Technical Report AC-TR-17-002

April 2017

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This is the authors' copy of a paper that appeared in Soft Computing, volume 21(1), pages 79–95, 2017.

manuscript No.
(will be inserted by the editor)

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Stefano Aguzzoli · Simone Bova · Diego Valota

In memory of Franco Montagna

Abstract We give a combinatorial description of the finitely generated free weak nilpotent minimum algebras, and provide explicit constructions of normal forms.

1 Introduction

A *triangular norm* T is a binary, associative and commutative $[0,1]$-valued operation on the unit square $[0,1]^2$ that is monotone ($b \leq c$ implies $T(a,b) \leq T(a,c)$ for all $a,b,c \in [0,1]$), has 1 as identity ($T(a,1) = a$ for all $a \in [0,1]$), and (thus) has 0 as annihilator ($T(a,0) = 0$ for all $a \in [0,1]$). In the theory of fuzzy sets, triangular norms and their duals, triangular conorms, model respectively intersections and unions of fuzzy sets, and hence provide natural interpretations for conjunctions and disjunctions of propositions whose truth values range over the unit interval. If a triangular norm T is left continuous, then the operation $R(a,b) = \max\{c \mid T(a,c) \leq b\}$, called the *residual* of T, is the unique binary $[0,1]$-valued operation on the unit square that satisfies the residuation equivalence,

$$T(a,b) \leq c \text{ if and only if } a \leq R(b,c),$$

for all $a,b,c \in [0,1]$, and hence arguably acts as the logical implication induced by the interpretation of T as a logical conjunction (for instance, it implies right distributivity of R over T).

It is known that the class of all left continuous triangular norms and their residuals, intended as the algebraic structures obtained by equipping the unit interval $[0,1]$ with a distributive bounded integral lattice structure ($\wedge$, $\vee$, 0, and 1) together with a triangular norm and its residual ($\cdot$ and $\to$), generates a certain variety of residuated lattices, *MTL-algebras*, which forms in fact the algebraic counterpart of a many-valued propositional logic called *monoidal triangular norm* logic, *MTL-logic*; for a discussion and an axiomatization of MTL-logic we refer the reader to [10, 14].

Adopting this logical interpretation, if

$$\mathbb{A} = (A, \wedge, \vee, \cdot, \to, 0, 1)$$

is a MTL-algebra, then the unary operation term defined by

$$a' \fallingdotseq a \to 0,$$

for all $a \in A$, is intended as a *negation* operation. Interestingly, the class of unary operation $': [0,1] \to [0,1]$ arising as negation operations of MTL-algebras over $[0,1]$ coincides with the class of weak negation operations [16]; that is, unary operations over $[0,1]$ such that, for all $a,b \in [0,1]$: $0' = 1$; $a \leq b$ implies $b' \leq a'$; and, $a \leq a''$.

Given a weak negation $': [0,1] \to [0,1]$, it is possible to equip $[0,1]$ with a particular MTL-algebraic structure by defining the norm operation as follows, for all

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$a, b \in [0, 1]$:

$$a \cdot b = \begin{cases} 0 & \text{if } a \leq b', \\ a \wedge b & \text{otherwise.} \end{cases} \tag{1}$$

For instance, Figure 1 displays the first four members of the family of weak negations $\{f_n \mid n = 0, 1, 2, \dots\}$, where f_n is the step function over $[0, 1]$ that maps 0 to 1, and $((i-1)/n, i/n]$ to $(n-i)/n$ for $i = 1, 2, \dots, n$, so that f_n has 2^n discontinuities. The top part displays the graphs of f_0, f_1, f_2, and f_3, and the bottom part displays the triangular norms induced (1).

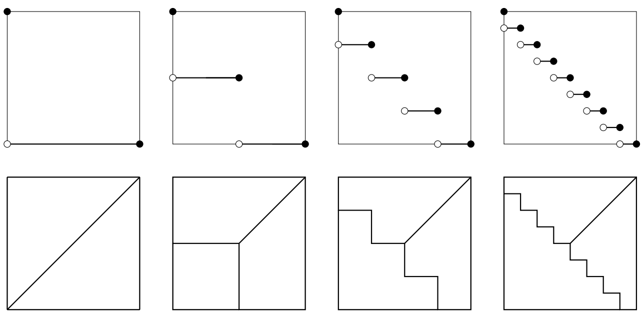

Fig. 1

In fact, the class of all weak negations, intended as the MTL-algebraic structures over $[0, 1]$ described above, generates a subvariety of MTL-algebras, namely the variety of *weak nilpotent minimum* algebras, or, for short, *WNM-algebras*. The naming refers to the *nilpotent minimum* triangular norm, introduced by Fodor [11], which corresponds via (1) to the special weak negation $a' = 1 - a$ for all $a \in [0, 1]$, which is *involutive*, that is $a'' = a$ for all $a \in [0, 1]$. See Figure 2. Actually, the family $\{f_n \mid n = 0, 1, 2, \dots\}$ is sufficient to generate all WNM-algebras [16]. WNM-algebras have been extensively studied in Carles Noguera's PhD dissertation [16]. We refer the reader to this monograph for background.

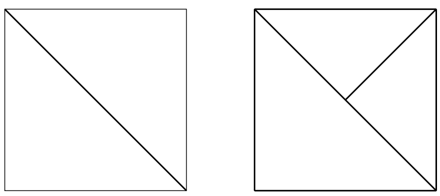

Fig. 2: The graphs of the involutive weak negation $a \mapsto 1 - a$ (on the left) and its triangular norm (on the right), the nilpotent minimum triangular norm by Fodor [11].

In this note, we give a concrete, combinatorial description of free finitely generated free algebras in the variety of WNM-algebras. Knowledge of the structure of the free WNM-algebras is interesting for both logical and algebraic reasons.

On the logical side, the elements of the free algebra, which we explicitly construct, are exactly the truth functions of the corresponding propositional logic. The result then launches further investigation of various features of the deductive system, such as interpolation, unification, and admissibility; it is worth to mention that in [9], Ciabattoni et al. present a uniform method for generating analytic logical calculi from given axiom schemata, and the WNM-logic represents a hard case (in a sense that can be made precise) where the method succeeds. In the recent work [2] the authors use a WNM-chain to solve an open problem posed by Franco Montagna in [15], namely that, for extensions of the logic MTL, the single chain completeness does not imply the strong single chain completeness.

On the algebraic side, the problem is non-trivial because it requires a description of finitely generated WNM-chains, nice enough to study a certain subalgebra of their direct product. Exploiting the fact that WNM-algebras are locally finite, a combinatorial description of WNM-chains is reachable, in sharp contrast with MTL-algebras, where a nice description of chains is unknown (and hard). Certain special cases of WNM-algebras have been studied, namely the variety generated by f_0 and f_1 in Figure 1, respectively Gödel [13] and RDP-algebras [18,6], and the variety generated by the involutive negation in Figure 2, NM-algebras, together with NMG-algebras [5]. The paper [12] classifies all subvarieties of NM-algebras, while [8,4] determines the structure of free NM-algebras. In this note, in the veine of [3], we generalize such results to the entire class of WNM-algebras.

We conclude the introduction by making precise the background notions and facts about finitely presented algebras and weak nilpotent minimum algebras used in the above discussion. For further standard background in universal algebra, we refer the reader to [7].

1.1 Finitely Generated Free Algebras

Let σ be a finite algebraic signature, that is, a finite set of operation symbols with an arity function $\mathrm{ar}\colon \sigma \to \{0\} \cup \mathbb{N}$. Let $X = \{x_1, x_2, \dots\}$ be a countable set of variables; $x, y, z, \dots$ denote arbitrary pairwise distinct variables in X. The set of σ-terms is the smallest set T such that: $X \cup \{f \in \sigma \mid \mathrm{ar}(f) = 0\} \subseteq T$; for all $f \in \sigma$, if $\mathrm{ar}(f) = k \geq 1$ and $s_1, \dots, s_k \in T$, then $f(s_1, \dots, s_k) \in T$. For $Y \subseteq X$, we let T_Y denote the set of σ-terms on variables Y; in short, we write T_n instead of $T_{\{x_1,\dots,x_n\}}$; if $t \in T_n$, we also write $t(x_1,\dots,x_n)$.

Equations (on σ) are first-order σ-formulas of the form $s = t$ with $s, t \in T$; we say that $s = t$ is in T_Y if $s, t \in T_Y$; if Ξ is a set of equations, we say that Ξ is in T_Y if each equation in Ξ is in T_Y. If $Y \subseteq X$ is finite and Ξ is a finite set of equations in T_Y, we denote by $\wedge\Xi$ the conjunction of all equations in Ξ.

A σ-algebra $\mathbb{A} = (A, (f^{\mathbb{A}})_{f\in\sigma})$ is a non-empty set A equipped with a family of operations indexed by σ, such that $f^{\mathbb{A}} \colon A^{\mathrm{ar}(f)} \to A$ for all $f \in \sigma$; in particular, $f^{\mathbb{A}} \in A$ if $\mathrm{ar}(f) = 0$. $\mathbb{A}$ is trivial if $|A| = 1$. If $t \in T_Y$ and $g \colon X \to A$, then the evaluation of t in $\mathbb{A}$ under g, in symbols $t^{\mathbb{A}}(g) \in A$, is defined inductively on t as follows: $t^{\mathbb{A}}(g) = g(x)$ if $t = x \in Y$; $t^{\mathbb{A}}(g) = f^{\mathbb{A}}(s_1^{\mathbb{A}}(g),\dots,s_{\mathrm{ar}(f)}^{\mathbb{A}}(g))$ if $t = f(s_1,\dots,s_{\mathrm{ar}(f)})$ with $f \in \sigma$ and $s_1,\dots,s_{\mathrm{ar}(f)} \in T_Y$; in particular, $t^{\mathbb{A}}(g) = f^{\mathbb{A}}$ if $t = f$ and $\mathrm{ar}(f) = 0$. We write $\mathbb{A}, g \models s = t$ iff $s^{\mathbb{A}}(g) = t^{\mathbb{A}}(g)$.

A class of σ-algebras $\mathcal{V}$ is an (algebraic) variety if and only if, there exists a set of equations Ξ such that $\mathbb{A} \in \mathcal{V}$ iff $\mathbb{A}, g \models s = t$ for all $g \colon X \to A$ and $s = t$ in Ξ [7]; if $\mathcal{V}$ is the class of models of Ξ we also write $\mathcal{V}_\Xi$.

As usual, an *n-generated σ-algebra* $\mathbb{A}$ is an algebra on a signature $\sigma_{\{1,\dots,n\}}$ (in short, σ_n) extending σ with n new constant symbols $x_1,\dots,x_n$, that is, $\sigma_n = (\sigma, x_1,\dots,x_n)$ with $\mathrm{ar}(x_i) = 0$ for $i = 1,\dots,n$, and

$$\mathbb{A} = (A, (f^{\mathbb{A}})_{f\in\sigma}, x_1^{\mathbb{A}},\dots,x_n^{\mathbb{A}}),$$

where for each $a \in A$ there is a term $t \in T_n$ such that $t^{\mathbb{A}} = a$. Then, if $\mathbb{A}$ and $\mathbb{B}$ are n-generated σ-algebras, we say that:

1. $\mathbb{A}$ is a *subalgebra* of $\mathbb{B}$ if there exists an injective σ_n-homomorphism from $\mathbb{A}$ to $\mathbb{B}$;
2. $\mathbb{A}$ is *isomorphic* to $\mathbb{B}$ if there exists a bijective σ_n-homomorphism from $\mathbb{A}$ to $\mathbb{B}$;
3. $\mathbb{B}$ is a *quotient* of $\mathbb{A}$ if there exists a surjective σ_n-homomorphism h from $\mathbb{A}$ to $\mathbb{B}$ and $\mathbb{B}$ is isomorphic to $\mathbb{A}/\equiv$, where $\equiv$ is the congruence relation on $\mathbb{A}$ defined as usual ($a \equiv b$ iff $h(a) = h(b)$ for all $a, b \in A$).

For $Y \subseteq X$, the σ-algebra

$$\mathbb{T}_Y \leftrightharpoons (T_Y, (f^{\mathbb{T}_Y})_{f\in\sigma})$$

where $f^{\mathbb{T}_Y}(s_1,\dots,s_{\mathrm{ar}(f)}) \leftrightharpoons f(s_1,\dots,s_{\mathrm{ar}(f)})$ for all $s_1,\dots,s_{\mathrm{ar}(f)} \in T_Y$ (in particular $f^{\mathbb{T}_Y} = f$ if $\mathrm{ar}(f) = 0$) is called the term algebra (on σ). Note that

$$\mathbb{T}_n \leftrightharpoons (T_n, (f^{\mathbb{T}_n})_{f\in\sigma_n}, x_1^{\mathbb{T}_n},\dots,x_n^{\mathbb{T}_n})$$

is in fact an n-generated σ-algebra with generators $x_i^{\mathbb{T}_n} = x_i$ for $i = 1,\dots,n$.

We define the notion of finitely presented algebra for $\mathcal{V}_\Xi$ a finitely axiomatized variety, that is, with Ξ finite. A presentation is a pair (Y, Σ) where Σ is a finite set of equations in T_Y; (Y, Σ) is finite if $Y = \{x_1,\dots,x_n\}$ for some $n \in \mathbb{N}$. A finite presentation $(\{x_1,\dots,x_n\}, \Sigma)$ defines the equivalence relation,

$$s \equiv t \text{ if and only if } \{\wedge\Xi, \wedge\Sigma\} \models s = t, \tag{2}$$

where $s, t \in T_n$ are related iff for all $\mathbb{A} \in \mathcal{V}_\Xi$ and $g \colon X \to A$, if $\mathbb{A}, g \models \Sigma$, then $\mathbb{A}, g \models s = t$. The relation $\equiv$ is a congruence relation on $\mathbb{T}_n$. In this setting, the algebra in $\mathcal{V}_\Xi$, finitely presented by $(\{x_1,\dots,x_n\}, \Sigma)$, is the quotient

$$\mathbb{T}_n/\equiv.$$

Conversely, a σ-algebra $\mathbb{A} \in \mathcal{V}_\Xi$ is finitely presented iff $\mathbb{A}$ is isomorphic to a quotient $\mathbb{T}_n/\equiv$, where $\equiv$ is the congruence defined as in (2) by some finite presentation.

If $\Sigma = \emptyset$, then we denote $\mathbb{T}_n/\equiv$ by

$$\mathbb{F}_n \leftrightharpoons (\{[t]_\equiv \mid t \in T_n\}, (f^{\mathbb{F}_n})_{f\in\sigma_n}, x_1^{\mathbb{F}_n},\dots,x_n^{\mathbb{F}_n})$$

and we refer to $\mathbb{F}_n$ as the σ-algebra in $\mathcal{V}_\Xi$ freely generated by $x_i^{\mathbb{F}_n} \leftrightharpoons [x_i]_\equiv$ for $i = 1,\dots,n$. In this case, by (2), if Θ is a finite set of equations in T_n and $h \colon \{x_1,\dots,x_n\} \to F_n$ is such that $x_i \mapsto [x_i]_\equiv$ for $i = 1,\dots,n$, then,

$$\mathbb{F}_n, h \models \Theta \text{ if and only if } \mathbb{A} \models \wedge\Theta \tag{3}$$

for all $\mathbb{A} \in \mathcal{V}_\Xi$.

Notation 1 *If $\mathbb{A} = (A, (f^{\mathbb{A}})_{f\in\sigma})$ is a σ-algebra, and $s, t \in T_n$, then we write*

$$\mathbb{A} \models s = t \textit{ if and only if } s^{\mathbb{A}} = t^{\mathbb{A}};$$

moreover, if $a \in A$ is such that $s^{\mathbb{A}} = a$, then we write $\mathbb{A} \models s = a$ instead of $\mathbb{A}, h \models s = x$, where $h \colon X \to A$ is such that $h(x) = a$.

1.2 Weak Nilpotent Minimum Algebras

Fix $\sigma = (\wedge, \cdot, \to, 0, 1)$ with $\mathrm{ar}(\circ) = 2$ for all $\circ \in \{\wedge, \cdot, \to\}$, and $\mathrm{ar}(0) = \mathrm{ar}(1) = 0$. We write x' instead of $x \to 0$, $x \vee y$ instead of $((x \to y) \to y) \wedge ((y \to x) \to x)$, and x^2 instead of $x \cdot x$. As usual, we adopt the infix notation for binary operation symbols. A *monoidal triangular norm based logic* algebra (in short, *MTL-algebra*) is a σ-algebra $\mathbb{A} = (A, \wedge^{\mathbb{A}}, \cdot^{\mathbb{A}}, \to^{\mathbb{A}}, 0^{\mathbb{A}}, 1^{\mathbb{A}})$ such that $(A, \wedge^{\mathbb{A}}, \vee^{\mathbb{A}}, 0^{\mathbb{A}}, 1^{\mathbb{A}})$ is a bounded lattice, $(A, \cdot^{\mathbb{A}}, 1^{\mathbb{A}})$ is a commutative monoid, $a \cdot^{\mathbb{A}} c \leq^{\mathbb{A}} b$ if and only if $c \leq^{\mathbb{A}} a \to^{\mathbb{A}} b$ for all $a, b, c \in A$ (residuation), which is true if and only if,

$$\mathbb{A} \models (x \to ((x \cdot y) \vee z)) \wedge y = y,$$
$$\mathbb{A} \models (y \vee z) \cdot x = (y \cdot x) \vee (z \cdot x),$$
$$\mathbb{A} \models (y \cdot (y \to x)) \vee x = x;$$

and $\mathbb{A} \models x \to y \vee y \to x = 1$ (prelinearity). Therefore there exists a finite set of equations Ξ in $T_{\{x,y,z\}}$ such that a σ-algebra $\mathbb{A}$ models Ξ if and only if $\mathbb{A}$ is a MTL-algebra [10]; we denote the variety of MTL-algebras by $\mathcal{MTL}$. We collect some known facts on MTL-algebras [10].

