}
&\mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_1)] = \mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_2)] \Leftrightarrow \\
&\frac{1}{2m}\alpha_1 + \frac{1}{2m}\sum_{k=1}^{m-1}\alpha_k + \frac{1}{2}\alpha_m = \frac{1}{2m}\alpha_1 + \frac{2}{2m} \cdot \alpha_2 + \\
&\frac{1}{2m}\sum_{k=1}^{m-2}\alpha_k + \frac{m-1}{2m} \cdot \alpha_{m-1} \Leftrightarrow \\
&\frac{1}{2m}\sum_{k=1}^{m-1}\alpha_k + \frac{1}{2}\alpha_m = \frac{2}{2m} \cdot \alpha_2 + \frac{1}{2m}\sum_{k=1}^{m-2}\alpha_k + \frac{m-1}{2m} \cdot \alpha_{m-1} \Leftrightarrow \\
&\frac{1}{2m}\alpha_{m-1} + \frac{1}{2}\alpha_m = \frac{2}{2m} \cdot \alpha_2 + \frac{m-1}{2m} \cdot \alpha_{m-1} \Leftrightarrow \\
&\frac{1}{2}\alpha_m = \frac{2}{2m} \cdot \alpha_2 + \frac{m-2}{2m} \cdot \alpha_{m-1} \Leftrightarrow \\
&\alpha_m = \frac{2}{m} \cdot \alpha_2 + \frac{m-2}{m} \cdot \alpha_{m-1}
\end{aligned}
$$

Because $\alpha_2 \geq \cdots \geq \alpha_{m-1} \geq \alpha_m$, the fact that $\alpha_m = \frac{2}{m} \cdot \alpha_2 + \frac{m-2}{m} \cdot \alpha_{m-1}$ implies $\alpha_2 = \cdots = \alpha_{m-1} = \alpha_m$. It follows that $\alpha_1 > \alpha_2 = \cdots = \alpha_{m-1} = \alpha_m$, and thus $\mathcal{F}$ corresponds to the plurality rule. □

**Proposition 19.** *No PSR can make Walsh's distribution unbiased.*

*Proof.* Suppose, for the sake of contradiction, that the Walsh's distribution $\pi_W$ is unbiased with respect to a given PSR $\mathcal{F}$ characterized by the positional score vector $\alpha = (\alpha_1, \ldots, \alpha_m)$. We can assume, w.l.o.g., that $\alpha_1 = 1$, $\alpha_m = 0$, and $\alpha_y \in [0,1]$ for every $1 < j < m$. By definition, for every candidates $x$ and $y$, we have $\mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x)] = \mathbb{E}_{\pi_W}[S^{\mathcal{F}}(y)]$, i.e., $\sum_{\succ_i \in \Pi^m} \pi_W(\succ_i) \cdot \alpha_{r_{\succ_i}(x)} = \sum_{\succ_i \in \Pi^m} \pi_W(\succ_i) \cdot \alpha_{r_{\succ_i}(y)}$, and thus $\sum_{\succ_i \in \Pi^m} \frac{1}{2^{m-1}} \cdot \alpha_{r_{\succ_i}(x)} = \sum_{\succ_i \in \Pi^m} \frac{1}{2^{m-1}} \cdot \alpha_{r_{\succ_i}(y)}$ which implies $\sum_{\succ_i \in \Pi^m} \alpha_{r_{\succ_i}(x)} = \sum_{\succ_i \in \Pi^m} \alpha_{r_{\succ_i}(y)}$.

Consider the extreme candidate $x_1$ and the median candidate $x_c := x_{\lceil \frac{m}{2} \rceil}$. We must have $\sum_{\succ_i \in \Pi^m} \alpha_{r_{\succ_i}(x_1)} = \sum_{\succ_i \in \Pi^m} \alpha_{r_{\succ_i}(c)}$. By Observation 3, candidate $c$ can never be ranked at a position worse than $\gamma := \lfloor \frac{m}{2} \rfloor + 1$, and thus we have $\sum_{\succ_i \in \Pi^m} \alpha_{r_{\succ_i}(c)} = \sum_{k=1}^{m} \mathscr{D}_m(x_c, k) \cdot \alpha_k = \sum_{k=1}^{\gamma} \mathscr{D}_m(x_c, k) \cdot \alpha_k$ where $\sum_{k=1}^{\gamma} \mathscr{D}_m(x_c, k) = 2^{m-1}$. Since $x_1$ is an extreme candidate, it is ranked last in half of the single-peaked orders. Therefore, by the fact that $\alpha_m = 0$, we have $\sum_{\succ_i \in \Pi^m} \alpha_{r_{\succ_i}(x_1)} = \sum_{k=1}^{m} \mathscr{D}_m(x_1, k) \cdot \alpha_k = \sum_{k=1}^{m-1} \mathscr{D}_m(x_1, k) \cdot \alpha_k$ where $\sum_{k=1}^{m-1} \mathscr{D}_m(x_1, k) \cdot \alpha_k = 2^{m-2}$. Let us now analyze the difference between $\sum_{\succ_i \in \Pi^m} \alpha_{r_{\succ_i}(x_c)}$ and $\sum_{\succ_i \in \Pi^m} \alpha_{r_{\succ_i}(x_1)}$:

$$
\begin{aligned}
&\sum_{\succ_i \in \Pi^m} \alpha_{r_{\succ_i}(x_c)} - \sum_{\succ_i \in \Pi^m} \alpha_{r_{\succ_i}(x_1)} \\
=&\sum_{k=1}^{\gamma} \mathscr{D}_m(x_c, k) \cdot \alpha_k - \sum_{k=1}^{m-1} \mathscr{D}_m(x_1, k) \cdot \alpha_k \\
=&\sum_{k=1}^{\gamma} (\mathscr{D}_m(x_c, k) - \mathscr{D}_m(x_1, k)) \cdot \alpha_k - \sum_{k=\gamma+1}^{m-1} \mathscr{D}_m(x_1, k) \cdot \alpha_k
\end{aligned}
$$

However, by Lemma 4, we have $\mathscr{D}_m(x_c, k) = 2^{k-2}(\binom{m-k}{c-1} + \binom{m-k}{c-k}) \geq 2^{k-2}$ for $k \in \{2, \ldots, \gamma\}$, while $\mathscr{D}_m(x_1, k) = 2^{k-2}(\binom{m-k}{0} + \binom{m-k}{1-k}) = 2^{k-2}$ for $k \in \{2, \ldots, \gamma\}$, and $\mathscr{D}_m(x_c, 1) = \binom{m-1}{c-1}$ and $\mathscr{D}_m(x_1, 1) = 1$. Therefore, $\mathscr{D}_m(x_c, k) - \mathscr{D}_m(x_1, k) \geq 0$ for every $k \in [\gamma]$.

Since $\alpha_1 \geq \alpha_2 \geq \ldots \alpha_m$, it follows that:

$$
\begin{aligned}
&\sum_{k=1}^{\gamma} (\mathscr{D}_m(x_c, k) - \mathscr{D}_m(x_1, k)) \cdot \alpha_k - \sum_{k=\gamma+1}^{m-1} \mathscr{D}_m(x_1, k) \cdot \alpha_k \\
\geq&\sum_{k=1}^{\gamma} (\mathscr{D}_m(x_c, k) - \mathscr{D}_m(x_1, k)) \cdot \alpha_\gamma - \sum_{k=\gamma+1}^{m-1} \mathscr{D}_m(x_1, k) \cdot \alpha_k \\
=&(2^{m-1} - (1 + \sum_{k=2}^{\lfloor m/2 \rfloor + 1} 2^{k-2}) \cdot \alpha_\gamma - \sum_{k=\gamma+1}^{m-1} \mathscr{D}_m(x_1, k) \cdot \alpha_k \\
=&(2^{m-1} - 2^{\lfloor m/2 \rfloor}) \cdot \alpha_\gamma - \sum_{k=\gamma+1}^{m-1} \mathscr{D}_m(x_1, k) \cdot \alpha_k \\
\geq&(2^{m-1} - 2^{\lfloor m/2 \rfloor}) \cdot \alpha_\gamma - \sum_{k=\gamma+1}^{m-1} \mathscr{D}_m(x_1, k) \cdot \alpha_\gamma \\
=&(2^{m-1} - 2^{\lfloor m/2 \rfloor}) \cdot \alpha_\gamma - (2^{m-2} - 2^{\lfloor m/2 \rfloor}) \cdot \alpha_\gamma \\
>&0
\end{aligned}
$$

Hence, we always have $\sum_{\succ_i \in \Pi^m} \alpha_{r_{\succ_i}(x_c)} > \sum_{\succ_i \in \Pi^m} \alpha_{r_{\succ_i}(x_1)}$, no matter the chosen positional score vector, a contradiction. □

Finally, the polarized distribution is also unbiased w.r.t. other rather natural PSRs.