Note that for all $\mathbb{A} \in \mathcal{MTL}$ and $a, b \in A$, by residuation and integrality, $a =^{\mathbb{A}} a \cdot^{\mathbb{A}} 1^{\mathbb{A}} \leq^{\mathbb{A}} b$ iff $1^{\mathbb{A}} \leq^{\mathbb{A}} a \to^{\mathbb{A}} b = 1^{\mathbb{A}}$, therefore $b <^{\mathbb{A}} a$ iff $a \to^{\mathbb{A}} b <^{\mathbb{A}} 1^{\mathbb{A}}$. Moreover,

$$\mathbb{A} \models x' = x'''. \tag{4}$$

Let $\mathbb{A}$ be a MTL-algebra. A filter on $\mathbb{A}$ is a non-empty upset $B \subseteq A$ closed under the operation $\cdot^{\mathbb{A}}$. A filter B on $\mathbb{A}$ is prime iff $B \subset A$ and $a \to^{\mathbb{A}} b \in B$ or $b \to^{\mathbb{A}} a \in B$ for all $a, b \in A$. The set of filters on $\mathbb{A}$, with intersection as meet operation and closure of union under $\cdot^{\mathbb{A}}$ as join operation, is a lattice. The lattice of congruences on $\mathbb{A}$ is isomorphic to the lattice of filters on $\mathbb{A}$, via the map that sends a congruence $\equiv$ on $\mathbb{A}$ to the filter $\{a \in A \mid a \equiv 1^{\mathbb{A}}\} \eqqcolon [1^{\mathbb{A}}]_{\equiv}$; the inverse map sends a filter $B \subseteq A$ to the congruence, $a \equiv b$ iff for all $a, b \in A$, $a \to^{\mathbb{A}} b \in B$ and $b \to^{\mathbb{A}} a \in B$. In fact, under such bijective correspondence, completely meet irreducible congruences maps to prime filters, which implies by universal algebraic facts that subdirectly irreducible MTL-algebras are chains. In fact, let $\mathbb{C} \eqqcolon \mathbb{A}/\equiv$. If $[1^{\mathbb{A}}]_{\equiv}$ is prime, then $a \to^{\mathbb{A}} b \in [1^{\mathbb{A}}]_{\equiv}$ or $b \to^{\mathbb{A}} a \in [1^{\mathbb{A}}]_{\equiv}$ for all $a, b \in A$; in the first case,

$$[a]_{\equiv} \to^{\mathbb{C}} [b]_{\equiv} \eqqcolon [a \to^{\mathbb{A}} b]_{\equiv} = [1^{\mathbb{A}}]_{\equiv}$$

implies $[a]_{\equiv} \leq^{\mathbb{C}} [b]_{\equiv}$; in the second case, similarly, $[b]_{\equiv} \leq^{\mathbb{C}} [a]_{\equiv}$. Then $\mathbb{C}$ is a chain. Similarly, if $\mathbb{C}$ is a chain, then $[1^{\mathbb{A}}]_{\equiv}$ is prime. It follows by universal algebraic facts, that the variety of MTL-algebras is generated by MTL-chains.

A MTL-algebra $\mathbb{A}$ is a *weak nilpotent minimum* algebra (in short, *WNM-algebra*) if

$$\mathbb{A} \models (x \cdot y)' \vee ((x \wedge y) \to x \cdot y) = 1; \tag{5}$$

we let $\mathcal{WNM}$ denote the variety of WNM-algebras. In particular, the variety $\mathcal{WNM}$ is generated by WNM-chains, and for all WNM-chains $\mathbb{C}$, the operations $\cdot^{\mathbb{C}}$ and $\to^{\mathbb{C}}$ are uniquely determined by the lattice and negation operations, as follows (for all $a, b \in C$):

$$a \cdot^{\mathbb{C}} b = \begin{cases} 0^{\mathbb{C}} & \text{if } a \leq^{\mathbb{C}} b'^{\mathbb{C}}, \\ a \wedge^{\mathbb{C}} b & \text{otherwise;} \end{cases} \tag{6}$$

$$a \to^{\mathbb{C}} b = \begin{cases} 1^{\mathbb{C}} & \text{if } a \leq^{\mathbb{C}} b, \\ a'^{\mathbb{C}} \vee^{\mathbb{C}} b & \text{otherwise.} \end{cases} \tag{7}$$

Direct inspection of the previous equations and (4) shows that finitely generated WNM-chains are finite, which implies that the variety $\mathcal{WNM}$ is locally finite, that is, finitely generated algebras are finite [16].

Let $\mathbb{A} \in \mathcal{WNM}$. Then $\mathbb{A}$ is: a *NMG-algebra* (notion introduced in [20], while the following one-variable axiomatisation is given in [1]), if

$$\mathbb{A} \models x'' \vee (x'' \to x) = 1;$$

a *RDP-algebra* (*revised drastic product* algebra) [19], if

$$\mathbb{A} \models x'' \vee (x \to x') = 1;$$

a *NM-algebra* (*nilpotent minimum* algebra), if $\mathbb{A}$ is a WNM-algebra (or an NMG-algebra) and

$$\mathbb{A} \models x'' = x;$$

a *Gödel* algebra, if $\mathbb{A}$ is a MTL-algebra (or a WNM-algebra) and

$$\mathbb{A} \models x = x^2.$$

Notice that in [1] it is proved that Gödel algebras, NM-algebras, and NMG-algebras can be axiomatised from MTL-algebras using only one-variable axioms. This is achieved replacing (5) with the following:

$$\mathbb{A} \models (x \cdot x)' \vee (x \to x \cdot x) = 1. \tag{8}$$

On the other hand, replacing (5) by (8) does not work for RDP-algebras: as a matter of fact, MTL-algebras satisfying (8) constitutes a subvariety properly larger than RDP-algebras, named GP-algebras in [1].

We apply routinely the following known facts [17].

Proposition 1 *For all WNM-chains $\mathbb{C}$ and $g \colon X \to C$:*

$$\mathbb{C}, g \models x \leq x'' = \bigwedge\{z \in C \mid x \leq z, z = z''\}, \tag{9}$$

$$\mathbb{C}, g \models x = x^2 \textit{ iff } \mathbb{C}, g \models x' < x \textit{ or } \mathbb{C}, g \models x = 0, \tag{10}$$

$$\mathbb{C}, g \models x \leq y \textit{ implies } \mathbb{C}, g \models y' \leq x', \tag{11}$$

$$\mathbb{C}, g \models x' < x \textit{ and } \mathbb{C}, g \models y' < y \textit{ implies } \mathbb{C}, g \models x' < y, \tag{12}$$

$$\mathbb{C}, g \models x \leq x' \textit{ and } \mathbb{C}, g \models y' < y \textit{ implies } \mathbb{C}, g \models x \leq y, \tag{13}$$

$$\mathbb{C}, g \models x' < x \textit{ and } \mathbb{C}, g \models y \leq y' \textit{ implies } \mathbb{C}, g \models x' < y'. \tag{14}$$

Organization. In this note, we provide an explicit description of finitely presented WNM-algebras. We provide an explicit direct decomposition of the WNM-algebra freely generated by $x_1, \ldots, x_n$, and we give an explicit construction of normal forms.

The paper is organized as follows. Let $n \geq 1$. In Section 2, we characterize the (finite) set

$$\mathcal{C}_n = \{\mathbb{C}_1, \ldots, \mathbb{C}_m\},$$

where each $\mathbb{C}_j$ is a subdirectly irreducible WNM-algebra n-generated by $x_i^{\mathbb{C}_j}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, and the $\mathbb{C}_j$'s are pairwise non σ_n-isomorphic. By universal algebraic facts, the WNM-algebra $\mathbb{F}_n$ freely generated by $x_i^{\mathbb{F}_n} = (x_i^{\mathbb{C}_1}, \ldots, x_i^{\mathbb{C}_m})$ for $i = 1, \ldots, n$ is (σ_n-isomorphic to) the subalgebra $\mathbb{A}$ of $\mathbb{C}_1 \times \cdots \times \mathbb{C}_m$ generated by $x_i^{\mathbb{A}} = (x_i^{\mathbb{C}_1}, \ldots, x_i^{\mathbb{C}_m})$ for $i = 1, \ldots, n$ [7]. In Section 3, we characterize factors in the direct decomposition of $\mathbb{F}_n$. In Section 4, we provide an explicit combinatorial description of $\mathbb{F}_n$.

2 Subdirectly Irreducible WNM-algebras

In this section, we describe the (finite) set

$$\mathcal{C}_n = \{\mathbb{C}_1, \ldots, \mathbb{C}_m\}$$

of (pairwise non σ_n-isomorphic) subdirectly irreducible n-generated WNM-algebras. Actually, the structure of subdirectly irreducible WNM-algebras is well-known, see for instance [16]. Being WNM-algebras a subvariety of MTL-algebras, the subdirectly irreducible WNM-algebras are chains, whose operations are completely determined by the choice of the negation operation, which is an arbitrary weak negation. Moreover, being WNM-algebras a locally finite variety, the n-generated subdirectly irreducible members coincide with the n-generated chains, which all have finite cardinality. In this section we classify σ_n-isomorphism classes of subfirectly irreducible n-generated WNM-algebras by subdividing the universe of n-generated chains into *blocks*. This representation turns out to be useful to characterise the direct factors of the free n-generated WNM-algebra, given in a later section.

Definition 1 (Blockwise Representation) Let $\mathbb{C}$ be a WNM-chain generated by $x_1^{\mathbb{C}}, \ldots, x_n^{\mathbb{C}} \in C$. Then

$$\mathrm{bk}(\mathbb{C}) \leftrightharpoons (\{B_1, \ldots, B_k\}, (f^{\mathrm{bk}(\mathbb{C})})_{f \in \sigma}, x_1^{\mathrm{bk}(\mathbb{C})}, \ldots, x_n^{\mathrm{bk}(\mathbb{C})})$$

(reads blockwise $\mathbb{C}$) is the n-generated WNM-chain such that:

1. the blocks $B_1, \ldots, B_k$ form a partition of $\{0, 1, x_i, x_i', x_i'' \mid i = 1, \ldots, n\}$;
2. the generator $x_i^{\mathrm{bk}(\mathbb{C})}$ is the block containing x_i for $i = 1, \ldots, n$;
3. $x, y \in B_j$ iff $\mathbb{C} \models x = y$ for $j = 1, \ldots, k$;
4. $B_j <^{\mathrm{bk}(\mathbb{C})} B_{j+1}$ iff $\mathbb{C} \models x < y$, where $x \in B_j$, $y \in B_{j+1}$, $j = 1, \ldots, k-1$;
5. $B_j'^{\mathrm{bk}(\mathbb{C})} = B_l$ iff $\mathbb{C} \models x' = y$, where $x \in B_j$, $y \in B_l$, $j = 1, \ldots, k$.

We also write,

$$\mathrm{bk}(\mathbb{C}) = B_1 < \cdots < B_k.$$

The n-generated WNM-chains $\mathrm{bk}(\mathbb{C})$ and $\mathbb{C}$ are σ_n-isomorphic, clearly. The next fact characterizes the class $\mathcal{C}_1$ of singly generated WNM-chains.

Proposition 2 $\mathcal{C}_1 = \{\mathbb{C}_i \mid i = 1, \ldots, 9\}$, *where:*

$$\begin{aligned}
\mathrm{bk}(\mathbb{C}_1) &= 0x_1x_1'' < x_1'1, \\
\mathrm{bk}(\mathbb{C}_2) &= 0 < x_1 < x_1'' < x_1' < 1, \\
\mathrm{bk}(\mathbb{C}_3) &= 0 < x_1x_1'' < x_1' < 1, \\
\mathrm{bk}(\mathbb{C}_4) &= 0 < x_1 < x_1'x_1'' < 1, \\
\mathrm{bk}(\mathbb{C}_5) &= 0 < x_1x_1'x_1'' < 1, \\
\mathrm{bk}(\mathbb{C}_6) &= 0 < x_1' < x_1 < x_1'' < 1, \\
\mathrm{bk}(\mathbb{C}_7) &= 0 < x_1' < x_1x_1'' < 1, \\
\mathrm{bk}(\mathbb{C}_8) &= 0x_1' < x_1 < x_1''1, \\
\mathrm{bk}(\mathbb{C}_9) &= 0x_1' < x_1x_1''1,
\end{aligned}$$

with slight liberality in the usage of the blockwise notation.

Proof Equations (4),(6) and (7) show that each singly generated σ_1-WNM-chain $\mathbb{A}$ contains at most 5 elements: its universe is the set $\{0^{\mathbb{A}}, x_1^{\mathbb{A}}, (x_1')^{\mathbb{A}}, (x_1'')^{\mathbb{A}}, 1\}$. Moreover, two such chains $\mathbb{A}$, $\mathbb{B}$ such that, for each pair of elements $c_1, c_2 \in \{0, x_1, x_1', x_1'', 1\}$ it holds that $c_1^{\mathbb{A}} \leq_{\mathbb{A}} c_2^{\mathbb{A}}$ iff $c_1^{\mathbb{B}} \leq_{\mathbb{B}} c_2^{\mathbb{B}}$, are clearly σ_1-isomorphic. Taking into account that $\mathbb{A} \models x \leq x''$, direct inspection now proves that each singly generated σ_1-WNM-chain is isomorphic with one in $\mathcal{C}_1$. Notice that all chains in $\mathcal{C}_1$ have negations that are restrictions of f_2. $\square$

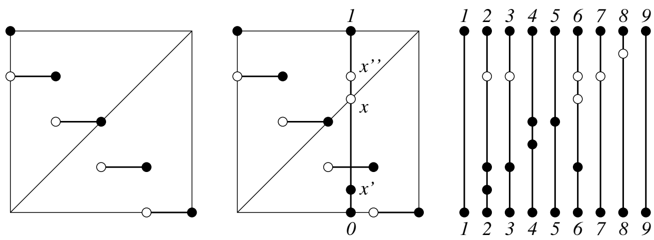

Fig. 3: The construction of $\mathcal{C}_1$ in Proposition 2. On the left, the graph of $x\colon [0,1] \to [0,1]$ and $x'\colon [0,1] \to [0,1]$, where $0' = 1$, $a' = 3/4$ for $a \in (0, 1/4]$, $a' = 1/2$ for $a \in (1/4, 1/2]$, $a' = 1/4$ for $a \in (1/2, 3/4]$, $a' = 0$ for $a \in (3/4, 1]$. The WNM-chain $\mathbb{C}_6$ (center) is generated by $x = 5/8$, so that $x' = 1/4$ and $x'' = 3/4$. On the right, the WNM-chains $\mathbb{C}_1, \ldots, \mathbb{C}_9$, numbered from 1 to 9, where solid, $\bullet$, and open, $\circ$, dots denote respectively idempotent and non-idempotent elements.

Let $\mathbb{C} \in \mathcal{C}_n$. For $i = 1, \ldots, n$, the *orbit* of x_i in $\mathbb{C}$ is the $\sigma_{\{i\}}$-subalgebra of $\mathbb{C}$ generated by $x_i^{\mathbb{C}}$. We define $\mathrm{orbit}(\mathbb{C}, 0) \leftrightharpoons 1$, $\mathrm{orbit}(\mathbb{C}, 1) \leftrightharpoons 9$, and for $i = 1, \ldots, n$,

$$\mathrm{orbit}(\mathbb{C}, x_i) \leftrightharpoons j,$$

iff the orbit of x_i in $\mathbb{C}$ is $\sigma_{\{i\}}$-isomorphic to $\mathbb{C}_j \in \mathcal{C}_1$, where $j \in \{1, \dots, 9\}$. Notice that the orbit of x_i in $\mathbb{C}$ is in $\mathcal{C}_1$, hence $\sigma_{\{i\}}$-isomorphic to $\mathbb{C}_j \in \mathcal{C}_1$ for some $j \in \{1, \dots, 9\}$.

Example 1 $\mathbb{C} \in \mathcal{C}_n$ is a Boolean (respectively, Gödel, NM, NMG, RDP) chain iff $\mathrm{orbit}(\mathbb{C}, x_i) \in \{1, 9\}$ (respectively, $\mathrm{orbit}(\mathbb{C}, x_i) \in \{1, 8, 9\}$, $\mathrm{orbit}(\mathbb{C}, x_i) \in \{1, 3, 5, 7, 9\}$, $\mathrm{orbit}(\mathbb{C}, x_i) \in \{1, 3, 5, 7, 8, 9\}$, $\mathrm{orbit}(\mathbb{C}, x_i) \in \{1, 4, 5, 8, 9\}$) for all $i = 1, \dots, n$.

Let

$$\mathcal{K}_n \subseteq \mathcal{C}_n$$

be such that $\mathbb{C} \in \mathcal{K}_n$ iff $\mathbb{C} \in \mathcal{C}_n$ and there does not exist $\mathbb{D} \in \mathcal{C}_n$ and a congruence $\equiv$ on $\mathbb{D}$ above the identity such that $\mathbb{C} = \mathbb{D}/\equiv$.

Proposition 3 $\mathbb{C} \in \mathcal{K}_n$ *if and only if* $\mathrm{orbit}(\mathbb{C}, x_i) \in \{2, 3, \dots, 7\}$ *for all* $i = 1, \dots, n$.

Proof Let $\mathrm{bk}(\mathbb{C}) = B_1 < \cdots < B_k$. *Then,* $\mathbb{C} \in \mathcal{K}_n$ *iff* $B_k = \{1\}$, *iff* $\mathrm{orbit}(\mathbb{C}, x_i) \neq 1, 8, 9$ *in* $\mathbb{C}$ *for* $i = 1, \dots, n$. $\square$

Example 2 ($n = 1$) By Proposition 3, $\mathcal{K}_1 = \{\mathbb{C}_2, \dots, \mathbb{C}_7\} \subseteq \mathcal{C}_1$. See Figure 4. In fact, $\mathbb{C}_1$ is a quotient of $\mathbb{C}_2$ via $[1]_\equiv \coloneqq \{x_1', 1\}$, $\mathbb{C}_8$ is a quotient of $\mathbb{C}_6$ via $[1]_\equiv \coloneqq \{x_1'', 1\}$, and $\mathbb{C}_9$ is a quotient of $\mathbb{C}_7$ and $\mathbb{C}_8$ via $[1]_\equiv \coloneqq \{x_1, x_1'', 1\}$.

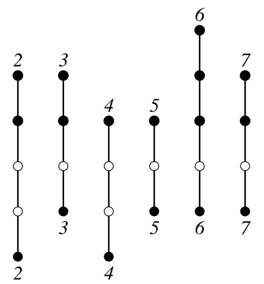

Fig. 4: $\mathcal{K}_1 = \{\mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4, \mathbb{C}_5, \mathbb{C}_6, \mathbb{C}_7\} \subseteq \mathcal{C}_1$.

Example 3 ($n = 2$) $\mathcal{K}_2$ is listed in Appendix A. For readability sake, we display the generators x_1 and x_2 as x and y respectively.

Proposition 4 *The free n-generated WNM-algebra $\mathbb{F}_n$ is (isomorphic to) the subalgebra of $\prod_{\mathbb{C} \in \mathcal{K}_n} \mathbb{C}$ generated by $(x_i^{\mathbb{C}})_{\mathbb{C} \in \mathcal{K}_n}$ for $i = 1, \dots, n$.*

Proof Let $\mathcal{C}_n = \{\mathbb{C}_1, \dots, \mathbb{C}_k, \mathbb{C}_{k+1}, \dots, \mathbb{C}_l\}$ and let $\mathcal{K}_n = \{\mathbb{C}_1, \dots, \mathbb{C}_k\}$. By universal algebraic facts [7], $\mathbb{F}_n$ is isomorphic to the subalgebra of $\prod_{\mathbb{C} \in \mathcal{C}_n} \mathbb{C}$ generated by $(x_i^{\mathbb{C}})_{\mathbb{C} \in \mathcal{C}_n}$ for $i = 1, \dots, n$. By Proposition 3, the latter is σ_n-isomorphic to the subalgebra of $\prod_{\mathbb{C} \in \mathcal{K}_n} \mathbb{C}$ generated by $(x_i^{\mathbb{C}})_{\mathbb{C} \in \mathcal{K}_n}$ for $i = 1, \dots, n$. $\square$

We establish the basic terminology and facts on WNM-chains.

Notation 2 *Let $\mathbb{D} \in \mathcal{K}_n$. We write,*

$$\begin{aligned}
D_0 &\coloneqq \{0\},\\
D_1 &\coloneqq \{x_i, x_i'' \mid \mathrm{orbit}(\mathbb{D}, x_i) \in \{2, 3\}, i = 1, \dots, n\}\\
&\quad \cup \{x_i' \mid \mathrm{orbit}(\mathbb{D}, x_i) \in \{6, 7\}, i = 1, \dots, n\},\\
D_2 &\coloneqq \{x_i \mid \mathrm{orbit}(\mathbb{D}, x_i) = 4, i = 1, \dots, n\},\\
D_3 &\coloneqq \{x_i', x_i'' \mid \mathrm{orbit}(\mathbb{D}, x_i) = 4, i = 1, \dots, n\}\\
&\quad \cup \{x_i, x_i', x_i'' \mid \mathrm{orbit}(\mathbb{D}, x_i) = 5, i = 1, \dots, n\},\\
D_4 &\coloneqq \{x_i' \mid \mathrm{orbit}(\mathbb{D}, x_i) \in \{2, 3\}, i = 1, \dots, n\}\\
&\quad \cup \{x_i, x_i'' \mid \mathrm{orbit}(\mathbb{D}, x_i) \in \{6, 7\}, i = 1, \dots, n\},\\
D_5 &\coloneqq \{1\}.
\end{aligned}$$

Also, we let $l_{\mathbb{D}}, g_{\mathbb{D}} \in D$ be such that,

$$\mathbb{D} \models l_{\mathbb{D}} = \bigwedge_{x \in D_4 \cup D_5} x,$$

$$\mathbb{D} \models g_{\mathbb{D}} = \bigvee_{x \in D_0 \cup D_1 \cup D_2 \cup D_3} x.$$

The following facts hold by inspection of $\mathcal{C}_1$. We write $p \prec q$ to mean that $p < q$ and there is no r such that $p < r < q$.

Fact 1 (Blocks) *Let $\mathbb{D} \in \mathcal{K}_n$. Then,*

(i) $x \in D_0$ iff $\mathbb{D} \models x = 0$ iff $x' \in D_5$;
(ii) $x \in D_1$ iff $\mathbb{D} \models 0 < x \leq x'' < x'$ iff $x' \in D_4$;
(iii) $x \in D_2$ iff $\mathbb{D} \models x < x'' = x'$, and $x \in D_2$ implies $x' \in D_3$;
(iv) $x \in D_3$ iff $\mathbb{D} \models x = x'' = x'$ iff $x' \in D_3$;
(v) $x \in D_4$ iff $\mathbb{D} \models x' < x < 1$ iff $x' \in D_1$;
(vi) $x \in D_5$ iff $\mathbb{D} \models x = 1$ iff $x' \in D_0$.

Also, $l_{\mathbb{D}}$ is the least element $x \in D$ such that $\mathbb{D} \models x' < x$, and $g_{\mathbb{D}}$ is the greatest element $x \in D$ such that $\mathbb{D} \models x \leq x'$, so $\mathbb{D} \models g_{\mathbb{D}} \prec l_{\mathbb{D}}$.

In words, $l_{\mathbb{D}}$ is the least idempotent element strictly above the bottom and $g_{\mathbb{D}}$ is the greatest non-idempotent element in $\mathbb{D}$; note that $\mathbb{D} \in \mathcal{K}_n$ implies that $g_{\mathbb{D}}$ exists. For instance, for each chain $\mathbb{D} \in \mathcal{K}_1$ in Figure 4, $l_{\mathbb{D}}$ and $g_{\mathbb{D}}$ are respectively the smallest solid dot (above the bottom) and the largest open dot.

Proposition 5 (Order Between Blocks) *Let $\mathbb{D} \in \mathcal{K}_n$. Then, $\mathbb{D} \models x < y$ for all $x \in D_i$ and $y \in D_j$ with $0 \leq i < j \leq 5$.*

Proof It is sufficient to show that $\mathbb{D} \models x < y$ for all $x \in D_i$ and $y \in D_{i+1}$ with $i = 1, 2, 3$. In all cases, clearly $\mathbb{D} \models x \neq y$. Assume for a contradiction $\mathbb{D} \models y < x$. If $x \in D_1$ and $y \in D_2$, then $\mathbb{D} \models y < x \leq x'' < x' \leq y' = y'' \leq x''$. If $x \in D_2$ and $y \in D_3$, then $\mathbb{D} \models y < x < x'' = x' \leq y' = y'' = y$. If $x \in D_3$ and $y \in D_4$, then $\mathbb{D} \models y' < y < x = x'' = x' \leq y'$. $\square$

Proposition 6 (Order Within Blocks) *Let $\mathbb{D} \in \mathcal{K}_n$. Then, if $x, y \in D_3$, then $\mathbb{D} \models x = y$.*

Proof If $x, y \in D_3$, assume for a contradiction $\mathbb{D} \models x < y$ (the case $\mathbb{D} \models y < x$ is symmetric). Then, $\mathbb{D} \models x' = x < y = y' \leq x'$. □

Let $\mathbb{D} \in \mathcal{K}_n$. In light of Proposition 5 and Proposition 6, $\mathrm{bk}(\mathbb{D})$ has the form,

$$D_0 < D_{1,1} < \cdots < D_{1,i_1} < D_{2,1} < \cdots < D_{2,i_2} < D_3 \leq \\ \leq D_3 < D_{4,1} < \cdots < D_{4,i_4} < D_5,$$

where $\{D_{j,1}, \ldots, D_{j,i_j}\}$ is a partition of D_j $(j = 1, 2, 4)$.

We prepare a technical fact for later use in Theorem 2. If $\mathbb{C} \in \mathcal{K}_n$ and $I \in \{\{0, 1\}, \{2, 3\}, \{4, 5\}\}$, we write $C_I \leftrightharpoons \cup_{i \in I} C_i$.

Proposition 7 *Let $\mathbb{C}, \mathbb{D} \in \mathcal{K}_n$. Let $I, J \in \{\{0, 1\}, \{2, 3\}, \{4, 5\}\}$, $x \in C_I$, $y \in C_J$, $w \in D_I$, $z \in D_J$.*

(i) $\mathbb{C} \models x \leq y$ and $\mathbb{D} \models z < w$ implies either $I = J = \{0, 1\}$, or $I = J = \{2, 3\}$, or $I = J = \{4, 5\}$.
(ii) $\mathbb{C} \models y < x$ implies either $x \in C_{\{0,1\}}$ and $y \in C_{\{0,1\}}$, or $x \in C_{\{2,3\}}$ and $y \in C_{\{0,1\}} \cup C_{\{2,3\}}$, or $x \in C_{\{4,5\}}$ and $y \in C_{\{0,1\}} \cup C_{\{2,3\}} \cup C_{\{4,5\}}$.
(iii) $\mathbb{C} \models x \leq y'$ and $\mathbb{D} \models z' < w$ implies either $(I, J) = (\{0, 1\}, \{4, 5\})$ or $(I, J) = (\{4, 5\}, \{0, 1\})$.
(iv) $\mathbb{C} \models y' < x$ implies either $x \in C_{\{0,1\}}$ and $y \in C_{\{4,5\}}$, or $x \in C_{\{2,3\}}$ and $y \in C_{\{4,5\}}$, or $x \in C_{\{4,5\}}$ and $y \in C_{\{0,1\}} \cup C_{\{2,3\}} \cup C_{\{4,5\}}$.

Proof *(i)* If $x \in C_{\{0,1\}}$ and $w \in D_{\{0,1\}}$, then $z \in D_{\{0,1\}}$ because $\mathbb{D} \models z < w$, and then $y \in C_{\{0,1\}}$. If $x \in C_{\{2,3\}}$ and $w \in D_{\{2,3\}}$, then $z \in D_{\{0,1\}} \cup D_{\{2,3\}}$ because $\mathbb{D} \models z < w$. Then $y \in C_{\{0,1\}} \cup C_{\{2,3\}}$. But $y \notin C_{\{0,1\}}$ because $\mathbb{C} \models x \leq y$. Then $y \in C_{\{2,3\}}$ and $z \in D_{\{2,3\}}$. If $x \in C_{\{4,5\}}$ and $w \in D_{\{4,5\}}$, then $y \in C_{\{4,5\}}$ because $\mathbb{C} \models x \leq y$, and then $z \in D_{\{2,3\}}$.

(ii) Clear.

(iii) By part *(i)* and Fact 1, $\mathbb{C} \models x \leq y'$ and $\mathbb{D} \models z' < w$ implies either $x \in C_{\{0,1\}}$, $y \in C_{\{4,5\}}$, $w \in D_{\{0,1\}}$, and $z \in D_{\{4,5\}}$, or $x, y \in C_{\{2,3\}}$ and $w, z \in D_{\{2,3\}}$, or $x \in C_{\{4,5\}}$, $y \in C_{\{0,1\}}$, $w \in D_{\{4,5\}}$, and $z \in D_{\{0,1\}}$. But $w, z \in D_{\{2,3\}}$ is impossible because $\mathbb{D} \models z' < w$.

(iv) By part *(ii)* noticing that $x, y \in C_{\{2,3\}}$ is impossible as $\mathbb{C} \models y' < x$. □

3 Direct Factors

In this section, we describe directly indecomposable n-generated WNM-algebras, in fact the direct factors of $\mathbb{F}_n$.

Definition 2 (Signature, $\mathbb{C} \sim \mathbb{D}$) $\mathbb{C}$ and $\mathbb{D}$ in $\mathcal{K}_n$ have the same *signature* (in symbols, $\mathbb{C} \sim \mathbb{D}$) iff:

(S_1) $C_i = D_i$ for $i = 1, 2, 3, 4$;
(S_2) $\mathbb{C} \models x \diamond y$ iff $\mathbb{D} \models x \diamond y$ for all $x, y \in C_2$ and all $\diamond \in \{<, =\}$.

The signature relation is an equivalence relation on $\mathcal{K}_n = \{\mathbb{C}_1, \ldots, \mathbb{C}_k\}$. In the next section, we prove that $\{\mathbb{C}_i \mid i \in I\}$ is a block in the partition induced by the signature relation over $\mathcal{K}_n$ iff the subalgebra of $\prod_{j \in I} \mathbb{C}_j$ generated by $(x_i^{\mathbb{C}_j})_{j \in I}$ for $i = 1, \ldots, n$ is a direct factor of $\mathbb{F}_n$.

Example 4 ($n = 1$) The signature relation partitions $\mathcal{K}_1$ into four blocks, namely $\{\mathbb{C}_2, \mathbb{C}_3\}$, $\{\mathbb{C}_4\}$, $\{\mathbb{C}_5\}$, and $\{\mathbb{C}_6, \mathbb{C}_7\}$.

Example 5 ($n = 2$) See Appendix A. The signature relation partitions $\mathcal{K}_2$ into 18 blocks $B_1, \ldots, B_{18}$, namely, for $j = 1, \ldots, 18$, $B_j = \{\mathbb{C}_k \mid k \in K_j\}$ with

$K_1 = \{1, 2, 3, 4, 13, 14, 35, 36, 41, 42, 43, 44, 53, 54\}$,
$K_2 = \{5, 15\}$, $K_3 = \{6, 16\}$,
$K_4 = \{7, \ldots, 12, 17, \ldots, 22\}$, $K_5 = \{23\}$,
$K_6 = \{24\}$, $K_7 = \{25, 26\}$, $K_8 = \{27, 28\}$,
$K_9 = \{29, \ldots, 34, 39, 40, 69, \ldots, 74\}$, $K_{10} = \{37\}$,
$K_{11} = \{38\}$, $K_{12} = \{45, 55\}$, $K_{13} = \{46, 56\}$,
$K_{14} = \{47, \ldots, 52, 57, \ldots, 62\}$, $K_{15} = \{63\}$,
$K_{16} = \{64\}$, $K_{17} = \{65, 66\}$, $K_{18} = \{67, 68\}$.

Definition 3 (Infix, $\mathrm{infix}(\mathbb{C}, \mathbb{D})$) Let $\mathbb{C} \in \mathcal{K}_n$. For an interval $I = B_1 < \cdots < B_k$ in $\mathrm{bk}(\mathbb{C})$, we write $x \in I$ iff $x \in B_1 \cup \cdots \cup B_k$.

An *infix* of $\mathbb{C}$ is an interval I in $\mathrm{bk}(\mathbb{C})$ such that:

(I_1) There exists $x \in I$ such that $\mathbb{C} \models x = g_{\mathbb{C}}$ or $\mathbb{C} \models x = l_{\mathbb{C}}$.

Let $\mathbb{C} \sim \mathbb{D}$ in $\mathcal{K}_n$. Then, $\mathrm{infix}(\mathbb{C}, \mathbb{D})$ is the greatest common infix I of $\mathbb{C}$ and $\mathbb{D}$ such that:

(I_2) $x_i \in I$ and $x_i', x_i'' \notin I$, or $x_i, x_i', x_i'' \in I$ for all $i = 1, \ldots, n$.

Example 6 ($n = 1$) If $\mathcal{K}_1$ is partitioned as in Example 4, $\mathrm{infix}(\mathbb{C}_2, \mathbb{C}_3) = \mathrm{infix}(\mathbb{C}_6, \mathbb{C}_7) = \emptyset$.