**Proposition 28.** *The polarized distribution is unbiased with respect to:*

1. *the PSR characterized by the score vector* $(2, 1, \ldots, 1, 0)$,
2. *the* $m/2$*-approval rule, when* $m$ *is even.*

*Proof.* 1. Consider the PSR rule characterized the positional score vector $(2, 1, \ldots, 1, 0)$. Under the polarized distribution, candidates $x_1$ and $x_m$ can be ranked either at the first or at the last position, with equal probability, and all other candidates are surely ranked at another position. It follows that the expected score of candidates $x_1$ and $x_m$ is equal to $\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0 = 1$ and the expected score of each candidate $x_j$, where $1 < j < m$, is equal to $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 1$.

2. In the $\frac{m}{2}$-approval rule, where $m$ is even, each candidate gains one point per preference order where it is ranked among the first $m/2$ candidates, and zero points otherwise. Under the polarized distribution, each candidate $x_j$ can be ranked either at position $j$, and thus among the top $m/2$ iff $j \leq m/2$, or at position $m - j + 1$, and thus among the top $m/2$ iff $j > m/2$, with equal probability. It follows that the expected score of each candidate $x_j$ is equal to $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{2}$.

□

## F Other Structured Distributions

**Lemma 29** (DKW inequality). *Let $X_1, \ldots, X_n$ be some independent and identical random variables distributed with a law $F$. Let $F_n(x, \omega) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{X_i(\omega) \leq x\}}$ then $\mathbb{P}(\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| > \varepsilon) \leq 2e^{-2n\varepsilon^2}, \quad \forall \varepsilon > 0$.*

**Proposition 22.** *For a unimodal preference distribution $\pi$, the probability that all PSRs and Condorcet-consistent rules agree is lower bounded by $B_\pi := 1 - 2exp(-2n\varepsilon^2)$, for $\varepsilon := \min_{\succ_i, \succ_j \in \Pi^m} |\pi(\succ_i) - \pi(\succ_j)|$.*

*Proof.* We remark that for $\varepsilon$ sufficiently small, i.e. $\varepsilon = \min_{\{\succ_i, \succ_i'\}} |\mathbb{P}_\pi(\succ_i) - \mathbb{P}_\pi(\succ_i')|$, we have $\{||F_n(., \omega) - F||_\infty \leq \varepsilon\} = \{F_n \text{ is unimodal}\}$. Applying Lemma 29 on the contrary event and using theorem 4.1 from Chatterjee and Storcken [13], we get that the described voting rules agree with probability at least $1 - 2e^{-2n\varepsilon^2}$. □

**Definition 4** (Dirichlet law). *Let $d \geq 2$ be an integer. Let $\Sigma$ be the $(d-1)$-dimensional simplex*

$$\Sigma = \left\{ (x_1, \ldots, x_d) \in [0, 1]^d \mid \sum_{k=1}^{d} x_k = 1 \right\}$$

*then*

$$f(x_1, \ldots, x_d)\, d\Sigma(x_1, \ldots, x_d)$$
$$= f\left(x_1, \ldots, x_{d-1}, 1 - \sum_{k=1}^{d-1} x_k\right) \mathbb{1}_{\{x \in [0,1]^{d-1}, \sum_{k=1}^{d-1} x_k \leq 1\}}\, dx_1 \cdots dx_{d-1}$$

*for any continuous function $f$.*

**Lemma 30** (Asymptotic convergence of Pólya-Eggenberger urn [3]). *Let $d \geq 2$ and $R \geq 1$ be an integer. Let also $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}^d \setminus \{0\}$. Let $(P_n)_{n \geq 0}$ be the $d$-color Pólya-Eggenberger urn random process having $R$ as reinforcement parameter and $\beta$ as initial composition. Then, almost surely and in any $L^t$, $t \geq 1$,*

$$\frac{P_n}{nR} \xrightarrow[n \to \infty]{} V$$

*where $V$ is a $d$-dimensional Dirichlet-distributed random vector, with parameters $\left(\frac{\beta_1}{R}, \ldots, \frac{\beta_d}{R}\right)$.*

**Remark 31.** *Let us recall that the convergence in $L^t$, $t \geq 1$ implies the convergence in law.*

**Proposition 23.** *Under the Pólya-Eggenberger urn culture, the probability that all PSRs asymptotically agree is lower bounded by $\frac{1}{2}$ if $r < \frac{2}{3}$ and $m = 3$, and by $\frac{1}{4}$ if $r < \frac{1}{6}$ and $m = 4$.*

*Proof.* Let us recall (see Lemma 30) that a Pólya-Eggenberger urn asymptotically converges to the Dirichlet law (see Definition 4). Thus, we can calculate the probability that a specific distribution of preferences occurs.