Example 7 ($n = 2$) See Appendix A. Let $\{B_1, \ldots, B_{18}\}$ be the partition of $\mathcal{K}_2$ in Example 5. We list $\mathrm{infix}(\mathbb{C}, \mathbb{D})$ for every $\mathbb{C} \sim \mathbb{D}$ in $\mathcal{K}_2$. By direct computation:

$\mathrm{infix}(\mathbb{C}_2, \mathbb{C}_{13}) = y < y'' < y'$,
$\mathrm{infix}(\mathbb{C}_4, \mathbb{C}_{14}) = yy'' < y'$,
$\mathrm{infix}(\mathbb{C}_{42}, \mathbb{C}_{53}) = x < x'' < x'$,
and $\mathrm{infix}(\mathbb{C}_{44}, \mathbb{C}_{54}) = xx'' < x'$ (in B_1);
$\mathrm{infix}(\mathbb{C}_5, \mathbb{C}_{15}) = y < y'y''$ (in B_2);
$\mathrm{infix}(\mathbb{C}_6, \mathbb{C}_{16}) = yy'y''$ (in B_3);

$\text{infix}(\mathbb{C}_9, \mathbb{C}_{19}) = y' < y < y''$,
$\text{infix}(\mathbb{C}_7, \mathbb{C}_{17}) = \text{infix}(\mathbb{C}_7, \mathbb{C}_9) = \text{infix}(\mathbb{C}_{17}, \mathbb{C}_{19}) = \text{infix}(\mathbb{C}_7, \mathbb{C}_{19}) = \text{infix}(\mathbb{C}_{17}, \mathbb{C}_9) = y$,
$\text{infix}(\mathbb{C}_{12}, \mathbb{C}_{22}) = y' < yy''$,
$\text{infix}(\mathbb{C}_{11}, \mathbb{C}_8) = x < x'' < x'$,
and $\text{infix}(\mathbb{C}_{21}, \mathbb{C}_{18}) = xx'' < x'$ (in B_4);
$\text{infix}(\mathbb{C}_{25}, \mathbb{C}_{26}) = x < x'x''$ (in B_7);
$\text{infix}(\mathbb{C}_{27}, \mathbb{C}_{28}) = xx'x''$ (in B_8);
$\text{infix}(\mathbb{C}_{30}, \mathbb{C}_{31}) = x' < x < x''$,
$\text{infix}(\mathbb{C}_{29}, \mathbb{C}_{32}) = \text{infix}(\mathbb{C}_{29}, \mathbb{C}_{30}) = \text{infix}(\mathbb{C}_{29}, \mathbb{C}_{31}) = \text{infix}(\mathbb{C}_{30}, \mathbb{C}_{32}) = \text{infix}(\mathbb{C}_{31}, \mathbb{C}_{32}) = x$,
$\text{infix}(\mathbb{C}_{33}, \mathbb{C}_{34}) = x' < xx''$,
$\text{infix}(\mathbb{C}_{70}, \mathbb{C}_{71}) = y' < y < y''$,
$\text{infix}(\mathbb{C}_{69}, \mathbb{C}_{72}) = \text{infix}(\mathbb{C}_{70}, \mathbb{C}_{69}) = \text{infix}(\mathbb{C}_{70}, \mathbb{C}_{72}) = \text{infix}(\mathbb{C}_{69}, \mathbb{C}_{71}) = \text{infix}(\mathbb{C}_{71}, \mathbb{C}_{72}) = y$,
and $\text{infix}(\mathbb{C}_{73}, \mathbb{C}_{74}) = y' < yy''$ (in B_9);
$\text{infix}(\mathbb{C}_{45}, \mathbb{C}_{55}) = x < x'x''$ (in B_{12});
$\text{infix}(\mathbb{C}_{46}, \mathbb{C}_{56}) = xx'x''$ (in B_{13});
$\text{infix}(\mathbb{C}_{49}, \mathbb{C}_{59}) = x' < x < x''$,
$\text{infix}(\mathbb{C}_{47}, \mathbb{C}_{57}) = \text{infix}(\mathbb{C}_{47}, \mathbb{C}_{49}) = \text{infix}(\mathbb{C}_{47}, \mathbb{C}_{59}) = \text{infix}(\mathbb{C}_{57}, \mathbb{C}_{49}) = \text{infix}(\mathbb{C}_{57}, \mathbb{C}_{59}) = x$,
$\text{infix}(\mathbb{C}_{52}, \mathbb{C}_{62}) = x' < xx''$,
$\text{infix}(\mathbb{C}_{51}, \mathbb{C}_{48}) = y < y'' < y'$,
and $\text{infix}(\mathbb{C}_{61}, \mathbb{C}_{58}) = yy'' < y'$ (in B_{14});
$\text{infix}(\mathbb{C}_{65}, \mathbb{C}_{66}) = y < y'y''$ (in B_{17});
$\text{infix}(\mathbb{C}_{67}, \mathbb{C}_{68}) = yy'y''$ (in B_{18}).

Theorem 1 *Let $t \in \mathbb{T}_n$ and $\mathbb{C} \sim \mathbb{D}$ in $\mathcal{K}_n$.*
(i) $\mathbb{C} \models t' < t$ iff $\mathbb{D} \models t' < t$.
(ii) For all $x \in \text{infix}(\mathbb{C}, \mathbb{D})$, $\mathbb{C} \models t = x$ iff $\mathbb{D} \models t = x$.

The statement says that the partition of $\mathcal{K}_n$ under the equivalence relation $\sim$ yields in fact subsets $\{\mathbb{D}_1, \dots, \mathbb{D}_l\} \subseteq \mathcal{K}_n$ maximal under inclusion (in the powerset of $\mathcal{K}_n$) such that the subalgebra of $\mathbb{D}_1 \times \cdots \times \mathbb{D}_l$ generated by $(x_i^{\mathbb{D}_1}, \dots, x_i^{\mathbb{D}_1})$ for $i = 1, \dots, n$ lacks nontrivial direct factors.

Before proving the theorem, we illustrate the idea with two examples, which we then formalize in Section 4.3.

Example 8 ($n = 1$) See Figure 5.

Example 9 ($n = 2$) See Figure 6.

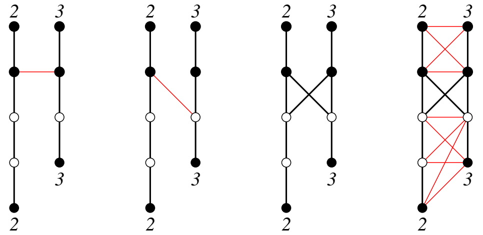

Fig. 5: Consider $\mathbb{C}_2 \sim \mathbb{C}_3$ in $\mathcal{K}_1$. The first diagram displays the term $x_1' \in T_1$ as a pair in the subalgebra of $\mathbb{C}_2 \times \mathbb{C}_3$ generated by $(x_1^{\mathbb{C}_2}, x_1^{\mathbb{C}_3})$; note that $\mathbb{C}_2, \mathbb{C}_3 \models x_1'' < x_1'$. The second diagram displays a maximal antichain in the disjoint union of $\mathbb{C}_2$ and $\mathbb{C}_3$, with $a_2 \in C_2$ and $a_3 \in C_3$, that is not realizable by a term $t \in T_1$, in the sense that there not exists a term $t \in T_1$ such that $\mathbb{C}_2 \models t = a_2$ and $\mathbb{C}_3 \models t = a_3$. In fact, $\mathbb{C}_2 \models x_1'' = t' < t = x_1'$ but $\mathbb{C}_3 \models x_1 = x_1'' = t < t' = x_1'$, impossible by Theorem 1(i). The third diagram presents a poset $\mathbf{P}$, extending the disjoint union of $\mathbb{C}_2$ and $\mathbb{C}_3$ with two new cover relations, such that there not exists a maximal antichain in $\mathbf{P}$ not realizable by a term $t \in T_1$ in the above sense; Theorem 3 proves that in fact each such maximal antichain is realized by a term $t \in T_1$.

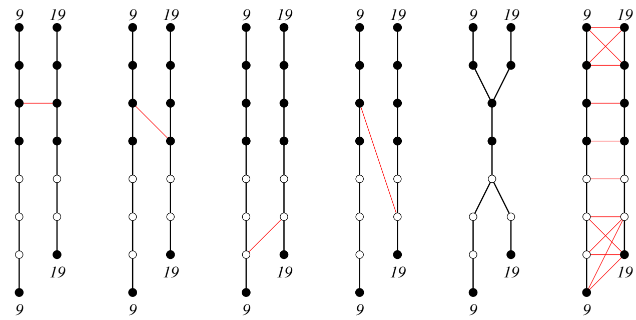

Fig. 6: Consider $\mathbb{C}_9 \sim \mathbb{C}_{19}$ in $\mathcal{K}_2$. Note that by Example 7, $\text{infix}(\mathbb{C}_9, \mathbb{C}_{19}) = y' < y < y''$. The configurations in the first and third diagrams are consistent with Theorem 1. The configurations in the second and fourth diagrams are inconsistent with Theorem 1, respectively violating (ii) and (i). All configurations in the sixth diagram, corresponding to maximal antichains in the poset in the fifth diagram, are consistent with Theorem 1; in fact, Theorem 3 shows that they are all realizable by terms in T_1.

3.1 Proof of Theorem 1

Proof By induction on t, we prove,

(i') for all $I \in \{\{0,1\}, \{2,3\}, \{4,5\}\}$, there exists $x \in \cup_{i \in I} C_i = C_I$ such that $\mathbb{C} \models t = x$ iff there exists $y \in \cup_{i \in I} D_i = D_I$ such that $\mathbb{D} \models t = y$,

and part (ii). Part (i) follows directly from (i') noticing that if $x \in C_{\{0,1\}} \cup C_{\{2,3\}}$ and $y \in D_{\{0,1\}} \cup D_{\{2,3\}}$, then by Fact 1, $\mathbb{C}, \mathbb{D} \models t \leq t'$, and if $x \in C_{\{4,5\}}$ and $y \in C_{\{4,5\}}$, then by Fact 1, $\mathbb{C}, \mathbb{D} \models t' < t$.

Base Case. $t \in \{0, 1, x_i \mid i = 1, \dots, n\}$.

Case $t \in \{0,1\}$: If $t = 1$, then $\mathbb{C} \models t = x$ iff $x = 1 \in C_5$, and $\mathbb{D} \models t = y$ iff $y = 1 \in D_5$. This settles (i). For (ii), if $x \in \text{infix}(\mathbb{C}, \mathbb{D})$, then $\mathbb{C} \models 1 = x$ iff $x = 1$ iff $\mathbb{D} \models 1 = x$. The case $t = 0$ is similar.

Case $t = x_i$ ($i \in \{1, \dots, n\}$): As $\mathbb{C} \in \mathcal{K}_n$, $x_i \in C_1 \cup C_2 \cup C_3 \cup C_4$, say $x_i \in C_j$. By Proposition 5, if $\mathbb{C} \models t = x$, then $x \in C_j$. By (S_1), $C_i = D_i$ for $i = 1, 2, 3, 4$. Then, letting $y = x_i \in D_j$ we have $\mathbb{D} \models t = y$. The converse is symmetric. This settles (i). For (ii), if $x \in \text{infix}(\mathbb{C}, \mathbb{D})$

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and $\mathbb{C} \models t = x_i = x$, then $\mathbb{D} \models t = x_i = x$ by definition. The converse is symmetric. This settles (ii).

Inductive Case. $t \in \{u \wedge v, u \cdot v, u \to v\}$.

Case $t = u \wedge v$: We distinguish four cases.

Case $\mathbb{C} \models u \leq v$ and $\mathbb{D} \models u \leq v$: Then, $\mathbb{C} \models t = u$ and $\mathbb{D} \models t = u$ and both parts follow by induction hypothesis.

Case $\mathbb{C} \models u \leq v$ and $\mathbb{D} \models u > v$: Let $\mathbb{C} \models u = x$ with $x \in C_I$ and $\mathbb{C} \models v = y$ with $y \in C_J$, so that by induction, $\mathbb{D} \models u = w$ with $w \in D_I$ and $\mathbb{D} \models v = z$ with $z \in D_J$ $(I, J \in \{\{0,1\}, \{2\}, \{3\}, \{4,5\}\})$.

In this case, by Proposition 7, it holds that either $x, y \in C_{\{0,1\}}$ and $w, z \in D_{\{0,1\}}$, or $x, y \in C_{\{2,3\}}$ and $w, z \in D_{\{2,3\}}$, or $x, y \in C_{\{4,5\}}$ and $w, z \in D_{\{4,5\}}$. Then part (i) follows, because $\mathbb{C} \models t = u \wedge v = x \wedge y \in \{x, y\}$ and $\mathbb{C} \models t = u \wedge v = w \wedge z \in \{w, z\}$.

For part (ii), we prove that if $\mathbb{C} \models x = u \leq v = y$ and $\mathbb{D} \models w = u > v = z$, so that $\mathbb{C} \models t = u = x$ and $\mathbb{D} \models t = v = z$, and $\mathbb{D} \models z \neq x$, then $x \notin \text{infix}(\mathbb{C}, \mathbb{D})$. Assume for a contradiction that $x \in \text{infix}(\mathbb{C}, \mathbb{D})$. Then $\mathbb{C} \models u = x$ implies $\mathbb{D} \models u = x$ inductively.

Suppose first that $x, y \in C_I$ and $w, z \in D_I$ with $I \in \{\{0,1\}, \{2,3\}\}$. Then, $y \in \text{infix}(\mathbb{C}, \mathbb{D})$ by (I_1), because $\mathbb{C} \models x \leq y \leq g_{\mathbb{C}} \prec l_{\mathbb{C}}$ and $x \in \text{infix}(\mathbb{C}, \mathbb{D})$. Then, $\mathbb{C} \models v = y$ implies $\mathbb{D} \models v = y$ inductively. Since $\mathbb{C} \models x \leq y$ and $x, y \in \text{infix}(\mathbb{C}, \mathbb{D})$, we have $\mathbb{D} \models u = x \leq y = v$, contradicting $\mathbb{D} \models u > v$.

Suppose now that $x, y \in C_{\{4,5\}}$ and $w, z \in D_{\{4,5\}}$. In this case, $\mathbb{D} \models g_{\mathbb{D}} \prec l_{\mathbb{D}} \leq z = v < w = u$, because $\mathbb{D} \models z' < z$. Observe that $z \notin \text{infix}(\mathbb{C}, \mathbb{D})$, because otherwise $\mathbb{D} \models v = z$ implies $\mathbb{C} \models v = z$ inductively, which implies that $\mathbb{C} \models x > z = v = y$ (contradicting $\mathbb{C} \models x \leq y$), since $\mathbb{D} \models x = u > z$ and $x, z \in \text{infix}(\mathbb{C}, \mathbb{D})$. Then, by (I_1), $w \notin \text{infix}(\mathbb{C}, \mathbb{D})$, a contradiction since $\mathbb{D} \models w = u = x$ and $x \in \text{infix}(\mathbb{C}, \mathbb{D})$.

Conversely, we prove that if $\mathbb{D} \models w = u > v = z$ and $\mathbb{C} \models x = u \leq v = y$, so that $\mathbb{D} \models t = v = z$ and $\mathbb{C} \models t = u = x$, and $\mathbb{C} \models x \neq z$, then $z \notin \text{infix}(\mathbb{C}, \mathbb{D})$. Assume for a contradiction that $z \in \text{infix}(\mathbb{C}, \mathbb{D})$. Then $\mathbb{D} \models v = z$ implies $\mathbb{C} \models v = z$ inductively.

Suppose first that $x, y \in C_I$ and $w, z \in D_I$ with $I \in \{\{0,1\}, \{2,3\}\}$. Then, $w \in \text{infix}(\mathbb{C}, \mathbb{D})$ by (I_1), because $\mathbb{D} \models z = v < u = w \leq g_{\mathbb{D}} \prec l_{\mathbb{D}}$. Then, $\mathbb{D} \models u = w$ implies $\mathbb{C} \models u = w$ inductively. Since $\mathbb{D} \models w > z$ and $w, z \in \text{infix}(\mathbb{C}, \mathbb{D})$, we have $\mathbb{C} \models u = w > z = v$, contradicting $\mathbb{C} \models u \leq v$.

Suppose now that $x, y \in C_{\{4,5\}}$ and $w, z \in D_{\{4,5\}}$. In this case, $\mathbb{C} \models g_{\mathbb{C}} \prec l_{\mathbb{C}} \leq x \leq y$. Observe that $x \notin \text{infix}(\mathbb{C}, \mathbb{D})$, because otherwise $\mathbb{D} \models u = x$ inductively, which implies that $\mathbb{D} \models u = x \leq z = v$ (contradicting $\mathbb{D} \models u > v$), since $\mathbb{C} \models x = u \leq v = z$ and $x, z \in \text{infix}(\mathbb{C}, \mathbb{D})$. Then, by (I_1), $y \notin \text{infix}(\mathbb{C}, \mathbb{D})$, a contradiction since $\mathbb{C} \models y = v = z$ and $z \in \text{infix}(\mathbb{C}, \mathbb{D})$.

This settles (ii).

Case $\mathbb{C} \models u > v$ and $\mathbb{D} \models u \leq v$: Swap $\mathbb{C}$ and $\mathbb{D}$ in the previous case.

Case $\mathbb{C} \models u > v$ and $\mathbb{D} \models u > v$: Then, $\mathbb{C} \models t = v$ and $\mathbb{D} \models t = v$ and both parts follow by induction hypothesis.

Case $t = u \to v$: We distinguish four cases.

Case $\mathbb{C} \models u \leq v$ and $\mathbb{D} \models u \leq v$: $\mathbb{C} \models t = x$ iff $x = 1 \in C_5$, and $\mathbb{D} \models t = y$ iff $y = 1 \in D_5$. This settles both (i) and (ii).

Case $\mathbb{C} \models u \leq v$ and $\mathbb{D} \models u > v$: Let $\mathbb{C} \models u = x$ with $x \in C_I$ and $\mathbb{C} \models v = y$ with $y \in C_J$, so that by induction, $\mathbb{D} \models u = w$ with $w \in D_I$ and $\mathbb{D} \models v = z$ with $z \in D_J$ $(I, J \in \{\{0,1\}, \{2,3\}, \{4,5\}\})$.

In this case by Proposition 7, it holds that either $x, y \in C_{\{0,1\}}$ and $w, z \in D_{\{0,1\}}$, or $x, y \in C_{\{2,3\}}$ and $w, z \in D_{\{2,3\}}$, or $x, y \in C_{\{4,5\}}$ and $w, z \in D_{\{4,5\}}$. For part (i), we have $\mathbb{C} \models t = x$ iff $x = 1 \in C_5$. If $w, z \in D_{\{0,1\}}$, then $\mathbb{D} \models t = u' \vee v = w' \vee z = w'$ with $w' \in D_{\{4,5\}}$; and if $w, z \in D_{\{4,5\}}$, then $\mathbb{D} \models t = u' \vee v = w' \vee z = z$ with $z \in D_{\{4,5\}}$. This settles (i).

For part (ii), we have $\mathbb{C} \models t = x$ with $x = 1 \in C_5$ and, $\mathbb{D} \models t = w'$ if $x, y \in C_I$ and $w, z \in D_I$ for $I \in \{\{0,1\}, \{2,3\}\}$, or $\mathbb{D} \models t = z$ if $x, y \in C_{\{4,5\}}$ and $w, z \in D_{\{4,5\}}$. We prove that $w' \notin \text{infix}(\mathbb{C}, \mathbb{D})$ in the first case, and $z \notin \text{infix}(\mathbb{C}, \mathbb{D})$ in the second case; both imply that $1 \notin \text{infix}(\mathbb{C}, \mathbb{D})$ by (I_1). This settles (ii).

Assume for a contradiction $w' \in \text{infix}(\mathbb{C}, \mathbb{D})$, with $w' \in D_4$. By (I_2), $w \in \text{infix}(\mathbb{C}, \mathbb{D})$. Then, $\mathbb{D} \models w = u$ implies $\mathbb{C} \models w = u$ inductively. Then, $\mathbb{C} \models w = u = x \leq y = v$. As $y \in C_I$, we have $\mathbb{C} \models y \leq y'$, then $\mathbb{C} \models w \leq y \leq g_{\mathbb{C}} \prec l_{\mathbb{C}}$ so that $y \in \text{infix}(\mathbb{C}, \mathbb{D})$ by (S_2). Then, $\mathbb{C} \models y = v$ implies $\mathbb{D} \models y = v$ inductively. Then, $\mathbb{C} \models w = x \leq y$ with $w, y \in \text{infix}(\mathbb{C}, \mathbb{D})$ implies $\mathbb{D} \models u = w = x \leq y = z = v$, contradicting $\mathbb{D} \models u > v$.

Assume for a contradiction $z \in \text{infix}(\mathbb{C}, \mathbb{D})$, with $z \in D_4$. Then, $\mathbb{D} \models z = v$ implies $\mathbb{C} \models z = v$ inductively. As $y \in C_{\{4,5\}}$, we have $\mathbb{C} \models y' < y$. Then $\mathbb{C} \models g_{\mathbb{C}} \prec l_{\mathbb{C}} \leq x \leq y = v = z$ and $z \in \text{infix}(\mathbb{C}, \mathbb{D})$, so that $x \in \text{infix}(\mathbb{C}, \mathbb{D})$ by (S_2). Then, $\mathbb{C} \models x = u$ implies $\mathbb{D} \models x = u$ inductively. Then, $\mathbb{C} \models x \leq y$ with $x, y \in \text{infix}(\mathbb{C}, \mathbb{D})$ implies $\mathbb{D} \models u = x \leq y = v$, a contradiction.

Case $\mathbb{C} \models u > v$ and $\mathbb{D} \models u \leq v$: Swap $\mathbb{C}$ and $\mathbb{D}$ in the previous case.

Case $\mathbb{C} \models u > v$ and $\mathbb{D} \models u > v$: Let $\mathbb{C} \models u = x$ with $x \in C_I$ and $\mathbb{C} \models v = y$ with $y \in C_J$, so that by induction, $\mathbb{D} \models u = w$ with $w \in D_I$ and $\mathbb{D} \models v = z$ with $z \in D_J$ $(I, J \in \{\{0,1\}, \{2,3\}, \{4,5\}\})$. By (7), we have $\mathbb{C} \models t = u' \vee v$ and $\mathbb{D} \models t = u' \vee v$.

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For part (i), by Proposition 7 and Fact 1 we have: $\mathbb{C} \models t = x' \vee y \in C_{\{4,5\}}$ and $\mathbb{D} \models t = w' \vee z \in D_{\{4,5\}}$ if $I = \{0,1\}$; $\mathbb{C} \models t = x' \vee y \in C_{\{2,3\}}$ and $\mathbb{D} \models t = w' \vee z \in D_{\{2,3\}}$ if $I = \{2,3\}$; $\mathbb{C} \models t = x' \vee y \in C_J$ and $\mathbb{D} \models t = w' \vee z \in D_J$ if $I = \{4,5\}$.

For part (ii), we have $\mathbb{C} \models t = u' \vee v$ and $\mathbb{D} \models t = u' \vee v$. If $\mathbb{C}, \mathbb{D} \models t = u' \vee v = u'$ and $u' \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$, or $\mathbb{C}, \mathbb{D} \models t = u' \vee v = v$ and $v \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$, we are done.

If $\mathbb{C} \models t = u' \vee v = u'$, $\mathbb{D} \models t = u' \vee v = v$, and $\mathbb{D} \models v > u'$, we claim that $u' \notin \mathrm{infix}(\mathbb{C}, \mathbb{D})$. Assume for a contradiction that $u' \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$. Note that in this case $I = \{4,5\}$ and $J = \{0,1\}$. Then, $\mathbb{D} \models u' < v \leq g_{\mathbb{D}} \prec l_{\mathbb{D}}$, and $v \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$ by (I_1). But $\mathbb{C} \models v \leq u'$ and $\mathbb{D} \models v > u'$ with $u', v \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$ is a contradiction. Analogously, if $\mathbb{D} \models t = v$ and $\mathbb{C} \models t = u' > v$, then $v \notin \mathrm{infix}(\mathbb{C}, \mathbb{D})$, because otherwise $\mathbb{C} \models v \leq u' \leq g_{\mathbb{C}} \prec l_{\mathbb{C}}$, so that $u' \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$, impossible because $\mathbb{C} \models v \leq u'$ and $\mathbb{D} \models v > u'$.

If $\mathbb{C} \models t = v$ and $\mathbb{D} \models t = u' > v$, we reason along the lines of the previous case swapping $\mathbb{C}$ and $\mathbb{D}$.

This settles (ii).

Case $t = u \cdot v$: We distinguish four cases.

Case $\mathbb{C} \models u \leq v'$ and $\mathbb{D} \models u \leq v'$: $\mathbb{C} \models t = x$ iff $x = 0 \in C_0$, and $\mathbb{D} \models t = y$ iff $y = 0 \in D_0$. This settles both (i) and (ii).

Case $\mathbb{C} \models u \leq v'$ and $\mathbb{D} \models u > v'$: Let $\mathbb{C} \models u = x$ with $x \in C_I$ and $\mathbb{C} \models v = y$ with $y \in C_J$, so that by induction, $\mathbb{D} \models u = w$ with $w \in D_I$ and $\mathbb{D} \models v = z$ with $z \in D_J$ $(I, J \in \{\{0,1\}, \{2,3\}, \{4,5\}\})$.

For part (i), by Proposition 7, the pair (I, J) is either $(\{0,1\}, \{4,5\})$ or $(\{4,5\}, \{0,1\})$. By (6), we have $\mathbb{C} \models t = 0$ and $\mathbb{D} \models t = u \wedge v$. Then, $\mathbb{D} \models t = u$ with $u \in D_{\{0,1\}}$, or $\mathbb{D} \models t = v$ with $v \in D_{\{0,1\}}$, which settles (i).

For part (ii), if $0 \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$, then $1 \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$, then $\mathbb{C} = \mathbb{D}$, impossible. Conversely, we show that if $\mathbb{D} \models t = u > 0$ then $u \notin \mathrm{infix}(\mathbb{C}, \mathbb{D})$, and if $\mathbb{D} \models t = v > 0$ then $v \notin \mathrm{infix}(\mathbb{C}, \mathbb{D})$.

Assume for a contradiction that $\mathbb{D} \models 0 < u = u \wedge v = t$ and $u \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$. From part (i), we have $u \in D_{\{0,1\}}$ and $v \in D_{\{4,5\}}$, so that $\mathbb{C} \models u \leq v' \leq g_{\mathbb{C}} \prec l_{\mathbb{C}}$, which implies $v' \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$, a contradiction since $\mathbb{C} \models u \leq v'$ and $\mathbb{D} \models u > v'$ with $u, v' \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$.

Next assume for a contradiction that $\mathbb{D} \models 0 < v = u \wedge v = t$ and $v \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$. From part (i), we have $v \in D_{\{0,1\}}$ and $u \in D_{\{4,5\}}$, so that $\mathbb{D} \models v < u' \leq g_{\mathbb{C}} \prec l_{\mathbb{C}}$, which implies $u' \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$, a contradiction since $\mathbb{C} \models u \leq v'$ and $\mathbb{D} \models u > v'$ with $u, v' \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$.

Case $\mathbb{C} \models u > v'$ and $\mathbb{D} \models u \leq v'$: Swap $\mathbb{C}$ and $\mathbb{D}$ in the previous case.

Case $\mathbb{C} \models u > v'$ and $\mathbb{D} \models u > v'$: Let $\mathbb{C} \models u = x$ with $x \in C_I$ and $\mathbb{C} \models v = y$ with $y \in C_J$, so that by induction, $\mathbb{D} \models u = w$ with $w \in D_I$ and $\mathbb{D} \models v = z$ with $z \in D_J$ $(I, J \in \{\{0,1\}, \{2,3\}, \{4,5\}\})$. We have $\mathbb{C} \models t = u \wedge v$ and $\mathbb{D} \models t = u \wedge v$ by (6).

By Proposition 7 and Fact 1, we have: $\mathbb{C} \models t = x \wedge y \in C_I$ and $\mathbb{D} \models t = w \wedge z \in D_I$ if $I \in \{\{0,1\}, \{2,3\}\}$; and $\mathbb{C} \models t = x \wedge y \in C_J$ and $\mathbb{D} \models t = w \wedge z \in D_J$ if $I = \{4,5\}$, which settles part (i).

For part (ii), if $\mathbb{C}, \mathbb{D} \models t = u$ with $u \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$, or $\mathbb{C}, \mathbb{D} \models t = v$ with $v \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$, the claim holds. If $\mathbb{C} \models t = u$ and $\mathbb{D} \models t = v < u$, we claim that $v \notin \mathrm{infix}(\mathbb{C}, \mathbb{D})$. Otherwise, assume $v \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$ for a contradiction. If $I, J \in \{\{0,1\}, \{2,3\}\}$, then $\mathbb{D} \models v < u \leq g_{\mathbb{D}} \prec l_{\mathbb{D}}$, then $u \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$, a contradiction since $\mathbb{C} \models u \leq v$ and $\mathbb{D} \models u > v$. If $I = \{4,5\}$, then $\mathbb{C} \models g_{\mathbb{D}} \prec l_{\mathbb{D}} \leq u \leq v$, then $u \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$, again a contradiction. The case $\mathbb{C} \models t = v$ and $\mathbb{D} \models t = u < v$ is symmetric. This settles part (ii).

4 Free Algebra

In this section, we describe the free n-generated WNM-algebra $\mathbb{F}_n$.

Definition 4 Let

$$A \subseteq \prod_{\mathbb{C} \in \mathcal{K}_n} C \tag{15}$$

be such that $a \in A$ iff, for all $\mathbb{C} \sim \mathbb{D}$ in $\mathcal{K}_n$:

(i) $\mathbb{C} \models \pi_{\mathbb{C}}(a)' < \pi_{\mathbb{C}}(a)$ iff $\mathbb{D} \models \pi_{\mathbb{D}}(a)' < \pi_{\mathbb{D}}(a)$;
(ii) If $x \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$, then $\mathbb{C} \models \pi_{\mathbb{C}}(a) = x$ iff $\mathbb{D} \models \pi_{\mathbb{D}}(a) = x$.

Proposition 8 *A in (15) is a subuniverse of the $(\wedge, \vee, x_1, \ldots, x_n)$-reduct of $\prod_{\mathbb{C} \in \mathcal{K}_n} \mathbb{C}$.*

Proof Let $\mathbb{B}$ denote the $(\wedge, \vee, x_1, \ldots, x_n)$-reduct of the product $\prod_{\mathbb{C} \in \mathcal{K}_n} \mathbb{C}$. Clearly $x_i^{\mathbb{B}} \in A$ for $i = 1, \ldots, n$. Let $a, b \in A$. We show that $a \wedge^{\mathbb{B}} b \in A$ (a similar argument shows that $a \vee^{\mathbb{B}} b \in A$). Note that for all $\mathbb{E} \in \mathcal{K}_n$, we have $\mathbb{E} \models \pi_{\mathbb{E}}(a) = \pi_{\mathbb{E}}(a \wedge^{\mathbb{B}} b)$ or $\mathbb{E} \models \pi_{\mathbb{E}}(b) = \pi_{\mathbb{E}}(a \wedge^{\mathbb{B}} b)$. Let $\mathbb{C} \sim \mathbb{D}$ in $\mathcal{K}_n$.

For part (i), we distinguish four cases. First, $\mathbb{C} \models \pi_{\mathbb{C}}(a) = \pi_{\mathbb{C}}(a \wedge^{\mathbb{B}} b)$ and $\mathbb{D} \models \pi_{\mathbb{D}}(a) = \pi_{\mathbb{D}}(a \wedge^{\mathbb{B}} b)$. Since $a \in A$, we have that $\mathbb{C} \models \pi_{\mathbb{C}}(a \wedge^{\mathbb{B}} b)' < \pi_{\mathbb{C}}(a \wedge^{\mathbb{B}} b)$ iff $\mathbb{D} \models \pi_{\mathbb{D}}(a \wedge^{\mathbb{B}} b)' < \pi_{\mathbb{D}}(a \wedge^{\mathbb{B}} b)$. Second, $\mathbb{C} \models \pi_{\mathbb{C}}(a) = \pi_{\mathbb{C}}(a \wedge^{\mathbb{B}} b)$ and $\mathbb{D} \models \pi_{\mathbb{D}}(b) = \pi_{\mathbb{D}}(a \wedge^{\mathbb{B}} b)$. If $\mathbb{C} \models \pi_{\mathbb{C}}(a)' < \pi_{\mathbb{C}}(a)$ and $\mathbb{D} \models \pi_{\mathbb{D}}(b)' < \pi_{\mathbb{D}}(b)$, then $\mathbb{C} \models \pi_{\mathbb{C}}(a \wedge^{\mathbb{B}} b)' < \pi_{\mathbb{C}}(a \wedge^{\mathbb{B}} b)$ and $\mathbb{D} \models \pi_{\mathbb{D}}(a \wedge^{\mathbb{B}} b)' < \pi_{\mathbb{D}}(a \wedge^{\mathbb{B}} b)$, and we are done. Similarly, if $\mathbb{C} \models \pi_{\mathbb{C}}(a) \leq \pi_{\mathbb{C}}(a)'$ and $\mathbb{D} \models \pi_{\mathbb{D}}(b) \leq \pi_{\mathbb{D}}(b)'$, we are done. The case $\mathbb{C} \models \pi_{\mathbb{C}}(a)' < \pi_{\mathbb{C}}(a)$ and $\mathbb{D} \models \pi_{\mathbb{D}}(b) \leq \pi_{\mathbb{D}}(b)'$ is impossible, because $\mathbb{C} \models \pi_{\mathbb{C}}(b)' \leq \pi_{\mathbb{C}}(a)' < \pi_{\mathbb{C}}(a) \leq \pi_{\mathbb{C}}(b)$ implies $\mathbb{D} \models \pi_{\mathbb{D}}(b)' < \pi_{\mathbb{D}}(b)$ since $b \in A$. Similarly, the case $\mathbb{C} \models$

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$\pi_{\mathbb{C}}(a)' < \pi_{\mathbb{C}}(a)$ and $\mathbb{D} \models \pi_{\mathbb{D}}(b)' < \pi_{\mathbb{D}}(b)$ is impossible. The remaining two cases are similar.

For part (ii), let $x \in \text{infix}(\mathbb{C}, \mathbb{D})$ be such that $\mathbb{C} \models \pi_{\mathbb{C}}(a \wedge^{\mathbb{B}} b) = x$. We show that if $\mathbb{C} \models \pi_{\mathbb{C}}(a) = x$, then $\mathbb{D} \models \pi_{\mathbb{D}}(a \wedge^{\mathbb{B}} b) = x$; a symmetric argument shows that if $\mathbb{C} \models \pi_{\mathbb{C}}(b) = x$, then $\mathbb{D} \models \pi_{\mathbb{D}}(a \wedge^{\mathbb{B}} b) = x$.

Assume $\mathbb{C} \models \pi_{\mathbb{C}}(a) = x$. Then $\mathbb{D} \models \pi_{\mathbb{D}}(a) = x$ since $a \in A$. If there exists $y \in \text{infix}(\mathbb{C}, \mathbb{D})$ such that $\mathbb{C} \models \pi_{\mathbb{C}}(b) = y$, then since $\mathbb{C} \models x \leq y$ implies $\mathbb{D} \models x \leq y$, we have $\mathbb{D} \models \pi_{\mathbb{D}}(a \wedge^{\mathbb{B}} b) = x$, and we are done. If there does not exist $y \in \text{infix}(\mathbb{C}, \mathbb{D})$ such that $\mathbb{C} \models \pi_{\mathbb{C}}(b) = y$, then $\mathbb{C} \models \pi_{\mathbb{C}}(b)' < \pi_{\mathbb{C}}(b)$ by (I_1), so that $\mathbb{D} \models \pi_{\mathbb{D}}(b)' < \pi_{\mathbb{D}}(b)$ by part (i). Then $\mathbb{D} \models \pi_{\mathbb{D}}(a) = x \leq \pi_{\mathbb{D}}(b)$ by (I_1). Then $\mathbb{D} \models \pi_{\mathbb{D}}(a \wedge^{\mathbb{B}} b) = x$, and we are done. □

By Proposition 8, the $(\wedge, \vee, x_1, \ldots, x_n)$-structure

$$\mathbb{A} = (A, \wedge^{\mathbb{A}}, \vee^{\mathbb{A}}, x_1^{\mathbb{A}}, \ldots, x_n^{\mathbb{A}}), \tag{16}$$

with operations and constants inherited from the $(\wedge, \vee, x_1, \ldots, x_n)$-reduct of $\prod_{\mathbb{C} \in \mathcal{K}_n} \mathbb{C}$, is a distributive lattice. In this setting, we note that $a \in A$ is join irreducible iff there exists $\mathbb{C} \in \mathcal{K}_n$ and $c \in \mathbb{C}$ such that a is the least element in $\mathbb{A}$ such that $\mathbb{C} \models \pi_{\mathbb{C}}(a) = c$; in this case, we say that a *corresponds* to c and we write $a_{\mathbb{C}}^c$.