We now need to describe the event $D_m$ where all positional scoring rules agree. We use the known fact that all positional scoring rules will agree if all $k$-approval voting rules agree with each other [43] and get for example for $m = 3$: $D_3 = \{(p_1, p_2, p_3, p_4, p_5, p_6) \in \Sigma \mid p_1 + p_2 > p_3 + p_4, p_1 + p_2 > p_5 + p_6, p_2 + p_5 > p_4 + p_6, p_1 + p_3 > p_4 + p_6\}$ using the following notation $(p_1, p_2, p_3, p_4, p_5, p_6)$ for the proportion of each preference in the election in the following order $(a \succ b \succ c), (a \succ c \succ b), (b \succ a \succ c), (b \succ c \succ a), (c \succ a \succ b), (c \succ b \succ a)$.

We now come back to our initial question which is to compute $\lim_{n\to\infty} \mathbb{P}_{P-E}(D_3)$. Using Lemma 30, we are able to identify the limit law and to compute $\mathbb{P}_V(D)$, for every $0 < R \leqslant 4$. Since the analytical is fastidious, we use the Monte-Carlo method with a very high precision ($n = 10,000,000$) to compute the integral and get the desired result. We recover the result on $r$ by doing the change of variable.

We follow the exact same framework for $m = 4$, compute $D_4$ which is much more complicated and get the desired result. $\square$

**Proposition 24.** *Under the Pólya-Eggenberger urn culture, the probability that plurality and Borda asymptotically agree is lower bounded by $\frac{3}{4}$ if $r < \frac{2}{3}$ and $m = 3$, and by $\frac{3}{5}$ if $r < \frac{1}{6}$ and $m = 4$.*

*Proof.* We follow the exact same steps as in the previous proof but we need to construct a different space to find where Plurality and Borda agree. For example for $m = 3$, $D_3 = \{(p_1, p_2, p_3, p_4, p_5, p_6) \in \Sigma \mid p_1+p_2 > p_3+p_4, p_1+p_2 > p_5+p_6, p_1+2\cdot p_2+p_5 > p_3+2\cdot p_4+p_6, 2\cdot p_1+p_2+p_3 > p_4+p_5+2\cdot p_6\}$ using the following notation $(p_1, p_2, p_3, p_4, p_5, p_6)$ for the proportion of each preference in the election in the following order $(a \succ b \succ c), (a \succ c \succ b), (b \succ a \succ c), (b \succ c \succ a), (c \succ a \succ b), (c \succ b \succ a)$. $\square$

**Proposition 25.** *If the election is drawn with a Pólya-Eggenberger urn culture with $R < 4$ then every pair of positional scoring rules $\mathcal{F}_1$ and $\mathcal{F}_2$ asymptotically disagree with a positive probability, i.e., $\lim_{n\to\infty} \mathbb{P}(\mathcal{F}_1(\succ) \neq \mathcal{F}_2(\succ)) > 0$.*

*Proof.* Let $\mathcal{F}_1, \mathcal{F}_2$ be two positional scoring rules. There exist two positional score vectors $\alpha^1$ and $\alpha^2$ corresponding to these two rules. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are different, $\alpha^1$ and $\alpha^2$ differ on at least one component, i.e., there exists $i \in [m]$ such that $\alpha^1_i \neq \alpha^2_i$. Let us denote $\varepsilon = \alpha^1_i - \alpha^2_i > 0$. We will show that there exists a profile $\succ$ such that $\lim_{n\to\infty} \mathbb{P}(\mathcal{F}_1(\succ) \neq \mathcal{F}_2(\succ)) > 0$. Specifically, to build such a profile we consider an arbitrary profile such that the $j^{\text{th}}$ candidate has an asymptotic score of 0, then we slowly increase the proportion of one preference such that candidate $j$ is ranked in position $i$ until $\mathcal{F}_1(\succ) \neq \mathcal{F}_2(\succ)$. By doing so, we find that we can still increase this proportion from $\delta < \varepsilon$ and keep the disagreement between the two rules. Thus, there exists a non negligible set where $\mathcal{F}_1(\succ) \neq \mathcal{F}_2(\succ)$. Finally, we identify the limit law of a Pólya-Eggenberger urn as the Dirichlet random variable thanks to Lemma 30 and conclude that $\lim_{n\to\infty} \mathbb{P}(\mathcal{F}_1(\succ) \neq \mathcal{F}_2(\succ)) > 0$ because this is a continuous density on a non negligible set. $\square$