We prove the two key facts. Theorem 2 proves that for every term $t \in T_n$ there exists a tuple $a \in A$ that corresponds to t, and Corollary 1 in Section 4 proves that for every tuple $a \in A$ there exists a term $t \in T_n$ that corresponds to a.

Theorem 2 *For all $t \in T_n$, there exists $a \in A$ such that $\prod_{\mathbb{C} \in \mathcal{K}_n} \mathbb{C} \models t = a$.*

Proof Let $t \in T_n$. Let $a \coloneqq (t^{\mathbb{C}})_{\mathbb{C} \in \mathcal{K}_n} \in \prod_{\mathbb{C} \in \mathcal{K}_n} C$. Then $\prod_{\mathbb{C} \in \mathcal{K}_n} \mathbb{C} \models t = a$ by definition. We claim that $a \in A$.

Let $\mathbb{C} \sim \mathbb{D}$ in $\mathcal{K}_n$. Then by Theorem 1, $\mathbb{C} \models t' < t$ iff $\mathbb{D} \models t' < t$; and for all $x \in \text{infix}(\mathbb{C}, \mathbb{D})$, $\mathbb{C} \models t = x$ iff $\mathbb{D} \models t = x$. The statement follows immediately because $\mathbb{C} \models \pi_{\mathbb{C}}(a) = t^{\mathbb{C}} = t$ and $\mathbb{D} \models \pi_{\mathbb{D}}(a) = t^{\mathbb{C}} = t$ by definition of a. □

We now prove the second key fact in a sequence of lemmas.

Theorem 3 (Join Irreducible) *Let $a \in A$ be join irreducible. Then, there exists a term $t_a \in T_n$ such that $\prod_{\mathbb{C} \in \mathcal{K}_n} \mathbb{C} \models t_a = a$.*

Lemma 1 *Let $\mathbb{C} \in \mathcal{K}_n$, let $c \in C$ such that $\mathbb{C} \models l_{\mathbb{C}} \leq c$, and let $a_{\mathbb{C}}^c \in A$ be the join irreducible element corresponding to c. Then, there exists a term $t_{\mathbb{C}}^c \in T_n$ such that $\prod_{\mathbb{C} \in \mathcal{K}_n} \mathbb{C} \models t_{\mathbb{C}}^c = a_{\mathbb{C}}^c$.*

Proof (Proof of Lemma 1) See Section 4.1.

Lemma 2 *Let $\mathbb{C} \in \mathcal{K}_n$, let $c \in C$ such that $\mathbb{C} \models c \leq g_{\mathbb{C}}$, and let $a_{\mathbb{C}}^c \in A$ be the join irreducible element in $\mathbb{A}$ corresponding to c. Then, there exists a term $t_{\mathbb{C}}^c \in T_n$ such that $\prod_{\mathbb{C} \in \mathcal{K}_n} \mathbb{C} \models t_{\mathbb{C}}^c = a_{\mathbb{C}}^c$.*

Proof (Proof of Lemma 2) See Section 4.2.

Proof (Proof of Theorem 3) Immediate by Lemma 1 and Lemma 2, noticing that $a \in A$ is join irreducible iff exactly one of the following two cases occurs: there exists $\mathbb{C} \in \mathcal{K}_n$ and $c \in \mathbb{C}$ such that $\mathbb{C} \models l_{\mathbb{C}} \leq c$; or, there exists $\mathbb{C} \in \mathcal{K}_n$ and $c \in \mathbb{C}$ such that $\mathbb{C} \models c \leq g_{\mathbb{C}}$.

Corollary 1 (Normal Forms) *For all $a \in A$, there exists a term $t \in T_n$ such that $\prod_{\mathbb{C} \in \mathcal{K}_n} \mathbb{C} \models t = a$.*

Proof Let $a \in A$. Then there exist join irreducible $a_1, \ldots, a_k \in A$ such that $\prod_{\mathbb{C} \in \mathcal{K}_n} \mathbb{C} \models a = a_1 \vee \cdots \vee a_k$. For $t_{a_1}, \ldots, t_{a_k}$ the terms given by Theorem 3, let $t = t_{a_1} \vee \cdots \vee t_{a_k}$. Then, $\prod_{\mathbb{C} \in \mathcal{K}_n} \mathbb{C} \models t = a$. □

Using Theorem 2 and Corollary 1, expand the $(\wedge, \vee, x_1, \ldots, x_n)$-algebra $\mathbb{A}$ in (16) to a σ_n-algebra,

$$\mathbb{A} = (A, \wedge^{\mathbb{A}}, \vee^{\mathbb{A}}, \cdot^{\mathbb{A}}, \to^{\mathbb{A}}, 0^{\mathbb{A}}, 1^{\mathbb{A}}, x_1^{\mathbb{A}}, \ldots, x_n^{\mathbb{A}}), \tag{17}$$

by putting:

1. if $a \in A$ is such that $\prod_{\mathbb{C} \in \mathcal{K}_n} \mathbb{C} \models 0 = a$ (respectively, $\prod_{\mathbb{C} \in \mathcal{K}_n} \mathbb{C} \models 1 = a$), then $0^{\mathbb{A}} = a$ (respectively, $1^{\mathbb{A}} = a$);
2. for all $\circ \in \{\cdot, \to\}$ and $a, b \in A$, if $r, s \in T_n$ are such that $\prod_{\mathbb{C} \in \mathcal{K}_n} \mathbb{C} \models r = a$ and $\prod_{\mathbb{C} \in \mathcal{K}_n} \mathbb{C} \models s = b$ and $c \in A$ is such that $\prod_{\mathbb{C} \in \mathcal{K}_n} \mathbb{C} \models r \circ s = c$, then $a \circ^{\mathbb{A}} b = c$.

Theorem 4 (Free Algebra) *The σ_n-algebra $\mathbb{A}$ in (17) is σ_n-isomorphic to the free n-generated WNM-algebra $\mathbb{F}_n$.*

Proof Let $\mathbb{B}$ denote the σ_n-algebra $\prod_{\mathbb{C} \in \mathcal{K}_n} \mathbb{C}$. Then $\mathbb{A}$ is σ_n-isomorphic to the σ_n-subalgebra of $\mathbb{B}$ generated by $x_i^{\mathbb{B}}$ for $i = 1, \ldots, n$, via the σ_n-isomorphism that sends $x_i^{\mathbb{A}}$ to $x_i^{\mathbb{B}}$ for $i = 1, \ldots, n$. □

Example 10 $(n = 1)$ $\mathbb{F}_1$ is displayed in Figure 7.

Example 11 $(n = 2)$ $\mathbb{F}_2$ is displayed in Appendix A.

4.1 Lemma 1

Definition 5 Define the binary operations,

$$x <_1 y \coloneqq (y \to x) \to y' \wedge x \to y, \tag{18}$$

$$x <_2 y \coloneqq (y \to x) \to (y \to x)', \tag{19}$$

$$x <_4 y \coloneqq (y \to x) \to y, \tag{20}$$

$$x =_i y \coloneqq (x \to y \wedge y \to x)^2, \tag{21}$$

where $i \in \{1, 2, 4\}$.

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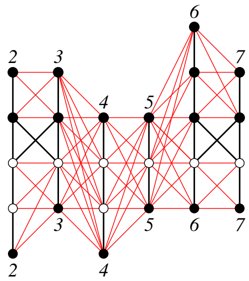

Fig. 7: $|\mathbb{F}_1| = 1200$.

Proposition 9 *Let* $\mathbb{D} \in \mathcal{K}_n$ *and* $\diamond \in \{<, =\}$. *Then:*

(i) *if* $x, y \in D_1$, *then*

$$\mathbb{D} \models x \diamond_1 y = \begin{cases} 1 & \textit{if } \mathbb{D} \models x \diamond y, \\ x' \wedge y' & \textit{otherwise;} \end{cases}$$

(ii) *if* $x, y \in D_2$, *then*

$$\mathbb{D} \models x \diamond_2 y = \begin{cases} 1 & \textit{if } \mathbb{D} \models x \diamond y, \\ 0 & \textit{otherwise;} \end{cases}$$

(iii) *if* $x, y \in D_4$, *then*

$$\mathbb{D} \models x \diamond_4 y = \begin{cases} 1 & \textit{if } \mathbb{D} \models x \diamond y, \\ x \wedge y & \textit{otherwise.} \end{cases}$$

Proof (Proof of Proposition 9) (i) Let $x, y \in D_1$. If $\mathbb{D} \models x < y < y' \leq x'$, then $\mathbb{D} \models x <_1 y = (y' \vee x) \to y' \wedge 1 = y' \to y' = 1$; if $\mathbb{D} \models y = x < x' = y'$, then $\mathbb{D} \models x <_1 y = 1 \to y' \wedge 1 = y' = x' \wedge y'$; if $\mathbb{D} \models y < x < x' \leq y'$, then $\mathbb{D} \models x <_1 y = 1 \to y' \wedge (x' \vee y) = x' \wedge y'$.

If $\mathbb{D} \models x = y$, then $\mathbb{D} \models x =_1 y = 1^2 = 1$; if $\mathbb{D} \models x < y < y' \leq x'$, then $\mathbb{D} \models x =_1 y = (1 \wedge (y' \vee x))^2 = (y')^2 = x' \wedge y'$; if $\mathbb{D} \models y < x < x' \leq y'$, then $\mathbb{D} \models x =_1 y = ((x' \vee y) \wedge 1)^2 = (x')^2 = x' \wedge y'$.

(ii) Let $x, y \in D_2$. If $\mathbb{D} \models x < y < y'' = y' = x'' = x'$, then $\mathbb{D} \models x <_2 y = (y' \vee x) \to (y' \vee x)' = y' \to y'' = 1$; if $\mathbb{D} \models y \leq x$, then $\mathbb{D} \models x <_2 y = 1 \to 1' = 1 \to 0 = 0$.

If $\mathbb{D} \models x = y$, then $\mathbb{D} \models x =_2 y = 1^2 = 1$; if $\mathbb{D} \models x < y < y'' = y' = x'' = x'$, then $\mathbb{D} \models x =_2 y = (1 \wedge (y' \vee x))^2 = (y')^2 = 0$; if $\mathbb{D} \models y < x < x'' = x' = y'' = y'$, then $\mathbb{D} \models x =_2 y = ((x' \vee y) \wedge 1)^2 = (x')^2 = 0$.

(iii) Let $x, y \in D_4$. If $\mathbb{D} \models y' \leq x' < x < y$, then $\mathbb{D} \models x <_4 y = (y' \vee x) \to y = x \to y = 1$; if $\mathbb{D} \models y \leq x$, then $\mathbb{D} \models x <_4 y = 1 \to y = y = x \wedge y$.

If $\mathbb{D} \models x = y$, then $\mathbb{D} \models x =_4 y = 1^2 = 1$; if $\mathbb{D} \models x < y$, then $\mathbb{D} \models x =_4 y = (1 \wedge x)^2 = x^2 = x = x \wedge y$; if $\mathbb{D} \models y < x$, then $\mathbb{D} \models x =_4 y = (y \wedge 1)^2 = y^2 = y = x \wedge y$. $\square$

Proposition 10 *For* $I \in \{\{2,3\}, \{4\}, \{5\}, \{6,7\}\}$, *there exists a unary term* o_I *such that, for all* $i = 1, \ldots, n$ *and all* $\mathbb{D} \in \mathcal{K}_n$,

$$\mathbb{D} \models o_I(x_i) = \begin{cases} 1 & \textit{if } \mathrm{orbit}(\mathbb{D}, x_i) \in I, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof (Proof of Proposition 10) Let $i \in \{1, \ldots, n\}$ and $\mathbb{D} \in \mathcal{K}_n$. By direct computation,

$$\mathbb{D} \models o_{\{2,3\}}(x_i) \risingdotseq x_i'' <_2 x_i' = \begin{cases} 1 & \text{if } \mathrm{orbit}(\mathbb{D}, x_i) \in \{2,3\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathbb{D} \models o_{\{5\}}(x_i) \risingdotseq x_i =_2 x_i' = \begin{cases} 1 & \text{if } \mathrm{orbit}(\mathbb{D}, x_i) = 5, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbb{D} \models o_{\{6,7\}}(x_i) \risingdotseq x_i' <_2 x_i = \begin{cases} 1 & \text{if } \mathrm{orbit}(\mathbb{D}, x_i) \in \{6,7\}, \\ 0 & \text{otherwise,} \end{cases}$$

and $o_{\{4\}}(x_i) = (o_{\{2,3\}}(x_i) \vee o_{\{5\}}(x_i) \vee o_{\{6,7\}}(x_i))'$ are the desired terms. $\square$

We now prove Lemma 1. We want to show that if $\mathbb{C} \in \mathcal{K}_n$, $c \in C$ is such that $\mathbb{C} \models l_{\mathbb{C}} \leq c$, and $a_{\mathbb{C}}^c \in A$ is join irreducible element in $\mathbb{A}$ corresponding to c, then there exists a term $t_{\mathbb{C}}^c \in \mathbb{T}_n$ such that $\mathbb{A} \models t_{\mathbb{C}}^c = a_{\mathbb{C}}^c$.

Proof (Proof of Lemma 1) Let $\mathbb{C} \in \mathcal{K}_n$ and let $c \in C$ be such that $\mathbb{C} \models l_{\mathbb{C}} \leq c = z$ where $z \in \{0, 1, x_i, x_i', x_i'' \mid i = 1, \ldots, n\}$. Let,

$$r_{\mathbb{C}} \risingdotseq \bigwedge_{I \in \{\{2,3\},\{4\},\{5\},\{6,7\}\}} \; \bigwedge_{\{i \mid \mathrm{orbit}(\mathbb{C}, x_i) \in I\}} o_I(x_i), \tag{22}$$

$$e_{\mathbb{C}} \risingdotseq r_{\mathbb{C}} \wedge \bigwedge_{\diamond \in \{<,=\}} \; \bigwedge_{\{(x,y) \in C_2^2 \mid \mathbb{C} \models x \diamond y\}} x \diamond_2 y, \tag{23}$$

$$s_{\mathbb{C}} \risingdotseq \bigwedge_{i \in \{1,4\}} \; \bigwedge_{\diamond \in \{<,=\}} \; \bigwedge_{\{(x,y) \in C_i^2 \mid \mathbb{C} \models x \diamond y\}} x \diamond_i y. \tag{24}$$

Observe preliminarily that, for all $\mathbb{D} \in \mathcal{K}_n$,

$$\mathbb{D} \models e_{\mathbb{C}} = \begin{cases} 1 & \text{if } \mathbb{C} \sim \mathbb{D}, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$t_{\mathbb{C}}^1 = e_{\mathbb{C}} \wedge s_{\mathbb{C}}.$$

We claim that the term,

$$t_{\mathbb{C}}^c = z \wedge t_{\mathbb{C}}^1,$$

satisfies the statement. Let $a_{\mathbb{C}}^1 \in A$ be the join irreducible element in $\mathbb{A}$ corresponding to $1 \in C$. We show that $\mathbb{D} \models t_{\mathbb{C}}^1 = \pi_{\mathbb{D}}(a_{\mathbb{C}}^1)$ for all $\mathbb{D} \in \mathcal{K}_n$, so that $\mathbb{A} \models t_{\mathbb{C}}^1 = a_{\mathbb{C}}^1$. The rest follows directly. Let $\mathbb{D} \in \mathcal{K}_n$.

If $\mathbb{D} = \mathbb{C}$, by construction $\mathbb{D} \models t_{\mathbb{C}}^1 = 1 = \pi_{\mathbb{C}}(a_{\mathbb{C}}^1) = \pi_{\mathbb{D}}(a_{\mathbb{C}}^1)$. Assume $\mathbb{C} \neq \mathbb{D}$.

If $\mathbb{C} \not\sim \mathbb{D}$, then $\mathbb{D} \models t_{\mathbb{C}} = 0 = \pi_{\mathbb{D}}(a^1_{\mathbb{C}})$ by the preliminar observation. Assume $\mathbb{C} \sim \mathbb{D}$.

Let $I = \mathrm{infix}(\mathbb{C}, \mathbb{D})$. Let $C_{4,i} \in \mathrm{bk}(\mathbb{C})$ be the least block in $\mathrm{bk}(\mathbb{C})$ not in I, and let $D_{4,i} \in \mathrm{bk}(\mathbb{D})$ be the least block in $\mathrm{bk}(\mathbb{D})$ not in I. If such pair $C_{4,i}$ and $D_{4,i}$ does not exist, then C_5 and D_5 are in I, that is, $1 \in I$, so that $0 \in I$ by (I_2), and therefore $\mathbb{C} = \mathbb{D}$. Clearly, $\mathbb{D} \models y = \pi_{\mathbb{D}}(a^1_{\mathbb{C}})$, where $y \in D_{4,i}$. Then, it is sufficient to show that $\mathbb{D} \models t^1_{\mathbb{C}} = y$.

If $C_{4,i} \neq D_{4,i}$, then there exist $x \in C_{4,i} \setminus D_{4,i}$ and $y \in D_{4,i}$ (notice that by definition, $C_{4,i}, D_{4,i} \neq \emptyset$). By (S_1), $C_4 = D_4$. Note that there exists $D_{4,i} < D_{4,j}$ such that $x \in D_{4,j}$ so that $\mathbb{D} \models y < x$. If $y \in C_{4,i}$, since $\mathbb{C} \models y = x$, the term $t^1_{\mathbb{C}}$ contains the conjunct $y =_4 x$. Then, $\mathbb{D} \models t^1_{\mathbb{C}} \leq (y =_4 x) = y \wedge x = y$. If $y \notin C_{4,i}$, then there exists $C_{4,i} < C_{4,k}$ such that $y \in C_{4,k}$ and $\mathbb{C} \models x < y$, so that the term $t^1_{\mathbb{C}}$ contains the conjunct $x <_4 y$. Since $\mathbb{D} \models y < x$, we have $\mathbb{D} \models t^1_{\mathbb{C}} \leq (x <_4 y) = x \wedge y = y$. Since $\mathbb{C} \models t^1_{\mathbb{C}} = 1$, and $D_{4,i}$ is the least block in $\mathrm{bk}(\mathbb{D})$ not in I. Assume for a contradiction that $\mathbb{D} \models z = t^1_{\mathbb{C}} < y$. Since $\mathbb{C} \models 0 = (t^1_{\mathbb{C}})' < t^1_{\mathbb{C}} = 1$, by Theorem 1$(i)$ we have $\mathbb{D} \models (t^1_{\mathbb{C}})' < t^1_{\mathbb{C}}$, therefore, $\mathbb{D} \models l_{\mathbb{D}} \leq z \leq y$ with $z \in I$ by (I_1), so that by Theorem 1(ii) we have $\mathbb{C} \models z = t^1_{\mathbb{C}}$. But, $\mathbb{C} \models z < x < 1$, a contradiction. So, $\mathbb{D} \models y \leq t_{\mathbb{C}}$, and therefore $\mathbb{D} \models t^1_{\mathbb{C}} = y$.

If $C_{4,i} = D_{4,i}$ but $D_{4,i} \notin I$, by (I_2) it is the case that there exists $y' \in D_{4,i} \cap \{x'_i, x''_i \mid i = 1, \ldots, n\}$ such that if $y \in D_{1,j}$; indeed, if $z \in \{x_i \mid i = 1, \ldots, n\}$ for all $z \in D_{4,i}$, then $C_{4,i} = D_{4,i}$ implies $D_{4,i} \in I$. Then, the interval $D_{1,j} < \cdots < D_{4,i}$ is not a common infix to $\mathbb{C}$ and $\mathbb{D}$. In this case, there exist $v, w \in C_1 = D_1$ such that either $\mathbb{C} \models y \leq v = w \leq g_{\mathbb{C}} \prec l_{\mathbb{C}}$ and $\mathbb{D} \models y \leq v < w \leq g_{\mathbb{D}} \prec l_{\mathbb{D}}$, or $\mathbb{C} \models y \leq v < w \leq g_{\mathbb{C}} \prec l_{\mathbb{C}}$ and $\mathbb{D} \models y \leq v = w \leq g_{\mathbb{D}} \prec l_{\mathbb{D}}$.

In the first case, the term $t^1_{\mathbb{C}}$ contains the conjunct $v =_1 w$. Since $\mathbb{D} \models v < w$, we have $\mathbb{D} \models t^1_{\mathbb{C}} \leq (v =_1 w) = v' \wedge w' = w' \leq y'$. In the second case, the term $t_{\mathbb{C}}$ contains the conjunct $v <_1 w$. Since $\mathbb{D} \models v = w$ so that $\mathbb{D} \models w' \leq v'$, we have $\mathbb{D} \models t^1_{\mathbb{C}} \leq (v <_1 w) = v' \wedge w' = w' \leq y'$. Since $\mathbb{C} \models t^1_{\mathbb{C}} = 1$, and and $D_{4,i}$ is the least block in $\mathrm{bk}(\mathbb{D})$ not in I, we have $\mathbb{D} \models y' \leq t^1_{\mathbb{C}}$ by Theorem 1. So, $\mathbb{D} \models t^1_{\mathbb{C}} = y'$. $\square$

Example 12 Let $a_1, \ldots, a_9 \in F_1$ be the idempotent join irreducible elements in $\mathbb{F}_1$, depicted in Figure 8 from left to right and top to bottom respectively. Let:

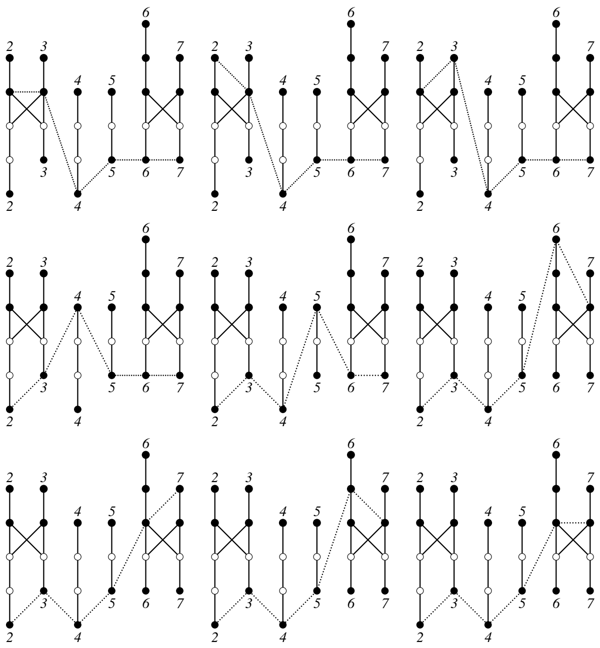

Fig. 8: The 9 idempotent join irreducible elements in $\mathbb{F}_1$, respectively $a_1, \ldots, a_9 \in F_1$ from left to right, and top to bottom.

$$
\begin{aligned}
t_1 &\coloneqq x' \wedge o_{\{2,3\}}(x),\\
t_2 &\coloneqq o_{\{2,3\}}(x) \wedge (x <_1 x''),\\
t_3 &\coloneqq o_{\{2,3\}}(x) \wedge (x =_1 x''),\\
t_4 &\coloneqq o_{\{4\}}(x),\\
t_5 &\coloneqq o_{\{5\}}(x),\\
t_6 &\coloneqq o_{\{6,7\}}(x) \wedge (x <_4 x''),\\
t_7 &\coloneqq o_{\{6,7\}}(x) \wedge (x =_4 x''),\\
t_8 &\coloneqq x'' \wedge o_{\{6,7\}}(x) \wedge (x <_4 x''),\\
t_9 &\coloneqq x \wedge o_{\{6,7\}}(x),
\end{aligned}
$$

so that $t_i \in \mathbb{T}_1$ correspond to $a_i \in F_1$ as by Lemma 1, that is,

$$\mathbb{F}_1 \models t_i = a_i,$$

for $i = 1, \ldots, 9$.

4.2 Lemma 2

We want to show that if $\mathbb{C} \in \mathcal{K}_n$, $c \in C$ is such that $\mathbb{C} \models c \leq g_{\mathbb{C}}$, and $a^c_{\mathbb{C}} \in A$ is the join irreducible element in $\mathbb{A}$ corresponding to c, then there exists a term $t^c_{\mathbb{C}} \in \mathbb{T}_n$ such that $\mathbb{A} \models t^c_{\mathbb{C}} = a^c_{\mathbb{C}}$.

Proof (Proof of Lemma 2) Let $\mathbb{C} \in \mathcal{K}_n$ and let $c \in C$ be such that $\mathbb{C} \models c \leq g_{\mathbb{C}}$. Note $c \in C_0 \cup C_1 \cup C_2 \cup C_3$. Let $a^c_{\mathbb{C}} \in A$ be join irreducible element of $\mathbb{A}$ corresponding to $c \in C$. We define $t^c_{\mathbb{C}} \in \mathbb{T}_n$ such that $\mathbb{A} \models t^c_{\mathbb{C}} = a^c_{\mathbb{C}}$. Two cases.

Case 1: $c \in C_2 \cup C_3$.

If $c \in C_2$, then let $i \in \{1, \ldots, n\}$ be such that $\mathbb{C} \models c = x_i$. Let $e_{\mathbb{C}} \in \mathbb{T}_n$ be as in (23). Let,

$$t^c_{\mathbb{C}} \risingdotseq e_{\mathbb{C}} \wedge x_i.$$

If $c \in C_3$, then let $i \in \{1, \ldots, n\}$ be such that $\mathbb{C} \models c = x'_i$. Let,

$$t^c_{\mathbb{C}} \risingdotseq e_{\mathbb{C}} \wedge x'_i.$$

We claim,

$$\mathbb{A} \models t^c_{\mathbb{C}} = a^c_{\mathbb{C}}.$$

Case 2: $c \in C_0 \cup C_1$. Two subcases.

Subcase 2.1: $\mathbb{C} \models c = d'$ for some $d \in C_4 \cup C_5$. Without loss of generality, let $d \in C_4 \cup C_5$ be the largest in $\mathbb{C}$ such that $\mathbb{C} \models c = d'$.

For $\mathbb{E} \in \mathcal{K}_n$ and $e \in E$ be such that $\mathbb{E} \models l_{\mathbb{E}} \leq e$, we let $a^e_{\mathbb{E}} \in A$ be the join irreducible element of $\mathbb{A}$ corresponding to $e \in E$ and, by Lemma 1, we let $t^e_{\mathbb{E}} \in \mathbb{T}_n$ be such that $\mathbb{A} \models t^e_{\mathbb{E}} = a^e_{\mathbb{E}}$. We define

$$t^c_{\mathbb{C}} \risingdotseq \left(t^d_{\mathbb{C}} \vee \bigvee_{\mathbb{C} \sim \mathbb{D}, d \notin \mathrm{infix}(\mathbb{C}, \mathbb{D})} t^1_{\mathbb{D}} \vee \bigvee_{\mathbb{C} \not\sim \mathbb{D}} t^1_{\mathbb{D}} \right)',$$

and we claim,

$$\mathbb{A} \models t^c_{\mathbb{C}} = a^c_{\mathbb{C}}.$$

We prove the claim. Note that $\mathbb{C} \models t^c_{\mathbb{C}} = d' = c$ by construction. Assume for a contradiction that there exists $\mathbb{D} \in \mathcal{K}_n$ such that $\mathbb{D} \models \pi_{\mathbb{D}}(a^c_{\mathbb{C}}) < t^c_{\mathbb{C}}$. If $\mathbb{D} \not\sim \mathbb{C}$, then $\mathbb{D} \models 0 = t^c_{\mathbb{C}}$ by construction; so, $\mathbb{D} \sim \mathbb{C}$. If $\mathbb{D} \sim \mathbb{C}$ and $d \notin \mathrm{infix}(\mathbb{C}, \mathbb{D})$, then $\mathbb{D} \models 0 = t^c_{\mathbb{C}}$ by construction. So, $\mathbb{D} \sim \mathbb{C}$ and $d \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$. If $d' = c \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$, then $\mathbb{D} \models c = t^c_{\mathbb{C}}$ by Theorem 1, and $\mathbb{D} \models c = \pi_{\mathbb{D}}(a^c_{\mathbb{C}})$. So, $d' = c \notin \mathrm{infix}(\mathbb{C}, \mathbb{D})$. By (I_2), it is the case that for some $i \in \{1, \ldots, n\}$, $\mathbb{C} \models d = x_i$ and $\mathbb{C} \models d' = x'_i$ with $x'_i \notin \mathrm{infix}(\mathbb{C}, \mathbb{D})$. Note that $\mathbb{C} \models x_i < x''_i$ and $\mathbb{D} \models x_i < x''_i$ and $x''_i \notin \mathrm{infix}(\mathbb{C}, \mathbb{D})$, because $d = x_i = x''_i \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$ implies $x'_i \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$ by (I_2). Clearly, $\mathbb{C} \models (x''_i)' = x'_i = c$, but $\mathbb{C} \models d = x_i$, a contradiction with the fact that $d \in C_4 \cup C_5$ is the largest in $\mathbb{C}$ such that $\mathbb{C} \models c = d'$. This settles the claim.

Subcase 2.2: $\mathbb{C} \models c \neq d'$ for all $d \in C_4 \cup C_5$. Then, there exists $i \in \{1, \ldots, n\}$ such that $\mathbb{C} \models c = x_i < x''_i < l_{\mathbb{C}} \leq x'_i$. We define

$$t^{x''_i}_{\mathbb{C}} \risingdotseq \left(t^{x'_i}_{\mathbb{C}} \vee \bigvee_{\mathbb{C} \sim \mathbb{D}, x'_i \notin \mathrm{infix}(\mathbb{C}, \mathbb{D})} t^1_{\mathbb{D}} \vee \bigvee_{\mathbb{C} \not\sim \mathbb{D}} t^1_{\mathbb{D}} \right)',$$

and by the previous part we note that

$$\mathbb{A} \models t^{x''_i}_{\mathbb{C}} = a^{x''_i}_{\mathbb{C}}.$$

Then we define,

$$t^c_{\mathbb{C}} \risingdotseq x_i \wedge t^{x''_i}_{\mathbb{C}},$$

and we claim,

$$\mathbb{A} \models t^c_{\mathbb{C}} = a^c_{\mathbb{C}}.$$

We prove the claim. We have $\mathbb{C} \models t^c_{\mathbb{C}} = x_i = c$ by construction. Assume for a contradiction that there exists $\mathbb{D} \in \mathcal{K}_n$ such that $\mathbb{D} \models \pi_{\mathbb{D}}(a^c_{\mathbb{C}}) < t^c_{\mathbb{C}}$. If $\mathbb{D} \sim \mathbb{C}$ then $\mathbb{D} \models t^c_{\mathbb{C}} = 0$, then $\mathbb{D} \not\sim \mathbb{C}$. If $x'_i \notin \mathrm{infix}(\mathbb{C}, \mathbb{D})$, then $\mathbb{D} \models t^c_{\mathbb{C}} = 0$, then $x'_i \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$. Then $x_i \in \mathrm{infix}(\mathbb{C}, \mathbb{D})$ by (I_2). Then $\mathbb{D} \models t^c_{\mathbb{C}} = x_i = c$ by Theorem 1 and $\mathbb{D} \models \pi_{\mathbb{D}}(a^c_{\mathbb{C}}) = c$, contradiction. This settles the claim.

4.3 Poset Representation

In this section, we describe an explicit combinatorial construction of the universe of the free n-generated WNM-algebra $\mathbb{F}_n$, which we call *poset representation.* We recall some standard terminology and notation on posets; the combinatorial notation (summations, coefficients, et cetera) is standard.

A *(finite) poset* is a structure $\mathbf{P} = (P, \leq^{\mathbf{P}})$ such that P is a (finite) set and $\leq^{\mathbf{P}}$ is a reflexive, antisymmetric, and transitive binary relation on P. Let $\mathbf{P} = (P, \leq^{\mathbf{P}})$ be a poset. We let $\mathbf{P}^{\partial} \risingdotseq (P, \{(q, p) \mid p \leq^{\mathbf{P}} q\})$ denote the poset *dual* to $\mathbf{P}$. Let $p, q \in P$. We say that p and q are *comparable* (in $\mathbf{P}$) if $p \leq^{\mathbf{P}} q$ or $q \leq^{\mathbf{P}} p$, and *incomparable* otherwise, and we write $p \parallel q$. An *antichain* in $\mathbf{P}$ is a set of elements in P pairwise incomparable in $\mathbf{P}$. An antichain Q in $\mathbf{P}$ is *maximal* if there does not exist an antichain in $\mathbf{P}$ that includes Q properly. We let $\mathrm{maxant}(\mathbf{P})$ denote the set of maximal antichains in the poset $\mathbf{P}$.

Let $\mathbf{1}$ denote the one element poset (unique up to isomorphism). Let $\mathbf{P}$ and $\mathbf{Q}$ be posets with $P \cap Q = \emptyset$. We let $\mathbf{P} + \mathbf{Q}$ denote the *horizontal sum* of $\mathbf{P}$ and $\mathbf{Q}$, that is, the poset $\mathbf{P} + \mathbf{Q} \risingdotseq (P \cup Q, \leq^{\mathbf{P}+\mathbf{Q}})$ where $p \leq^{\mathbf{P}+\mathbf{Q}} q$ if and only if either $p, q \in P$ and $p \leq^{\mathbf{P}} q$, or $p, q \in Q$ and $p \leq^{\mathbf{Q}} q$. We let $\mathbf{P} \oplus \mathbf{Q}$ denote the *vertical sum* of $\mathbf{P}$ and $\mathbf{Q}$, that is, the poset $\mathbf{P} \oplus \mathbf{Q} \risingdotseq (P \cup Q, \leq^{\mathbf{P} \oplus \mathbf{Q}})$ where $p \leq^{\mathbf{P} \oplus \mathbf{Q}} q$ if and only if either $p, q \in P$ and $p \leq^{\mathbf{P}} q$, or $p, q \in Q$ and $p \leq^{\mathbf{Q}} q$, or $p \in P$ and $q \in Q$. In the following, we always assume that posets involved in horizontal and vertical sums are disjoint, by taking isomorphic copies of operands when necessary. Moreover, for all $n \in \mathbb{N}$, we freely write $\mathbf{n}$ instead of $((\cdots(\mathbf{1} \oplus \mathbf{1}) \oplus \cdots \oplus \mathbf{1}) \oplus \mathbf{1})$, where $\mathbf{1}$ occurs n times, and we freely write $n\mathbf{P}$ instead of $((\cdots(\mathbf{P}+\mathbf{P})+\cdots+\mathbf{P})+\mathbf{P})$, where $\mathbf{P}$ occurs n times.

The class of *series-parallel* posets is the smallest class of posets that contains $\mathbf{1}$ and is closed under horizontal and vertical sums.

We define recursively a map poset: $\mathbb{N} \to \mathcal{FSP}$, from the set $\mathbb{N}$ of natural numbers to the class $\mathcal{FSP}$ of finite series-parallel posets. For all $n \in \mathbb{N}$, there exists a bijection $r\colon A \to \mathrm{maxant}(\mathrm{poset}(n))$, between the set A in (15), that is, by Theorem 4, the universe of the free n-generated WNM-algebra $\mathbb{F}_n$, and the set of maximal antichains in $\mathrm{poset}(n)$.

Definition 6 **(**poset: $\mathbb{N} \to \mathcal{FSP}$**)** Let $n \in \mathbb{N}$. Then,

$$\mathrm{poset}(n) = \sum_{\substack{a,b,c \in \{0\} \cup \mathbb{N},\\ a+b+c=n}} \binom{n}{a,b,c} \mathbf{X}_{a,b,c},$$

$$\mathbf{X}_{a,b,c} = \begin{cases} \mathbf{Y}^{\partial}_{a,c} \oplus \mathbf{Z}_{a,c} & \text{if } b = 0,\\ \sum\limits_{i=1}^{b} i! \left\{ {b \atop i} \right\} \left(\left(\mathbf{Y}^{\partial}_{a,c} \oplus \mathbf{i} \oplus \mathbf{Z}_{a,c} \right) + \right. & \\ \left. + \left(\mathbf{Y}^{\partial}_{a,c} \oplus \mathbf{i} \oplus \mathbf{1} \oplus \mathbf{Z}_{a,c} \right) \right) & \text{otherwise;} \end{cases}$$

$$\mathbf{Y}_{a,c} = \begin{cases} \mathbf{1} & \text{if } a = c = 0,\\ \sum\limits_{i=0}^{a-1} \sum\limits_{j=0}^{c-1} \binom{a}{i} \binom{c}{j} \left(\mathbf{S}_{i,j} + \mathbf{S}^{-}_{i,j} \right) + & \\ \sum\limits_{j=0}^{c-1} \binom{c}{j} \left(\mathbf{S}_{a,j} + \mathbf{S}^{+}_{a,j} \right) + & \\ \sum\limits_{i=0}^{a-1} \binom{a}{i} \left(\mathbf{S}_{i,c} + \mathbf{S}^{-}_{i,c} \right) & \text{otherwise;} \end{cases}$$

$$\mathbf{Z}_{a,c} = \begin{cases} \mathbf{1} & \text{if } a = c = 0,\\ \sum\limits_{i=0}^{a-1} \sum\limits_{j=0}^{c-1} \binom{a}{i} \binom{c}{j} \left(\mathbf{T}_{i,j} + \mathbf{T}^{-}_{i,j} \right) + & \\ \sum\limits_{j=0}^{c-1} \binom{c}{j} \left(\mathbf{T}_{a,j} + \mathbf{T}^{+}_{a,j} \right) + & \\ \sum\limits_{i=0}^{a-1} \binom{a}{i} \left(\mathbf{T}_{i,c} + \mathbf{T}^{-}_{i,c} \right) & \text{otherwise;} \end{cases}$$

$$\mathbf{S}_{i,j} = \mathbf{1} \oplus \mathbf{Y}_{i,j},$$

$$\mathbf{S}^{+}_{i,j} = \begin{cases} \mathbf{2} & \text{if } i = j = 0,\\ \mathbf{S}_{i,j} + \mathbf{Y}_{i,j} & \text{otherwise;} \end{cases}$$

$$\mathbf{S}^{-}_{i,j} = (\mathbf{1} \oplus \mathbf{S}_{i,j}) + \sum_{k=0}^{i-1} \binom{i}{k} \left((\mathbf{1} \oplus \mathbf{S}_{k,j}) + (\mathbf{1} \oplus \mathbf{S}^{-}_{k,j}) \right),$$

$$\mathbf{T}_{i,j} = \mathbf{1} \oplus \mathbf{Z}_{i,j},$$

$$\mathbf{T}^{+}_{i,j} = \begin{cases} \mathbf{3} & \text{if } i = j = 0,\\ \mathbf{1} \oplus (\mathbf{T}_{i,j} + \mathbf{Z}_{i,j}) & \text{otherwise;} \end{cases}$$

$$\mathbf{T}^{-}_{i,j} = \mathbf{T}_{i,j} + \sum_{k=0}^{i-1} \binom{i}{k} \left(\mathbf{T}_{k,j} + \mathbf{T}^{-}_{k,j} \right).$$

Theorem 5 *There is a bijective correspondence between the universe of* $\mathbb{F}_n$ *and the set*

$$\mathrm{maxant}(\mathrm{poset}(n)).$$

Actually, $\mathrm{poset}(n)$ is built together with a correspondence mapping each chain in $\mathcal{K}_n$ to a maximal chain of $\mathrm{poset}(n)$ (see Fig. 10, for the case $n = 2$). This makes then easy to equip $\mathrm{maxant}(\mathrm{poset}(n))$ with a structure of WNM-algebra, mimicking the operations of each chain of $\mathcal{K}_n$ over its image in $\mathrm{poset}(n)$. The resulting WNM-algebra constitutes then an isomorphic representation of $\mathbb{F}_n$. Furthermore, a recurrence computing the cardinality of $\mathbb{F}_n$ can be easily deduced from Definition 6.

In this paper we do not prove Theorem 5. The proof, which will be a lengthy and complex but purely combinatorial argument, will be the focus of a future work. However, we invite the reader to check that poset(1) and poset(2) actually correspond to those posets given in the corresponding examples (see Fig. 7, Fig. 9 and Fig. 10). We finally remark that the poset in Fig. 10 has been first (and thus, independently) generated by a brute-force algorithm. Subsequently, the correctness of the output of the computation of $\mathrm{poset}(n)$ has been checked with a brute-force algorithm for all $n \leq 5$.

Example 13 ($n = 1$) The Hasse diagram of poset(1) is displayed in Figure 9.

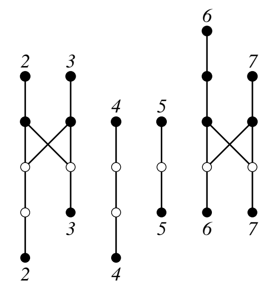

Fig. 9: poset(1).

Example 14 ($n = 2$) The Hasse diagram in Appendix A is poset(2).

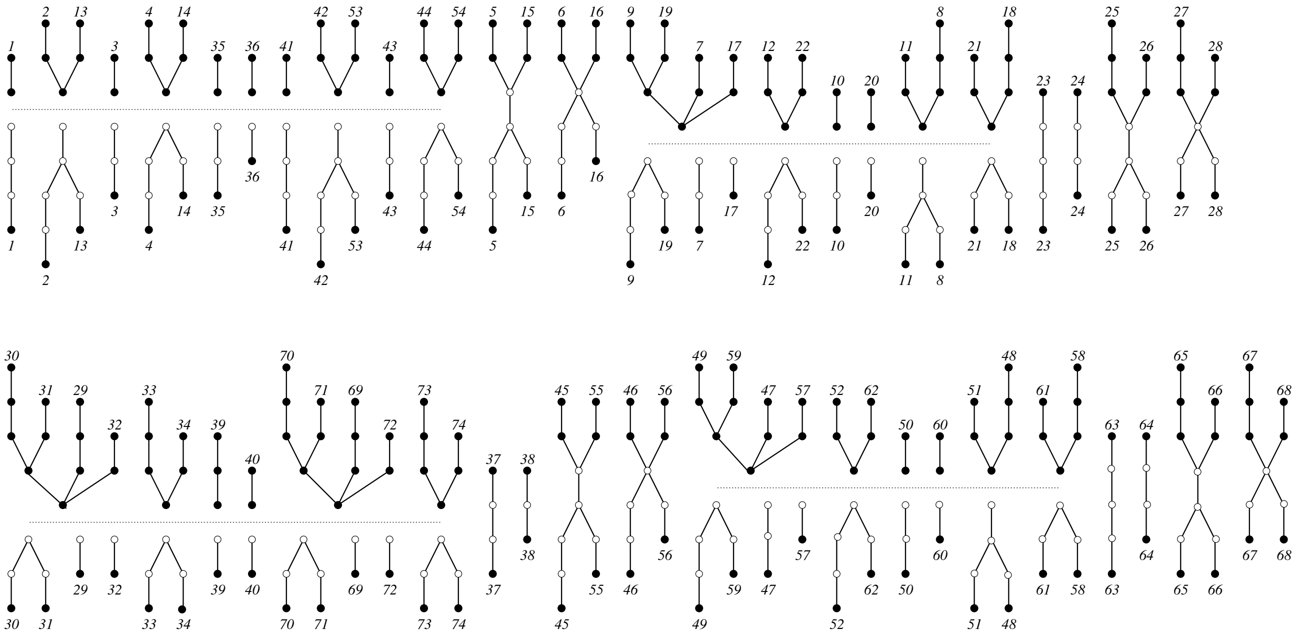

Fig. 10: poset(2). $|\mathbb{F}_2| = |\mathrm{maxant}(\mathrm{poset}(2))| = 12,632,396,375,864,721,690,604,339,200,000,000 \approx 1.26324 \cdot 10^{34}$. In the actual Hasse diagram, each point immediately below the dotted line edges each point immediately above the dotted line.

A The 2-generated free WNM-algebra

In the following list, an item of the form,

$$3 > 2 : \mathbb{C}_{43} = 0 < y < y''xx'' < y'x' < 1$$

represents a WNM-chain generated by x and y, and specifies that $\mathbb{C}_{43} \models x > y$, $\mathrm{orbit}(\mathbb{C}_{43}, x) = 3$, and $\mathrm{orbit}(\mathbb{C}_{43}, y) = 2$. In this format, the set,

$$\mathcal{K}_2 = \{\mathbb{C}_i \mid i = 1, 2, \ldots, 74\},$$

is as follows:

$2 < 2$: $\mathbb{C}_1 = 0 < x < y < y''x'' < y'x' < 1$;
$2 < 2$: $\mathbb{C}_2 = 0 < x < x'' < y < y'' < y' < x' < 1$;
$2 < 3$: $\mathbb{C}_3 = 0 < x < x''yy'' < x'y' < 1$;
$2 < 3$: $\mathbb{C}_4 = 0 < x < x'' < yy'' < y' < x' < 1$;
$2 < 4$: $\mathbb{C}_5 = 0 < x < x'' < y < y'y'' < x' < 1$;
$2 < 5$: $\mathbb{C}_6 = 0 < x < x'' < yy'y'' < x' < 1$;
$2 < 6$: $\mathbb{C}_7 = 0 < x < x''y' < y < x'y'' < 1$;
$2 < 6$: $\mathbb{C}_8 = 0 < y' < x < x'' < x' < y < y'' < 1$;
$2 < 6$: $\mathbb{C}_9 = 0 < x < x'' < y' < y < y'' < x' < 1$;
$2 < 7$: $\mathbb{C}_{10} = 0 < x < x''y' < x'yy'' < 1$;
$2 < 7$: $\mathbb{C}_{11} = 0 < y' < x < x'' < x' < yy'' < 1$;
$2 < 7$: $\mathbb{C}_{12} = 0 < x < x'' < y' < yy'' < x' < 1$;
$3 < 2$: $\mathbb{C}_{13} = 0 < xx'' < y < y'' < y' < x' < 1$;
$3 < 3$: $\mathbb{C}_{14} = 0 < xx'' < yy'' < y' < x' < 1$;
$3 < 4$: $\mathbb{C}_{15} = 0 < xx'' < y < y'y'' < x' < 1$;
$3 < 5$: $\mathbb{C}_{16} = 0 < xx'' < yy'y'' < x' < 1$;
$3 < 6$: $\mathbb{C}_{17} = 0 < xx''y' < y < x'y'' < 1$;
$3 < 6$: $\mathbb{C}_{18} = 0 < y' < xx'' < x' < y < y'' < 1$;
$3 < 6$: $\mathbb{C}_{19} = 0 < xx'' < y' < y < y'' < x' < 1$;
$3 < 7$: $\mathbb{C}_{20} = 0 < xx''y' < x'yy'' < 1$;
$3 < 7$: $\mathbb{C}_{21} = 0 < y' < xx'' < x' < yy'' < 1$;
$3 < 7$: $\mathbb{C}_{22} = 0 < xx'' < y' < yy'' < x' < 1$;
$4 < 4$: $\mathbb{C}_{23} = 0 < x < y < x'x''y'y'' < 1$;
$4 < 5$: $\mathbb{C}_{24} = 0 < x < x'x''yy'y'' < 1$;
$4 < 6$: $\mathbb{C}_{25} = 0 < y' < x < x'x'' < y < y'' < 1$;
$4 < 7$: $\mathbb{C}_{26} = 0 < y' < x < x'x'' < yy'' < 1$;
$5 < 6$: $\mathbb{C}_{27} = 0 < y' < xx'x'' < y < y'' < 1$;
$5 < 7$: $\mathbb{C}_{28} = 0 < y' < xx'x'' < yy'' < 1$;
$6 < 6$: $\mathbb{C}_{29} = 0 < x'y' < x < y < x''y'' < 1$;
$6 < 6$: $\mathbb{C}_{30} = 0 < y' < x' < x < x'' < y < y'' < 1$;
$6 < 7$: $\mathbb{C}_{31} = 0 < y' < x' < x < x'' < yy'' < 1$;
$6 < 7$: $\mathbb{C}_{32} = 0 < x'y' < x < x''yy'' < 1$;
$7 < 6$: $\mathbb{C}_{33} = 0 < y' < x' < xx'' < y < y'' < 1$;
$7 < 7$: $\mathbb{C}_{34} = 0 < y' < x' < xx'' < yy'' < 1$;
$2 = 2$: $\mathbb{C}_{35} = 0 < xy < x''y'' < x'y' < 1$;
$3 = 3$: $\mathbb{C}_{36} = 0 < xx''yy'' < x'y' < 1$;
$4 = 4$: $\mathbb{C}_{37} = 0 < xy < x'x''y'y'' < 1$;
$5 = 5$: $\mathbb{C}_{38} = 0 < xx'x''yy'y'' < 1$;
$6 = 6$: $\mathbb{C}_{39} = 0 < x'y' < xy < x''y'' < 1$;
$7 = 7$: $\mathbb{C}_{40} = 0 < x'y' < xx''yy'' < 1$;
$2 > 2$: $\mathbb{C}_{41} = 0 < y < x < x''y'' < x'y' < 1$;
$2 > 2$: $\mathbb{C}_{42} = 0 < y < y'' < x < x'' < x' < y' < 1$;
$3 > 2$: $\mathbb{C}_{43} = 0 < y < y''xx'' < y'x' < 1$;
$3 > 2$: $\mathbb{C}_{44} = 0 < y < y'' < xx'' < x' < y' < 1$;
$4 > 2$: $\mathbb{C}_{45} = 0 < y < y'' < x < x'x'' < y' < 1$;
$5 > 2$: $\mathbb{C}_{46} = 0 < y < y'' < xx'x'' < y' < 1$;
$6 > 2$: $\mathbb{C}_{47} = 0 < y < y''x' < x < y'x'' < 1$;
$6 > 2$: $\mathbb{C}_{48} = 0 < x' < y < y'' < y' < x < x'' < 1$;
$6 > 2$: $\mathbb{C}_{49} = 0 < y < y'' < x' < x < x'' < y' < 1$;
$7 > 2$: $\mathbb{C}_{50} = 0 < y < y''x' < y'xx'' < 1$;
$7 > 2$: $\mathbb{C}_{51} = 0 < x' < y < y'' < y' < xx'' < 1$;
$7 > 2$: $\mathbb{C}_{52} = 0 < y < y'' < x' < xx'' < y' < 1$;
$2 > 3$: $\mathbb{C}_{53} = 0 < yy'' < x < x'' < x' < y' < 1$;
$3 > 3$: $\mathbb{C}_{54} = 0 < yy'' < xx'' < x' < y' < 1$;
$4 > 3$: $\mathbb{C}_{55} = 0 < yy'' < x < x'x'' < y' < 1$;
$5 > 3$: $\mathbb{C}_{56} = 0 < yy'' < xx'x'' < y' < 1$;
$6 > 3$: $\mathbb{C}_{57} = 0 < yy''x' < x < y'x'' < 1$;
$6 > 3$: $\mathbb{C}_{58} = 0 < x' < yy'' < y' < x < x'' < 1$;
$6 > 3$: $\mathbb{C}_{59} = 0 < yy'' < x' < x < x'' < y' < 1$;
$7 > 3$: $\mathbb{C}_{60} = 0 < yy''x' < y'xx'' < 1$;
$7 > 3$: $\mathbb{C}_{61} = 0 < x' < yy'' < y' < xx'' < 1$;
$7 > 3$: $\mathbb{C}_{62} = 0 < yy'' < x' < xx'' < y' < 1$;
$4 > 4$: $\mathbb{C}_{63} = 0 < y < x < y'y''x'x'' < 1$;

$5 > 4$: $\mathbb{C}_{64} = 0 < y < y'y''xx'x'' < 1$;
$6 > 4$: $\mathbb{C}_{65} = 0 < x' < y < y'y'' < x < x'' < 1$;
$7 > 4$: $\mathbb{C}_{66} = 0 < x' < y < y'y'' < xx'' < 1$;
$6 > 5$: $\mathbb{C}_{67} = 0 < x' < yy'y'' < x < x'' < 1$;
$7 > 5$: $\mathbb{C}_{68} = 0 < x' < yy'y'' < xx'' < 1$;
$6 > 6$: $\mathbb{C}_{69} = 0 < y'x' < y < x < y''x'' < 1$;
$6 > 6$: $\mathbb{C}_{70} = 0 < x' < y' < y < y'' < x < x'' < 1$;
$7 > 6$: $\mathbb{C}_{71} = 0 < x' < y' < y < y'' < xx'' < 1$;
$7 > 6$: $\mathbb{C}_{72} = 0 < y'x' < y < y''xx'' < 1$;
$6 > 7$: $\mathbb{C}_{73} = 0 < x' < y' < yy'' < x < x'' < 1$;
$7 > 7$: $\mathbb{C}_{74} = 0 < x' < y' < yy'' < xx'' < 1$.

Acknowledgements The second author is supported by the FWF Austrian Science Fund (Parameterized Compilation, P26200). The third author is supported by a Marie Curie INdAM-COFUND Outgoing Fellowship.

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